Invariant Curves of Mappings with Averaged Small Twist

Rafael Ortega
Departamento de Matemática Aplicada, Facultad de Ciencias
Universidad de Granada, 18071 Granada, Spain.
E-mail: rortega@goliat.ugr.es

Abstract

The Invariant Curve Theorem asserts the existence of invariant curves for certain planar mappings of the type $\theta_1 = \theta + \omega + \delta\alpha(r) + \ldots$, $r_1 = r + \ldots$, where α satisfies the twist condition $\alpha'(r) \neq 0$. This paper discusses the possibility of obtaining variants of this Theorem for mappings of the more general type $\theta_1 = \theta+\omega+\delta\ell_1(\theta, r)+\ldots$, $r_1 = r + \ell_2(\theta, r) + \ldots$. It is well known that if ω satisfies a diophantine condition then the twist condition can be replaced by $\int_0^{2\pi} \frac{\partial \ell_1}{\partial r}(\theta, r)d\theta \neq 0$. In this paper it will be shown that this is also the case for any number ω which is not commensurable with 2π (without imposing any arithmetic condition).
As an application of this result to differential equations we shall discuss the problem of boundedness for a class of piecewise linear forced oscillators.

Key Words: Invariant Curves; Twist Condition; Intersection Property; Forced Oscillators.

1 Introduction

Let C be an infinite cylinder with polar coordinates (θ, r), $\theta \equiv \theta + 2\pi$, $-\infty < r < \infty$, and let A be the compact region $\{a \leq r \leq b\}$. The numbers a and b are fixed. An invariant curve for a mapping $f : A \subset C \to C$ is a Jordan curve Γ in A which is homotopic to the circle $r = a$ and satisfies

$$f(\Gamma) = \Gamma.$$

The intersection property for f means that $f(\Gamma) \cap \Gamma \neq \emptyset$ for every Jordan curve in A which is homotopic to $r = a$.

Given a fixed $\omega \in \mathbf{R}$, the rotation of angle ω is

$$R_\omega(\theta, r) = (\theta + \omega, r).$$

This mapping has the intersection property and all circles $r = constant$ are invariant under R_ω. Consider now a family of smooth mappings depending on the parameter $\delta \in [0, 1]$,

$$f_\delta : A \subset C \to C, \quad (\theta, r) \mapsto (\theta_1, r_1)$$

satisfying the intersection property and such that

$$f_\delta \to R_\omega \quad \text{as } \delta \to 0^+.$$

Is it true that f_δ has invariant curves if δ is small? As stated, the answer to this question is negative and additional conditions on the family $\{f_\delta\}$ are required. The Small Twist Theorem of Moser [12] says that the answer is yes if f_δ has a twist. This means that f_δ can be expressed in the form

$$\begin{cases} \theta_1 = \theta + \omega + \delta\alpha(r) + \dots \\ r_1 = r + \dots \end{cases} \tag{1}$$

where

$$\alpha'(r) > 0 \quad \forall r \in [a, b] \tag{2}$$

and the remainders indicated by dots are of the order $o(\delta)$.

Assume now that the expansion of f_δ with respect to δ has the general form

$$\begin{cases} \theta_1 = \theta + \omega + \delta\ell_1(\theta, r) + \dots \\ r_1 = r + \delta\ell_2(\theta, r) + \dots \end{cases} \tag{3}$$

where ℓ_1 and ℓ_2 are fixed functions. In this paper I discuss the question of how to interpret and extend the twist condition (2) to this general setting. As will be seen, the answer depends on whether or not the number ω is commensurable with 2π. The case $\omega = 2\pi$ was studied in [14, 15]. For this angle the condition

$$\frac{\partial \ell_1}{\partial r}(\theta, r) > 0 \quad \forall(\theta, r)$$

is not sufficient and the existence of invariant curves depends on the behaviour of the continuous dynamical system

$$\dot\theta = \ell_1(\theta, r), \quad \dot r = \ell_2(\theta, r).$$

When $\omega = 2\pi\frac{p}{q}$ the results in [14, 15] can be applied to the power f_δ^q. This allows to derive conclusions for f_δ^q similar to those for f_δ in the case $\omega = 2\pi$. The functions ℓ_i must be replaced by the discrete averages

$$\ell_i^{\#}(\theta, r) = \frac{1}{q}\sum_{k=0}^{q-1} \ell_i(\theta + k\omega, r).$$

All this follows easily from [14, 15]. (See also the comments in this paper in Example 1 of Section 2). Once we know that f_δ^q has an invariant curve, the existence of an invariant curve for f_δ follows by a topological argument based on the properties of simply connected domains in the plane (see the proof of Proposition 2 in [6]).

