

# Remarks on Nonlinear Elliptic Problems on Long Domains

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## Abstract

In this paper, we discuss the behaviour of solutions near a turning point for weakly non-linear elliptic equations on long domains. We are looking for results independent of the domain length. In the process we obtain some local uniqueness and non-degeneracy results for homoclinic solutions on infinite strips.

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## 1 Introduction

In this short paper, we modify some ideas in [5] to greatly improve results in [4] on the behaviour near the turning point of positive solutions of problems of the form

$$\begin{aligned} -\Delta u &= \lambda f(u) \text{ in } \Omega \times (-n, n) \\ u &= 0 \text{ on } \partial(\Omega \times (-n, n)) \end{aligned}$$

where  $f$  is convex or convex like and  $\Omega$  is a smooth bounded domain in  $R^k$ . We prove there is a neighbourhood of the turning point uniform in  $n$  such that in this neighbourhood the branch consists of a single curve with all turning points very close (with distance tending to zero as  $n \rightarrow \infty$ ) to the main turning point and possibly some extra small branches confined to a small neighbourhood  $N$  of the main turning point and possibly branching off

the main branch where the size of  $N$  tends to zero as  $n \rightarrow \infty$ . A more precise statement appears later.

We expect these small oddities do not occur but it seems very technical to prove this. We discuss this later.

In the process, as the key new step, we prove a local uniqueness and non-degeneracy (modulo translations) for solutions of our equation on the infinite strip as they bifurcate from solutions independent of  $x_{k+1}$ . This is of independent interest because there are very few non-degeneracy results known for homoclinic solutions. Moreover if  $f$  is real analytic one can frequently combine this with analytic bifurcation theory as in [5] or [6] to obtain non-degenerate homoclinic solutions far from the bifurcation point. As in [5], the main difficulty is that the asymptotic behaviour of the solutions depends very strongly on the direction.

The main reason this problem is technical is that zero is in the essential spectrum of the natural limit problem. We overcome this by a partial Liapounov-Schmitt reduction to reduce our problem to an ordinary differential equation (except for higher order terms) and then a rescaling to avoid the essential spectrum. (This was motivated by [5]), though similar ideas were used earlier by Magnus and Stuart)

## 2 Small homoclinic solutions

In this section, we study homoclinic solutions of

$$-\Delta u = \lambda f(u) \tag{1}$$

in  $\Omega \times R$  with homogeneous Dirichlet boundary conditions on  $\partial\Omega \times R$  which are even in  $x_{k+1}$  (where  $\Omega \subseteq R^k$ ) and strictly decreasing in  $x_{k+1}$  for  $x_{k+1} \geq 0$  where  $\lambda$  is near  $\lambda_0$  and  $u$  is uniformly close to  $u_0(x')$  on  $S = \Omega \times R$ . Here  $x = (x', x_{k+1})$  and  $\Delta' u_0 = \lambda_0 f(u_0)$  where  $\Delta'$  is the Laplacian on  $\Omega \subseteq R^k$  for Dirichlet boundary conditions. We assume  $f$  is  $C^3$  (though  $f$  is  $C^2$  would suffice for most results).

We consider the case where the principal eigenvalue  $\tilde{\gamma}$  of the eigenvalue problem (for  $\gamma$ )

$$\begin{aligned} -\Delta' h &= \lambda_0 f'(u_0)h + \gamma h \text{ in } \Omega \\ h &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2}$$

is zero. Let  $\phi_1$  be the corresponding positive eigenfunction. The case where  $\tilde{\gamma} > 0$  is much easier and is considered (implicitly) in [4]. As we will see below  $\tilde{\gamma} \geq 0$  always (unless there are no such solution  $u$ ).

Before proceeding, we make a few remarks. Suppose that  $\lim_{x_{k+1} \rightarrow \pm\infty} u(x', x_{k+1})$  exist and are equal,  $u(x', x_{k+1})$  is greater than or equal to this limit on  $S$  and a technical condition (condition N2 below) holds in the case when  $\lim_{x_{k+1} \rightarrow \infty} u(x', x_{k+1})$  is a degenerate solution of  $-\Delta' u = f(u)$  on  $\Omega$ . Then either  $u$  is independent of  $x_{k+1}$  or, up to a translation,  $u$  is even in  $x_{k+1}$  and  $\frac{\partial u}{\partial x_{k+1}} < 0$  for  $x' \in \Omega, x_{k+1} > 0$ . This is proved by sliding plane arguments (cp [1] or [4]). The only technical difficulty is to obtain a good asymptotic expansion of  $u$  near  $x_{k+1} = \infty$ . This is discussed in the appendix. Moreover,

$\lim_{|x_{k+1}| \rightarrow \infty} u(x', x_{k+1})$  is a weakly stable solution of

$$\begin{aligned} -\Delta' u &= \lambda f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{3}$$

where by weakly stable we mean that the principal eigenvalue  $\hat{\gamma}$  of

$$\begin{aligned} -\Delta' h &= \lambda f'(\lim_{|x_{k+1}| \rightarrow \infty} u(x', x_{k+1}))h + \gamma h \text{ in } \Omega \\ h &= 0 \end{aligned}$$

is non-negative. This is proved in [4] (and is used in the proof of the previous results).

