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Asymptotic Estimates and Qualitative Properties of an Elliptic Problem in Dimension Two

Khalil El Mehdi

Faculte des Sciences et Techniques
Universite de Nouakchott
BP 5026, Nouakchott, Mauritania
e-mail:khalil@univ-nkc.mr

Massimo Grossi*

Dipartimento di Matematica
Universitá di Roma "La Sapienza"
P.le A. Moro, Roma, Italia
e-mail:grossi@mat.uniroma1.it

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Abstract

In this paper we study a semilinear elliptic problem on a bounded domain in $\mathbb{R}^2$ with large exponent in the nonlinear term. We consider positive solutions obtained by minimizing suitable functionals. We prove some asymptotic estimates which enable us to associate a "limit problem" to the initial one. Using these estimates we prove some qualitative properties of the solutions, namely characterization of the level sets and nondegeneracy.

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1 Introduction and main results

In this paper we consider the following elliptic problem

$$(P_{\lambda,p}) \quad \begin{cases} -\Delta u + \lambda u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $\lambda \geq 0$ and p is a large positive parameter. We will focus on the solutions to $(P_{\lambda,p})$ obtained by the following variational method: We define on $H_0^1(\Omega)\setminus\{0\}$ the C^2-functional

$$J_\lambda(u) = \frac{\int_\Omega |\nabla u|^2 + \lambda \int_\Omega u^2}{(\int_\Omega |u|^{p+1})^{2/(p+1)}}$$

and we consider the following minimizing problem

$$c_{\lambda,p}^2 := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} J_\lambda(u). \tag{1.1}$$

A standard variational argument shows that $c_{\lambda,p}^2$ can be achieved by a positive function. Then after a multiplicative constant we find a positive function $u_{\lambda,p}$ which solves $(P_{\lambda,p})$ and satisfies

$$c_{\lambda,p}^2 = \frac{\int_\Omega |\nabla u_{\lambda,p}|^2 + \lambda \int_\Omega u_{\lambda,p}^2}{(\int_\Omega |u_{\lambda,p}|^{p+1})^{2/(p+1)}}. \tag{1.2}$$

In the remainder of this paper we denote by $u_{\lambda,p}$ the least energy solution of $(P_{\lambda,p})$ obtained in this way.

The aim of this paper is to study qualitative properties of the solution $u_{\lambda,p}$ for $\lambda \geq 0$ and p large. An essential tool in the proof of these results is to have information on the asymptotic behavior of $u_{\lambda,p}$ as p becomes large. The asymptotic behavior of the solutions of $P_{\lambda,p}$ was initially studied by Ren and Wei when $\lambda = 0$. More precisely, in [16] and [17] the authors proved the following result:

Theorem 1.1 *([16], [17]) Let Ω be a smooth bounded domain and $\lambda = 0$ in $(P_{\lambda,p})$. Let us denote by $u_{0,p}$ a least energy solution of $J_0(u)$. Then, for any sequence u_{p_n} of $u_{0,p}$ with $p_n \to +\infty$, there exists a subsequence of u_{p_n}, still denoted by u_{p_n}, such that*

i. *$u_{p_n}^{p_n}(\int_\Omega u_{p_n}^{p_n})^{-1} \to \delta_{x_0}$ in the sense of distribution, where δ_{x_0} is the Dirac function at point x_0.*

ii. *$u_{p_n}^{p_n}(\int_\Omega u_{p_n}^{p_n})^{-1} \to G(x, x_0)$ in $W^{1,q}(\Omega)$ weakly for any $1 < q < 2$, where G is the Green's function of $-\Delta$ with Dirichlet boundary condition. Furthermore, for any compact set $K \subset \bar{\Omega}\setminus\{x_0\}$, we have $v_{p_n} \to G(., x_0)$ in $C^{2,\alpha}(K)$.*

iii. *x_0 is a critical point of the Robin function R defined by $R(x) = g(x, x)$, where*

$$g(x, y) = G(x, y) + \frac{1}{2\pi} Log|x - y|$$

is the regular part of the Green's function.

Moreover, in [16] it was also showed that

$$0 < C_1 \leq ||u_{0,p}||_{L^\infty(\Omega)} \leq C_2 \tag{1.3}$$

for some constants C_1, C_2 and for p large enough. From these results we can see that when p gets large, the least energy solution $u_{0,p}$ looks like a single spike.

One of the results of this paper is to obtain asymptotic estimates for the least energy solution $u_{\lambda,p}$, but of different type than the corresponding one due to Ren and Wei. To describe our results we need to introduce the following problem

$$\begin{cases} -\Delta u = e^u & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u < +\infty. \end{cases} \tag{1.4}$$

In [5] it was proved that any solution of 1.4 is given by

$$U_{\mu,y}(x) = Log\left(\frac{8\mu^2}{(1+\mu^2|x-y|^2)^2}\right) \tag{1.5}$$

with $\mu \in \mathbb{R}$ and $y \in \mathbb{R}^2$.

Now we can claim the following:

Theorem 1.2 *Let Ω be a smooth bounded domain of $\mathbb{R}^2$, $\lambda \geq 0$ and let $u_{\lambda,p}$ be a least energy solution of $(P_{\lambda,p})$. Then, we have*
i. *$||u_{\lambda,p}||_\infty^{p-1} \to +\infty$ as $p \to +\infty$.*
ii. *If $\varphi_{\lambda,p}$ is the function defined for $x \in \Omega_{\lambda,p} := ||u_{\lambda,p}||_\infty^{(p-1)/2}(\Omega - x_{\lambda,p})$*

$$\varphi_{\lambda,p}(x) = (p-1)Log\Big(\frac{u_{\lambda,p}}{||u_{\lambda,p}||_\infty}\big(x_{\lambda,p} + \frac{x}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}}\big)\Big)$$

where $x_{\lambda,p} \in \Omega$ is such that $||u_{\lambda,p}||_\infty = u_{\lambda,p}(x_{\lambda,p})$, then, for any sequence φ_{p_n} of φ_p with $p_n \to \infty$, there exists a subsequence of φ_{p_n}, still denoted by φ_{p_n}, such that $\varphi_{p_n} \to U_{\bar{\mu},0}$ in $C^2_{loc}(\mathbb{R}^2)$, where $\bar{\mu}^2 = 1/8$ and $U_{\bar{\mu},0}$ is given by 1.5.

Since $||u_{\lambda,p}||_\infty^{(p-1)/2} \to \infty$ and $x_p \to x_0 \in \Omega$ (see Corollary 2.2 below), we have that $\Omega_p \to \mathbb{R}^2$ as $p \to \infty$. From this, we say that 1.4 is the "limit problem " of $(P_{\lambda,p})$ as $p \to \infty$.

A similar phenomenon (existence of a "limit problem ") occurs in several situations in higher dimensions. A typical example is the following problem

$$\begin{cases} -\Delta u = n(n-2)u^{\frac{n+2}{n-2}-\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \tag{1.6}$$

where ε is a small positive parameter and $n \geq 3$. Here it is well known that the limit problem associated to 1.6 is

$$\begin{cases} -\Delta u = n(n-2)u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \end{cases} \tag{1.7}$$

which admits the two parameters family of solutions

$$\delta_{\mu,y}(x) = \frac{\mu^{(n-2)/2}}{(1+\mu^2|x-y|^2)^{(n-2)/2}}.$$

Theorem 1.2 emphasizes some similarities between the problem $(P_{\lambda,p})$ when p is large and some corresponding problems in higher dimensions.

We remark that Theorem 1.2 is the starting point to obtain similar results as in singularity perturbed problems involving the critical Sobolev exponent, namely uniqueness or qualitative properties of solutions. Proof of Theorem 1.2 is given in Section 2.

