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Admissible Shapes of 4-Body Non-collinear Relative Equilibria

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Abstract

In this paper, we study possible shapes of 4-body non-collinear relative equilibria for any positive masses, and give estimates on their geometric quantities.

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1 Introduction

Let $\mathbf{R}$ and $\mathbf{R}^+$ denote the sets of real numbers and positive real numbers respectively. As usual (cf. [15], [13], [1]), we have

Definition 1.1 Given $m = (m_1, \ldots, m_n) \in (\mathbf{R}^+)^n$, a configuration $q = (q_1, \ldots, q_n) \in (\mathbf{R}^2)^n$ is a *relative equilibrium* (RE for short) for m if q satisfies $\sum_{i=1}^n m_i q_i = 0$, $q_i \neq q_j$ whenever $i \neq j$, and it is a solution of the system

$$\lambda q_i + \sum_{\substack{j \neq i \\ 1 \le j \le n}} \frac{m_j(q_j - q_i)}{r_{ij}^3} = 0, \qquad \text{for } 1 \le i \le n, \tag{1.1}$$

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for some constant $\lambda \in \mathbf{R}$, where $r_{ij} = |q_i - q_j|$. An RE $q = (q_1, \ldots, q_n)$ is *collinear*, if all the q_is are located on a line in $\mathbf{R}^2$. A non-collinear 4-body RE $q = (q_1, q_2, q_3, q_4)$ is *convex* if the four points q_1, q_2, q_3, and q_4 form a convex configuration in $\mathbf{R}^2$, and q is *concave* if one of the points q_1, q_2, q_3, and q_4 is located in the interior of the triangle formed by the other three points. Two REs q and p of m are *equivalent* if q and p differ by an SO(2) rotation followed by a scalar multiplication.

For general studies on REs we refer to [15], [13], [1] and [10]. Collinear REs have been deeply understood (cf. [15], [13], [10]). This paper is devoted to study the geometric properties and admissible shapes of non-collinear 4-body REs. More emphases are specially put on concave ones. Earlier studies on this problem can be found for example in [4], [8], [11], [9], [2], [3], [7], etc. Specially detailed results on convex 4-body REs can be found in W. D. MacMillan and W. Bartky [8]. But only a few results on concave REs are known so far. In this paper we carry out further studies on non-collinear 4-body REs and obtain the following results:

1° For all $m \in (\mathbf{R}^+)^4$ and any 4-body concave RE $q = (q_1, q_2, q_3, q_4)$ of m with q_2 located in the interior of the triangle $\Delta q_1 q_3 q_4$, fixing q_3 and q_4, in Theorem 3.1 below we construct admissible regions R_1 and R_2 in $\mathbf{R}^2$ such that $q_1 \in R_1$ and $q_2 \in R_2$ must hold. We also prove that every interior angle of $\Delta q_1 q_3 q_4$ is in $(30^o, 90^o)$, and each interior side divides an interior angle into two sub-interior angles which are in $(0^o, 75^o)$.

2° Fixing two opposite points q_2 and q_4 in any 4-body convex RE $\diamond q_1 q_2 q_3 q_4$ for m, in Theorem 4.1 below we construct admissible regions G_1 and G_3 as well as a semi-admissible region T in $\mathbf{R}^2$ such that $q_1 \in G_1$ and $q_3 \in G_3$ must hold, and that q_1 and q_3 can not belong to T simultaneously. We also prove that every interior angle of $\diamond q_1 q_2 q_3 q_4$ is in $(60^o, 150^o)$ and the diagonal divides each interior angle into two sub-interior angles both of which are in $(0^o, 90^o)$.

Remark 1.1 1° It is claimed (see [8], pp.857-858) that for any 4-body convex RE $q = (q_1, q_2, q_3, q_4)$ with mass $m \in (\mathbf{R}^+)^4$, any interior angle of the convex quadrilateral q belongs to $[60^o, 120^o]$, and that diagonals divides each interior angle into two sub-interior angles each of which is less than 60^o. Note that these two upper bounds 120^o and 60^o are incorrect by our Theorem 5.1 below, which also show that the estimate $(60^o, 150^o)$ in Theorem 4.1 for interior angles is sharp.

2° Note that our estimates on geometric quantities in these REs and the regions in $\mathbf{R}^2$ defined in Theorems 3.1 and 4.1 work for all $m \in (\mathbf{R}^+)^4$ and for all REs with mass m. This is rather different from results obtained in [8] and other papers. These regions restrict possible shapes for non-collinear 4-body REs.

2 Preliminary results

In $\mathbf{R}^2$, we denote by $\overline{AB}$ the segment connecting two points A and B, and by $\angle ABC$ the angle without orientation with vertex B and the points A and C located on the two sides respectively. We denote by ΔABC the triangle formed by three points A, B, and C in clockwise order, and by $\diamond ABCD$ the convex quadrilateral formed by four points A, B, C, and D in clockwise order. For reader's convenience, here we list some well known results on non-collinear REs which will be used in this paper.

Lemma 2.1 (Conley's perpendicular bisector theorem, cf. p.510 of [9]) *Let* $q = (q_1, \ldots, q_n)$ *be an* n*-body RE for* $m \in (\mathbf{R}^+)^n$. *Denote by* Q_1, Q_2, Q_3, *and* Q_4 *the four open quadrants in* $\mathbf{R}^2$ *in clockwise order obtained by deleting the line passing through some two points* q_i *and* q_j *of* q *and the perpendicular bisector of* $\overline{q_i q_j}$. *Then if there exists some* $q_k \in Q_1 \cup Q_3$, *there must exist at least another* $q_m \in Q_2 \cup Q_4$.