We assume that  $\int_{\Omega} f(u_0)\phi_1 \neq 0$  and  $\int_{\Omega} f''(u_0)\phi_1^3 \neq 0$ . For simplicity, we assume that this second integral is positive. We call these conditions Assumptions *N1* and *N2* respectively. The importance of these assumptions is that they ensure that the solutions of (3) near  $(u_0, \lambda_0)$  form a single continuous curve  $(u_0(t), \lambda_0 - t^2)$  where  $u_0$  is  $C^2$  in  $t$  and is increasing in  $t$ ,  $u_0(0) = u_0$  and  $u_0'(0) = p\phi$  where  $p > 0$ . Moreover these solutions are hyperbolic if  $t \neq 0$  and are stable if  $t < 0$  and unstable if  $t > 0$ . This is a standard application of classical Crandall-Rabinowitz theorems [2] and [3] except we have used a  $C^2$  change of variables on  $R$  to get the  $\lambda$  equation in a simpler form. (The change of variables simplifies later calculations slightly.) Note that Assumption *N1* holds in many cases. For example it holds if  $u_0$  is positive and if  $f(y) \geq 0$  for  $y \geq 0$  or if  $yf'(y) - f(y)$  has fixed sign or if  $k = 2$  and  $\Omega$  is a GNN domain in the sense of [5]. By our earlier remarks, only the solutions for  $t \leq 0$  will be of interest to us. We look for solutions of (1) of the form  $u = u_0(t) + w$ ,  $\lambda = \lambda_0 - t^2$  where  $t < 0$  and small,  $w$  is even,  $w \rightarrow 0$  as  $|x_{k+1}| \rightarrow \infty$  and  $w$  is small in  $L^\infty(S)$ . Then our equation for  $w$  is

$$\begin{aligned} \Delta w &= (\lambda_0 - t^2)f(u_0(t) + w) - (\lambda_0 - t^2)f(u_0(t)) \\ &= \lambda_0(f'(u_0(t))w + \frac{1}{2}f''(u_0(t))w^2 + \\ &\quad g(u_0(t), w) - t^2[f(u_0(t) + w) - f(u_0(t))] \end{aligned} \tag{4}$$

plus the boundary condition where

$$g(u_0(t), w) = f(u_0(t) + w) - f(u_0(t)) - f'(u_0(t))w - \frac{1}{2}f''(u_0(t))w^2 .$$

Here, as elsewhere, we always assume Dirichlet boundary conditions. Unfortunately, the operator  $-\Delta - \lambda_0 f'(u_0)I$  is not Fredholm in natural spaces on  $S$  so that we need to split into Fredholm and non-Fredholm parts. We write  $w = B\phi_1 + z$  where  $B$  is a function of  $x_{k+1}$  and  $z(x', x_{k+1})$  is orthogonal to  $\phi_1$  for each  $x_{k+1}$ . Let  $(I - P)$  be the natural projection corresponding to this decomposition, that is,  $(I - P)w = z$ . It is easy to see that  $I - P$  is continuous on  $L^\infty(S)$  and maps the closed subspace  $\hat{L}(S) = \{v \in L^\infty(S) : v \rightarrow 0 \text{ as } |x_{k+1}| \rightarrow \infty \text{ uniformly in } x'\}$  into itself. Now the operator  $W \equiv \Delta - (I - P)(f'(u_0)I)$  is invertible as a map of  $\hat{T} = \{u \in \hat{L}^\infty(S) \cap W_{loc}^{2,p}(S) : \Delta u \in L^\infty(S), u = 0 \text{ or } \partial S, Pu = 0\}$  into  $\{u \in \hat{L}(S) : Pu = 0\}$ . We explain this. First we can easily use