Let us now pass to the non-commensurable case. The main result of this paper says that if

$$\omega \notin 2\pi\mathbf{Q}$$

and ℓ_1 satisfies the averaged twist condition

$$\int_0^{2\pi} \frac{\partial \ell_1}{\partial r}(\theta, r)d\theta > 0$$

then f_δ has invariant curves for small δ. The reader may have noticed that this result is well known when ω satisfies a diophantine condition. For this case a C^∞-version of the theorem can be found in Section 3.3.4 of [17]. In this paper it will be shown that the conclusion holds without any requirements on the arithmetic of the number ω. The proof will consist in showing that there exists a change of variables transforming (3) into (1) with

$$\alpha(r) = \frac{1}{2\pi}\int_0^{2\pi} \ell_1(\theta, r)d\theta.$$

The construction of this change of variables will lead to the study of the linear difference equation

$$X(\theta + \omega, r) - X(\theta, r) = F(\theta, r) \tag{4}$$

where F is a given function defined on the cylinder and having zero average with respect to θ. The traditional approach consists in solving exactly this equation and it is here where the diophantine condition plays a role. As it is well known this problem has small divisors and X will not be smooth unless ω satisfies some arithmetic properties (diophantine conditions, numbers of constant type,...). However, as soon as ω is not commensurable with 2π this linear equation is always solvable in an approximate sense. This means that we can solve it for some $F^\star$ that is smooth and arbitrarily close to F. This will be sufficient for the construction of the change of variables.

The Small Twist Theorem was designed by Moser to prove the stability of elliptic fixed points of general type (see in particular [11]) but it has found many other consequences in Stability Theory [10, 16, 1] and also in the study of the Littlewood's problem on oscillators (see for instance [7, 13]). The result obtained in this paper is useful to simplify the use of the Twist Theorem in some applications. To illustrate this point I shall derive from it a new result on the boundedness of solutions of an asymmetric oscillator.

The main theorem of the paper is stated in Section 2. This section also contains some examples on how to apply the result. The proof of the theorem is postponed to Section 5. Sections 3 and 4 are devoted to obtain some auxiliary results that will be employed in Section 5. More specifically, the approximate solvability of the linear difference equation (4) is discussed in Section 3 while some consequences of the intersection property are discussed in Section 4. After the proof in Section 5 the paper is finished with an appendix. In this Appendix I include the precise version of the Small Twist Theorem that is employed in the paper. I also include some indications on how to deduce this version from the work of Herman in [3, 4].

During the preparation of this paper I have benefited from discussions on this problem with several people. Carles Simó gave me many useful suggestions in a meeting at Oberwolfach in 97. The same year, in another meeting at El Escorial, Angel Jorba posed me a related and interesting question. In may 98 I presented a preliminary version of these results in a seminar organized by Tongren Ding and Bin Liu at the University of Peking.

2 Main Theorem

Let $A = \mathbf{T}^1 \times [a,b]$ be a finite part of the cylinder $C = \mathbf{T}^1 \times \mathbf{R}$, where $\mathbf{T}^1 = \mathbf{R}/2\pi\mathbf{Z}$. The region A is normalized by the condition

$$b - a \geq 2.$$

A generic point in C will be denoted by $(\overline{\theta}, r)$ where $\overline{\theta} = \theta + 2\pi\mathbf{Z}$, $\theta \in \mathbf{R}$ and $r \in \mathbf{R}$. The universal covering space of A can be identified to $\mathbf{R} \times [a,b]$, with coordinates (θ, r). This fact will be used very often. Given a mapping $(\overline{\theta}, r) \in A \to (\overline{\theta}_1, r_1) \in C$, the lift will be denoted by $(\theta, r) \in \mathbf{R} \times [a,b] \to (\theta_1, r_1) \in \mathbf{R}^2$. Also, real valued functions defined over A will be identified to functions defined over $\mathbf{R} \times [a,b]$ and 2π-periodic with respect to θ.

Let $\mathcal{J}$ denote the class of Jordan curves in C that are homotopic to the circle $r = constant$. The subclass of $\mathcal{J}$ composed by those curves that lie in A will be denoted by $\mathcal{J}_A$. More precisely,

$$\mathcal{J}_A = \{\Gamma \in \mathcal{J} : \Gamma \subset A\}.$$

Let us consider a given mapping

$$f : A \subset C \to C.$$

It is said that f has the *intersection property* if

$$f(\Gamma) \cap \Gamma \neq \emptyset$$

for each $\Gamma \in \mathcal{J}_A$.