In Section 3 we give a first application of Theorem 1.2; we study the shape of the level sets of solutions $u_{\lambda,p}$ when p is large enough. Namely we have the following result:

Theorem 1.3 *Let $u_{\lambda,p}$ be a least energy solution to $P_{\lambda,p}$ satisfying (1.1). Let Ω be convex. Then there exists $p_0 \geq 1$ such that for any $p > p_0$, we have*

$$(x-x_p)\nabla u_{\lambda,p}(x) < 0, \quad \forall x \in \Omega\diagdown\{x_p\}$$

where $x_p \in \Omega$ satisfies $u_{\lambda,p}(x_p) = ||u_{\lambda,p}||_\infty$. In particular, x_p is the only critical point and the superlevels are strictly star shaped with respect to x_p for p large enough.

If Ω is also symmetric, the claim pf Theorem 1.3 follows by the well known Gidas-Ni-Nirenberg Theorem. This result was proved by Lin ([13]) if $\lambda = 0$ and $p > 1$ with different techniques.

In Section 4 we give another application of Theorem 1.2, proving a nondegeneracy result to $P_{\lambda,p}$ for domains which satisfy the assumption of the Gidas-Ni-Nirenberg Theorem.

Theorem 1.4 *Let Ω be a smooth bounded domain of $\mathbb{R}^2$ which is symmetric with respect to the plane $x_1 = 0$ and $x_2 = 0$ and convex with respect to the direction x_1 and x_2. Let $u_{\lambda,p}$ be a least energy solution of $P_{\lambda,p}$. Then there exist $p_0 \geq 1$ such that for any $p \geq p_0$ we have that $u_{\lambda,p}$ is nondegenerate, i.e. the problem*

$$\begin{cases} -\Delta v + \lambda v = p u_{\lambda,p}^{p-1} v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.8}$$

admits only the trivial solution $v \equiv 0$.

Similar ideas used in the proof of Theorem 1.4 could help to obtain uniqueness result for the least energy solution to $(P_{\lambda,p})$. It will be done in a forthcoming paper.

2 Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. Here we suppose that $\lambda > 0$ is fixed. We begin by proving some auxiliary lemmas.

Lemma 2.1 *There exists $c > 0$ such that $||u_{\lambda,p}||_\infty \geq c$, where $u_{\lambda,p}$ is a solution of $(P_{\lambda,p})$ and c is independent of p.*

Proof. Let λ_1 be the first eigenvalue of $-\Delta$ and e_1 be a corresponding positive eigenfunction. Then if $u_{\lambda,p}$ is a solution of $(P_{\lambda,p})$, we have

$$0 = \int_\Omega (u_{\lambda,p}\Delta e_1 - e_1 \Delta u_{\lambda,p}) = -\lambda_1 \int_\Omega u_{\lambda,p} e_1 + \int_\Omega e_1(u_{\lambda,p}^p - \lambda u_{\lambda,p}).$$

Thus

$$\int_\Omega e_1 u_{\lambda,p}^p = (\lambda + \lambda_1)\int_\Omega e_1 u_{\lambda,p}.$$

Hence

$$(\lambda+\lambda_1)\int_\Omega e_1 u_{\lambda,p} \leq ||u_{\lambda,p}||_\infty^{p-1}\int_\Omega e_1 u_{\lambda,p}.$$

Then

$$||u_{\lambda,p}||_\infty \geq (\lambda+\lambda_1)^{\frac{1}{p-1}} \geq \min\{\lambda_1, 1\}.$$

Therefore our lemma follows. □

Lemma 2.2 *For p large enough, there exists c such that*

$$c_{\lambda,p} \leq c p^{-1/2}$$

where $c_{\lambda,p}$ is defined in 1.1.

Proof. We follow the proof of Lemma 2.2 in [16]. Without loss of generality we can assume $0 \in \Omega$. Let $R > 0$ be such that $B(0,R) \subset \Omega$. For $0 < d < R$, we introduce the following Moser function

$$m_d(x) = \frac{1}{\sqrt{2\pi}}\begin{cases} (Log(R/d))^{1/2} & \text{if } 0 \leq |x| \leq d \\ Log(R/|x|)(Log(R/d))^{-1/2} & \text{if } d \leq |x| \leq R \\ 0 & \text{if } |x| \geq R. \end{cases}$$

Then $m_d \in H_0^1(\Omega)$ and $||\nabla m_d||_{L^2(\Omega)} = 1$. Observe that

$$\int_\Omega m_d^{p+1}(x)dx = I_1 + I_2$$

where

$$I_1 = \left(\frac{1}{\sqrt{2\pi}}(Log(R/d))^{1/2}\right)^{p+1}\pi d^2$$

and

$$I_2 = \left(\frac{1}{\sqrt{2\pi}}(Log(R/d))^{-1/2}\right)^{p+1}\int_{d<|x|<R}(Log(R/|x|))^{p+1}dx.$$

Thus

$$|m_d|_{L^{p+1}(\Omega)} \geq I_1^{1/(p+1)}.$$

Choosing $d = Re^{-(p+1)/4}$, we find

$$|m_d|^2_{L^{p+1}} \geq (p+1)(8\pi e)^{-1}(\pi R^2)^{2/(p+1)}.$$

Hence

$$\frac{\int |\nabla m_d|^2 + \lambda \int m_d^2}{|m_d|^2_{L^{p+1}}} \leq \Big(1 + \frac{\lambda'}{\lambda_1(B(0,R))}\Big)\frac{\int |\nabla m_d|^2}{|m_d|^2_{L^{p+1}}} \leq c(R)(p+1)^{-1}R^{-4/(p+1)}.$$

Then

$$c_{\lambda,p} \leq c(R)(p+1)^{-1/2}R^{2/(p+1)}.$$

Therefore our lemma follows. □

In addition, since our solution $u_{\lambda,p}$ satisfies (1.2) and

$$\int_\Omega |\nabla u_{\lambda,p}|^2 + \lambda \int_\Omega u^2_{\lambda,p} = \int_\Omega u^{p+1}_{\lambda,p}$$

we easily derive the following result.

Corollary 2.1 *For* p *large enough, there exists* $c > 0$ *such that*

$$p\int_\Omega u^{p+1}_{\lambda,p} \leq c \quad \textit{and} \quad p\left(\int_\Omega |\nabla u_p|^2 + \int_\Omega u_p^2\right) \leq c. \tag{1.9}$$

Now, we recall the following lemma (see [7], [9]).

Lemma 2.3 *([7] [9]) Let* u *be a solution of*

$$\begin{cases} -\Delta u = F(u) & \textit{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \textit{on } \partial\Omega \end{cases}$$

where Ω *is a bounded smooth domain and* F *is a* C^1*-function. Then, there exists a neighborhood* ω *of* $\partial\Omega$ *and* $C > 0$*, both depending only on* Ω*, such that*

$$||u||_{L^\infty(\omega)} \leq C||u||_{L^1(\Omega)}.$$

Corollary 2.2 *Let us denote by* $x_{\lambda,p}$ *the point where* $u_{\lambda,p}$ *achieves its maximum, that is* $||u_{\lambda,p}||_\infty = u_{\lambda,p}(x_{\lambda,p})$*. Then* $x_{\lambda,p} \to x_\lambda \in \Omega$*.*

Proof. From Lemma 2.1, we have $||u_{\lambda,p}||_\infty \geq c > 0$ and from Corollary 2.1, we derive that

$$\int_\Omega u_{\lambda,p} \to 0.$$

Using Lemma 2.3 we deduce that the point $x_{\lambda,p}$ is far away from the boundary. Thus the claim follows. □