eigenfunction expansions in terms of the eigenfunctions of  $-\Delta' - f'(u_0)I$  on  $\Omega$  to prove that this operator has zero kernel in the larger space  $T$  where in the definition of  $T$  we replace  $\hat{L}(S)$  by  $L^\infty(S)$ . (This uses essentially that  $\tilde{\gamma} \geq 0$ ). Since our operator is invariant under translations in  $x_{k+1}$  we can then argue as in the proof of (2.5) on pp. 136-137 of [5] to deduce that there exists  $c > 0$  such that  $\|Wu\|_\infty \geq c\|u\|_\infty$  if  $u \in T$ . Thus this inequality holds on  $\hat{T}$ . Hence we see that  $W$  has closed range on  $\hat{T}$ . Since  $W$  has zero kernel, we see that  $W|_{\hat{T}}$  is invertible if its range is dense in  $\hat{L}(S)$ . Thus it suffices to prove that  $h$ 's whose support is compact in  $\bar{S}$  are in the range of  $\hat{W}$ . Since such an  $h$  is in  $L^2(S)$ , we can easily use eigenfunction expansion (more precisely eigenfunction expansions in  $x'$ , Fourier transforms in  $x_{k+1}$ ) to obtain a  $W^{2,2}$  solution with zero trace on  $\partial S$ . We can then apply standard local estimates on  $\Omega \times [r, r + 1]$  to find that  $u$  is uniformly bounded on  $S$  and  $u \in W_{loc}^{2,p}(S)$ . Hence if we prove that  $u \rightarrow 0$  as  $|x_{k+1}| \rightarrow \infty$  uniformly in  $x'$ , we will have proved our claim. If not, we can use an argument as on pp. 136-137 again to prove that there exists a non-trivial bounded solution  $\hat{u}$  of  $Wz = 0$  in  $S, z = 0$  on  $\partial S$ . (Here we use a translation in  $x_{k+1}$  and use that  $h = 0$  if  $|x_{k+1}|$  is large.) As before, this is impossible. Hence we have proved the invertibility of  $W$  as a map of  $\hat{T}$  into  $\hat{L}(S) \cap \{w : Pw = 0\}$ . By standard regularity theorems, we can deduce that  $W^{-1}$  is continuous from  $\hat{L}(S) \cap \{w : Pw = 0\}$  into  $C^1(S)$ .

Now by projecting (4), we find that

$$Wz = (I - P)[\lambda_0(f'(u_0(t)) - f'(u_0))](\tilde{w}) + \frac{1}{2}f''(u_0(t))\tilde{w}^2 + g(u_0(t), \tilde{w}) - t^2f(u_0(t) + \tilde{w}) + t^2f(u_0(t))$$

where  $\tilde{w} = B\phi_1 + z$ .

By the Implicit Function Theorem and Taylor's theorem, we can solve this for  $z$  as a function of  $B$  (for  $z, B$  small)  $z = z(t, B)$  and by standard estimates, we find that

$$\|z\|_{C^1} \leq K(\|t\| \|B\|_\infty + (\|B\|_\infty)^2)$$

with the corresponding natural estimates for  $z(t, B_1) - z(s, B_2)$ . Moreover  $z$  is  $C^2$  in  $t$  and  $B$  (since  $f$  is). Hence (1) reduces locally to an equation for  $B$  which is easily seen to be

$$-B'' = \alpha(t)B + CB^2 + \text{Rem} \tag{5}$$

where

$$\alpha(t) = \lambda_0 \int_\Omega (f'(u_0(t)) - f'(u_0))\phi_1^2$$

$$C = \lambda_0 \int_\Omega f''(u_0)\phi_1^3 > 0$$

and  $\|\text{Rem}\|_\infty \leq o(\|B\|_\infty^2 + \|t\| \|B\|_\infty)$  with the corresponding estimates for first derivatives. Since we can write  $\alpha(t) = \alpha'(0)t + o(t)$ , we can rewrite (5) as

$$-B'' = \mu t B + CB^2 + \text{Rem} \tag{6}$$

at the expense of changing the term Rem. Here

$$\begin{aligned} \mu &= \lambda_0 \int_{\Omega} f''(u(0))\phi_1^3 \\ &= C \end{aligned}$$

since  $u'(0) = \phi_1$ .

Note that we only need to look for solutions where  $B$  is positive, decreasing and decaying to zero since  $B(x_{k+1}) = \int_{\Omega} w(x', x_{k+1})\phi_1(x')dx'$  and by our earlier comments on solutions on the strip.

Now our linearized operator at  $t = 0$  and  $B = 0$  is  $-B''$  which is not Fredholm on  $L^\infty(R)$ . To overcome this, we use rescaling (and check we do not lose any solutions).

We look for solutions  $B = -tY(\sqrt{-t}x)$ . Remember that  $t \leq 0$ . Here and for the rest of this section we write  $x$  for  $x_{k+1}$ .

Then our equation becomes

$$-Y'' = -CY + Y^2 + \tilde{R}em \tag{7}$$

where we expect  $Y$  to be of order 1 and  $\|\tilde{R}em\|_\infty = o(1)$  (and is zero if  $Y \equiv 0$ ) as long as  $\|Y\|_\infty \leq K$ . Thus our problem is a small perturbation of

$$-Y'' = -CY + Y^2. \tag{8}$$

Then to obtain a non-trivial solution of (7), it suffices to prove that the unique even positive solution  $Y_0$  of (8) decaying to zero is non-degenerate in the space of even bounded decaying functions with the sup norm because we can use the implicit function theorem. (From the start we could have worked in the space of functions even in  $x_{k+1}$  with only minor modifications of our arguments.) Note also that we can easily check  $\tilde{R}em$  satisfies a small Lifschitz condition in  $Y$  and that, by our earlier arguments, any solutions of (8) are translations of even solutions. We can easily use the first integral of (8) to prove the existence and uniqueness of the even positive solution of (8) such that  $Y(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

