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Global and non global solutions for a class of coupled parabolic systems

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Abstract: In the present paper, we investigate the global well-posedness and exponential decay for some coupled non-linear heat equations. Moreover, we discuss the global and non global existence of solutions using the potential well method.

Keywords: non-linear heat system, global well-posedness, decay, blow-up

MSC: 35K55

1 Introduction

This paper is concerned with the Cauchy problem for a coupled heat system with power-type non-linearities

$$\begin{cases} \dot{u}_j - \Delta u_j + \delta u_j = \mu \left(\sum_{k=1}^m a_{jk} |u_k|^p \right) |u_j|^{p-2} u_j; \\ u_j(0, \cdot) = \psi_j, \end{cases} \quad (1.1)$$

which models a broad variety of physical phenomena called reaction-diffusion equations, motivated by neuroscience, surface chemistry, gas dynamics or predator-prey interactions [3, 10, 11, 16].

Here and hereafter $u_j : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}$ for $j \in [1, m]$, $\delta \in \{0, 1\}$, $\mu = \pm 1$ and $a_{jk} = a_{kj}$ are positive real numbers. A solution $\mathbf{u} := (u_1, \dots, u_m)$ to (1.1) formally satisfies some decay of the energy

$$\begin{aligned} E^\delta(\mathbf{u}(t)) &:= \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} \left(|\nabla u_j|^2 + \delta |u_j|^2 - \frac{\mu}{p} \sum_{k=1}^m a_{jk} |u_j(t, x)|^p |u_k(t, x)|^p \right) dx \\ &= E^\delta(\mathbf{u}(0)) - \sum_{j=1}^m \int_0^t \|\dot{u}_j(s)\|_{L^2(\mathbb{R}^N)}^2 ds. \end{aligned}$$

The particular case $a_{jk} = \delta_j^k$ gives some classical scalar semi-linear heat equations. Thus, before going further, let us recall some historic facts about the semi-linear parabolic equation. The model case given by a pure power non-linearity is of particular interest. The question of well-posedness in the energy space of the following heat problem

$$(NLH)_p \quad \dot{u} - \Delta u \pm |u|^{p-1} u = 0, \quad p > 1, \quad u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R},$$

was widely investigated. This equation satisfies a scaling invariance. Indeed, if u is a solution to $(NLH)_p$ with datum u_0 , then $u_\lambda := \lambda^{\frac{2}{p-1}} \mathbf{u}(\lambda^2 \cdot, \lambda \cdot)$ is a solution to $(NLH)_p$ with data $\lambda^{\frac{2}{p-1}} u_0(\lambda \cdot)$. For $s_c := \frac{N}{2} - \frac{2}{p-1}$, the space \dot{H}^{s_c} whose norm is invariant under the dilatation $u \mapsto u_\lambda$ is relevant in this theory. The energy critical case $s_c = 1$ corresponds to the critical power $p_c := \frac{N+2}{N-2}$, for $N \geq 3$.

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Local well-posedness of $(NLH)_p$ holds in the energy critical case and the local existence interval does not depend only on $\|u_0\|_{H^1}$. Then, an iteration of the local well-posedness theory fails to prove global existence and a finite time blow-up of solutions may happen [9]. Now, the energy critical case of $(NLH)_p$ is known to be well-posed in some Besov spaces [15]. See [12] in the two space dimensions case and [4, 24, 25] in the scale of Lebesgue spaces $L^q(\mathbb{R}^N)$.

The topic of blow-up of solutions to bi-component parabolic systems with positive data on bounded domains have been attracting great attention. There have been numerous publications in the literature in this direction and we refer the interested reader to [6, 8, 13, 14, 18, 20, 22] and references therein.

This paper seems to be one of few works dealing with m-component coupled semi-linear heat systems. Moreover, to the author knowledge, the stability of standing waves was not treated in the case of non-linear heat equations. The parabolic system (1.1) is a generalization of the bi-component problem considered in [26], where the global existence, long time decay and finite time blowup of solutions were investigated using the potential-well method.

It is the purpose of this manuscript to obtain global well-posedness and exponential decay of solutions to the defocusing non-linear coupled heat system (1.1), in the energy space. In the focusing sign, using the concepts of invariant sets suggested by Payne and Sattinger in [19], global and non global existence of solutions are discussed, moreover an exponential decay in the energy space holds for any global solution under the potential well. Finally, the existence of infinitely many non global solutions near the ground state is obtained.

The rest of the paper is organized as follows. The next section contains the main results and some technical tools needed in the sequel. Sections three and four are devoted to proving well-posedness of the heat system (1.1) in the energy space. In section five, the existence of critical ground states is investigated. In section 6, global and non global existence of solutions to (1.1) are discussed. The last section contains a proof of strong instability for stationary solutions.

We define the product space

$$\mathcal{H} := H^1(\mathbb{R}^N) \times \dots \times H^1(\mathbb{R}^N) = [H^1(\mathbb{R}^N)]^m,$$

where $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the complete norm

$$\|u\|_{H^1(\mathbb{R}^N)} := \left(\|u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \right)^{\frac{1}{2}}.$$

We denote the real numbers

$$p^* := 1 + \frac{2}{N} \quad \text{and} \quad p^c := \begin{cases} \frac{N}{N-2} & \text{if } N > 2; \\ \infty & \text{if } N = 2 \end{cases}$$

and we assume here and hereafter that

$$\delta = 1 - \delta_p^{p^c} = \begin{cases} 0 & \text{if } p = p^c; \\ 1 & \text{if } p \neq p^c. \end{cases}$$

We mention that C will denote a constant which may vary from line to line and if A and B are non negative real numbers, $A \lesssim B$ means that $A \leq CB$. For $1 \leq r \leq \infty$ and $(s, T) \in [1, \infty) \times (0, \infty)$, we denote the Lebesgue space $L^r := L^r(\mathbb{R}^N)$ with the usual norm $\|\cdot\|_r := \|\cdot\|_{L^r}$, $\|\cdot\| := \|\cdot\|_2$ and

$$\|u\|_{L^s_r(L^r)} := \left(\int_0^T \|u(t)\|_r^s dt \right)^{\frac{1}{s}}, \quad \|u\|_{L^s(L^r)} := \left(\int_0^{+\infty} \|u(t)\|_r^s dt \right)^{\frac{1}{s}}.$$

For simplicity, we denote the usual Sobolev Space $W^{s,p} := W^{s,p}(\mathbb{R}^N)$ and $H^s := W^{s,2}$. If X is an abstract space $C_T(X) := C([0, T], X)$ stands for the set of continuous functions valued in X and X_{rd} is the set of radial elements in X , moreover for an eventual solution to (1.1), we denote $T^* > 0$ its lifespan.

2 Main results and background

In what follows, we give the main results and some estimates needed in the sequel.

2.1 Main results

First, local well-posedness of the heat problem (1.1) is claimed.

Theorem 2.1. *Let $2 \leq N \leq 3$, $2 \leq p < p^c$ and $\Psi \in \mathcal{H}$. Then, there exist $T^* > 0$ and a unique maximal solution to (1.1),*

$$\mathbf{u} \in C([0, T^*), \mathcal{H}).$$

Moreover,

- 1/ $\mathbf{u} \in (L^{\frac{4p}{N(p-1)}}([0, T^*], W^{1,2p}))^m$;
- 2/ \mathbf{u} satisfies decay of the energy;
- 3/ if $\mu = -1$, it follows that $T^* = \infty$ and there exists $\gamma > 0$ such that

$$\|\mathbf{u}(t)\|_{\mathcal{H}} = O(e^{-\gamma t}), \quad \text{when } t \rightarrow +\infty.$$

Remark 2.2. *The unnatural condition $p \geq 2$ seems to be technical and yields to the restriction $N \leq 3$.*

In the critical case, global existence of solutions to (1.1) holds in the energy space for small data.

Theorem 2.3. *Take $N = 3$ and assume that $p = p^c$. Then, there exists $\epsilon_0 > 0$ such that if $\Psi := (\psi_1, \dots, \psi_m) \in \mathcal{H}$ satisfies $\sum_{j=1}^m \int_{\mathbb{R}^N} |\nabla \psi_j|^2 dx \leq \epsilon_0$, the system (1.1) possesses a unique solution $\mathbf{u} \in C(\mathbb{R}_+, \mathcal{H})$.*

Second, we are interested on the focusing case ($\mu = 1$). For $\mathbf{u} := (u_1, \dots, u_m) \in \mathcal{H}$, $\alpha, \beta \in \mathbb{R}$, we call constraint

$$2K_{\alpha,\beta}^\delta(\mathbf{u}) := \sum_{j=1}^m ((2\alpha + (N - 2)\beta)\|\nabla u_j\|^2 + (2\alpha + N\beta)\delta\|u_j\|^2) - \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} (2p\alpha + N\beta)|u_j u_k|^p dx.$$

Definition 2.4. $\Psi := (\psi_1, \dots, \psi_m)$ is said to be a ground state solution to (1.1) if

$$\Delta \psi_j - \delta \psi_j + \sum_{k=1}^m a_{jk} |\psi_k|^p |\psi_j|^{p-2} \psi_j = 0, \quad 0 \neq \Psi \in \mathcal{H}_{rd} \tag{2.2}$$

and it minimizes the problem

$$m_{\alpha,\beta}^\delta := \inf_{0 \neq \mathbf{u} \in \mathcal{H}} \left\{ E^\delta(\mathbf{u}) \quad \text{s.t.} \quad K_{\alpha,\beta}^\delta(\mathbf{u}) = 0 \right\}. \tag{2.3}$$

Remark 2.5. *If Ψ satisfies (2.2), then it is a global solution to (1.1).*

The existence of ground states in the sub-critical case is known [21]. In the critical case, the situation is as follows.