Lemma 2.4 *There exist a sequence* $p_n \to \infty$ *such that*

$$\lim_{n\to\infty} ||u_{\lambda,p_n}||_\infty^{p_n-1} \to +\infty.$$

Proof. We argue by contradiction. Let us suppose that there exists $c > 0$ such that for any $p > 1$ and $\lambda > 0$ we have

$$||u_{\lambda,p}||_\infty^{p-1} \le c. \tag{1.10}$$

Let us consider the following function

$$\bar{u}_{\lambda,p}(X) = \frac{1}{||u_{\lambda,p}||_\infty} u_{\lambda,p}(x_p + \frac{X}{||u_{\lambda,p}||_\infty^{(p-1)/2}}) \quad \text{for } X \in \Omega_{\lambda,p} \tag{1.11}$$

where $\Omega_{\lambda,p} = ||u_{\lambda,p}||_\infty^{(p-1)/2}(\Omega - x_{\lambda,p})$ and $x_{\lambda,p} \in \Omega$ such that $u_{\lambda,p}(x_{\lambda,p}) = ||u_{\lambda,p}||_\infty$. It is easy to see that $\bar{u}_p$ satisfies

$$\begin{cases} -\Delta \bar{u}_{\lambda,p} = \bar{u}_{\lambda,p}^p - \frac{\lambda}{||u_{\lambda,p}||_\infty^{p-1}} \bar{u}_{\lambda,p} & \text{in } \Omega_{\lambda,p} \\ \bar{u}_{\lambda,p}(0) = 1, \quad 0 \le \bar{u}_{\lambda,p} \le 1 & \text{in } \Omega_{\lambda,p} \\ \bar{u}_{\lambda,p} = 0 & \text{on } \partial\Omega_{\lambda,p}. \end{cases} \tag{1.12}$$

Thus by the standard regularity theory we deduce that there exists a sequence $p_n \to \infty$ such that $\bar{u}_{p_n} \to \bar{u}$ in $C^1_{loc}(\mathbb{R}^2)$. Moreover by (1.10), up to a subsequence of p_n, $\Omega_{p_n} \to D := \gamma(\Omega - x_0)$ as $n \to +\infty$ with $\gamma = \lim_{n\to\infty} ||u_{p_n}||_\infty^{p_n - 1}$. Let us point out that, by $0 \le \bar{u}_p \le 1$, we derive that $\bar{u}_{p_n}^{p_n} \rightharpoonup \psi \ge 0$ weakly in $L^q(D)$ for any $q > 1$. Finally $\bar{u}$ satisfies

$$\begin{cases} -\Delta \bar{u} = \psi - \bar{\lambda}\bar{u} & \text{in } D \\ \bar{u}(0) = 1, \quad 0 \le \bar{u} \le 1 & \text{in } D \\ \bar{u} = 0 & \text{on } \partial D \end{cases}$$

where $\bar{\lambda} = \lim_{n\to+\infty} \lambda ||u_{p_n}||_\infty^{1-p_n} = \frac{\lambda}{\gamma}$. Thus

$$\int_D |\nabla \bar{u}|^2 + \bar{\lambda}\int_D \bar{u}^2 = \int_D \psi\bar{u}.$$

Observe that, by Lebesgue's Theorem and the definition of $\bar{u}$, we have

$$\int_D \psi\bar{u} = \lim_{n\to\infty} \int_{\Omega_{p_n}} \bar{u}_{p_n}^{p_n} . \bar{u}_{p_n} = \lim_{n\to\infty} \int_\Omega ||u_{p_n}||_\infty^{-2} u_{p_n}^{p_n+1}.$$

From Corollary 2.1 and Lemma 2.1, we derive

$$\int_D \psi\bar{u} = 0.$$

Hence

$$\int_D |\nabla \bar{u}|^2 + \bar{\lambda}\int_D \bar{u}^2 = 0.$$

Therefore $\bar{u} \equiv 0$ which gives a contradiction with $\bar{u}(0) = 1$ and our lemma follows. □

Lemma 2.5 *For any $x \in B(x_p, \frac{R}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})$ we have, for p large enough,*

$$u_{\lambda,p}(x) \geq \gamma_R > 0$$

where R is an arbitrary positive number and γ_R is a constant only depending on R.

Proof. For $X \in \Omega_{\lambda,p}$, we set

$$W_{\lambda,p}(X) = u_{\lambda,p}(x_p + \frac{X}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}}).$$

Thus $W_{\lambda,p}$ satisfies

$$-\Delta W_{\lambda,p}(X) = c_{\lambda,p}(X) W_{\lambda,p}(X)$$

where

$$c_{\lambda,p}(X) = \frac{u_{\lambda,p}^{p-1}}{(p-1)||u_{\lambda,p}||_\infty^{p-1}}(x_p + \frac{X}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{p-1}}) - \frac{\lambda}{(p-1)||u_{\lambda,p}||_\infty^{p-1}}.$$

Observe that, from Lemma 2.4, we deduce

$$|c_p(X)| \leq C \qquad \text{for } p \text{ large enough.}$$

From the standard Harnack inequality [11], we get

$$||u_{\lambda,p}||_\infty = \sup_{B(0,R)} W_p(X) \leq c_R \inf_{B(0,R)} W_p(X) \quad \text{for } p \text{ large enough.}$$

Thus

$$\inf_{B(0,R)} W_p \geq \frac{||u_{\lambda,p}||_\infty}{c_R}.$$

From Lemma 2.1, we deduce

$$\inf_{B(0,R)} W_p \geq \gamma_R$$

and therefore our lemma follows. □

Lemma 2.6 *Let us consider the function*

$$F_{\lambda,p}(X) = \frac{1}{||u_{\lambda,p}||_\infty^{p-1}} \frac{|\nabla u_{\lambda,p}(x_p + \frac{X}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})|^2}{u_{\lambda,p}^2(x_p + \frac{X}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})}, \quad \textit{for } X \in \Omega_p. \qquad (1.13)$$

Then, for any $R > 0$, we have, for p large enough

$$||F_p||_{L^\infty(B(0,R))} \leq C_R$$

where C_R is a constant only depending on R.

Proof. According to Lemma 2.5, it is enough to prove

$$\frac{1}{||u_{\lambda,p}||_\infty^{p-1}}\Big|\nabla u_{\lambda,p}\Big(x_p+\frac{X}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}}\Big)\Big|^2\le c.$$

For $X\in B(0,2R)$, let

$$f^i_{\lambda,p}(X)=\frac{1}{||u_{\lambda,p}||_\infty^{(p-1)/2}}\frac{\partial u_{\lambda,p}}{\partial x_i}\Big(x_p+\frac{X}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}}\Big),\quad \text{for } i=1,2.$$

It is sufficient to prove

$$|f^i_{\lambda,p}|_{L^\infty(B(0,R))}\le c,\quad \text{for } i=1,2 \text{ and c is independent of } p.$$

We point out that

$$-\Delta f^i_{\lambda,p}=c_p(X)f^i_{\lambda,p}$$

with

$$c_p(X)=\frac{p}{p-1}\frac{u^{p-1}_{\lambda,p}}{||u_{\lambda,p}||_\infty^{p-1}}(x_p+\frac{X}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})-\frac{\lambda}{(p-1)||u_{\lambda,p}||_\infty^{p-1}}.$$

From Lemma 2.4 we have

$$|c_p(X)|\le C\qquad \text{for } p \text{ large enough.}$$

Hence, by the standard weak Harnack inequalities (Theorem 8.17 of [11]), we have

$$||f^i_{\lambda,p}||_{L^\infty(B(0,R))}\le c||f^i_{\lambda,p}||_{L^2(B(0,R))}.$$

Observe that

$$\begin{aligned}||f^i_{\lambda,p}||^2_{L^2(B(0,2R))}&=\int_{B(0,2R)}\frac{1}{||u_{\lambda,p}||_\infty^{p-1}}|\frac{\partial u_{\lambda,p}}{\partial x_i}(x_p+\frac{X}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})|^2dX\\ &=(p-1)\int_{B(x_p,\frac{2R}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})}\Big|\frac{\partial u_{\lambda,p}}{\partial x_i}(x)\Big|^2dx\\ &\le(p-1)\int_\Omega|\nabla u_{\lambda,p}|^2.\end{aligned}$$

From Corollary 2.1, we derive

$$||f^i_{\lambda,p}||_{L^2(B(0,2R))}\le c.$$

Therefore our lemma follows □

Next we will prove Theorem 1.2.