Next we prove the non-degeneracy of the positive decaying even solution  $Y_0$  of (8) (in the space of even bounded functions). Note that  $Y'_0$  is a solution of the linearized equation of (8) which decay exponentially (but of course is odd not even). Now the linearized equation of (8) if of the form  $-h'' + q(x)Y$  where  $q \rightarrow \tilde{C}$  as  $x \rightarrow \pm\infty$ . It is well known (cp [7]) that such an equation can not have two linearly independent bounded solutions at  $+\infty$ . (This can also be easily be proved by Wronskian arguments). Hence non-degeneracy holds. Hence our claim follows, except that we need to prove our solution  $w$  is positive and we have to show that there are no other positive solutions.

To prove the first of these claims, note that, if  $y$  is a solution of (7) uniformly close to the unique positive solution  $Y$  of (8), then  $\|y\|_\infty \approx \|Y\|_\infty$  while  $\|y^-\|_\infty$  is small. Thus in the original variable,  $\|B\|_\infty \sim |t|$  while  $\|B^-\|_\infty = o(|t|)$ . Hence we see from the estimate for  $z$  that the corresponding solution  $w$  of (1) has the property that  $\|w\|_\infty \sim |t|$  while  $\|w^-\|_\infty$  is  $o(|t|)$ .

Now by standard perturbation theory (and using assumption N2), it is easy to see that the least eigenvalue of  $-\Delta - (\lambda_0 - t^2)f'(u_0(t))I$  on  $\Omega$  is asymptotically  $\alpha|t|$  for small

negative  $t$  where  $\alpha > 0$ . Remember that the least eigenvalue is zero and is simple for  $t = 0$ . Hence by separation of variables the result is true for the least point of the spectrum of the same operator on  $L^2(S)$ . Thus the least eigenvalue of the same operator on  $L^2(U)$  is at least  $\alpha|t|$  if  $U \subseteq S$  and for Dirichlet boundary conditions on  $\partial U$ . Now let  $U = \{x \in S : w(x) < 0\}$ . By our earlier estimates  $|w|_\infty = O(|t|)$  on  $U$  and hence  $f(u_0(t) + w) - f(u_0) = (f'(u_0(t)) + o(t))w$  on  $U$  (since  $w < 0$  on  $U$  and  $\|w^-\|_\infty = o(|t|)$ ). Hence by the results above and the standard minimum principle for the inf of the spectrum, we see that  $-\Delta - (\lambda_0 - t^2) \frac{f(u_0(t)+w)-f(u_0(t))}{w} I$  has its spectrum in  $L^2(U)$  strictly positive for small  $t < 0$ . (Note that since  $\alpha > 0$ ,  $o(t)$  terms will have a higher order effect). On the other hand if  $t \neq 0$ , standard estimates ensure that  $w$  decays exponentially as  $x \rightarrow \infty$  and hence  $w$  is an eigenfunction on  $U$  in  $L^2(U)$  corresponding to the eigenvalue zero and hence we have a contradiction. Thus  $w$  is positive. Note that  $t = 0$  is impossible because, if  $w$  is a positive solution,  $B$  is positive even and decreasing for  $x \geq 0$  while (6) (for  $t = 0$ ) implies that  $B'' < 0$  which easily gives a contradiction. Hence our claim follows.

It remains to prove that the above solution we have constructed is the only small positive solution. We proved already that  $t \leq 0$  and the argument of the previous paragraph shows that  $t = 0$  is impossible. As there, if  $w$  is positive, then  $B$  is positive, even and decreasing for  $x \geq 0$  and  $B \rightarrow 0$  as  $x \rightarrow \infty$ . Since  $B''(0) \leq 0$  (because  $B$  has a maximum at  $x = 0$ ), (6) implies that  $\|B\|_\infty \geq \mu|t|$  or equivalently  $\|Y\|_\infty \geq \mu$ . If  $Y_i$  are positive decreasing solutions for  $t = t_i$  of (7) with  $\mu \leq \|Y_i\|_\infty \leq K$  where  $t_i \rightarrow 0$  as  $i \rightarrow \infty$ , a standard compactness argument shows that a subsequence of  $Y_i$  converges uniformly on compact sets to a solution  $\tilde{Y}$  of (8) with  $\mu \leq \tilde{Y}(0) \leq K$  and  $\tilde{Y}$  positive and decreasing. Now it is easy to use the first integral to check either  $\tilde{Y}$  is the unique positive solution of (8) which tends to zero as  $t \rightarrow \infty$  or  $\tilde{Y} \equiv C$ . To show that the second case is impossible we choose  $x_i > 0$  such that  $Y_i(x_i) = \frac{1}{2}C$ . Since  $Y_i \rightarrow C$  uniformly on compact subsets,  $x_i \rightarrow \infty$  as  $i \rightarrow \infty$ . Hence a subsequence of  $Y_i(x - x_i)$  converges uniformly on compact sets to a positive bounded decreasing solution of  $-Y'' = -CY + Y^2$  on  $R$  with  $Y(0) = \frac{1}{2}C$ . It follows easily that  $Y(x) \rightarrow 0$  as  $x \rightarrow \infty$  and  $Y(x) \rightarrow C$  as  $x \rightarrow -\infty$  (since these are the only stationary points of the equation). It follows easily from the first integral that this does not occur. Hence  $Y_i$  converges uniformly on compact sets to  $Y_0$  as  $i \rightarrow \infty$ . Since  $Y_0(x) \rightarrow 0$  as  $x \rightarrow \infty$  and  $Y_i$  and  $Y_0$  are positive and decreasing for  $x > 0$ , it follows that  $Y_i \rightarrow Y_0$  uniformly as  $i \rightarrow \infty$ . Thus  $Y_i$  is the solution we constructed earlier by the implicit function theorem.