Proposition 2.6. *Take a couple of real numbers $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+ \cup \{(1, -\frac{2}{N})\}$ and $p = p^c$. Then,*

- 1/ $m^0 := m_{\alpha,\beta}^0$ is nonzero and independent of (α, β) ;
- 2/ there is a minimizer of (2.3) in the following meaning

$$0 \neq \Psi \in \mathcal{H}_{rd} \quad \text{and} \quad m^0 = E^0(\Psi). \tag{2.4}$$

Remark 2.7. *It is not proved that the minimizer of (2.3) is a solution to (2.2), because of a lack of uniqueness of such a solution. Despite, we will call Ψ as ground state.*

Using the potential well method [19], we discuss global existence and finite-time blow-up of solutions to the focusing problem (1.1). Let us define the sets

$$A_{\alpha,\beta}^{\delta,+} := \{\mathbf{u} \in \mathcal{H} \text{ s. t. } E^\delta(\mathbf{u}) < m_{\alpha,\beta}^\delta \text{ and } K_{\alpha,\beta}^\delta(\mathbf{u}) \geq 0\};$$

$$A_{\alpha,\beta}^{\delta,-} := \{\mathbf{u} \in \mathcal{H} \text{ s. t. } E^\delta(\mathbf{u}) < m_{\alpha,\beta}^\delta \text{ and } K_{\alpha,\beta}^\delta(\mathbf{u}) < 0\}.$$

For easy notation, we write

$$E := E^1, K_{\alpha,\beta} := K_{\alpha,\beta}^1, m_{\alpha,\beta} := m_{\alpha,\beta}^1, A_{\alpha,\beta}^+ := A_{\alpha,\beta}^{1,+}, A_{\alpha,\beta}^- := A_{\alpha,\beta}^{1,-}.$$

Theorem 2.8. *Take $2 \leq N \leq 3, 2 \leq p < p^c, (\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+$ and $\mu = \delta = 1$. Let $\mathbf{u} \in C_{T^*}(\mathcal{H})$ be a maximal solution to (1.1). Then,*

1/ *if there exists $t_0 \in [0, T^*)$ such that $\mathbf{u}(t_0) \in A_{\alpha,\beta}^+$, so \mathbf{u} is global. Moreover, for small $\|\mathbf{u}_0\|$, there exists $y > 0$ such that*

$$\|\mathbf{u}(t)\|_{\mathcal{H}} = O(e^{-yt}), \text{ when } t \rightarrow +\infty;$$

2/ *if there exists $t_0 \in [0, T^*)$ such that $\mathbf{u}(t_0) \in A_{\alpha,\beta}^-$, so \mathbf{u} blows-up in finite time.*

The last result concerns instability by blow-up for standing waves of the heat problem (1.1). Indeed, near a ground state, there exist infinitely many data giving non global solutions to (1.1).

Theorem 2.9. *Take $2 \leq N \leq 3, 2 \leq p < p^c$ and $\mu = \delta = 1$. Let Ψ be a ground state solution to (2.2). Then, for any $\varepsilon > 0$, there exists $\mathbf{u}_0 \in \mathcal{H}$ such that $\|\mathbf{u}_0 - \Psi\|_{\mathcal{H}} < \varepsilon$ and the maximal solution to (1.1) with data \mathbf{u}_0 is not global.*

In the next subsection, we give some standard estimates needed in the paper.

2.2 Tools

We start with some properties of the free heat kernel [4].

Proposition 2.10. *Denoting the free operator associated to the heat equation*

$$e^{t\Delta} \mathbf{u} := \mathcal{F}^{-1}(e^{-t|\cdot|^2}) * \mathbf{u} = \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}} e^{-\frac{|\cdot|^2}{4t}} * \mathbf{u}, \quad t > 0,$$

yields

1/ $e^{t\Delta} \Psi := (e^{t\Delta} \Psi_1, \dots, e^{t\Delta} \Psi_m)$ is the solution to the linear problem associated to (1.1);

2/ $e^{t\Delta} \Psi + \int_0^t e^{(t-s)\Delta} (-\delta u_1 + \mu u_1 |u_1|^{p-2} \sum_{k=1}^m a_{1k} |u_k|^p, \dots, -\delta u_m + \mu u_m |u_m|^{p-2} \sum_{k=1}^m a_{mk} |u_k|^p) ds$ is the solution to the problem (1.1);

3/ $(e^{t\Delta})^* = e^{t\Delta}$;

4/ $\|e^{t\Delta} \mathbf{u}\|_{L^y} \lesssim \frac{\|\mathbf{u}\|_{L^\beta}}{t^{\frac{N}{2}(\frac{1}{\beta} - \frac{1}{y})}}$, for all $1 \leq \beta \leq y \leq \infty, t > 0, \mathbf{u} \in L^\beta(\mathbb{R}^N)$.

Let us recall the so-called Strichartz estimate [7].

Definition 2.11. *A pair (q, r) of positive real numbers is said to be admissible if*

$$2 \leq q, r \leq \infty, (q, r, N) \neq (2, \infty, 2) \text{ and } \frac{2}{q} = N\left(\frac{1}{2} - \frac{1}{r}\right).$$

Proposition 2.12. *Let two admissible pairs (q, r) and (a, b) . Then, there exists a positive real number C such that for any $T > 0$,*

$$\|\mathbf{u}\|_{L_T^q(L^r)} \leq C\left(\|\mathbf{u}(0, \cdot)\| + \|\dot{\mathbf{u}} - \Delta \mathbf{u}\|_{L_T^{q'}(L^{b'})}\right). \tag{2.5}$$

Existence of a ground state solution to (1.1) was obtained recently [21].

Proposition 2.13. Take $N \geq 2, p_* < p < p^c$ and two real numbers $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+ \cup \{(1, -\frac{2}{N})\}$. Then,

- 1/ $m := m_{\alpha, \beta}$ is nonzero and independent of (α, β) ;
- 2/ there is a minimizer of (2.3), which is some nontrivial solution to (2.2);
- 3/ if we make the following assumptions

$$a_{jj} = \mu_j \text{ and } a_{jk} = \mu \text{ for } j \neq k \in [1, m]$$

then, at least two components of the minimizer are non zero if $\mu > 0$ is large enough;

- 4/ if $(N, \alpha) \neq (2, 0)$,

$$m_{\alpha, \beta} = \inf_{0 \neq \mathbf{u} \in \mathcal{H}} \{H_{\alpha, \beta}(\mathbf{u}) := (E - \frac{1}{2\alpha + N\beta} K_{\alpha, \beta})(\mathbf{u}) \text{ s. t. } K(\mathbf{u}) \leq 0\}.$$

In the rest of this subsection, we collect some standard estimates independent of the parabolic problem (1.1).

Proposition 2.14. Let $\varepsilon > 0$. There is no real function $G \in C^2(\mathbb{R}_+)$ satisfying

$$G(0) > 0, G'(0) > 0 \text{ and } GG'' - (1 + \varepsilon)(G')^2 \geq 0 \text{ on } \mathbb{R}_+.$$

Proof. Assume with contradiction, the existence of such a function. Then $(G^{-(1+\varepsilon)} G')' \geq 0$ and

$$\frac{G'}{G^{1+\varepsilon}} \geq \frac{G'(0)}{G(0)} > 0.$$

This is a Riccati inequality with blow-up time $T < \frac{1}{\varepsilon} \frac{G(0)}{G'(0)}$. This contradiction achieves the proof. □

Let us gather some useful Sobolev embeddings [1].

Proposition 2.15. The continuous injections hold

- 1/ $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ whenever $1 < p < q < \infty, s > 0$ and $\frac{1}{p} \leq \frac{1}{q} + \frac{s}{N}$;
- 2/ $W^{s,p_1}(\mathbb{R}^N) \hookrightarrow W^{s-N(\frac{1}{p_1} - \frac{1}{p_2}), p_2}(\mathbb{R}^N)$ whenever $1 \leq p_1 \leq p_2 \leq \infty$.

The following Gagliardo-Nirenberg inequality [17] will be useful.

Proposition 2.16. Take $N \geq 2$ and $1 < p \leq p^c$. Then, for any $(u_1, \dots, u_m) \in \mathcal{H}$,

$$\sum_{j,k=1}^m \int_{\mathbb{R}^N} |u_j u_k|^p dx \leq C \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{(p-1)N}{2}} \left(\sum_{j=1}^m \|u_j\|^2 \right)^{\frac{N-p(N-2)}{2}}. \tag{2.6}$$

We close this subsection with an absorption result.