Proof of Theorem 1.2. According to Lemma 2.4, it only remains to prove part ii. of the Theorem. To do this we introduce the following function

$$v_{\lambda,p}(X)=\varphi_{\lambda,p}(X)-Z_{\lambda,p}(X),\quad \text{for } X\in\Omega_p \tag{1.14}$$

where $Z_{\lambda,p}$ satisfies

$$\begin{cases} -\Delta Z_{\lambda,p} + \frac{\lambda}{||u_{\lambda,p}||_\infty^{p-1}} = F_{\lambda,p} & \text{in} \quad B(0,R) \\ Z_{\lambda,p} = 0 & \text{on} \quad \partial B(0,R) \end{cases} \tag{1.15}$$

where $\varphi_{\lambda,p}$ is defined in Theorem 1.2 and $F_{\lambda,p}$ is defined by 1.13. By the maximum principle we have that $Z_{\lambda,p} \ge 0$ From Lemmas 2.4 and 2.6 and the standard regularity theory, we derive that for any $R > 0$

$$||Z_p||_{C^1(B(0,R))} \le C_R \tag{1.16}$$

where C_R only depends on R. Thus setting

$$V_{\lambda,p}(X) = e^{Z_{\lambda,p}(X)} \tag{1.17}$$

we have that, for any $q \ge 1$,

$$\forall R > 0, \ \exists C_R > 0 \quad \text{such that } ||V_{\lambda,p}||_{L^q(B(0,R))} \le C_R.$$

By direct computation it is not difficult to see that $v_{\lambda,p}$ satisfies

$$\begin{cases} -\Delta v_{\lambda,p} = V_{\lambda,p}(x) e^{v_{\lambda,p}} & \text{in } B(0,R) \\ v_{\lambda,p} \le 0 & \text{in } B(0,R). \end{cases} \tag{1.18}$$

We claim that
a. $V_{\lambda,p} \ge 0$ in $B(0,R)$
b. $||V_{\lambda,p}||_{L^q(B(0,R))} \le C_R \quad \forall q \ge 1$
c. $\int_{B(0,R)} e^{qv_{\lambda,p}} \le C'_R \quad \forall q \ge 1$
Note that **a** and **b** follow by the definition of $V_{\lambda,p}(x)$. Concerning **c** we have that

$$\begin{aligned} &\int_{B(0,R)} e^{qv_{\lambda,p}} \le \int_{B(0,R)} e^{q\varphi_{\lambda,p}} \\ &= \int_{B(0,R)} \left[\frac{u_{\lambda,p}^{p-1}}{||u_{\lambda,p}^{p-1}||_\infty^{p-1}} \left(x_{\lambda,p} + \frac{X}{(p-1)^{1/2} ||u_{\lambda,p}||_\infty^{(p-1)/2}} \right) \right]^q \\ &\le \int_{\Omega_{\lambda,p}} \frac{u_{\lambda,p}^{p-1}}{||u_{\lambda,p}^{p-1}||_\infty^{p-1}} \Big(x_{\lambda,p} + \frac{X}{(p-1)^{1/2} ||u_{\lambda,p}||_\infty^{(p-1)/2}} \Big) \\ &= (p-1) \int_\Omega u_{\lambda,p}^{p-1} \le C. \end{aligned} \tag{1.19}$$

Thus, we are in the setting of Theorem 3 of Brezis-Merle [3] and we then have the following alternative :
either
(i) $v_{\lambda,p}$ is bounded in $L^\infty(B(0,R))$
or
(ii) $v_{\lambda,p} \to -\infty$ uniformly in $B(0,R)$

or
(iii) $v_{\lambda,p} \to -\infty$ uniformly in $B(0,R) \diagdown S$, where S is the blow-up set of $v_{\lambda,p}$, i.e. $S = \{x \in B(0,R)$ such that there exists a sequence $y_{\lambda,p} \in B(0,R)$ with $y_{\lambda,p} \to x$ and $v_{\lambda,p}(y_{\lambda,p}) \to +\infty\}$.

Since $v_{\lambda,p} \leq 0$, we derive $S = \emptyset$ and so (iii) does not occur. Let us also prove that (ii) cannot happen. From 1.16 it is sufficient to prove

$$\min_{B(0,R)} \varphi_{\lambda,p}(X) \geq -C_R.$$

Let us introduce the following function

$$\psi_{\lambda,p}(X) = \frac{u_{\lambda,p}^{p-1}}{||u_{\lambda,p}||_\infty^{p-1}} \Big(x_{\lambda,p} + \frac{X}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}} \Big), \quad \text{for } X \in \Omega_{\lambda,p}. \tag{1.20}$$

It is easy to see that $\psi_{\lambda,p}$ satisfies

$$-\Delta \psi_{\lambda,p} = \psi_{\lambda,p}^2 - \frac{\lambda}{||u_{\lambda,p}||_\infty^{p-1}} \psi_{\lambda,p} - \frac{p-2}{p-1} \frac{|\nabla \psi_{\lambda,p}|^2}{\psi_{\lambda,p}}.$$

Hence

$$-\Delta \psi_{\lambda,p}(X) \leq a(X) \psi_{\lambda,p}(X)$$

with $a(X) = \psi_{\lambda,p}(X) - \frac{\lambda}{||u_{\lambda,p}||_\infty^{p-1}} \in (-1, 1]$.

By standard weak Harnack inequality (see Theorem 8.17 of [11]), we derive

$$1 = \sup_{B(0,\frac{R}{2})} \psi_{\lambda,p}(X) \leq C_R \left(\int_{B(0,R)} \psi_{\lambda,p}^2 \right)^{1/2}.$$

Thus

$$\int_{B(0,R)} \psi_{\lambda,p}^2 \geq C_R^{-2}.$$

Hence

$$\int_{B(0,R)} e^{2\varphi_{\lambda,p}} \geq C_R^{-2}.$$

So (ii) also cannot occur.

Therefore $v_{\lambda,p}$ is bounded in $L^\infty(B(0,R))$. Then $\varphi_{\lambda,p}$ is also bounded in $L^\infty(B(0,R))$ since

$$-C_R \leq \varphi_{\lambda,p} = v_{\lambda,p} + Z_{\lambda,p} \leq 0 \quad \text{in } B(0,R).$$

Using the standard regularity theory, since $||Z_{\lambda,p}||_{L^\infty(B(0,R))} \leq C$ and $v_{\lambda,p}$ is bounded we derive from (1.15) and (1.18) that $Z_{\lambda,p}$ and $v_{\lambda,p}$ are both bounded in $C^1(B(0,R))$. Thus $||\varphi_{\lambda,p}||_{C^1(B(0,R))} \leq C$.

We note that $\varphi_{\lambda,p}$ satisfies

$$-\Delta\varphi_{\lambda,p}=-\frac{\lambda}{||u_{\lambda,p}||_\infty^{p-1}}+\frac{1}{p-1}|\nabla\varphi_{\lambda,p}|^2+e^{\varphi_{\lambda,p}}. \tag{1.21}$$

Again by the standard regularity theory we get $||\varphi_{\lambda,p}||_{C^2_{loc}(\mathbb{R}^2)}\le C$. Then, for any sequence $p_n\to\infty$ there exists a subsequence (denoted again by p_n) such that $\varphi_{p_n}\to\varphi_{\lambda,p}$ in $C^1_{loc}(\mathbb{R}^2)$.