Thus it remains to bound solutions of (7). It is convenient to instead bound small solutions of (6). Suppose by way of contradiction that  $t_i \rightarrow 0$  and  $B_i$  are positive even decreasing solutions (for  $x \geq 0$ ) of (6) such that  $\|B_i\|_\infty \rightarrow 0$  and  $-t_i^{-1}\|B_i\|_\infty \rightarrow \infty$  as  $i \rightarrow \infty$ . By our earlier estimates, we see that the remainder term is  $o(\|B_i\|_\infty)^2$ . Hence  $\hat{B}_i = (\|B_i\|_\infty)^{-1}B_i$ , satisfies

$$-\hat{B}_i'' = t_i \hat{B}_i + \|B_i\|_\infty (\hat{B}_i)^2 + o(\|B_i\|_\infty).$$

Hence if we rescale  $x$  by a factor  $(\|B_i\|_\infty)^{\frac{1}{2}}$ , we obtain a solution  $\tilde{B}_i$  which is positive decreasing (for  $X_i \geq 0$ ),  $\|\tilde{B}_i\|_\infty = 1$  and

$$-\tilde{B}_i = (\tilde{B}_i)^2 + o(1)$$

where we have used that  $t_i/\|B_i\|_\infty \rightarrow 0$  as  $i \rightarrow \infty$ . Here  $X_i$  is  $x$  after rescaling. Thus in the limit, we would obtain a positive decreasing (for  $X \geq 0$ ) function  $\bar{B}$  such that  $\|\bar{B}\|_\infty = 1$  and  $-\bar{B}'' = \bar{B}^2$ . Thus  $\bar{B}$  is concave and we easily obtain a contradiction. This completes the proof of the existence and uniqueness of the small homoclinic solution.

**Remark**

1. If  $t$  is close to zero, it is easy to modify our arguments to show that this solution is non-degenerate (modulo translations). The arguments here are a little simpler. The key points are that the reduction and scaling do not affect non-degeneracy. This is of interest because in general difficult to prove that homoclinic solutions are non-degenerate.

2. Note that there can be other types of solutions for example positive solutions periodic in  $x_{k+1}$  and solutions increasing in  $x_{k+1}$  for  $x_{k+1} \geq 0$ .

Summarizing this section, we have proved the following theorem.

**Theorem 1** *Assume that  $u_0$  is a solution of  $-\Delta'u = \lambda_0 f(u)$  on  $\Omega, u = 0$  on  $\partial\Omega$  such that the principal eigenvalue of (2) is zero and Assumptions N1 and N2 hold and the integral in Assumption N2 is strictly positive. For  $t$  small and negative, let  $u(t)$  denote the unique stable solution of  $-\Delta'u = (\lambda_0 - t^2)f(u)$  in  $\Omega, u = 0$  on  $\partial\Omega$  near  $u_0$ . Then there is an  $\epsilon > 0$  such that for  $t$  small and negative there is a unique solution  $\tilde{u}(t)$  of (1) for  $\lambda = \lambda_0 - t^2$  which is even in  $x_{k+1}, \tilde{u}(t) > u_0(t)$  on  $\Omega \times R, \|\tilde{u}(t) - u_0(t)\|_\infty \leq \epsilon$  and  $\tilde{u}(t) \rightarrow u_0(t)$  as  $x_{k+1} \rightarrow \infty$ . Moreover  $\tilde{u}(t)$  is non-degenerate modulo translations.*

**Remarks**

1. The result can almost certainly be proved by using the centre manifold reduction ideas of Kirchgassner on infinite strips rather than our Liapounov-Schmidt reduction. We have used the latter technique because it is also usable for the long domain problem in §2 where it seems to have the potential to produce more results.