Lemma 2.17. Let $T > 0$ and $X \in C([0, T], \mathbb{R}_+)$ such that

$$X \leq a + bX^\theta \text{ on } [0, T],$$

where $a, b > 0, \theta > 1, a < (1 - \frac{1}{\theta}) \frac{1}{(\theta b)^{\frac{1}{\theta}}}$ and $X(0) \leq \frac{1}{(\theta b)^{\frac{1}{\theta-1}}}$. Then

$$X \leq \frac{\theta}{\theta - 1} a \text{ on } [0, T].$$

Proof. The function $f(x) := bx^\theta - x + a$ is decreasing on $[0, (b\theta)^{\frac{1}{1-\theta}}]$ and increasing on $[(b\theta)^{\frac{1}{1-\theta}}, \infty)$. The assumptions imply that $f((b\theta)^{\frac{1}{1-\theta}}) < 0$ and $f(\frac{\theta}{\theta-1} a) \leq 0$. As $f(X(t)) \geq 0, f(0) > 0$ and $X(0) \leq (b\theta)^{\frac{1}{1-\theta}}$, we conclude the proof by a continuity argument. □

3 Local well-posedness

This section is devoted to prove Theorem 2.1. The proof contains three steps: local existence, uniqueness and global existence in the sub-critical case. In this section, we take $\mu = -1$, indeed the sign of the non-linearity has no local effect.

3.1 Local existence

We use a standard fixed point argument. For $T > 0$, $\frac{R}{2} := C\|\Psi\|_{\mathcal{H}}$ we denote the space

$$E_{T,R} := \left\{ \mathbf{u} \in C_T(\mathcal{H}) \cap \left(L_T^{\frac{4p}{N(p-1)}}(W^{1,2p}) \right)^m \text{ s. t. } \left(\|\mathbf{u}\|_{L_T^\infty(\mathcal{H})} + \|\mathbf{u}\|_{L_T^{\frac{4p}{N(p-1)}}(W^{1,2p})} \right) \leq R \right\}$$

endowed with the complete distance

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_T, \quad \|\mathbf{u}\|_T := \sum_{j=1}^m \left(\|u_j\|_{L_T^\infty(L^2)} + \|u_j\|_{L_T^{\frac{4p}{N(p-1)}}(L^{2p})} \right).$$

Define the function

$$\phi(\mathbf{u})(t) := e^{t\Delta}\Psi - \sum_{k=1}^m a_{jk} \int_0^t e^{(t-s)\Delta} \left(|u_k|^p |u_1|^{p-2} u_1 + u_1, \dots, |u_k|^p |u_m|^{p-2} u_m + u_m \right) ds.$$

We prove the existence of some small $T, R > 0$ such that ϕ is a contraction on the ball $B_T(R)$ with center zero and radius R . Take $\mathbf{u}, \mathbf{v} \in E_T$, applying the Strichartz estimate (2.5), we get for small $T > 0$,

$$\begin{aligned} d(\phi(\mathbf{u}), \phi(\mathbf{v})) &\lesssim \sum_{j,k=1}^m \left\| |u_k|^p |u_j|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j \right\|_{L^{\frac{4p}{p(4-N)+N}}(L^{\frac{2p}{2p-1}})} + \|\mathbf{u} - \mathbf{v}\|_{L_T^1(L^2)} \\ &\lesssim \sum_{j,k=1}^m \left\| |u_k|^p |u_j|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j \right\|_{L^{\frac{4p}{p(4-N)+N}}(L^{\frac{2p}{2p-1}})} + T \|\mathbf{u} - \mathbf{v}\|_{L_T^\infty(L^2)}. \end{aligned}$$

To derive the contraction, consider the function

$$f_{jk} : \mathbb{R}^m \rightarrow \mathbb{R}, \quad (u_1, \dots, u_m) \mapsto |u_k|^p |u_j|^{p-2} u_j.$$

With the mean value Theorem

$$|f_{jk}(\mathbf{u}) - f_{jk}(\mathbf{v})| \lesssim \max\{|u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2}, |v_k|^p |v_j|^{p-2} + |v_k|^{p-1} |v_j|^{p-1}\} \|\mathbf{u} - \mathbf{v}\|.$$

Using Hölder inequality, Sobolev embedding and denoting the quantity

$$(\mathcal{J}) := \|f_{jk}(\mathbf{u}) - f_{jk}(\mathbf{v})\|_{L_T^{\frac{4p}{p(4-N)+N}}(L^{\frac{2p}{2p-1}})},$$

we compute via a symmetry argument

$$\begin{aligned} (\mathcal{J}) &\lesssim \left\| (|u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2}) \|\mathbf{u} - \mathbf{v}\| \right\|_{L_T^{\frac{4p}{p(4-N)+N}}(L^{\frac{2p}{2p-1}})} \\ &\lesssim \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4p}{N(p-1)}}(L^{2p})} \left\| |u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2} \right\|_{L_T^{\frac{4p}{4p-2N(p-1)}}(L^{\frac{p}{p-1}})} \\ &\lesssim T^{\frac{2p-N(p-1)}{2p}} \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4p}{N(p-1)}}(L^{2p})} \left(\|u_k\|_{L_T^\infty(L^{2p})}^{p-1} \|u_j\|_{L_T^\infty(L^{2p})}^{p-1} + \|u_k\|_{L_T^\infty(L^{2p})}^p \|u_j\|_{L_T^\infty(L^{2p})}^{p-2} \right) \\ &\lesssim T^{\frac{2p-N(p-1)}{2p}} \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4p}{N(p-1)}}(L^{2p})} \left(\|u_k\|_{L_T^\infty(H^1)}^{p-1} \|u_j\|_{L_T^\infty(H^1)}^{p-1} + \|u_k\|_{L_T^\infty(H^1)}^p \|u_j\|_{L_T^\infty(H^1)}^{p-2} \right). \end{aligned}$$

Then

$$\sum_{k,j=1}^m \|f_{jk}(\mathbf{u}) - f_{jk}(\mathbf{v})\|_{L_T^{\frac{4p}{p(4-N)+N}}(L^{\frac{2p}{2p-1}})} \lesssim T^{\frac{2p-N(p-1)}{2p}} R^{2(p-1)} \|\mathbf{u} - \mathbf{v}\|_T.$$

Thus, for $T > 0$ small enough, ϕ is a contraction satisfying

$$d(\phi(\mathbf{u}), \phi(\mathbf{v})) \lesssim \left(T^{\frac{2p-N(p-1)}{2p}} R^{2(p-1)} + T \right) d(\mathbf{u}, \mathbf{v}).$$

Taking in the last inequality $\mathbf{v} = 0$, yields

$$\begin{aligned} \|\phi(\mathbf{u})\|_T &\lesssim \left(T^{\frac{2p-N(p-1)}{2p}} R^{2(p-1)} + T \right) R + \|\Psi\| \\ &\lesssim \left(T^{\frac{2p-N(p-1)}{2p}} R^{2(p-1)} + T \right) R + \frac{R}{2}. \end{aligned}$$

Moreover, thanks to Hölder inequality and Sobolev embedding, we obtain

$$\begin{aligned} \|\nabla \phi(\mathbf{u})\|_T &\lesssim \frac{R}{2} + \|\nabla \mathbf{u}\|_{L_T^{\frac{4p}{N(p-1)}}(L^{2p})} \left(\| |u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2} \right)_{L_T^{\frac{4p}{4p-2N(p-1)}}(L^{\frac{p}{p-1}})} + \|\nabla \mathbf{u}\|_{L_T^1(L^2)} \\ &\lesssim \frac{R}{2} + T^{\frac{2p-N(p-1)}{2p}} \|\mathbf{u}\|_T \left(\|u_k\|_{L_T^\infty(L^{2p})}^{p-1} \|u_j\|_{L_T^\infty(L^{2p})}^{p-1} + \|u_k\|_{L_T^\infty(L^{2p})}^p \|u_j\|_{L_T^\infty(L^{2p})}^{p-2} \right) + T \|\nabla \mathbf{u}\|_{L_T^\infty(L^2)} \\ &\lesssim \frac{R}{2} + T^{\frac{2p-N(p-1)}{2p}} \|\mathbf{u}\|_T \left(\|u_k\|_{L_T^\infty(H^1)}^{p-1} \|u_j\|_{L_T^\infty(H^1)}^{p-1} + \|u_k\|_{L_T^\infty(H^1)}^p \|u_j\|_{L_T^\infty(H^1)}^{p-2} \right) + T \|\mathbf{u}\|_T \\ &\lesssim \frac{R}{2} + T^{\frac{2p-N(p-1)}{2p}} R^{2p-1} + TR. \end{aligned}$$

Since $2 \leq p < p^c$, ϕ is a contraction of $X_{T,R}$ for some $T > 0$ small enough.