Let us show that $e^{\varphi_{p_n}}\to e^{\varphi}$ in $L^1_{loc}(\mathbb{R}^2)$. Since $\varphi_{p_n}\rightharpoonup\varphi$ in $H^1_{loc}(\mathbb{R}^2)$ we have

$$\begin{aligned}\int_{B(0,R)}|e^{\varphi_{p_n}}-e^{\varphi}| &= \int_{B(0,R)}|\int_0^1 e^{t\varphi_{p_n}+(1-t)\varphi}dt||\varphi_{p_n}-\varphi|\\ &\le \int_{B(0,R)} e^{\varphi_{p_n}+\varphi}|\varphi_{p_n}-\varphi|\\ &\le \left(\int_{B(0,R)} e^{2(\varphi_{p_n}+\varphi)}\right)^{1/2}\left(\int_{B(0,R)}|\varphi_{p_n}-\varphi|^2\right)^{1/2}\end{aligned}$$

and the claim follows since $|\varphi_{p_n}|\le C$ in $B(0,R)$. We also note that, from Corollary 2.1

$$\begin{aligned}\lim_{n\to\infty}\int_{B(0,R)} e^{\varphi_{p_n}} &= \lim_{n\to+\infty}\int_{B(0,R)}\frac{1}{||u_{p_n}||_\infty^{p_n-1}}u_{p_n}^{p_n-1}(x_{p_n}+\frac{X}{\sqrt{p_n-1}||u_{p_n}||_\infty^{(p_n-1)/2}})dx\\ &= \lim_{n\to\infty}(p_n-1)\int_{B(x_{p_n},\frac{R}{\sqrt{p_n-1}||u_{p_n}||_\infty^{(p_n-1)/2}})} u_{p_n}^{p_n-1}(x)dx\\ &\le \lim_{n\to\infty}(p_n-1)\int_\Omega u_{p_n}^{p_n-1}(x)dx\\ &\le C\end{aligned}$$

where C does not depend on R. Then from Fatou's Lemma, we derive

$$\int_{\mathbb{R}^2}e^{\varphi}\le\lim_{n\to\infty}\int_{B(O,R)}e^{\varphi_{p_n}}\le C.$$

Passing to the limit in 1.21 and using Lemma 2.4, we deduce that φ satisfies

$$\begin{cases}-\Delta\varphi=e^{\varphi} & \text{in } \mathbb{R}^2\\ \varphi(0)=0,\ \varphi\le 0 & \text{in } \mathbb{R}^2\\ \int_{\mathbb{R}^2}e^{\varphi}dx<\infty.\end{cases}$$

According to Chen-Li [5], we derive $\varphi=U_{\bar\mu,0}$, where $U_{\bar\mu,0}$ is defined in 1.5. Then our proposition follows. □

3 Proof of Theorem 1.3

Let us start by recalling the following result which is a particular case of a general theorem due to Grossi-Molle [12].

Theorem 3.1 *Let Ω be a smooth domain in $\mathbb{R}^n$, with $n \geq 1$, and $f \in C^1(\Omega, \mathbb{R}^+)$. Suppose that $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\begin{cases} -\Delta u + \lambda u = u^p & in \quad \Omega \\ u > 0 & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

for $p > 1$ and $\lambda \in \mathbb{R}$. Let x_0 be a maximum point of u and assume that Ω is convex. If there exist an open set $W \subset \Omega$ containing x_0 such that
i. $(x - x_0)\nabla u(x) < 0, \quad \forall x \in W \setminus \{x_0\}$
ii. $(x - x_0)\nabla u(x) + \frac{2}{p-1}u(x) < 0, \quad \forall x \in \partial W$
iii. $\lambda_1(-\Delta - pu^{p-1}) > 0$ *in* $H_0^1(\Omega \setminus W)$*(λ_1 is the first eigenvalue of $-\Delta - pu^{p-1}$) then*

$$(x - x_0)\nabla u(x) < 0 \quad x \in \Omega \setminus \{x_0\}.$$

In particular, x_0 is the only critical point for u in Ω and the superlevel sets are strictly star shaped with respect to x_0.

For sake of completeness, we recall the proof of Theorem 3.1. *Proof of Theorem 3.1.* Arguing by contradiction let us suppose that there exists $\bar{x} \in \Omega \setminus \{x_0\}$ such that $(\bar{x} - x_0)\nabla u(\bar{x}) \geq 0$. By assumption i. $\bar{x} \notin W$. Let us consider

$$w(x) = (x - x_0)\nabla u(x) + (2/(p-1))u(x).$$

It turns out that $w(\bar{x}) \geq 0$.

Now let us call D the connected component of the set $\{x \in \Omega | w(x) > 0\}$ containing $\bar{x}$. By assumption ii. $w < 0$ on ∂W and so $W \cap \partial D = \emptyset$. Moreover if $z \in \partial\Omega$, we have

$$w(z) = (z - x_0)\nu(z)\frac{\partial u}{\partial \nu}(z).$$

Since $\frac{\partial u}{\partial \nu}(z) < 0$ and using the convexity of Ω we deduce that $w \leq 0$ on $\partial\Omega$. Thus $w \in H_0^1(D)$.
Now, it is easy to see that w satisfies the following equation

$$\begin{cases} -\Delta w - (pu^{p-1} + \lambda)w = -2\lambda u \leq 0 \quad \text{in } D \\ w \in H_0^1(D). \end{cases} \tag{1.22}$$

Since $\lambda_1(-\Delta_{\lambda,p}u^{p-1}I) > 0$ in $H_0^1(\Omega \setminus W)$ and $D \subset \Omega \setminus W$, we get $\lambda_1(-\Delta - pu^{p-1}I) > 0$ in $H_0^1(D)$. This implies that the maximum principle holds in D. Hence by 1.22, we have that $w \leq 0$ in D and this gives a contradiction. □

In order to prove Theorem 1.3, we will apply Theorem 3.1. Thus we only need to check that the assumptions of Theorem 3.1 are true. Let us start by proving the following result.

Proposition 3.1 *For any $R > 0$, we have*

$$(x - x_{\lambda,p})\nabla u_{\lambda,p}(x) < 0, \quad \forall x \in B(x_{\lambda,p}, \frac{R}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}}) \setminus \{x_{\lambda,p}\}$$

for p large enough.

Proof. For $x \in B(x_{\lambda,p}, \frac{R}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})$ and $X \in B(0,R)$, we have

$$(x - x_{\lambda,p}) \cdot \nabla u_{\lambda,p}(x) = \frac{1}{\sqrt{p-1}} u_{\lambda,p}(x) X \cdot \nabla \varphi_{\lambda,p}(X)$$

where $\varphi_{\lambda,p}$ is defined in Theorem 1.2. Thus it is sufficient to prove that

$$X \cdot \nabla \varphi_{\lambda,p}(X) < 0, \quad \forall X \in B(0,R) \diagdown \{0\}$$

for p large enough.

Arguing by contradiction, let us suppose that there exist R_0, a sequence $p_n \to +\infty$ and a sequence $\{X_n\}$ in $B(0,R)$ such that

$$X_n \cdot \nabla \varphi_{p_n}(X_n) \geq 0. \tag{1.23}$$

From Theorem 1.2, we know that $\varphi_{p_n} \to U_{\bar{\mu},0}$ in $C^2_{loc}(\mathbb{R}^2)$ where $U_{\bar{\mu},0}$ is defined 1.5. Since $X_n \in B(0,R)$, we can assume that there exists $X_0 \in B(0, 2R_0)$ such that $X_n \to X_0$ as $n \to +\infty$. Thus two cases may occur

Case 1. $X_0 \neq 0$. Then, in this case, it follows by the above convergence that

$$X_n \cdot \nabla \varphi_{p_n}(X_n) \to X_0 \cdot \nabla U_{\bar{\mu},0}(X_0) = -\frac{4\bar{\mu}^2|X_0|^2}{1+\bar{\mu}^2|X_0|^2} < 0 \quad \text{as } n \to +\infty$$

and this is a contradiction to 1.23. Thus this case cannot happen.