Lemma 2.2 (cf. [8]) *For any* 4*-body concave RE of* $m \in (\mathbf{R}^+)^4$, *all exterior sides are longer than the interior ones, and there exists* $r_0 > 0$ *depending on* m *and* q *such that* r_0 *is less than or equal to the lengths of all the exterior sides, and greater than or equal to the lengths of all the interior sides. Furthermore, the longest (or shortest) exterior side lies opposite to the longest (shortest, respectively) interior side.*

Lemma 2.3 (cf. [8]) 1° *For any* 4*-body convex RE* $q = (q_1, q_2, q_3, q_4)$ *with mass* $m \in (\mathbf{R}^+)^4$, *all exterior sides are shorter than the diagonals. Furthermore, there exists an* $r_0 > 0$ *depending on* m *and* q *such that the lengths of all the exterior sides are less than or equal to* r_0, *and the lengths of all the diagonals are greater than or equal to* r_0. *Among the exterior sides the shortest and the longest sides have to face each other.*

2° *Any interior angle of the convex quadrilateral* q *is not less than* 60^o.

Lemma 2.4 (Lemma 2.4 of [7]) *No three points among* q_1, q_2, q_3, *and* q_4 *can be collinear for any non-collinear* 4*-body RE* $q = (q_1, q_2, q_3, q_4)$ *of* $m \in (\mathbf{R}^+)^4$.

Proof. We give a new proof for this lemma which is different from that given in [7].

Assume that q_1, q_2, and q_3 are on a straight line with q_2 located between q_1 and q_3. Then because q is non-collinear, by Lemma 2.1, q_4 must locate on the perpendicular bisector lines of the segments $\overline{q_1 q_2}$ and $\overline{q_2 q_3}$, and not on these two segments. But these two lines are parallel to each other and have no common intersection points. This contradiction proves the lemma. ∎

3 4-body concave REs

Given an $m = (m_1, m_2, m_3, m_4) \in (\mathbf{R}^+)^4$, we study 4-body concave RE $q = (q_1, q_2, q_3, q_4)$ for m in this section. In this case, by Lemma 2.4, we suppose that q_2 is located in the interior of $\Delta q_1 q_3 q_4$ as in Figure 3.1. We start from an auxiliary lemma.

Lemma 3.1 *Let* $q = (q_1, q_2, q_3, q_4)$ *be a concave RE for* m *with* q_2 *located inside the triangle* $\Delta q_1 q_3 q_4$. *Then* $\Delta q_1 q_3 q_4$ *is an acute triangle, i.e., every interior angle* $\angle q_1 q_3 q_4$, $\angle q_3 q_4 q_1$, *or* $\angle q_4 q_1 q_3$ *is strictly less than* 90^o.

Proof. Assuming one of these three angles, say $\alpha \equiv \angle q_3 q_4 q_1 \geq 90^o$, we prove the lemma by contradiction.

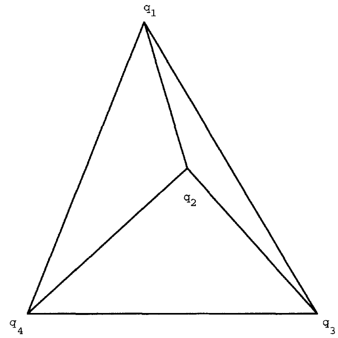

Figure 3.1: A 4-body concave RE $q = (q_1, q_2, q_3, q_4)$

Let A be the intersection point of the perpendicular bisectors L_{34} and L_{14} of the segments $\overline{q_3q_4}$ and $\overline{q_1q_4}$ respectively. Then A locates on the side of $\overline{q_1q_3}$ opposite to q_4 and outside $\Delta q_1q_3q_4$ when $\alpha > 90^o$, or on the segment $\overline{q_1q_3}$ when $\alpha = 90^o$. In both cases, by Lemma 2.1, the point q_2 should locates both on the q_3 side of L_{34} and on the q_1 side of L_{14}. But they have no common intersection point with the interior of $\Delta q_1q_3q_4$. This contradiction proves the lemma. ∎

Now, given a concave RE $q = (q_1, q_2, q_3, q_4)$ for m with q_2 located in the interior of $\Delta q_1q_3q_4$, we fix q_3 and q_4 as in Figure 3.2 and construct the admissible regions R_1 and R_2 in $\mathbf{R}^2$ so that $q_1 \in R_1$ and $q_2 \in R_2$ must hold.

Let $r = r_{34}$. The Figure 3.2 is obtained as follows.

⟨1⟩ Fix q_4 and q_3 from left to right on a horizontal line in $\mathbf{R}^2$. Let Q be the middle point of $\overline{q_3q_4}$.

⟨2⟩ We draw upper half circles C_3, C_Q, and C_4 with the same radius $r/2$ and centered at q_3, Q, and q_4 respectively. Denote by B and B' the intersection points of the upper half circles of C_4 and C_3 with C_Q respectively.

⟨3⟩ We draw upper half circles D_3 and D_4 with the same radius r and centered at q_3 and q_4 respectively. Denote by K the intersection point of upper half circles of D_3 and D_4.

⟨4⟩ We draw a circle C_K centered at K with radius r which passes through q_3 and q_4.