3.2 Uniqueness

In what follows, we prove uniqueness of solution to the Cauchy problem (1.1). Let $T > 0$ be a positive time, $\mathbf{u}, \mathbf{v} \in C_T(\mathcal{H})$ two solutions to (1.1) and $(w_1, \dots, w_m) = \mathbf{w} := \mathbf{u} - \mathbf{v}$. Then

$$\dot{w}_j - \Delta w_j + \delta w_j = \sum_{k=1}^m \left(-|u_k|^p |u_j|^{p-2} u_j + |v_k|^p |v_j|^{p-2} v_j \right), \quad w_j(0, \cdot) = 0.$$

Applying Strichartz estimate with the admissible pair $(q, r) = (\frac{4p}{N(p-1)}, 2p)$ and denoting for simplicity $L_T^q(L^r)$ the norm of $(L_T^q(L^r))^m$, we get for small $T > 0$,

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|_{L_T^q(L^r) \cap L_T^\infty(L^2)} &\lesssim \sum_{j,k=1}^m \|f_{jk}(\mathbf{u}) - f_{jk}(\mathbf{v})\|_{L_T^{q'}(L^{r'})} + \|\mathbf{u} - \mathbf{v}\|_{L_T^1(L^2)} \\ &\lesssim \sum_{j,k=1}^m \|f_{jk}(\mathbf{u}) - f_{jk}(\mathbf{v})\|_{L_T^{q'}(L^{r'})} + T \|\mathbf{u} - \mathbf{v}\|_{L_T^\infty(L^2)} \\ &\lesssim \sum_{j,k=1}^m \|f_{jk}(\mathbf{u}) - f_{jk}(\mathbf{v})\|_{L_T^{q'}(L^{r'})}. \end{aligned}$$

Taking $T > 0$ small enough, with a continuity argument, we may assume that

$$\max_{j=1, \dots, m} \|u_j\|_{L_T^\infty(H^1)} \leq 1.$$

Using previous computation with

$$(J) := \|f_{jk}(\mathbf{u}) - f_{jk}(\mathbf{v})\|_{L_T^{q'}(L^{r'})} = \| |u_k|^p |u_j|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j \|_{L_T^{q'}(L^{r'})},$$

we have

$$\begin{aligned}
 (\mathcal{J}) &\lesssim \left\| \left(|u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2} \right) \mathbf{u} - \mathbf{v} \right\|_{L_T^{\frac{4p}{p(4-N)+N}} (L^{\frac{2p}{2p-1}})} \\
 &\lesssim \left\| \mathbf{u} - \mathbf{v} \right\|_{L_T^{\frac{4p}{p(4-N)+N}} (L^{2p})} \left\| |u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2} \right\|_{L_T^\infty (L^{\frac{p}{p-1}})} \\
 &\lesssim T^{\frac{(4-N)p+N}{4p}} \left\| \mathbf{u} - \mathbf{v} \right\|_{L_T^{\frac{4p}{N(p-1)}} (L^{2p})} \left(\|u_k\|_{L_T^\infty(H^1)}^{p-1} \|u_j\|_{L_T^\infty(H^1)}^{p-1} + \|u_k\|_{L_T^\infty(H^1)}^p \|u_j\|_{L_T^\infty(H^1)}^{p-2} \right).
 \end{aligned}$$

Then

$$\|\mathbf{w}\|_{L_T^q(L^r)} \lesssim T^{\frac{(4-N)p+N}{4p}} \|\mathbf{w}\|_{L_T^q(L^r)}.$$

Uniqueness follows for small time and then for all time with a translation argument.

3.3 Global existence in the defocusing sub-critical case

The global existence is a consequence of the energy decay and previous calculations. Let $\mathbf{u} \in C([0, T^*), \mathcal{H})$ be the unique maximal solution of (1.1). We prove that \mathbf{u} is global. By contradiction, suppose that $T^* < \infty$. Consider for $0 < s < T^*$, the problem

$$(\mathcal{P}_s) \begin{cases} \dot{v}_j - \Delta v_j + v_j = - \left(\sum_{k,j=1}^m |v_k|^p \right) |v_j|^{p-2} v_j; \\ v_j(s, \cdot) = u_j(s, \cdot). \end{cases}$$

Using the same arguments of local existence, we can prove a real $\tau > 0$ and a solution $\mathbf{v} = (v_1, \dots, v_m)$ to (\mathcal{P}_s) on $C([s, s + \tau], \mathcal{H})$. Thanks to the decay of energy we see that τ does not depend on s . Thus, if we let s be close to T^* such that $T^* < s + \tau$, this fact contradicts the maximality of T^* .

3.4 Exponential decay

This subsection is devoted to prove that $u \in C(\mathbb{R}_+, \mathcal{H})$, the global solution to (1.1) for $\delta = -\mu = 1$ and $1 < p < p^c$ satisfies an exponential decay in the energy space.

Denoting the quantity $K(\mathbf{u}(t)) := \sum_{j=1}^m \|u_j(t)\|_{H^1}^2 + \sum_{1 \leq j, k \leq m} a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx$, yields

$$E(\mathbf{u}(t)) \leq K(\mathbf{u}(t)) \leq 2pE(\mathbf{u}(t)).$$

On the other hand, for $T > 0$,

$$\begin{aligned}
 \int_t^T K(\mathbf{u}(s)) ds &= \frac{1}{2} \sum_{j=1}^m (\|u_j(t)\|^2 - \|u_j(T)\|^2) \\
 &\leq \frac{1}{2} \sum_{j=1}^m \|u_j(t)\|^2 \\
 &\leq E(\mathbf{u}(t)).
 \end{aligned}$$

So,

$$\int_t^T E(\mathbf{u}(s)) ds \lesssim \int_t^T K(\mathbf{u}(s)) ds \lesssim E(\mathbf{u}(t)).$$

Thus, for some positive real number $T_0 > 0$,

$$\begin{aligned} y(t) &:= \int_t^\infty E(\mathbf{u}(s)) \, ds \\ &\lesssim E(\mathbf{u}(t)) \\ &\leq -T_0 y'(t) \end{aligned}$$

This implies that, for $t \geq T_0$,

$$y(t) \leq y(T_0)e^{1-\frac{t}{T_0}} \leq T_0 E(\mathbf{u}(T_0))e^{1-\frac{t}{T_0}}.$$

Taking account of the monotonicity of the energy, for large $T > 0$,

$$\int_t^T E(\mathbf{u}(s)) \, ds \geq \int_t^{t+T_0} E(\mathbf{u}(s)) \, ds \geq T_0 E(\mathbf{u}(t + T_0)).$$

Then,

$$E(\mathbf{u}(t + T_0)) \leq E(\mathbf{u}(T_0))e^{1-\frac{t}{T_0}}.$$

Finally, because $\mu = -1$,

$$\|\mathbf{u}(t + T_0)\|_{\mathcal{H}^c}^2 \lesssim E(\mathbf{u}(t + T_0)) \leq E(\mathbf{u}(T_0))e^{1-\frac{t}{T_0}}.$$

The proof is finished.

4 Global existence in the critical case

In this section $N = 3$ and $\delta = 0$. We establish global existence of a solution to (1.1) in the critical case $p = p^c$ for small data as claimed in Theorem 2.3.

Letting $I \subset \mathbb{R}$ a time interval, we define the norms

$$\begin{aligned} \|\mathbf{u}\|_{M(I)} &:= \|\nabla \mathbf{u}\|_{L^{\frac{2(N+2)}{N-2}}(I, L^{\frac{2N(N+2)}{N^2+4}})}; \\ \|\mathbf{u}\|_{S(I)} &:= \|\mathbf{u}\|_{L^{\frac{2(N+2)}{N-2}}(I, L^{\frac{2(N+2)}{N-2}})}. \end{aligned}$$

Let $M(\mathbb{R})$ be the completion of $C_c^\infty(\mathbb{R}^{N+1})$ endowed with the norm $\|\cdot\|_{M(\mathbb{R})}$, and $M(I)$ be the set consisting of the restrictions to I of functions in $M(\mathbb{R})$. An important quantity closely related to the mass and the energy, is the functional ξ defined for $\mathbf{u} \in \mathcal{H}$ by

$$\xi(\mathbf{u}) = \sum_{j=1}^m \int_{\mathbb{R}^N} |\nabla u_j|^2 \, dx.$$

We give an auxiliary result.