2. Our techniques could also be used to prove the uniqueness of the small positive even homoclinic solution of  $-\Delta u = \lambda f(u)$  on  $S$  which decays at infinity if  $\lambda$  is close to but less than  $\lambda_0$ . Here  $f(0) = 0, f'(0) = 1, f''(0) > 0$ .

### 3 The bounded domain case

In this section, we combine the result of the last section with results in [4] to study the large bounded domain problem. We assume that  $\tilde{\Omega}$  is a smooth bounded domain in  $R^k$ . Then we are interested in positive solutions of

$$\begin{aligned} \Delta u &= \lambda f(u) \text{ on } \Omega_n \\ u &= 0 \text{ on } \partial\Omega_n \end{aligned} \tag{9}$$

for  $n$  large where  $\Omega_n = \tilde{\Omega} \times (-n, n)$ . We assume that  $u_0$  is a neutrally stable positive solution of

$$\begin{aligned} -\Delta'u &= \lambda_0 f(u) \text{ on } \tilde{\Omega} \\ u &= 0 \text{ on } \partial\tilde{\Omega} \end{aligned} \tag{10}$$

such that the end equation

$$\begin{aligned}
 -\Delta u &= \lambda_0 f(u) \text{ on } \tilde{\Omega} \times [0, \infty) \\
 u &= 0 \text{ on } (\tilde{\Omega} \times \{0\}) \cup (\partial\tilde{\Omega} \times [0, \infty)) \\
 u &\longrightarrow u_0 \text{ as } x_{k+1} \longrightarrow \infty \text{ uniformly in } x' \\
 u &> 0 \text{ on } \tilde{\Omega} \times (0, \infty) .
 \end{aligned} \tag{11}$$

has a positive solution  $\hat{u}$ . The existence of a positive solution of this end equation is discussed extensively in [4] and the uniqueness is proved. In particular, it always exists if  $f(0) > 0$  and  $u_0$  is the minimal non-negative solution of (10). This is a little more than is proved in [4] because we remove a non-degeneracy hypotheses. The changes necessary are discussed in the appendix. Note that the end equation always has at most one solution and a necessary condition for a solution to exist is that  $u_0$  is a weakly stable solution of (10). Note also that by the theory in [4] the existence of a solution of the end equation is a necessary condition to be able to find solutions of (9) near  $u_0$  on compact subsets of  $\Omega_n$  unless there exists a stable non-negative solution  $u$  of (10) with  $E(u) = E(u_0)$  and  $0 \leq u < u_0$ .

With these assumptions, we obtain a good understanding of the solutions  $u$  of (9) for  $n$  large,  $\lambda$  near  $\lambda_0$  and  $u$  uniformly close to  $u_0$  on compact subsets of  $\Omega_n$ .

First note that it is easy to check by continuity that the conditions of Proposition 2 of [4] hold for the solution  $(u(t), \lambda_0 - t^2)$  of (9) where  $t$  is small and negative. We also use Proposition 3 of [4] here. Thus by Proposition 2 in [4] and our remarks in the appendix, the end equation has a solution for  $\lambda_0$  replace by  $\lambda_0 - t^2$  and  $u_0$  replaced by  $u_0(t)$  with  $t$  small negative. It is also easy to see this solution depends continuously on  $t$ . We can now apply Theorem 2 in [4], the theory on §1 of [4] and Theorem 1 here to deduce that there exists  $\epsilon_0 > 0$  such that for  $t$  small and negative, and  $n$  large (depending on  $t$ ), (9) has exactly 2 solutions within  $\epsilon_0$  of  $u_0(t)$  on compact sets in  $\text{cl}(\tilde{\Omega}) \times R$ . One of these  $u_{0,t,n}$  is stable and uniformly close to  $u_0(t)$  on compact subsets of  $\text{cl}(\tilde{\Omega}) \times R$  if  $n$  is large while the other  $u_{1,t,n}$  is unstable and is uniformly close to  $u_1(t)$  on compact sets where  $u_1(t)$  is the solution in Theorem 1. Note that we have also used Remark 1 before the statement of Theorem 1 here. Hence we see that we have exactly 2 positive solutions in our neighbourhood which are easily seen to depend continuously on  $t$  except if  $-\epsilon(n) < t < \epsilon(n)$  where  $\epsilon(n) \longrightarrow 0$  as  $n \longrightarrow \infty$ . It is convenient to reformulate this slightly differently.

Let  $Z_n$  denote the function which is  $\ell_n u_0 + (1 - \ell_n)\hat{u}$  where  $\ell_n(x_{k+1}) = 1$  if  $|x_{k+1}| \leq \frac{1}{2}n, \ell_n(x_{k+1}) = 0$  if  $|x_{k+1}| > \frac{2}{3}n, 0 \leq \ell \leq 1$  and  $\sup_{|x_{k+1}| \leq \frac{2}{3}n} |\ell'_n(y)| \longrightarrow 0$  as  $n \longrightarrow \infty$ . We then see from what we have proved above, the theory in [4] and standard degree theory that the following theorem holds.