⟨5⟩ We draw two lines L_3 and L_4 perpendicular to $\overline{q_3q_4}$ passing through q_3 and q_4 respectively. Denote by G and G' the intersection points of L_4 and L_3 with the upper half circle of C_K respectively.

⟨6⟩ We draw a line L_Q perpendicular to $\overline{q_3q_4}$ and passing through Q. This line L_Q passes through K. Denote by H and A the intersection points of L_Q with the upper half circles of C_Q and C_K respectively.

⟨7⟩ We draw a line M_3 passing through the points q_3 and B. The line M_3 intersects L_4 at a point E and the upper half circle of D_3 at a point F.

⟨8⟩ We draw a line M_4 passing through the points q_4 and B'. The line M_4

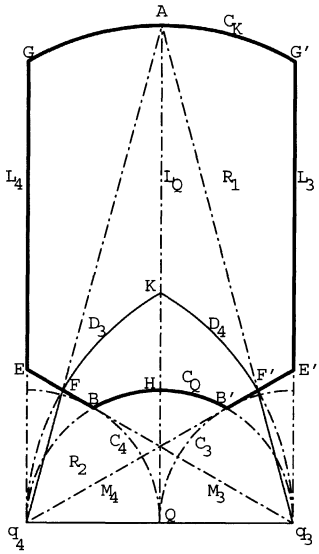

Figure 3.2: Admissible regions R_1 and R_2 for 4-body concave REs

intersects L_3 at a point E' and the upper half circle of D_4 at a point F'.

⟨9⟩ We connect q_3 to F' by the segment $\overline{q_3F'}$, and q_4 to F by the segment $\overline{q_4F}$.

⟨10⟩ We denote by R_1 the open region bounded by the upper arc of C_K from G to G', L_3 from G' to E', M_4 from E' to B', the upper arc of C_Q from B' to B, M_3 from B to E, and then L_4 from E to G. The region R_1 is surrounded by thick curves in Figure 3.2.

⟨11⟩ We denote by T_1 the closed region bounded by the upper arc of D_4 from K to q_3, $\overline{q_3q_4}$ from q_3 to q_4, and the upper arc of D_3 from q_4 to K. Then let $T_2 = T_1 \setminus \overline{q_3q_4}$.

⟨12⟩ We denote by R_2' the open region bounded by the upper arc of D_4 from K to F', the segment $\overline{F'q_3}$, the segment $\overline{q_3q_4}$, the segment $\overline{q_4F}$, and then the upper arc of D_3 from F to K. Let R_2 be the union of R_2' with the arc from F to K in the upper half circle of D_3 without the point F, and with the arc from F' to K in the upper half circle of D_4 without the point F'. The region R_2 is surrounded by thin curves in Figure 3.2.

Note that R_2 is a proper subset of T_2. Note also that the regions R_1, T_2, and R_2 are independent of the choice of $m \in (\mathbf{R}^+)^4$.

We need the following

Lemma 3.2 *In Figure 3.2 the points q_4, F, and A are collinear, and so are the points q_3, F', and A.*

Proof. In the triangle Δq_3q_4B, because $|q_4 - B| = r_{34}/2$ and B locates on C_Q, we have $\angle q_4q_3F = \angle q_4q_3B = 30^o$. Because q_4 and F are on D_3, we then have

$\angle q_3q_4F = \angle q_3Fq_4 = 75^o$.

On the other hand, because ΔKq_3q_4 is equilateral, $\angle q_3Kq_4 = 60^o$. Thus $\angle q_3Aq_4 = 30^o$ and

$$\angle Aq_4q_3 = \angle Aq_3q_4 = 75^o. \tag{3.1}$$

Therefore q_4, F, and A are collinear, and similarly so are q_3, F', and A. Thus the lemma is proved. ∎

The first main result of this paper is the following

Theorem 3.1 *For any $m \in (\mathbf{R}^+)^4$, let $q = (q_1, q_2, q_3, q_4)$ be a concave RE for m with q_2 located inside the triangle $\Delta q_1q_3q_4$. Fixing q_3 and q_4 in Figure 3.2, the following results always hold.*

1° *There must hold*

$$q_1 \in R_1. \tag{3.2}$$

2° *Each of the three interior angles of $\Delta q_1q_3q_4$ must be strictly greater than 30^o and strictly less than 90^o, i.e., there holds*

$$\begin{aligned} 30^o &< \min\{\angle q_1q_3q_4, \angle q_3q_4q_1, \angle q_4q_1q_3\} \\ &\le \max\{\angle q_1q_3q_4, \angle q_3q_4q_1, \angle q_4q_1q_3\} < 90^o. \end{aligned} \tag{3.3}$$

3° *There hold*

$$\frac{\sqrt{3}}{2} r_{34} < |q_1 - \overline{q_3q_4}| < (1 + \frac{\sqrt{3}}{2}) r_{34}, \tag{3.4}$$

$$\begin{aligned} \frac{\sqrt{3}}{4} \max\{r_{34}^2, r_{13}^2, r_{14}^2\} &< \text{area}(\Delta q_1q_3q_4) \\ &< \frac{2+\sqrt{3}}{4} \min\{r_{34}^2, r_{13}^2, r_{14}^2\}, \end{aligned} \tag{3.5}$$

where $|q_1 - \overline{q_3q_4}|$ denotes the distance between q_1 and the segment $\overline{q_3q_4}$.