Proposition 4.1. *Let $p = p^c$, $\Psi := (\psi_1, \dots, \psi_m) \in \dot{H} := (\dot{H}^1)^m$ and $A := \|\Psi\|_{\dot{H}}$. There exists $\delta := \delta_A > 0$ such that for any interval $I = [0, T]$, if*

$$\sum_{j=1}^m \|e^{tA} \psi_j\|_{S(I)} < \delta,$$

then there exists a unique solution $\mathbf{u} \in C(I, \mathcal{H})$ of (1.1) which satisfies $\mathbf{u} \in (M(I) \cap L^{\frac{2(N+2)}{N-2}}(I \times \mathbb{R}^N))$. Moreover,

$$\sum_{j=1}^m \|u_j\|_{S(I)} \leq 2\delta.$$

Proof. The proposition follows from a contraction mapping argument. Define the set

$$X_{a,b} := \left\{ \mathbf{u} \in M(I); \sum_{j=1}^m \|u_j\|_{M(I)} \leq a, \sum_{j=1}^m \|u_j\|_{S(I)} \leq b \right\}$$

where $a, b > 0$ are sufficiently small to fix later. We let the function

$$\phi(\mathbf{u})(t) := e^{t\Delta}\Psi - \sum_{k=1}^m \int_0^t e^{(t-s)\Delta} \left(|u_k|^{\frac{N}{N-2}} |u_1|^{\frac{4-N}{N-2}} u_1, \dots, |u_k|^{\frac{N}{N-2}} |u_m|^{\frac{4-N}{N-2}} u_m \right) ds.$$

Using Strichartz estimate, we get

$$\|\phi(\mathbf{u}) - \phi(\mathbf{v})\|_{M(I)} \lesssim \sum_{j,k=1}^m \|\nabla(f_{jk}(\mathbf{u}) - f_{jk}(\mathbf{v}))\|_{L_T^2(L^{\frac{2N}{N+2}})}.$$

As previously

$$\begin{aligned} \|\nabla(f_{jk}(\mathbf{u}) - f_{jk}(\mathbf{v}))\|_{L_T^2(L^{\frac{2N}{N+2}})} &\leq \left\| \sum_{i,l=1}^m \partial_i(u_l - v_l) \partial_l(f_{jk})(\mathbf{u}) \right\|_{L_T^2(L^{\frac{2N}{N+2}})} \\ &+ \left\| \sum_{i,l=1}^m \partial_i v_l \left(\partial_l(f_{jk})(\mathbf{u}) - \partial_l(f_{jk})(\mathbf{v}) \right) \right\|_{L_T^2(L^{\frac{2N}{N+2}})} \\ &\leq (J_1) + (J_2). \end{aligned}$$

Using Hölder inequality and Sobolev embedding, yields

$$\begin{aligned} (J_1) &\lesssim \left\| \nabla(\mathbf{u} - \mathbf{v}) \left(|u_k|^{\frac{2}{N-2}} |u_j|^{\frac{2}{N-2}} + |u_k|^{\frac{N}{N-2}} |u_j|^{\frac{4-N}{N-2}} \right) \right\|_{L_T^2(L^{\frac{2N}{N+2}})} \\ &\lesssim \|\nabla(\mathbf{u} - \mathbf{v})\|_{L_T^{\frac{2(N+2)}{N-2}}(L^{\frac{2N(N+2)}{N^2+4}})} \left(\|u_k\|_{L_T^{\frac{2(N+2)}{N-2}}(L^{\frac{2(N+2)}{N-2}})} \|u_j\|_{L_T^{\frac{2(N+2)}{N-2}}(L^{\frac{2(N+2)}{N-2}})} \right. \\ &+ \left. \|u_k\|_{L_T^{\frac{N}{N-2}}(L^{\frac{2(N+2)}{N-2}})} \|u_j\|_{L_T^{\frac{4-N}{N-2}}(L^{\frac{2(N+2)}{N-2}})} \right) \\ &\lesssim \|\mathbf{u} - \mathbf{v}\|_{M(I)} \left(\|u_k\|_{S(I)}^{\frac{2}{N-2}} \|u_j\|_{S(I)}^{\frac{2}{N-2}} + \|u_k\|_{S(I)}^{\frac{N}{N-2}} \|u_j\|_{S(I)}^{\frac{4-N}{N-2}} \right) \\ &\lesssim \|\mathbf{u} - \mathbf{v}\|_{M(I)} \|\mathbf{u}\|_{S(I)}^{\frac{4}{N-2}}. \end{aligned}$$

Using Hölder inequality and Sobolev embedding, yields

$$\begin{aligned} (J_2) &\lesssim \left\| \nabla \mathbf{v} \left(|u_k|^{\frac{4-N}{N-2}} |u_j|^{\frac{2}{N-2}} + |u_k|^{\frac{N}{N-2}} |u_j|^{\frac{6-2N}{N-2}} \right) \right\|_{L_T^2(L^{\frac{2N}{N+2}})} \\ &\lesssim \|\nabla \mathbf{v}\|_{L_T^{\frac{2(N+2)}{N-2}}(L^{\frac{2N(N+2)}{N^2+4}})} \|\mathbf{u} - \mathbf{v}\|_{S(I)}^m \left(\|u_k\|_{L_T^{\frac{4-N}{N-2}}(L^{\frac{2(N+2)}{N-2}})} \|u_j\|_{L_T^{\frac{2(N+2)}{N-2}}(L^{\frac{2(N+2)}{N-2}})} \right. \\ &+ \left. \|u_k\|_{L_T^{\frac{N}{N-2}}(L^{\frac{2(N+2)}{N-2}})} \|u_j\|_{L_T^{\frac{6-2N}{N-2}}(L^{\frac{2(N+2)}{N-2}})} \right) \\ &\lesssim \|\mathbf{v}\|_{M(I)} \|\mathbf{u} - \mathbf{v}\|_{S(I)} \|\mathbf{u}\|_{S(I)}^{\frac{6-N}{N-2}}. \end{aligned}$$

Then, thanks to Sobolev injections

$$\begin{aligned} \|\phi(\mathbf{u}) - \phi(\mathbf{v})\|_{M(I)} &\lesssim a^{\frac{4}{N-2}} \|\mathbf{u} - \mathbf{v}\|_{M(I)} + ba^{\frac{6-N}{N-2}} \|\mathbf{u} - \mathbf{v}\|_{S(I)} \\ &\lesssim (a^{\frac{4}{N-2}} + ba^{\frac{6-N}{N-2}}) \|\mathbf{u} - \mathbf{v}\|_{M(I)}. \end{aligned}$$

Moreover, taking in the previous inequality $\mathbf{v} = \mathbf{0}$, we get for small $\delta > 0$,

$$\begin{aligned} \|\phi(\mathbf{u})\|_{S(I)} &\leq \delta + Ca^{\frac{4}{N-2}}; \\ \|\phi(\mathbf{u})\|_{M(I)} &\leq CA + Cba^{\frac{4}{N-2}}. \end{aligned}$$

With a classical Picard argument, for small $a = 2\delta$, $b > 0$, there exists $\mathbf{u} \in X_{a,b}$, a solution to (1.1) satisfying

$$\|\mathbf{u}\|_{S(I)} \leq 2\delta.$$

With Strichartz estimate and arguing as previously, the solution $\mathbf{u} \in C(I, \mathcal{H})$. □

We are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Using the previous proposition via the fact that

$$\|e^{t\Delta}\Psi\|_{S(I)} \lesssim \|e^{t\Delta}\Psi\|_{M(I)} \lesssim \|\Psi\|_{\dot{H}},$$

it suffices to prove that $\|\mathbf{u}\|_{\dot{H}}$ remains small on the whole interval of existence of \mathbf{u} . Write with conservation of the energy and Sobolev’s inequality

$$\begin{aligned} \|\mathbf{u}\|_{\dot{H}}^2 &= 2E(\Psi) + \frac{N-2}{N} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |u_j(x,t)|^{\frac{N}{N-2}} |u_k(x,t)|^{\frac{N}{N-2}} dx \\ &\leq C(\xi(\Psi) + \xi(\Psi)^{\frac{N}{N-2}}) + C\left(\sum_{j=1}^m \|\nabla u_j\|_2^2\right)^{\frac{N}{N-2}} \\ &\leq C(\xi(\Psi) + \xi(\Psi)^{\frac{N}{N-2}}) + C\|\mathbf{u}\|_{\dot{H}}^{\frac{2N}{N-2}}. \end{aligned}$$

So by Lemma 2.17, if $\xi(\Psi)$ is sufficiently small, then \mathbf{u} stays small in the \dot{H} norm. Global existence is established. □

5 Existence of critical ground state

In this section we prove the existence of a ground state solution to (2.4) in the critical case. Precisely, we establish Proposition 2.6.

For $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+ \cup \{(1, -\frac{2}{N})\}$ and $\Psi \in \mathcal{H}$, recall the quantities

$$\begin{aligned} 2K_{\alpha,\beta}^\delta(\mathbf{u}) &:= \sum_{j=1}^m ((2\alpha + (N-2)\beta)\|\nabla u_j\|^2 + (2\alpha + N\beta)\delta\|u_j\|^2) \\ &\quad - \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} (2p\alpha + N\beta)|u_j u_k|^p dx. \end{aligned}$$

$$\begin{aligned} H_{\alpha,\beta}(\mathbf{u}) &:= E(\mathbf{u}) - \frac{1}{2\alpha + N\beta} K_{\alpha,\beta}(\mathbf{u}) \\ &= \frac{1}{2\alpha + N\beta} \left[\sum_{j=1}^m \beta \|\nabla u_j\|^2 + \alpha \left(1 - \frac{1}{p}\right) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \right]. \end{aligned}$$

Write also

$$\begin{aligned} 2K_{\alpha,\beta}^0(\Psi) &:= 2\mathcal{L}_{\alpha,\beta} E^0(\Psi) \\ &= ((2\alpha + (N-2)\beta) \sum_{j=1}^m \|\nabla \psi_j\|^2 - (2\alpha + \frac{N\beta}{p^c}) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^c} dx) \\ &= (2\alpha + \frac{N\beta}{p^c}) \left(\sum_{j=1}^m \|\nabla \psi_j\|^2 - \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^c} dx \right) \end{aligned}$$

and the operator

$$\begin{aligned} 2H_{\alpha,\beta}^0(\Psi) &:= (2E^0 - \frac{1}{(2\alpha p^c + N\beta)} K_{\alpha,\beta}^0)(\Psi) \\ &= \frac{2}{N} \sum_{j=1}^m \|\nabla \psi_j\|^2. \end{aligned}$$

Let the real number

$$d_{\alpha,\beta}^0 := \inf_{0 \neq \phi \in \mathcal{H}} \{H_{\alpha,\beta}^0(\Psi) \text{ s. t. } K_{\alpha,\beta}^0(\Psi) < 0\}.$$

Claim. $d_{\alpha,\beta}^0 = m_{\alpha,\beta}^0$.