Case 2. $X_0 = 0$. In this case let us consider the following function

$$g_n(t) = \varphi_{p_n}(tX_n), \quad \text{for } t \in [0,1].$$

It yields that g_n has a maximum at 0 and another critical point in [0,1] by 1.23(because $g_n'(1) = X_n \cdot \nabla \varphi_{p_n}(X_n) \geq 0$ and $g_n'(0) = 0$). Therefore there exists $\bar{t}_n \in [0,1]$ such that $g_n''(\bar{t}_n) = 0$. Now let $n \to +\infty$, from the above convergence and from the assumption $X_0 = 0$, it follows that 0 is a degenerate critical point for g_n and this is not true because $D^2U_{\bar{\mu},0}(0) = -cId$, with $c > 0$. Therefore this case also cannot happen and our proposition follows. □

Now we are able to prove Theorem 1.3. *Proof of Theorem 1.3.* We will prove that the assumption of Theorem 3.1 are true for $W = B(x_{\lambda,p}, \frac{R}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})$. Proposition 3.1 guarantees that assumption i holds. Note that

$$(x - x_{\lambda,p}) \cdot \nabla u_{\lambda,p}(x) + \frac{2}{p-1} u_{\lambda,p}(x) = \frac{1}{\sqrt{p-1}} (X \cdot \nabla \varphi_{\lambda,p}(X) + \frac{2}{p-1}) u_{\lambda,p}(x)$$

where $\varphi_{\lambda,p}$ is defined in Theorem 1.2. By the convergence of $\varphi_{\lambda,p}$ to $U_{\bar{\mu},0}$ and some easy computations we have that

$$X \cdot \nabla \varphi_{\lambda,p}(X) + \frac{2}{p-1} \to \frac{-4\bar{\mu}^2|X|^2}{1+\bar{\mu}^2|X|^2} \quad \text{as } p \to +\infty$$

uniformly on $\partial B(0,R)$. Thus

$$h_{\lambda,p}(x) := (x-x_{\lambda,p})\cdot\nabla u_{\lambda,p}(x)+\frac{2}{p-1}u_{\lambda,p}(x) < 0 \quad \text{on } \partial B(x_{\lambda,p}, \frac{R}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})$$

and then ii holds. Finally we have that $h_{\lambda,p}$ satisfies

$$-\Delta h_{\lambda,p} - pu^{p-1}h_{\lambda,p} + \lambda h_{\lambda,p} = -2\lambda u_{\lambda,p} < 0. \tag{1.24}$$

Since $\nabla u_{\lambda,p}(x_{\lambda,p}) = 0$, we also have $h_{\lambda,p}(x_{\lambda,p}) > 0$. Thus there exists a nodal region $C_{\lambda,p}$ of $h_{\lambda,p}$ in $B(x_{\lambda,p}, \frac{R}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})$ where $h_{\lambda,p}$ is positive. We derive from (1.24) that in $C_{\lambda,p}$ the first eigenvalue of linearized operator $L_{\lambda,p} = -\Delta - pu^{p-1} + \lambda$ is negative. Hence, since $u_{\lambda,p}$ is of index 1, the first eigenvalue of $L_{\lambda,p}$ in $\Omega\setminus C_{\lambda,p}$ is positive. Thus the first eigenvalue of $L_{\lambda,p}$ in $\Omega\setminus B(x_{\lambda,p}, \frac{R}{\sqrt{p-1}||u_{\lambda,p}||_\infty^{(p-1)/2}})$ is positive. Thus using Theorem 3.1, our theorem follows. □

Remark 3.1 It is not difficult to check that Theorem 3.1 holds if Ω is star shaped with respect to x_0. Hence, if Ω satisfies the assumptions of the Gidas-Ni-Nirenberg Theorem then Theorem 1.3 holds again.

Remark 3.2 It is easy to check that if $u_{\lambda,p}$ is a solution to $P_{\lambda,p}$, then the first eigenvalue of the linearized operator $-\Delta + (\lambda - pu_{\lambda,p}^{p-1})I_d)$ is negative.

The case where the linearized operator has only nonnegative eigenvalues was considered by various authors. More precisely, if we consider a solution u of

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega\subset\mathbb{R}^2 \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.25}$$

with Ω convex and $\lambda_1(\Delta - f'(u)I_d) \geq 0$ (the so called semistable solution), Cabré-Chanillo ([4]), Payne ([14]) and Sperb ([18]) showed the uniqueness of the critical point of u.

4 A nondegeneracy results

We start this section by recalling the following lemma due to Ren and Wei ([16]).

Lemma 4.1 *For every $t \geq 2$ there is D_t such that $||u||_{L^t(\Omega)} \leq D_t t^{\frac{1}{2}}||\nabla u||_{L^2(\Omega)}$ for all $u \in H_0^1(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^2$; furthermore*

$$\lim_{t\to\infty} D_t = (8\pi e)^{-\frac{1}{2}}. \tag{1.26}$$

From the previous lemma we derive the following estimate, which was showed in [16] for $\lambda = 0$.

Lemma 4.2 *We have, for p large enough and $\lambda \in [0,\lambda']$*

$$||u_{\lambda,p}||_\infty \leq C. \tag{1.27}$$

Proof. The proof is the same as in the case $\lambda = 0$ ([16], pp. 755-756). Let

$$\gamma_{\lambda,p} = \max_{x\in\bar\Omega} u_{\lambda,p}(x), \quad \mathcal{A} = \{x : \frac{\gamma_{\lambda,p}}{2} < u_{\lambda,p}(x)\}, \quad \Omega_t = \{x : t < u_{\lambda,p}(x)\}. \tag{1.28}$$

By Lemma 4.1 and Corollary 2.1

$$\Big(\int_\Omega u_{\lambda,p}^{2p}\Big)^{\frac{1}{2p}} \le C\sqrt{p}\Big(\int_\Omega |\nabla u_{\lambda,p}|^2 + \lambda\int_\Omega u_{\lambda,p}^2\Big)^{\frac12} \le C \tag{1.29}$$

for p large and C depending only on λ''. Hence

$$\Big(\frac{\gamma_{\lambda,p}}{2}\Big)^{2p}|\mathcal{A}| \le \int_\Omega u_{\lambda,p}^{2p} \le C^{2p}. \tag{1.30}$$

On the other hand

$$\int_{\Omega_t} u_{\lambda,p}^p = -\int_{\Omega_t} \Delta u_{\lambda,p} + \lambda\int_{\Omega_t} u_{\lambda,p} \ge \int_{\partial\Omega_t} |\nabla u_{\lambda,p}|ds. \tag{1.31}$$

Using the co-area formula ([8]) and the isoperimetric inequality we have

$$-\frac{d}{dt}|\Omega_t|\int_{\Omega_t} u_{\lambda,p}^p \ge \int_{\partial\Omega_t}\frac{ds}{|\nabla u_{\lambda,p}|}\int_{\partial\Omega_t}|\nabla u_{\lambda,p}|ds \ge |\partial\Omega_t|^2 \ge 4\pi|\Omega_t|. \tag{1.32}$$

From this point we can repeat step by step the proof of ([16], p.756) and we derive

$$\gamma_{\lambda,p} \le C \tag{1.33}$$

with C depending only on λ' for p large. □

In the next lemma we study the structure of the solutions of the linearized problem of P_{λ_p} "at infinity". The corresponding result in higher dimensions is well known ([2], [15],[1]). Here we use some ideas of [1] and [6].

Lemma 4.3 *Let $v \in L^\infty(\mathbb{R}^2)\cap C^2(\mathbb{R}^2)$ be a solution of the following problem*

$$-\Delta v = \frac{1}{(1+\frac{|x|^2}{8})^2}v \quad in\ \mathbb{R}^2. \tag{1.34}$$

Then

$$v(x) = \sum_{i=1}^2 a_i\frac{x_i}{1+\frac{|x|^2}{8}} + b\frac{8-|x|^2}{8+|x|^2}. \tag{1.35}$$

Proof. We write v as $v = \sum_{k=1}^{\infty}\psi_k(r)Y_k(\theta)$ where

$$\psi_k(r) = \int_{S^1} v(r,\theta)Y_k(\theta)d\theta, \tag{1.36}$$

and $Y_k(\theta)$ denotes the $k-th$ harmonic spheric satisfying

$$-\Delta_{S^1} Y_k(\theta) = k^2 Y_k(\theta). \tag{1.37}$$

Thus (1.34) becomes

$$(-\psi_k''(r) - \frac{1}{r}\psi_k'(r))Y_k(\theta) - \Delta_{S^1} Y_k(\theta)\frac{\psi_k(r)}{r^2} = \frac{1}{(1+\frac{r^2}{8})^2} Y_k(\theta)\psi_k(r) \tag{1.38}$$

and then

$$-\psi_k''(r) - \frac{1}{r}\psi_k'(r) + k^2\frac{\psi_k(r)}{r^2} = \frac{1}{(1+\frac{r^2}{8})^2}\psi_k(r). \tag{1.39}$$

Since v is smooth at the origin we deduce that $\psi_k(0) = 0$ for $k \geq 1$. Moreover since $v \in L^\infty(\mathbb{R}^2)$ we have that $\psi_k \in L^\infty(\mathbb{R})$ for any $k \geq 0$.