4° *There must hold*

$$q_2 \in R_2. \tag{3.6}$$

5° *Each of the two base interior angles of $\Delta q_2q_3q_4$ must be strictly greater than 0^o and strictly less than 75^o, i.e., there holds*

$$0^o < \min\{\angle q_2q_3q_4, \angle q_2q_4q_3\} \le \max\{\angle q_2q_3q_4, \angle q_2q_4q_3\} < 75^o. \tag{3.7}$$

6° *There hold*

$$0 < |q_2 - \overline{q_3q_4}| \le \frac{\sqrt{3}}{2} r_{34}, \tag{3.8}$$

$$0 < \text{area}(\Delta q_2q_3q_4) \le \frac{\sqrt{3}}{4} r_{34}^2. \tag{3.9}$$

Proof. The proof is carried out in four steps.

Step 1. We claim that

$$q_2 \in T_2. \tag{3.10}$$

Note that this is a weaker claim than 4°. In fact, if q_2 locates out side of T_2, then at least one of r_{23} and r_{24} is greater than $r \equiv r_{34}$. This violates Lemma 2.2.

Step 2. Proof of 1° *to* 3°.

⟨1⟩ By Lemma 3.1, the right inequality in (3.3) holds. Thus q_1 must locate in the open strip between the lines L_3 and L_4.

⟨2⟩ We claim that q_1 can not locate on or below the upper half circle of C_4. In fact, otherwise, by Lemma 2.1, q_2 must locate on the right hand side of the line L_Q. Thus we have $r_{14} \leq r/2 < r_{24}$. This contradicts to Lemma 2.2 and proves the claim.

By the same argument, q_1 can not locate on or below the upper half circle of C_3.

⟨3⟩ Note that q_1 can not locate on or below the upper half circle of C_Q, because otherwise, $\Delta q_1q_3q_4$ would not be an acute triangle and contradicts to Lemma 3.1.

⟨4⟩ We claim that q_1 can not locate on or above the upper half circle of C_K and inside the open region between lines L_3 and L_4.

In fact, if q_1 is strictly above the upper half circle of C_K, then by (3.10) we obtain $r_{12} > |A - K| = r = r_{34}$. This contradicts to Lemma 2.2.

If q_1 is on the upper arc of C_K between G and A, and $q_1 \neq G$ and A, then by Lemma 2.1 and (3.10), q_2 must locate on the right hand side of L_Q and inside T_2. Thus we have $r_{12} > |A - K| = r = r_{34}$. This contradicts to Lemma 2.2.

Similarly, q_1 can not locate on the upper arc of C_K between A and G', and $q_1 \neq A$ and G'.

If $q_1 = A$, then q_2 must locate on $L_Q \cap T_2$ by Lemma 2.1 and (3.10). Because $|A - K| = r = |K - q_4|$, the perpendicular bisector L_{14} of the segment $\overline{q_1q_4}$ passing through the middle point of $\overline{q_1q_4}$ and K. Thus $L_Q \cap T_2$ contains no points on the left hand side of L_{14}. Because q_3 is on the right hand side of L_{14}, this violates Lemma 2.1.

⟨5⟩ Note that if q_1 is on the upper arc of C_K between G and G', then $\angle q_3q_1q_4 = 30^o$. Thus by the above proofs in ⟨1⟩ to ⟨4⟩ for q_1 located in between the two lines L_3 and L_4, and below the upper arc of C_K between G and G', there must hold $\angle q_3q_1q_4 > 30^o$. Permuting q_1, q_3, and q_4, we obtain

$$\min\{\angle q_1q_3q_4, \angle q_1q_4q_3\} > 30^o. \tag{3.11}$$

Thus the left inequality in (3.3) holds.

⟨6⟩ By (3.11) we obtain that q_1 can not locate on or below the lines M_3 and M_4. The above ⟨1⟩ to ⟨5⟩ yield 1° and 2°.

Now, 3° follows from 1° and an elementary computation.

Step 3. Proof of 4°.

If q_1 is on the left hand side of L_Q, by 1° we have $q_1 \in R_1$. This implies $\angle q_1q_3q_4 < \angle Aq_3q_4$. Therefore q_2 can not locate on $\overline{q_3A}$ or on the right hand side of $\overline{q_3A}$.

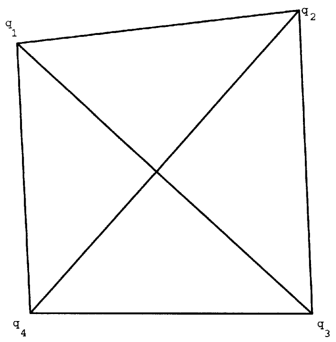

Figure 4.1: A convex RE q

If q_1 locates on L_Q or on the right hand side of L_Q, then q_2 locates on L_Q or on the left hand side of L_Q by Lemma 2.1, and thus can not locate on $\overline{q_3A}$ or on the right hand side of $\overline{q_3A}$.

Similarly, we obtain that q_2 can not locate on $\overline{q_4A}$ or on the left hand side of $\overline{q_4A}$. This proves 4° by Lemma 3.2 and (3.10).

Step 4. Proof of 5° *and* 6°.