Since $K_{\alpha,\beta}^0 = 0$ implies that $E^0 = H_{\alpha,\beta}^0$, it follows that $m_{\alpha,\beta}^0 \geq d_{\alpha,\beta}^0$.

Conversely, take $0 \neq \phi \in \mathcal{H}$ such that $K_{\alpha,\beta}^0(\phi) < 0$. Thus, when $0 < \lambda \rightarrow 0$, we get

$$\begin{aligned} K_{\alpha,\beta}^0(\lambda\Psi) &= (2\alpha + \frac{N\beta}{p^c}) \left(\lambda^2 \sum_{j=1}^m \|\nabla \psi_j\|^2 - \lambda^{p^c} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^c} dx \right) \\ &\simeq (2\alpha + \frac{N\beta}{p^c}) \lambda^2 \sum_{j=1}^m \|\nabla \psi_j\|^2 > 0. \end{aligned}$$

So, there exists $\lambda \in (0, 1)$ satisfying $K_{\alpha,\beta}^0(\lambda\Psi) = 0$ and

$$m_{\alpha,\beta}^0 \leq H_{\alpha,\beta}^0(\lambda\Psi) = \lambda^2 H_{\alpha,\beta}^0(\Psi) \leq H_{\alpha,\beta}^0(\Psi).$$

Thus, $m_{\alpha,\beta}^0 \leq d_{\alpha,\beta}^0$ and $m_{\alpha,\beta}^0 = d_{\alpha,\beta}^0$. The claim is proved.

Because of the definitions of $K_{\alpha,\beta}^0$ and $H_{\alpha,\beta}^0$, it is clear that $m_{\alpha,\beta}^0$ is independent of (α, β) and

$$m := m_{\alpha,\beta}^0 = \inf_{0 \neq \Psi \in \mathcal{H}} \left\{ \frac{1}{N} \sum_{j=1}^m \|\nabla \psi_j\|^2 \text{ s. t. } \sum_{j=1}^m \|\nabla \psi_j\|^2 < \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^c} dx \right\}.$$

Taking the scaling $\lambda\phi$, yields

$$\begin{aligned} m &= \inf_{0 \neq \Psi \in \mathcal{H}} \left\{ \frac{1}{N} \lambda^2 \sum_{j=1}^m \|\nabla \psi_j\|^2 \text{ s. t. } \lambda^{2-2p^c} \sum_{j=1}^m \|\nabla \psi_j\|^2 < \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^c} dx \right\} \\ &= \inf_{0 \neq \Psi \in \mathcal{H}} \left\{ \frac{1}{N} \sum_{j=1}^m \|\nabla \psi_j\|^2 \left(\frac{\sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^c} dx}{\sum_{j=1}^m \|\nabla \psi_j\|^2} \right)^{\frac{1}{1-p^c}} \right\} \\ &= \frac{1}{N} \inf_{0 \neq \Psi \in \mathcal{H}} \left\{ \left(\frac{(\sum_{j=1}^m \|\nabla \psi_j\|^2)^{\frac{1}{2}}}{(\sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^c} dx)^{\frac{1}{p^c}}} \right)^N \right\} \\ &= \frac{1}{N} (C^*)^{-N}. \end{aligned}$$

Here, C^* denotes the best constant of the Sobolev injection

$$\left(\sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^c} dx \right)^{\frac{1}{p^c}} \leq C^* \left(\sum_{j=1}^m \|\nabla \psi_j\|^2 \right)^{\frac{1}{2}}.$$

6 Invariant sets and applications

In this section, we prove Theorem 2.8 about global and non global existence of solutions to (1.1) in the energy space. We suppose in all this section that $\mu = 1$. First, we give some stable sets.

Lemma 6.1. *The sets $A_{\alpha,\beta}^{\delta,+}$ and $A_{\alpha,\beta}^{\delta,-}$ are invariant under the flow of (1.1).*

Proof. Let $\Psi \in A_{\alpha,\beta}^{\delta,+}$ and $\mathbf{u} \in C_{T^*}(\mathcal{H})$ be the maximal solution to (1.1). Assume that $\mathbf{u}(t_0) \notin A_{\alpha,\beta}^{\delta,+}$, for some time $t_0 \in (0, T^*)$. Since the energy is decreasing, we have $K_{\alpha,\beta}^\delta(\mathbf{u}(t_0)) < 0$. So, with a continuity argument, there exists a positive time $t_1 \in (0, t_0)$ such that $K_{\alpha,\beta}^\delta(\mathbf{u}(t_1)) = 0$. This contradicts the definition of m . The proof is similar in the case of $A_{\alpha,\beta}^{\delta,-}$. □

The fact that $m_{\alpha,\beta}$ is independent of (α, β) implies that some sets are also independent of (α, β) .

Lemma 6.2. *The sets $A_{\alpha,\beta}^{\delta,+}$ and $A_{\alpha,\beta}^{\delta,-}$ are independent of (α, β) .*

Proof. Let (α, β) and (α', β') in $\mathbb{R}_+^* \times \mathbb{R}_+ \cup \{(1, -\frac{2}{N})\}$. By the Propositions 2.6-2.13, the reunion $A_{\alpha,\beta}^{\delta,+} \cup A_{\alpha,\beta}^{\delta,-}$ is independent of (α, β) . So, it is sufficient to prove that $A_{\alpha,\beta}^{\delta,+}$ is independent of (α, β) . If $E^\delta(\mathbf{u}) < m$ and $K_{\alpha,\beta}^\delta(\mathbf{u}) = 0$, then $\mathbf{u} = 0$. So, $A_{\alpha,\beta}^{\delta,+}$ is open. The rescaling $\mathbf{u}^\lambda := e^{\alpha\lambda}\mathbf{u}(e^{-\beta\lambda} \cdot)$ implies that a neighborhood of zero is in $A_{\alpha,\beta}^{\delta,+}$. Moreover, this rescaling with $\lambda \rightarrow -\infty$ gives that $A_{\alpha,\beta}^{\delta,+}$ is contracted to zero and so it is connected. Now, write

$$A_{\alpha,\beta}^{\delta,+} = A_{\alpha,\beta}^{\delta,+} \cap (A_{\alpha',\beta'}^{\delta,+} \cup A_{\alpha',\beta'}^{\delta,-}) = (A_{\alpha,\beta}^{\delta,+} \cap A_{\alpha',\beta'}^{\delta,+}) \cup (A_{\alpha,\beta}^{\delta,+} \cap A_{\alpha',\beta'}^{\delta,-}).$$

Since by the definition, $A_{\alpha,\beta}^{\delta,-}$ is open and $0 \in A_{\alpha,\beta}^{\delta,+} \cap A_{\alpha',\beta'}^{\delta,-}$, using a connectivity argument, we have $A_{\alpha,\beta}^{\delta,+} = A_{\alpha',\beta'}^{\delta,+}$. □

Now, we prove the main result of this section.

Proof of Theorem 2.8.

1/ By Lemmas 6.2-6.1, $\mathbf{u}(t) \in A_{1,1}^+$ for any $t \in [0, T^*)$. Then,

$$m > E^\delta(\mathbf{u}) > H_{1,1}^\delta(\mathbf{u}).$$

Thus, $\mathbf{u}(t)$ is bounded in $(\dot{H}^1)^m$. Precisely

$$\sup_{t \in (0, T^*)} \|\nabla \mathbf{u}(t)\| < \infty.$$

Moreover, since $\partial_t(\|\mathbf{u}(t)\|^2) = -K_{1,0}(\mathbf{u}) < 0$, the L^2 norm of \mathbf{u} is decreasing and so $\|\mathbf{u}(t)\| \leq \|\mathbf{u}_0\|$. Thus

$$\sup_{t \in (0, T^*)} \|\mathbf{u}(t)\|_{\mathcal{H}} < \infty.$$

The global existence follows with classical methods since T^* depends only on the quantity $\|\mathbf{u}_0\|_{\mathcal{H}}$.