Let us consider the case $k = 0$. We have that $\psi_0(r)$ satisfies

$$-\psi_0''(r) - \frac{1}{r}\psi_0'(r) = \frac{1}{(1+\frac{r^2}{8})^2}\psi_0(r). \tag{1.40}$$

A direct computation shows that $\zeta_0(r) = \frac{8-r^2}{8+r^2}$ is a bounded solution of (1.40). Let us prove that if w is a second linearly independent solution of (1.40) then w is not bounded. We write $w(r) = c(r)\zeta_0(r)$. We get from (1.40)

$$-(c''\zeta_0 + 2c'\zeta_0' + c\zeta_0'') - \frac{1}{r}(c'\zeta_0 + c\zeta_0') = \frac{1}{(1+\frac{r^2}{8})^2}c\zeta_0 \tag{1.41}$$

and because ζ_0 is a solution of (1.40) we get

$$-c''\zeta_0 - c'(2\zeta_0' - \frac{1}{r}\zeta_0) = 0. \tag{1.42}$$

Setting $z = c'$ we obtain

$$z(r) = \frac{C}{r\zeta_0^2(r)} = C\frac{(8+r^2)^2}{r(8-r^2)^2} \sim \frac{C}{r} \quad \text{for } r \text{ large} \tag{1.43}$$

where C is a constant. This implies $c(r) \sim \log(r)$ for r large. Hence $c \notin L^\infty(\mathbb{R})$ and *a fortiori*, $w \notin L^\infty(\mathbb{R})$. Then $\zeta_0(r)$ is the unique bounded solution of (1.41).

Now we consider the case $k = 1$ in (1.39). Here we have that $\zeta_1(r) = \frac{r}{1+\frac{r^2}{8}}$ is a solution of (1.39). Repeating the same argument as in the case $k = 0$ we obtain that a second linearly independent solution w verifies

$$w(r) \sim r \quad \text{for } r \text{ large}. \tag{1.44}$$

Hence again $w \notin L^\infty(\mathbb{R})$ and then ζ_1 is the unique bounded solution of (1.39) for $k \geq 1$.

Now let us show that (1.39) has no nontrivial solution for $k \geq 2$. For $k \geq 1$ we set

$$A_k(\psi) = -\psi_k'' - \frac{1}{r}\psi_k' + k^2\frac{\psi_k}{r^2} - \frac{1}{(1+\frac{r^2}{8})^2}\psi_k. \tag{1.45}$$

By contradiction let us suppose that there exists $\bar\psi \not\equiv 0$ such that $A_k(\bar\psi) = 0$ for some $k \geq 2$. We claim that

$$\bar\psi > 0 \quad \text{in } \mathbb{R}. \tag{1.46}$$

Indeed if $\bar\psi$ changes sign we can select an interval $[x_1, x_2]$ with $0 \leq x_1 < x_2 < +\infty$ satisfying:

$$\bar\psi > 0 \quad \text{in }]x_1, x_2[. \tag{1.47}$$

By (1.47) we have that $\lambda_1(A_k) = 0$ in $[x_1, x_2]$. On the other hand we have that $A_k(\zeta_1) > 0$ in R and then the maximum principle holds in $[x_1, x_2]$ for A_k. Hence $\lambda_1(A_k) > 0$ and this gives a contradiction. Thus (1.46) holds. Moreover we have that

$$\lim_{r\to\infty} \bar\psi'(r) = 0. \tag{1.48}$$

In fact from (1.39)

$$(r\bar\psi')' - \frac{k^2}{r}\bar\psi = -\frac{r}{(1+\frac{r^2}{8})^2}\bar\psi \tag{1.49}$$

and then, for $r > 1$,

$$\begin{aligned} |r\bar\psi'(r)| &\leq |\bar\psi'(1)| + k^2\int_1^r \frac{|\bar\psi(t)|}{t}dt + \int_1^r \frac{t|\bar\psi(t)|}{(1+\frac{t^2}{8})^2}dt \leq \\ &\leq \bar\psi'(1) + k^2||\bar\psi||_\infty \log r + ||\bar\psi||_\infty \int_1^\infty \frac{t}{(1+\frac{t^2}{8})^2}dt \end{aligned} \tag{1.50}$$

and thus (1.50) implies (1.48).

Let us introduce the function $\eta(r) = r(\zeta_1\bar\psi' - \zeta_1'\bar\psi)$. It is easy to verify that

$$\eta'(r) = (1-k^2)\frac{\bar\psi\zeta_1}{r} < 0 \tag{1.51}$$

and

$$\lim_{r\to\infty} \eta(r) = 0. \tag{1.52}$$

Thus, if we show that $\lim\limits_{r\to 0} \eta(r) = 0$, using (1.51) and (1.52) we deduce a contradiction. By the definition of η we get, as $r \to 0$,

$$\eta(r) = \frac{r^2\bar\psi'(r)}{1+\frac{r^2}{8}} - \bar\psi(r)\frac{r(1-\frac{r^2}{8})}{(1+\frac{r^2}{8})^2} = \frac{r^2\bar\psi'(r)}{1+\frac{r^2}{8}} + o(r) \tag{1.53}$$

and since (1.49) implies $\lim\limits_{r\to 0} r^2\bar\psi'(r) = 0$ we have the claim. So there exists no solution to (1.39) as $k \geq 2$. Recalling that $Y_0(\theta) = constant$ and $Y_1(\theta) = x_1, x_2$, by (1.36) we derive (1.35). $\square$

Proof of Theorem 1.4. By contradiction let us assume that there exist sequences $p_n \to \infty$ and $v_n \equiv v_{\lambda,p_n} \in H_0^1(\Omega)$, $v_n \not\equiv 0$ satisfying

$$\begin{cases} -\Delta v_n + \lambda v_n = p_n u_{\lambda,p_n}^{p_n-1} v_n & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.54}$$

Since Ω satisfies the assumptions of the Gidas-Ni-Nirenberg Theorem we have that $u_{\lambda,p_n}(0) = ||u_{\lambda,p_n}||_\infty$ and u_{λ,p_n} is even in x_1 and x_2. Thus we may assume that

$$v_n(x_1,x_2) = v_n(-x_1,x_2) = v_n(x_1,-x_2). \tag{1.55}$$

Set $\tilde v_n(x) = v_n\Big(\frac{x}{\sqrt{p_n-1}||u_{\lambda,p_n}||_\infty^{\frac{p-1}{2}}}\Big)$ and $u_n(x) = \frac{1}{||u_{\lambda,p_n}||_\infty} u_{\lambda,p_n}\Big(\frac{x}{\sqrt{p_n-1}||u_{\lambda,p_n}||_\infty^{\frac{p-1}{2}}}\Big)$.