The left hand side of (3.7) follows from Lemma 2.4. The right hand side of (3.7) follows from (3.1), 4°, and Lemma 3.2. Thus 5° holds. Now, 6° follows from 4° and an elementary computation. The proof of Theorem 3.1 is complete. ∎

Remark 3.1 Note that our Figure 3.2 is different from the Figure 6 on p.858 of [8] in the sense that Figure 3.2 and our results in Theorem 3.1 work for any choice of $m \in (\mathbf{R}^+)^4$ and any concave RE for m, but the Figure 6 of [8] only works for one concave RE of a given m because the number r_0 on p.858 of [8] depends on both m and q. Note also that the properties of the vertical lines L_3 and L_4, and the half circle C_Q used in the construction of the region R_1, the properties of the segments $\overline{q_4F}$ and $\overline{q_3F'}$ in the construction of the region R_2,and the estimates in (3.3)-(3.5), and (3.7)-(3.9) are all new.

4 4-body convex REs

Given $m = (m_1, m_2, m_3, m_4) \in (\mathbf{R}^+)^4$, we study 4-body convex RE $q = (q_1,q_2,q_3,q_4)$ for m in this section. The convex quadrilateral $\diamond q_1q_2q_3q_4$ is shown in the Figure 4.1.

Now, given a convex RE $q = (q_1, q_2, q_3, q_4)$ for m, fixing the opposite vertices q_2 and q_4 on a straight line in $\mathbf{R}^2$, we construct the admissible regions G_1 and G_3, and the semi-admissible region T in $\mathbf{R}^2$ as in Figure 4.2 such that $q_1 \in G_1$ and $q_3 \in G_3$ must hold, and q_1 and q_3 can not locate in T simultaneously.

Let $r = r_{24}$. The Figure 4.2 is obtained as follows.

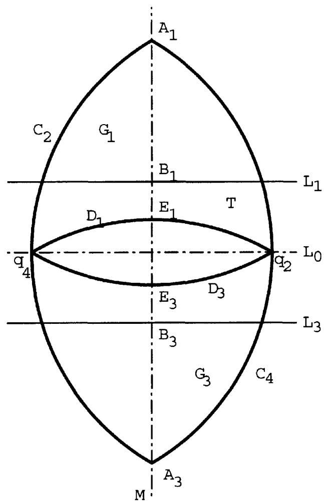

Figure 4.2: Regions G_1, G_3, and T for 4-body convex REs

⟨1⟩ Fix q_2 and q_4 from right to left on a horizontal line L_0 in $\mathbf{R}^2$.

⟨2⟩ We draw two circles C_2 and C_4 with radius r and centered at q_2 and q_4 respectively. Denote the two intersection points of C_2 and C_4 by A_1 and A_3 located above and below L_0 respectively.

⟨3⟩ We draw a vertical line M passing through A_1 and A_3. M intersects L_0 at a point Q.

⟨4⟩ We draw two lines L_1 and L_3 parallel to L_0 such that both of them have distance $r\sqrt{3}/6$ from L_0, and locate above and below L_0 respectively. Denote by B_1 and B_3 the intersection points of L_1 with L_0 and L_3 with L_0 respectively.

⟨5⟩ We draw two circles D_1 and D_3 with radius r and centered at A_3 and A_1 respectively. Then both of them pass through q_2 and q_4. Denote the intersection points of D_1 with M by E_1 and D_3 with M by E_3.

⟨6⟩ We denote by G_1 the open region bounded by the arc of the circle C_4 from A_1 down to q_2, the arc of the circle D_1 from q_2 through E_1 to q_4, and then the arc of the circle C_2 from q_4 up to A_1.

⟨7⟩ We denote by G_3 the open region bounded by the arc of the circle C_2 from A_3 up to q_4, the arc of the circle D_3 from q_4 through E_3 to q_2, and then the arc of the circle C_4 from q_2 down to A_3.

⟨8⟩ We denote by T the intersection of the set $G_1 \cup G_3$ and the closed strip between the lines L_1 and L_3.

Note that the regions G_1, G_3, and T are independent of the choice of $m \in (\mathbf{R}^+)^4$. The regions G_1 and G_3 are surrounded by thick curves in Figure 4.2.

The second main result of this paper is the following

Theorem 4.1 *For any $m \in (\mathbf{R}^+)^4$, let $q = (q_1, q_2, q_3, q_4)$ be a convex RE for m. Fixing q_2 and q_4 in Figure 4.2, the following results always hold.*

1° Every interior angle of $\diamond q_1q_2q_3q_4$ is in $(60^o, 150^0)$, i.e., there hold

$$\begin{aligned} 60^o &< \min\{\angle q_1q_2q_3, \angle q_2q_3q_4, \angle q_3q_4q_1, \angle q_4q_1q_2\} \\ &\leq \max\{\angle q_1q_2q_3, \angle q_2q_3q_4, \angle q_3q_4q_1, \angle q_4q_1q_2\} < 150^o. \end{aligned} \tag{4.1}$$

Specially, a convex RE for m can not have two interior angles greater than or equal to 120^o.

2° Every diagonal of $\diamond q_1q_2q_3q_4$ divides an interior angle into two sub-interior angles formed by this diagonal and a side, both of which are in $(0^o, 90^0)$.

3° There must hold

$$q_1 \in G_1, \qquad q_3 \in G_3. \tag{4.2}$$

4° q_1 and q_3 can not locate in T simultaneously.