Now, we prove an exponential decay of the solution. For small $\|\mathbf{u}_0\|$, since $\sup_t \|\mathbf{u}(t)\|_{\mathcal{H}} \lesssim 1$, thanks to Proposition 2.16, we get

$$\begin{aligned} K_{1,0}(\mathbf{u}(t)) &= \sum_{j=1}^m \|u_j(t)\|_{H^1}^2 - \sum_{1 \leq j, k \leq m} \int_{\mathbb{R}^N} a_{jk} |u_j u_k|^p dx \\ &\geq \sum_{j=1}^m \left[\|u_j(t)\|_{H^1}^2 - C \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{(p-1)N}{2}} \left(\sum_{j=1}^m \|u_j\|^2 \right)^{\frac{N-p(N-2)}{2}} \right] \\ &\geq \sum_{j=1}^m \|u_j(t)\|^2 + \sum_{j=1}^m \|u_j(t)\|_{H^1}^2 \left[1 - C \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{(p-1)N}{2}-2} \left(\sum_{j=1}^m \|u_j\|^2 \right)^{\frac{N-p(N-2)}{2}} \right] \\ &\geq \sum_{j=1}^m \|u_j(t)\|^2 + \sum_{j=1}^m \|u_j(t)\|_{H^1}^2 \left[1 - C \|\mathbf{u}_0\|^{N-p(N-2)} \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{(p-1)N}{2}-2} \right] \\ &\geq C \|\mathbf{u}(t)\|_{\mathcal{H}}^2 \\ &\geq CE(\mathbf{u}(t)). \end{aligned}$$

On the other hand

$$\begin{aligned}
 E(\mathbf{u}(t)) &= \frac{1}{2} \sum_{j=1}^m \|u_j(t)\|_{H^1}^2 - \frac{1}{2p} \sum_{1 \leq j, k \leq m} a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \\
 &= \frac{1}{2} \sum_{j=1}^m \|u_j(t)\|_{H^1}^2 - \frac{1}{2p} \left(\sum_{j=1}^m \|u_j(t)\|_{H^1}^2 - K_{1,0}(\mathbf{u}(t)) \right) \\
 &= \frac{p-1}{2p} \sum_{j=1}^m \|u_j(t)\|_{H^1}^2 + \frac{1}{2p} K_{1,0}(\mathbf{u}(t)) \\
 &\geq C \max\{K_{1,0}(\mathbf{u}(t)), \|\mathbf{u}(t)\|_{\mathcal{H}^c}\}.
 \end{aligned}$$

Moreover, for $T > 0$,

$$\begin{aligned}
 \int_t^T K_{1,0}(\mathbf{u}(s)) ds &= \frac{1}{2} (\|\mathbf{u}(t)\|^2 - \|\mathbf{u}(T)\|^2) \\
 &\leq \frac{1}{2} \|\mathbf{u}(t)\|^2 \\
 &\lesssim E(\mathbf{u}(t)).
 \end{aligned}$$

So,

$$\int_t^T E(\mathbf{u}(s)) ds \lesssim \int_t^T K_{1,0}(\mathbf{u}(s)) ds \lesssim E(\mathbf{u}(t)).$$

Thus, for some positive real number $T_0 > 0$,

$$\begin{aligned}
 y(t) &:= \int_t^\infty E(\mathbf{u}(s)) ds \\
 &\lesssim E(\mathbf{u}(t)) \\
 &\leq -T_0 y'(t)
 \end{aligned}$$

This implies that, for $t \geq T_0$,

$$y(t) \leq y(T_0) e^{1-\frac{t}{T_0}} \leq T_0 E(\mathbf{u}(T_0)) e^{1-\frac{t}{T_0}}.$$

Taking account of the monotonicity of the energy, for large $T > 0$,

$$\int_t^T E(\mathbf{u}(s)) ds \geq \int_t^{t+T_0} E(\mathbf{u}(s)) ds \geq T_0 E(\mathbf{u}(t+T_0)).$$

Then,

$$E(\mathbf{u}(t+T_0)) \leq E(\mathbf{u}(T_0)) e^{1-\frac{t}{T_0}}.$$

Finally,

$$\|\mathbf{u}(t+T_0)\|_{\mathcal{H}^c}^2 \lesssim E(\mathbf{u}(t+T_0)) \leq E(\mathbf{u}(T_0)) e^{1-\frac{t}{T_0}}.$$

The proof is finished.

- 2/ With a translation argument, we can assume that $t_0 = 0$. Thus, $E^\delta(\mathbf{u}(t)) \leq E^\delta(\mathbf{u}_0) < m$. Moreover, with Lemma 6.1, $\mathbf{u}(t) \in A_{\alpha,\beta}^{\delta,-}$ for any $t \in [0, T^*)$. Take the real function

$$L(t) := L(\mathbf{u}(t)) = \frac{1}{2} \sum_{j=1}^m \int_0^t \|u_j(s)\|^2 ds, \quad t \in [0, T^*).$$

Using the equation (1.1), a direct computation gives

$$L''(t) = \sum_{j=1}^m \int \dot{u}_j u_j \, dx = - \sum_{j=1}^m \|u_j(t)\|_{\dot{H}^1}^2 - \delta \sum_{j=1}^m \|u_j(t)\|^2 + \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx.$$

We discuss two cases.

1/ First case: $E^\delta(\mathbf{u}_0) > 0$. By Lemmas 6.1-6.2, we get for any $\lambda > 0$,

$$H_{1,\lambda}(\mathbf{u}) = \frac{1}{2 + N\lambda} \left[\sum_{j=1}^m 2\lambda \|\nabla u_j\|^2 + \left(1 - \frac{1}{p}\right) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \, dx \right] > m.$$

Thus, for any $\varepsilon > 0$,

$$\begin{aligned} L'' &= -\delta \sum_{j=1}^m \|u_j\|^2 + \varepsilon \sum_{j=1}^m \|\nabla u_j\|^2 - (1 + \varepsilon) \sum_{j=1}^m \|\nabla u_j\|^2 + \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx \\ &> -\delta \sum_{j=1}^m \|u_j\|^2 + \frac{\varepsilon}{2\lambda} \left[(2 + N\lambda)m - \left(1 - \frac{1}{p}\right) \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx \right] \\ &\quad - (1 + \varepsilon) \sum_{j=1}^m \|\nabla u_j\|^2 + \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx. \end{aligned}$$

Taking account of the identities

$$\begin{aligned} E^\delta(\mathbf{u}) &= \frac{1}{2} \sum_j^m \|u_j\|_{\dot{H}^1}^2 + \frac{\delta}{2} \sum_j^m \|u_j\|^2 - \frac{1}{2p} \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx \\ &= - \sum_{j=1}^m \int_0^t \|\dot{u}_j(s)\|^2 \, ds + E^\delta(\mathbf{u}_0), \end{aligned}$$

we obtain

$$\begin{aligned} L'' &> \frac{\varepsilon}{2\lambda} \left[(2 + N\lambda)m - \left(1 - \frac{1}{p}\right) \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx \right] - \delta \sum_j^m \|u_j\|^2 + \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx \\ &\quad - 2(1 + \varepsilon) \left[- \sum_{j=1}^m \int_0^t \|\dot{u}_j(s)\|^2 \, ds + E^\delta(\mathbf{u}_0) - \frac{\delta}{2} \sum_j^m \|u_j\|^2 + \frac{1}{2p} \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx \right] \\ &> \left[\varepsilon \left(\frac{N}{2} + \frac{1}{\lambda} \right) m - 2(1 + \varepsilon) E^\delta(\mathbf{u}_0) \right] + 2(1 + \varepsilon) \sum_{j=1}^m \int_0^t \|\dot{u}_j(s)\|^2 \, ds \\ &\quad + \left(1 - \frac{1}{p} - \frac{\varepsilon}{2\lambda} \left(1 - \frac{1}{p} + \frac{2\lambda}{p} \right) \right) \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx \\ &:= (I) + (II) + (III). \end{aligned}$$

Write

$$\begin{aligned} (I) &= \varepsilon \left(\frac{N}{2} + \frac{1}{\lambda} \right) m - 2(1 + \varepsilon) E^\delta(\mathbf{u}_0) \\ &= \varepsilon \left(\left(\frac{N}{2} + \frac{1}{\lambda} \right) m - 2E^\delta(\mathbf{u}_0) \right) - 2E^\delta(\mathbf{u}_0). \end{aligned}$$

On the other hand,

$$(III) = \left(1 - \frac{1}{p} - \frac{\varepsilon}{2\lambda} \left(1 - \frac{1}{p} + \frac{2\lambda}{p} \right) \right) \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p \, dx.$$

Take $\lambda = 0^+$ and

$$\frac{1}{\left(\frac{N}{2} + \frac{1}{\lambda}\right) \frac{m}{2E^\delta(\mathbf{u}_0)} - 1} < \varepsilon < \frac{2\lambda(p-1)}{p-1+2\lambda}.$$

This choice implies that the terms (I) and (III) are non negative. Thus,

$$L'' > 2(1 + \varepsilon) \sum_{j=1}^m \int_0^t \|\dot{u}_j(s)\|^2 ds.$$

Thanks to Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} LL'' &> (1 + \varepsilon) \|\dot{\mathbf{u}}\|_{L_t^2(L^2)}^2 \|\mathbf{u}\|_{L_t^2(L^2)}^2 \\ &> (1 + \varepsilon) \|\mathbf{u}\dot{\mathbf{u}}\|_{L_t^1(L^1)}^2 \\ &> (1 + \varepsilon)L'^2. \end{aligned}$$

Moreover, if $L(t) = 0$ for some positive time, we get $K_{1,0}(\mathbf{u}(t)) = 0$, which contradicts Lemma 6.1. Thus

$$(L^{-\varepsilon})'' = -\varepsilon L^{-\varepsilon-2} [L''L - (1 + \varepsilon)(L')^2] > 0.$$

Taking account of Propostion 2.14, for some finite time $T > 0$,

$$\lim_{t \rightarrow T} \int_0^t \|\mathbf{u}(s)\|^2 ds = \infty.$$

Thus, $T^* < \infty$ and u is not global. This ends the proof.