We have that $\tilde v_n$ satisfies

$$\begin{cases} -\Delta \tilde v_n + \frac{\lambda}{(p_n-1)||u_n||_\infty^{p-1}}\tilde v_n = \frac{p_n}{p_n-1} u_n^{p_n-1}\tilde v_n & \text{in } \Omega_n \\ v = 0 & \text{on } \partial\Omega_n \end{cases} \tag{1.56}$$

with $\Omega_n = \sqrt{p_n-1}||u_n||_\infty^{\frac{p-1}{2}} \cdot \Omega$. Finally we set

$$z_n = \frac{\tilde v_n}{||\tilde v_n||_\infty}. \tag{1.57}$$

Of course z_n satisfies

$$\begin{cases} -\Delta z_n + \frac{\lambda}{(p_n-1)||u_n||_\infty^{p-1}} z_n = \frac{p_n}{p_n-1} u_n^{p_n-1} z_n & \text{in } \Omega_n \\ |z_n| \leq 1 & \text{in } \Omega_n \\ z_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{1.58}$$

We want to pass to the limit in (1.58). Since u_n is a solution of P_{λ,p_n} we get, computing $u_n(x)$ at $x=0$,

$$\lambda \leq ||u_n||_\infty^{p_n-1}. \tag{1.59}$$

Then $\frac{\lambda}{(p_n-1)||u_n||_\infty^{p-1}} \leq \frac{1}{p_n-1} \to 0$ as $n \to \infty$. Now let us show the following estimates

$$\int_\Omega |\nabla z_n|^2 + \frac{\lambda}{(p_n-1)||u_n||_\infty^{p_n-1}} \int_{\Omega_n} z_n^2 \leq C_0, \tag{1.60}$$

where C_0 is a constant independent of n. Indeed from (1.58) and (1.9) we derive

$$\int_{\Omega_n} |\nabla z_n|^2 + \frac{\lambda}{(p_n-1)||u_n||_\infty^{p_n-1}} \int_{\Omega_n} z_n^2 \leq 2\int_{\Omega_n} u_n^{p_n-1} = 2(p_n-1)\int_\Omega u_n^{p_n-1} \leq C_0. \tag{1.61}$$

Moreover, from the classical Sobolev inequality

$$\int_\Omega |\nabla\phi|^2 + \lambda\int_\Omega |\phi|^2 \ge C(\lambda,\Omega)\left(\int_\Omega |\phi|^p\right)^{\frac{2}{p}}$$

we deduce

$$\begin{aligned}&\int_{\Omega_n} |\nabla z_n|^2 + \frac{\lambda}{(p_n-1)||u_n||_\infty^{p_n-1}}\int_{\Omega_n} z_n^2\\ &\ge C(\lambda,\Omega)\left(\frac{1}{(p_n-1)||u_n||_\infty^{p_n-1}}\right)^{\frac{2}{p_n}}\left(\int_{\Omega_n} |z_n|^{p_n}\right)^{\frac{2}{p_n}}. \qquad (1.62)\end{aligned}$$

From Lemma 2.1 and (1.60) we get

$$\left(\int_{\Omega_n} |z_n|^{p_n}\right)^{\frac{2}{p_n}} \le C \qquad (1.63)$$

where C is a constant independent of n.

Using (1.60) and the standard regularity theory we deduce the existence of a function $z \in C^2(\mathbb{R}^2)$, $|z| \le 1$, such that $z_n \to z$ in $C^2_{loc}(\mathbb{R}^2)$. Moreover z satisfies

$$\begin{cases} -\Delta z = \frac{1}{(1+\frac{|x|^2}{8})^2} z & \text{in } \mathbb{R}^2\\ |z| \le 1 & \text{in } \mathbb{R}^2\\ \int_{\mathbb{R}^2} |\nabla z|^2 \le C_0. \end{cases} \qquad (1.64)$$

From Lemma 4.3 it follows that

$$z(x) = \sum_{i=1}^{2} a_i \frac{x_i}{1+\frac{|x|^2}{8}} + b\frac{8-|x|^2}{8+|x|^2}. \qquad (1.65)$$

Step 1: $a_1 = a_2 = 0$ in (1.65).

By (1.55) we derive that $z(x)$ is even in x_1 and x_2. Hence by (1.65) we deduce $a_1 = a_2 = 0$.

Step 2: $b = 0$ in (1.65). From the previous step we have

$$z(x) = b\frac{8-|x|^2}{8+|x|^2}. \qquad (1.66)$$

If $b \ne 0$ we get that $\int_{\mathbb{R}^2} |\nabla z|^2 = +\infty$ which is not possible. So $b = 0$.

Step 3: the contradiction.

In this step we prove the claim of Theorem 1.4. We point out that in this step we will use Theorem 1.3.

By Step 1 and 2 we get that

$$z(x) \equiv 0 \quad \text{in } \mathbb{R}^2. \qquad (1.67)$$

Since $||z_n||_\infty = 1$ we can assume that there exists $x_n \in \Omega_n$ such that $z_n(x_n) = 1$. Since $z_n \to 0$ in $C^2(\mathbb{R}^2)$ we obtain that $|x_n| \to \infty$. By Theorem 1.3 and Theorem 1.2 we deduce that $u_n^{p_n-1}(x_n) \to 0$ as $n \to \infty$. Otherwise, if by contradiction $u_n^{p_n-1}(x_n) \geq C > 0$ we derive the existence of a point y_n such that $\nabla u_n(y_n) = 0$, a contradiction with Theorem 1.3.

Setting

$$\bar{z}_n(x) = z_n(x + x_n) \tag{1.68}$$

we get that $\bar{z}_n$ verifies

$$\begin{cases} -\Delta \bar{z}_n + \frac{\lambda}{(p_n-1)||u_n||_\infty^{p-1}} \bar{z}_n = \frac{p_n}{p_n-1} u_n^{p_n-1}(x + x_n)\bar{z}_n & \text{in } \Omega_n - \{x_n\} \\ \bar{z}_n \leq 1 & \text{in } \Omega_n - \{x_n\} \\ \bar{z}_n(0) = 1 & \\ \int_{\mathbb{R}^2} |\nabla \bar{z}_n|^2 \leq C_0. & \end{cases} \tag{1.69}$$

Passing to the limit in (1.69) we derive that $\bar{z}_n \to \bar{z}$ in $C^2_{loc}(\mathbb{R}^2)$ where $\bar{z}$ satisfies

$$\begin{cases} \Delta \bar{z} = 0 & \text{in } D \\ |\bar{z}| \leq 1 & \text{in } D, \end{cases} \tag{1.70}$$

where D is an half space if $dist(x_n, \Omega_n) \leq K$ or $D = \mathbb{R}^2$ if $\lim\limits_{n\to\infty} dist(x_n, \Omega_n) = +\infty$. In both case, by Liouville's Theorem we have that

$$z \equiv C \text{ in } D. \tag{1.71}$$

If D is an half space, using that $\bar{z}_n(x) = 0$ for $x \in \partial\Omega_n - x_n$ we get $\bar{z}(x) = 0$ for $x \in \partial D$. Standard arguments ([10]) leads a contradiction with $\bar{z}_n(x_n) = 1$.

Thus $D = \mathbb{R}^2$ and $\bar{z}(0) = 1$. Moreover, since $\bar{z}_n \to 1$ in $C^2(B(0,1))$, we get

$$||z_n||_{L^{p_n}(\Omega_n)} \geq ||\bar{z}_n||_{L^{p_n}(B(0,1))} > \frac{1}{2} \tag{1.72}$$

for $n > n_1$. On the other hand since $z_n(x_n) = 1$ and $z_n = 0$ on the boundary of Ω_n we get that there exists a point $x_{2,n}$ with $|x_{2,n}| \to \infty$ such that $z_n(x_{2,n}) = \frac{1}{2}$. Setting

$$\bar{\bar{z}}_n(x) = z_n(x + x_{2,n}) \tag{1.73}$$

and repeating the same procedure of above we derive

$$||z_n||_{L^{p_n}(\Omega_n)} \geq ||\bar{\bar{z}}_n||_{L^{p_n}(B(0,1))} > \frac{1}{2} \tag{1.74}$$

for $n > n_2$. Iterating this procedure, after a finite number of steps we reach a contradiction with (1.61). □

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