5° There hold

$$\frac{\sqrt{3}}{6} r_{24} < \max\{|q_1 - \overline{q_2q_4}|, |q_3 - \overline{q_2q_4}|\} < \frac{\sqrt{3}}{2} r_{24}, \tag{4.3}$$

$$(1 - \frac{\sqrt{3}}{2}) r_{24} < \min\{|q_1 - \overline{q_2q_4}|, |q_3 - \overline{q_2q_4}|\}, \tag{4.4}$$

$$\frac{3-\sqrt{3}}{6} \max\{r_{24}^2, r_{13}^2\} < \text{area}(\diamond q_1q_2q_3q_4) < \frac{\sqrt{3}}{2} \min\{r_{24}^2, r_{13}^2\}. \tag{4.5}$$

Proof. For any $m \in (\mathbf{R}^+)^4$, let $q = (q_1, q_2, q_3, q_4)$ be a convex RE for m. The proof is carried out in six steps.

Step 1. Proof of the left inequality of (4.1)

Assume that an interior angle of $\diamond q_1q_2q_3q_4$ satisfies $\angle q_1q_2q_3 < 60^o$. Then one of the two sub-interior angles $\angle q_2q_3q_1$ and $\angle q_2q_1q_3$ must be greater than 60^o. Therefore one of the two exterior sides r_{12} and r_{23} must be strictly longer than the diagonal r_{13}. This contradicts to 1° of Lemma 2.3, and yields a new proof of 2° of Lemma 2.3.

If there is an interior angle satisfying $\angle q_1q_2q_3 = 60^o$, without loss of generality, we assume that two sides of $\angle q_1q_2q_3$ satisfy $r_{12} \leq r_{23}$. Let ξ be the perpendicular bisector line of $\overline{q_1q_2}$. Note that q_4 can not locate on ξ and the line η passing through q_1 and q_2. Thus either q_3 locates on ξ or in the same quadrant of $\mathbf{R}^2 \setminus (\xi \cup \eta)$ with q_4. In each case, it contradicts to Lemma 2.1. Therefore the left inequality of (4.1) is proved.

Specially, if an RE $\diamond q_1q_2q_3q_4$ possesses two interior angles greater than or equal to 120^o, then one of the other two interior angles must be less than or equal to 60^o. This contradicts to the left inequality of (4.1).

Step 2. Proof of 2°.

Assume that a sub-interior angle of $\diamond q_1q_2q_3q_4$ satisfies $\angle q_1q_2q_4 \geq 90^o$. Then both of points q_3 and q_4 locate in one quadrant of $\mathbf{R}^2$ canceling the line passing through $\overline{q_1q_2}$ and the perpendicular bisector line of $\overline{q_1q_2}$. This contradicts Lemma 2.1 and proves 2°.

Now fix q_2 and q_4 in Figure 4.2.

Step 3. Let F_1 be the open region bounded by the arc of C_2 from q_4 up to A_1, the arc of C_4 from A_1 down to q_2, and $\overline{q_2q_4}$. Let F_3 be the open region bounded by the arc of C_2 from q_4 down to A_3, the arc of C_4 from A_3 up to q_2, and $\overline{q_2q_4}$.

Claim: $q_1 \in F_1$ *and* $q_3 \in F_3$.

In fact, if q_1 locates out side C_2 and C_4 and above L_0, then we have $\max\{r_{12}, r_{14}\} > r \equiv r_{24}$. This contradicts 1° of Lemma 2.3. Thus q_1 must locate on or inside the circles C_2 and C_4. If q_1 locates on C_2 from q_4 to A_1, then $r_{12} = r_{24}$ and hence the perpendicular bisector of the segment $\overline{q_1q_4}$ must pass through q_2. Then the location of q_3 violates Lemma 2.1. Similarly q_1 can not locate on the arc of C_4 from A_1 to q_2 either.

Thus, $q_1 \in F_1$ by Lemma 2.4. Similarly we must have $q_3 \in F_3$ and the claim is proved.

Step 4. Claim: q_1 *can not locate on or below the arc* ζ *of* D_1 *from* q_2 *through* E_1 *to* q_4.

In fact, assume that q_1 locates on or below ζ. Denote by γ the straight line passing through A_3 and q_1. Denote the intersection point of γ with ζ by q_0 which is q_1 if $q_1 \in \zeta$. Then the perpendicular bisector line L_{04} of $\overline{q_0q_4}$ passes through the point A_3. Then we have

$$\angle q_4q_1A_3 \geq \angle q_4q_0A_3 = \angle q_0q_4A_3 \geq \angle q_1q_4A_3.$$

Thus $|A_3 - q_4| \geq |A_3 - q_1|$. Therefore the perpendicular bisector line L_{14} of $\overline{q_1q_4}$ must intersect $\overline{q_4A_3}$. Similarly the perpendicular bisector line L_{12} of $\overline{q_1q_2}$ must intersect $\overline{q_2A_3}$. Therefore the intersection point H of L_{14} and L_{12} must locate on A_3 or satisfy $|H - L_0| > |A_3 - L_0|$.

Because q_2 locates on the right hand side of L_{14}, q_4 locates on the left hand side of L_{12}, by Lemma 2.1, q_3 should locate both on the left hand side of L_{14} and the right hand side of L_{12}. But these two sides have no common intersections with the region F_3 which q_3 should also locate in by Step 3. This contradiction proves our claim.

Similarly, we have that q_3 *can not locate on or above the arc of* D_3 *from* q_2 *through* E_3 *to* q_4.

Thus by Steps 3 and 4, we obtain $q_1 \in G_1$ and $q_3 \in G_3$, i.e., 3° holds.