**Theorem 2** *Assume that the above conditions hold and assumptions N1 and N2 hold. There exist  $\epsilon_1, \epsilon_2 > 0$  and  $\epsilon(n) > 0$  with  $\epsilon(n) \longrightarrow 0$  as  $n \longrightarrow \infty$  such that the only positive solutions of (9) with  $\lambda_0 - \epsilon_1 \leq \lambda \leq \lambda_0 - \epsilon(n)$  and  $\|u - Z_n\|_\infty \leq \epsilon_2$  are  $(u_{0,t,n}, \lambda_0 - t^2), (u_{1,t,n}, \lambda_0 - t^2)$  where  $t < 0$  and  $\lambda_0 - \epsilon_1 \leq \lambda_0 - t^2 \leq \lambda_0 - \epsilon(n)$ . Any other positive solution of (9) with  $\|u - Z_n\|_\infty \leq \epsilon_2, \lambda - \epsilon(n) \leq \lambda \leq \lambda_0 + \epsilon_1$  satisfies  $\lambda \longrightarrow \lambda_0$  and  $\|u - Z_n\|_\infty \longrightarrow 0$  as  $n \longrightarrow \infty$ . Moreover, there is a continuum of positive solutions of (9) joining  $(u_{o,m,n}, \lambda_0 - \epsilon(n)), (u_{1,m,n}, \lambda_0 - \epsilon(n))$  in  $\lambda \geq \lambda_0 - \epsilon(n)$*

where  $m = \epsilon(n)^{\frac{1}{2}}$ .

**Remarks**

1. Note that any bad behaviour of the continuum is concentrated very close to  $(Z_n, \lambda_0)$ . We have not excluded bad behaviour there including secondary bifurcations and extra very small continua though they seem unlikely. Note that if  $f$  is convex on  $[0, \|u_0\|_\infty + \delta]$ , where  $\delta > 0$  we can use theory for convex maps as in [3] show that the minimal solution  $u_{0,t,n}$  continues to exist up to  $\lambda = \tilde{\lambda}_n$  where  $\tilde{\lambda}_n > \lambda_0$ , there are no solutions near  $u_0$  if  $\lambda > \tilde{\lambda}_n$ , exactly one in a neighbourhood  $U_n$  of the minimal solution for  $\lambda = \tilde{\lambda}_n$  and exactly 2 in  $U_n$  if  $\tilde{\lambda}_n - \delta_n^1 < \lambda < \tilde{\lambda}_n$ . However  $\delta_n^1$  and  $U_n$  may depend on  $n$  and could possibly be small for  $n$  large.

2. If there is a weakly stable solution  $\hat{u}$  of (10) with  $\hat{u} < u_0$  and  $E(\hat{u}) = E(u_0)$ , then the theory will change because there is a possibility of solutions which for  $x_n \geq 0$  look like a patching together of  $u_0$  (or  $u_1$ ) with a heteroclinic solution of (4) joining  $\hat{u}$  and  $u_0$  and solutions of the end equation with  $u_0$  replaced by  $\hat{u}$ . Indeed since the end equation (11) has no solution in this case (by a result of the appendix) solutions of the type we were looking for in Theorem 2 do not exist.

3. We suspect that in Theorem 2 the solutions form a connected curve with only one turning point. However this seems technical to prove. With care, we can modify part of the proof of Theorem 1 to reduce our problem to an ordinary differential equation with higher order (possibly non local) terms. However, this equation seems much more difficult to analyse, partly because of the effects of the ends. Indeed, the case where  $u_0 \equiv 0$  in Remark 2 after Theorem 1 is much easier (because the end equation has no effect) and it can be shown with care that the natural result holds in this case. To prove this one uses a similar reduction to that in the proof of Theorem 1, various rescalings and similar arguments to that in the proof of Theorem 1.

## 4 Appendix

We sketch how to improve slightly several results of [4] which we have used in the main text. These are all about weakening non degeneracy hypotheses.