2/ Second case: $E^\delta(\mathbf{u}_0) \leq 0$. Compute for $\varepsilon > 0$ near to zero,

$$\begin{aligned} L'' &= -\sum_{j=1}^m \|u_j(t)\|_{H^1}^2 - \delta \sum_{j=1}^m \|u_j(t)\|^2 + \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p dx \\ &\geq (2 + \varepsilon) \left(\frac{1}{2p} \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p dx - \frac{1}{2} \sum_{j=1}^m \|u_j(t)\|_{H^1}^2 - \frac{\delta}{2} \sum_{j=1}^m \|u_j(t)\|^2 \right) \\ &+ \left(1 - \frac{2 + \varepsilon}{2p} \right) \sum_{j,k=1}^m \int a_{jk} |u_j u_k|^p dx \\ &\geq -(2 + \varepsilon) E^\delta(\mathbf{u}). \end{aligned}$$

So, thanks to the identity $\dot{E}^\delta(\mathbf{u}) = -\sum_{j=1}^m \|\dot{u}_j\|^2$, we get

$$L'' \geq (2 + \varepsilon) \left(\|\dot{\mathbf{u}}\|_{L_t^2(L^2)}^2 - E(\mathbf{u}_0) \right). \tag{6.7}$$

Now, the proof goes by contradiction assuming that $T^* = \infty$.

Claim 1: There exists $t_1 > 0$ such that $\int_0^{t_1} \|\dot{\mathbf{u}}(s)\|^2 ds > 0$.

Indeed, otherwise $\mathbf{u}(t) = \mathbf{u}_0$ almost everywhere and solves the elliptic stationary equation $-\Delta u_j + \delta u_j = \sum_{k=1}^m a_{jk} |u_k|^p |u_j|^{p-2} u_j$. Therefore, $\sum_{j=1}^m \|u_j\|_{H^1}^2 + \delta \sum_{j=1}^m \|u_j\|^2 = \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_k u_j|^p dx$ and

$$\left(1 - \frac{1}{p}\right) \|\mathbf{u}_0\|_{H^1}^2 = 2E^\delta(\mathbf{u}_0) \leq 0.$$

So, $\mathbf{u}_0 = 0$, which contradicts the fact that $K_{1,0}(\mathbf{u}_0) < 0$.

Claim 2: For any $0 < \alpha < 1$, there exists $t_\alpha > 0$ such that

$$(L' - L'(0))^2 \geq \alpha L'^2, \quad \text{on } (t_\alpha, \infty).$$

The claim immediately follows from the first one and (6.7) observing that

$$\lim_{t \rightarrow \infty} L(t) = \lim_{t \rightarrow \infty} L'(t) = +\infty.$$

Claim 3: One can choose $\alpha = \alpha(\varepsilon)$ such that

$$LL'' \geq (1 + \alpha)L'^2, \quad \text{on } (t_\alpha, \infty).$$

Indeed, we have

$$\begin{aligned} LL'' &\geq \frac{2 + \varepsilon}{2} \|\mathbf{u}\|_{L_t^2(L^2)}^2 \|\dot{\mathbf{u}}\|_{L_t^2(L^2)}^2 \\ &\geq \frac{2 + \varepsilon}{2} \|\mathbf{u}\dot{\mathbf{u}}\|_{L_t^1(L^1)}^2 \\ &\geq \frac{2 + \varepsilon}{2} (L' - L'(0))^2 \\ &\geq \frac{(2 + \varepsilon)\alpha}{2} L'^2, \end{aligned}$$

where we used (6.7) in the first estimate, Cauchy-Schwarz inequality in the second and Claim 2 in the last one. Now choosing α such that $1 < \frac{(2+\varepsilon)\alpha}{2} := 1 + \varepsilon'$, we get

$$LL'' > (1 + \varepsilon')L'^2, \quad \text{for large time.}$$

Thanks to Proposition 2.14, this ordinary differential inequality blows up in finite time and contradicts our assumption that the solution is global. This ends the proof. □

7 Strong instability

This section is devoted to prove Theorem 2.9 about strong instability of standing waves, so, we take here and hereafter $\delta = \mu = 1$. Denote the scaling $\mathbf{u}_\lambda := \lambda^{\frac{N}{2}} \mathbf{u}(\lambda)$. Let us write an auxiliary result.

Lemma 7.1. *Let $\mathbf{u} \in \mathcal{H}$ such that $K_{1, -\frac{2}{N}}(\mathbf{u}) \leq 0$. Then, there exists $\lambda_0 \leq 1$ such that*

- 1/ $K_{1, -\frac{2}{N}}(\mathbf{u}_{\lambda_0}) = 0$;
- 2/ $\lambda_0 = 1$ if and only if $K_{1, -\frac{2}{N}}(\mathbf{u}) = 0$;
- 3/ $\frac{\partial}{\partial \lambda} E(\mathbf{u}_\lambda) > 0$ for $\lambda \in (0, \lambda_0)$ and $\frac{\partial}{\partial \lambda} E(\mathbf{u}_\lambda) < 0$ for $\lambda \in (\lambda_0, \infty)$;
- 4/ $\lambda \rightarrow E(\mathbf{u}_\lambda)$ is concave on (λ_0, ∞) ;
- 5/ $\frac{\partial}{\partial \lambda} E(\mathbf{u}_\lambda) = \frac{N}{2\lambda} K_{1, -\frac{2}{N}}(\mathbf{u}_\lambda)$.

Proof. We have

$$K_{1, -\frac{2}{N}}(\mathbf{u}_\lambda) = \sum_{j=1}^m \frac{2\lambda^2}{N} \|\nabla u_j\|^2 - (1 - \frac{1}{p}) \lambda^{N(p-1)} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx.$$

Moreover, with a direct computation

$$\begin{aligned} \partial_\lambda E(\mathbf{u}_\lambda) &= \lambda \sum_{j=1}^m \|\nabla u_j\|^2 - \frac{N}{2} (1 - \frac{1}{p}) \lambda^{N(p-1)-1} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \\ &= \frac{N}{2\lambda} K_{1, -\frac{2}{N}}(\mathbf{u}_\lambda) \end{aligned}$$

which proves (5). Now

$$\begin{aligned} K_{1, -\frac{2}{N}}(\mathbf{u}_\lambda) &= \frac{2\lambda^2}{N} \sum_{j=1}^m \|\nabla \mathbf{u}_j\|^2 - (1 - \frac{1}{p})\lambda^{N(p-1)} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \\ &= \frac{2\lambda^2}{N} \sum_{j=1}^m \left[\|\nabla \mathbf{u}_j\|^2 - \frac{N}{2} (1 - \frac{1}{p})\lambda^{N(p-1)-2} \sum_{k=1}^m a_k \int_{\mathbb{R}^N} |u_j u_k|^p dx \right]. \end{aligned}$$

Since $p > p^*$, a monotony argument closes the proof of (1), (2) and (3). For (4), it is sufficient to compute using (3). \square

The next intermediate result reads

Lemma 7.2. *Let Ψ to be a ground state solution of (2.2), $\lambda > 1$ a real number close to one and $\mathbf{u}_\lambda \in C([0, T^*], \mathcal{H})$ the solution to (1.1) with data Ψ_λ . Then, for any $t \in (0, T^*)$,*

$$E(\mathbf{u}_\lambda(t)) < E(\Psi) \quad \text{and} \quad K_{1, -\frac{2}{N}}(\mathbf{u}_\lambda(t)) < 0.$$

Proof. By Lemma 7.1, we have

$$E(\Psi_\lambda) < E(\Psi) \quad \text{and} \quad K_{1, -\frac{2}{N}}(\Psi_\lambda) < 0.$$

Moreover, thanks to the decay of energy, it follows that for any $t > 0$,

$$E(\mathbf{u}_\lambda(t)) \leq E(\Psi_\lambda(t)) < E(\Psi).$$

Then $K_{1, -\frac{2}{N}}(\mathbf{u}_\lambda(t)) \neq 0$ because Ψ is a ground state. Finally $K_{1, -\frac{2}{N}}(\mathbf{u}_\lambda(t)) < 0$ with a continuity argument. \square

Now, we are ready to prove the instability result.

Take $\mathbf{u}_\lambda \in C_T(\mathcal{H})$ the maximal solution to (1.1) with data Ψ_λ , where $\lambda > 1$ is close to one and Ψ is a ground state solution to (2.2). With the previous Lemma, we get

$$\mathbf{u}_\lambda(t) \in A_{1, -\frac{2}{N}}^-, \quad \text{for any } t \in (0, T^*).$$

Then, using Theorem 2.8, it follows that

$$\lim_{t \rightarrow T^*} \|\mathbf{u}_\lambda(t)\|_{\mathcal{H}} = \infty.$$

The proof is finished via the fact that

$$\lim_{\lambda \rightarrow 1} \|\Psi_\lambda - \Psi\|_{\mathcal{H}} = 0.$$

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