Because $\angle q_2Mq_4 = 150^o$ for any point M in the arc ζ in the claim of Step 4, we have $\angle q_2q_1q_4 < 150^o$ when $q_1 \in G_1$. Then permuting the vertices yields the right inequality of (4.1) and 1° is proved.

Step 5. Proof of 4°.

Assume that q_1 and q_3 locate in the strip T simultaneously. If one of q_1 and q_3 is not on the perpendicular bisector line M of $\overline{q_2q_4}$, by 3° and Lemma 2.1, both of q_1 and q_3 must locate not on M, and must locate on the same side of M. Therefore one of the two angles $\angle q_1q_2q_3$ and $\angle q_1q_4q_3$ is strictly less than $\angle B_1q_2B_3 = \angle B_1q_4B_3 = 60^o$. This contradicts our 1°.

If both of q_1 and q_3 are on the vertical line M, then we must have $q_1 = B_1$ and $q_3 = B_3$ by 3° and Lemma 2.1. Because now $r_{13} = r_{23} = r\sqrt{3}/3$, the point q_3 is on the perpendicular bisector of $\overline{q_1q_2}$. Then the location of q_4 violates Lemma 2.1. Therefore 4° holds.

Step 6. 5° follows from 3° and 4° by elementary computations. Here special attention should be paid to the fact that the left hand side of (4.5) is the sum of the two triangles whose areas come from the left hand sides of (4.3) and (4.4) respectively. The proof is complete. ∎

Remark 4.1 Note that our Theorem 4.1 and Figure 4.2 work for any choice of $m \in (\mathbf{R}^+)^4$ and any convex RE for m, and are different from results of [8]. Most of the results in Theorem 4.1 are new.

There is a long standing open conjecture on whether $^\#R(n,m) < +\infty$ holds for any given $m \in (\mathbf{R}^+)^n$ (cf. §360 of [15] and [14]). Our Theorems 3.1 and 4.1 can be used to give a new proof of the following result which is related to this conjecture by a well-known theorem of M. Shub (cf. [12]). Note that the 45^o-theorem of R. Moeckel in [9] of 1990 gives a proof of Corollary 4.1 for any n-body collinear REs in $R(n,m)$.

Corollary 4.1. *For any given $m \in (\mathbf{R}^+)^4$, 4-body collinear REs for m are isolated in $R(4,m)$.*

Proof. The claim follows from 1°-3° of Theorem 3.1 for concave REs and 1°-3° of Theorem 4.1 for convex REs. Note that proofs of Theorems 3.1 and 4.1 depend only on Lemmas 2.1-2.5. The corollary is proved. ∎

5 4-body convex REs for angular estimates

In this section, we construct examples of 4-body convex REs possessing a prescribed interior angle in $(60^o, 150^o)$. These examples show that the estimates in inequalities of (4.1) are sharp, and also show that the theorem on p.858 claiming sub-interior angles are less than 60^o and the upper bound estimate 120^o for interior angles in the theorem on p.857 of §16 of [8] are incorrect.

Note that $\tan 60^o = \sqrt{3}$ and $\tan 15^o = 2-\sqrt{3}$. By 1° of Theorem 4.1, we have the corresponding restrictions on the possible values of u and v in Theorem 5.1. As usual, a 4-body convex configuration $q \in (\mathbf{R}^2)^4$ is a *kite* if it has two opposite pairs of equal sides.

Theorem 5.1 *For $2-\sqrt{3} < u \le v < \sqrt{3}$, and $s \in \mathbf{R}$, we construct a kite configuration $q = (q_1, q_2, q_3, q_4)$ in $\mathbf{R}^2$ depending on u, v, and s as in Figure 5.1 by setting $q_1 = (0, s+u)$ and $q_3 = (0, s-v)$ on the y-axis in $\mathbf{R}^2$, and $q_2 = (1, s)$, $q_4 = (-1, s)$, and $Q = (0, s)$ on a horizontal line. For every $\alpha \in (60^o, 150^o)$, let*

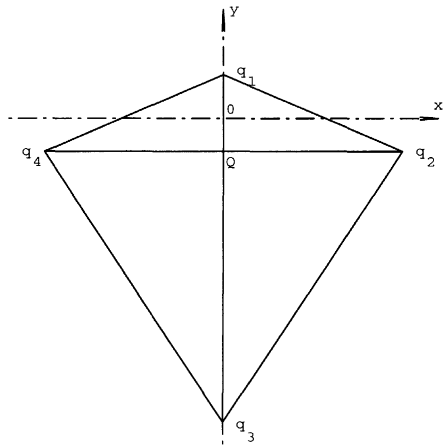

Figure 5.1: 4-body kite REs with a prescribed interior angle

$u = \tan(90^o - \alpha/2) \in (2-\sqrt{3}, \sqrt{3})$. *Then we have the following non-empty interval*

$$I \equiv (\max\{\frac{1-u^2}{2u}, \sqrt{1+u^2}-u\}, \sqrt{3}\,) \cap [u, \sqrt{3}\,) \neq \emptyset, \tag{5.1}$$

and that for every $v \in I$, the masses $m_1 > 0$, $m_3 > 0$, and the $s \in \mathbf{R}$ can be uniquely determined by u and v so that q is a convex kite RE for $m = (m_1, 1, m_3, 1)$ with $\angle q_2q_1q_4 = \alpha$.

Proof. The proof is carried out in three steps.

Step 1. Necessary and sufficient conditions for q to be an RE.