Firstly, we want to prove that if  $u$  is a solution of (1) such that  $\lim u(x', x_{k+1}) = u_0(x')$  uniformly in  $x'$ , if  $u(x', x_{k+1}) \geq u_0(x')$  always and if  $u_0$  satisfies Assumption N2, then, up to a translation in  $x_{k+1}$ ,  $u$  is even in  $x_{k+1}$  and  $\frac{\partial u}{\partial x_{k+1}} < 0$  if  $x_{k+1} > 0$ . As in [1] or [4], this is a very standard application of the moving plane method (applied to  $u(x', x_{k+1}) - u_0(x')$ ) if we obtain good asymptotics of  $u(x', x_{k+1}) - u_0(x')$  as  $x_{k+1} \rightarrow \infty$  (and similarly as  $x_{k+1} \rightarrow -\infty$ ). We can easily apply the theory in [1] (cp also Lemma 2 3 on p41 of [4]) to prove that  $u(x', x_{k+1}) = u_0(x') + B(x_{k+1})\phi_1 + r(x', x_{k+1})$  where  $r$  is orthogonal to  $\phi_1$  for each  $x_{k+1}$ ,  $r$  decays exponentially in  $x_{k+1}$  in the  $C^1$  norm and  $B$  decays slower than any exponential. Hence, we see that for  $x_{k+1}$  large  $u(x', x_{k+1}) - u_0(x') = (B(x_{k+1}) + o(1))\phi_1$  where  $o(1)$  decays exponentially. Thus we need only determine the asymptotics of  $B$ . By using our expansion of  $u$ , our equation for  $u$  and Assumption N2, it is an easy calculation to prove that  $B'' = (C + 0(B))B^2$  for large  $x_{k+1}$  where  $C$  is defined in §1 and  $C > 0$ . We use that  $f$  is  $C^3$  here. We prove that this

implies  $B(x_{k+1}) = \frac{C_1}{x_{k+1}^2} + \frac{C_3}{x_{k+1}^3} + o(x_{k+1}^{-3})$  for large  $x_{k+1}$  where  $C_1 > 0$  which gives the required asymptotics and hence the result. By a rescaling, we may assume  $C = 1$ . Since  $B > 0$  and  $B \rightarrow 0$  as  $x_{k+1} \rightarrow \infty$ , we easily deduce that  $B'' > 0$  and  $B' < 0$  for  $x_{k+1}$  large. If  $\mu > 1$ ,  $(\frac{1}{2}B'^2 - \frac{1}{3}\mu B^3)' = B'(B'' - \mu B^2) > 0$  for  $x_{k+1}$  large and hence  $\frac{1}{2}(B')^2 - \frac{1}{3}\mu B^3 < 0$  for large  $x_{k+1}$ . Similarly  $\frac{1}{2}B' - \frac{1}{3}\tau B^3 > 0$  for large  $x_{k+1}$  if  $\tau < 1$ . Since  $B' < 0$ , we deduce that  $B'/B^{\frac{3}{2}} \rightarrow -\sqrt{\frac{2}{3}}$  as  $x_{k+1} \rightarrow \infty$ . Hence  $B = C_1(x_{k+1})^{-2} + o(x_{k+1}^{-2})$  where  $C_1 = \frac{1}{6}$ . It is then easy to return to the  $B$  equation and obtain the estimate for the error. (We prove  $\frac{1}{2}(B')^2 - \frac{1}{3}B^3 = O(x_{k+1}^{-8})$  for large  $x_{k+1}$ ). Note that we could also handle some more degenerate cases if  $f$  is smooth.

Secondly, we generalize some results of [4] for the end equation.

**Proposition** *Assume that  $u_0$  is a weakly stable solution of (10) which is isolated from below (which is certainly true when Assumption N2 holds) and  $f(0) \geq 0$ . Then*

- (i) *The end equation (11) has at most one solution.*
- (ii) *The end equation has a solution if there does not exist a weakly stable solution  $\hat{u}$  of (10) with  $0 \leq \hat{u} < u_0$  and  $E(\hat{u}) \leq E(u_0)$ .*
- (iii) *The end equation has no solution if there exists a weakly stable solution  $\hat{u}$  of (10) with  $0 \leq \hat{u} < u_0$  and  $E(\hat{u}) \geq E(u_0)$  and Assumption N2 holds.*

*Proof.* We assume throughout the proof that  $u_0$  is degenerate. Otherwise the results in [4] apply.

(i) If  $u_1$  and  $u_2$  are distinct solutions of the end equation then as in [4], p22,  $\sup \{u_1, u_2\}$  is a subsolution of the end equation. Since  $u_0$  is a supersolution, then as in [4], p22, there is a solution  $u_3$  of the end equation with  $u_3 \geq \sup \{u_1, u_2\}$ . Thus if there is a non-uniqueness there must exist two solutions  $u_1, u_3$  of the end equation with  $u_3 > u_1$ . By the maximum principle applied to the equation for  $u_3 - u_1$ ,  $\frac{\partial u_3}{\partial n} > \frac{\partial u_1}{\partial n}$  on  $\partial\Omega \times \{0\}$ . However, by the first integral, we see as on page 34 of [4] that  $E(u_0) = -\frac{1}{2} \int_{\Omega} (\frac{\partial u_i}{\partial x_{k+1}})(x', 0))^2 dx'$ .

Hence we have a contradiction.

- (ii) This follows the proof of Proposition 2 in [4] which does not use the nondegeneracy.
- (iii) Part (iii) follows the proof of Proposition 3 in [4]. The only difference is in Step 3 of the proof there where in the sliding hump argument we have to use the asymptotics in the first part of this appendix rather than the exponential asymptotics in [4]. For this, we need Assumption N2 (though this assumption could be weakened).

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