Note that $r_{12} = \sqrt{1+u^2}$ and $r_{23} = \sqrt{1+v^2}$ in Figure 5.1. Using xy-coordinates of the points q_1, q_2, q_3, and q_4 introduced there, the system (1.1) becomes exactly:

$$\lambda(s+u) - \frac{u}{r_{12}^3} - \frac{m_3}{(u+v)^2} - \frac{u}{r_{12}^3} = 0, \tag{5.2}$$

$$\lambda s + \frac{m_1 u}{r_{12}^3} - \frac{m_3 v}{r_{23}^3} = 0, \tag{5.3}$$

$$\lambda - \frac{m_1}{r_{12}^3} - \frac{m_3}{r_{23}^2} - \frac{1}{4} = 0, \tag{5.4}$$

$$\lambda(s-v) + \frac{m_1}{(u+v)^2} + \frac{v}{r_{23}^3} + \frac{v}{r_{23}^3} = 0. \tag{5.5}$$

Substracting (5.3) from (5.2) and (5.5) from (5.2) respectively yield

$$\lambda u - \frac{m_1 u}{r_{12}^3} + m_3\left(\frac{v}{r_{23}^3} - \frac{1}{(u+v)^2}\right) = \frac{2u}{r_{12}^3}, \tag{5.6}$$

$$\lambda(u+v) - \frac{m_1}{(u+v)^2} - \frac{m_3}{(u+v)^2} = 2\left(\frac{u}{r_{12}^3} + \frac{v}{r_{23}^3}\right). \tag{5.7}$$

Subtracting (5.4) times u from (5.6) yields

$$m_3(u+v)\left(\frac{1}{r_{23}^3}-\frac{1}{(u+v)^3}\right)=2u\left(\frac{1}{r_{12}^3}-\frac{1}{8}\right). \tag{5.8}$$

Subtracting (5.4) times $(u+v)$ from (5.7) yields

$$\begin{aligned} m_1\left(\frac{u+v}{r_{12}^3}-\frac{1}{(u+v)^2}\right) &+ m_3\left(\frac{u+v}{r_{23}^3}-\frac{1}{(u+v)^2}\right) \\ &= 2\left(\frac{u}{r_{12}^3}+\frac{v}{r_{23}^3}\right)-\frac{u+v}{4}. \end{aligned} \tag{5.9}$$

Plugging (5.8) into (5.9) yields

$$m_1(u+v)\left(\frac{1}{r_{12}^3}-\frac{1}{(u+v)^3}\right)=2v\left(\frac{1}{r_{23}^3}-\frac{1}{8}\right). \tag{5.10}$$

Then (5.8) and (5.10) give necessary and sufficient conditions for existence of positive masses m_1 and m_3 so that q is an RE for $m=(m_1,1,m_3,1)$.

Step 2. Compatibility of u and v.

Note that from $u<\sqrt{3}$, we have $r_{12}=\sqrt{1+u^2}<2$. Thus The right hand side of (5.8) is positive. So $m_3>0$ exists if and only if

$$\sqrt{1+v^2}\equiv r_{23}<u+v. \tag{5.11}$$

If $\sqrt{1+u^2}\equiv r_{12}\geq u+v$, we get $v(v+2u)\leq 1$. Thus $v<1$ and $r_{23}=\sqrt{1+v^2}<\sqrt{2}$. Therefore if the left hand side of (5.10) is non-positive, the right hand side of (5.10) must be positive. This contradiction shows that $m_1>0$ exists if and only if both sides of (5.10) must be positive, that is $r_{23}<2$, which is $v<\sqrt{3}$ and holds by our initial assumption, and

$$\sqrt{1+u^2}\equiv r_{12}<u+v. \tag{5.12}$$

Therefore for $2-\sqrt{3}<u\leq v<\sqrt{3}$, the system (5.8) and (5.10) possesses positive solutions m_1 and m_3 if and only if (5.11) and (5.12) hold.

Note that (5.11) and (5.12) are equivalent to

$$v>\frac{1-u^2}{2u}, \qquad v>\sqrt{1+u^2}-u. \tag{5.13}$$

Note that for $u>0$, we always have $\sqrt{3}>\sqrt{1+u^2}-u$. Note also that $u>2-\sqrt{3}$ is equivalent to $(u+\sqrt{3})^2-4>0$, and hence equivalent to $\sqrt{3}>(1-u^2)/(2u)$. Therefore $u\in(2-\sqrt{3},\sqrt{3})$ implies the existence of v satisfying (5.13) and $u\leq v<\sqrt{3}$, i.e., (5.1) holds.

Step 3. Existences of an RE with a prescribed interior angle.

Given $\alpha \in (60^o, 150^o)$, we let $u = \tan(90^o - \alpha/2)$. Then we obtain $2 - \sqrt{3} = \tan 15^o < u < \tan 60^o = \sqrt{3}$. By our discussion in Step 2, the interval I defined in (5.1) is non-empty.

Fix any $v \in I$. By our discussion in Step 2, solutions $m_1 > 0$ and $m_3 > 0$ of (5.8) and (5.10) exist uniquely according to the above chosen u and v. Then $\lambda > 0$ is uniquely determined by (5.4), and $s \in \mathbf{R}$ is uniquely determined by (5.3). Therefore by the system (5.2)-(5.5), q defined in the theorem is an RE for $m = (m_1, 1, m_3, 1)$ with $\angle q_2 q_1 q_4 = 180^o - 2 \arctan u = \alpha$. The proof is complete. ∎

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