An *invariant curve* of f will be a curve Γ in $\mathcal{J}_A$ such that

$$f(\Gamma) = \Gamma.$$

We shall be interested in the existence of invariant curves for a one-parameter family of mappings $\{f_\delta\}_{\delta\in[0,1]}$ with $f_\delta : A \subset C \to C$. It will be assumed that f_δ has the intersection property for each δ. Moreover, the lift can be expressed in the form

$$\begin{cases} \theta_1 = \theta + \omega + \delta\ell_1(\theta, r) + \delta\varphi_1(\theta, r; \delta) \\ r_1 = r + \delta\ell_2(\theta, r) + \delta\varphi_2(\theta, r; \delta). \end{cases} \tag{5}$$

Here ω is a fixed real number and the functions ℓ_i and φ_i satisfy the following conditions:

$$\ell_1, \ell_2 \in C^4(A), \tag{6}$$

$$\varphi_1, \varphi_2 \in C^{4,0}(A \times [0,1]), \tag{7}$$

$$\varphi_1(\theta, r; 0) = \varphi_2(\theta, r; 0) = 0, \ \ \forall(\overline{\theta}, r) \in A. \tag{8}$$

Theorem 1 *In the previous setting assume that*

$$\omega \notin 2\pi\mathbf{Q} \tag{9}$$

and

$$\int_0^{2\pi} \frac{\partial \ell_1}{\partial r}(\theta, r)d\theta \neq 0 \ \ \forall r \in [a, b]. \tag{10}$$

Then there exists a positive number Δ such that f_δ has an invariant curve if

$$0 < \delta < \Delta.$$

It will follow from the proof that f_δ has infinitely many invariant curves in A.

To illustrate the result we apply it to the study of two special families of mappings.

Example 1. For each $\delta > 0$ let $f_\delta : C \to C$ be the diffeomorphism defined by

$$\begin{cases} \theta_1 = \theta + \omega + \delta r \\ r_1 = r + \delta\Phi(\theta + \delta r) \end{cases}$$

where $\Phi \in C^4(\mathbf{T}^1)$. This mapping preserves the differential form $d\theta \wedge dr$ and it has the intersection property if

$$\int_0^{2\pi} \Phi(\theta) d\theta = 0.$$

From now on this condition is always assumed.

We will look for invariant curves in the finite region

$$A: \quad -1 \leq r \leq 1.$$

To apply the previous theorem the lift of f_δ is rewritten in the form (5) with

$$\ell_1(\theta, r) = r, \;\; \ell_2(\theta, r) = \Phi(\theta), \;\; \varphi_1 = 0, \;\; \varphi_2(\theta, r; \delta) = \Phi(\theta + \delta r) - \Phi(\theta).$$

Thus, if ω satisfies (9), we can deduce the existence of invariant curves in $|r| \leq 1$ if δ is sufficiently small.

When ω belongs to $2\pi\mathbf{Q}$ the previous conclusion may be false. An example for $\omega = 2\pi$ was presented in [14]. The strategy of [14] will now be employed to construct examples for every ω commensurable with 2π. Given $\omega \in 2\pi\mathbf{Q}$ with

$$\omega = 2\pi\frac{p}{q},$$

the iteration f_δ^q has the expansion

$$\begin{cases} \theta_q = \theta + 2\pi p + \delta q r + O(\delta^2) \\ r_q = r + \delta q \Phi^{\#}(\theta) + O(\delta^2) \end{cases}$$

where

$$\Phi^{\#}(\theta) = \frac{1}{q}\sum_{k=0}^{q-1} \Phi(\theta + k\omega).$$

We associate to this mapping the differential equation

$$\dot{\theta} = qr, \quad \dot{r} = q\Phi^{\#}(\theta). \tag{11}$$

It follows from the theorem in [14] that f_δ^q has no invariant curves for small δ (in the finite region $|r| \le 1$) if (11) has a solution $(\theta(t), r(t))$ with

$$\inf r(t) < -1, \quad \sup r(t) > 1.$$

Since a curve which is invariant under f_δ is also invariant under f_δ^q, we conclude that f_δ has no invariant curves in $|r| \le 1$.

Example 2. Consider the asymmetric oscillator

$$\ddot{x} + ax^+ - bx^- = p(t) \tag{12}$$

where $a, b > 0$, $a \ne b$, and $p \in C^4(\mathbf{T}^1)$. We shall apply the previous theorem to deduce that all the solutions of this differential equation are bounded if the conditions below hold,

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \notin \mathbf{Q}, \tag{13}$$

$$\int_0^{2\pi} p(t)dt \ne 0. \tag{14}$$

The case $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbf{Q}$ was analyzed in [2] and [9].

We use the same notation of [13]. The successor mapping S associated to (12) was defined in the proof of Theorem 4.1 in [13]. It is easy to describe it in an intuitive form. Given $(\tau_0, v_0) \in \mathbf{R}^2$ with v_0 positive and large, we consider the solution of (12) satisfying $x(\tau_0) = 0$, $\dot{x}(\tau_0) = v_0$. We follow this solution until reaching the next zero where $x(t)$ is again increasing; that is, $\tau_1 > \tau_0$, $x(\tau_1) = 0$, $\dot{x}(\tau_1) = v_1 > 0$. Then we define

$$S : \mathbf{T}^1 \times [\nu, \infty) \subset C \to C, \quad (\overline{\tau_0}, v_0) \mapsto (\overline{\tau_1}, v_1),$$

where $\nu > 0$ depends on $p(t)$. This mapping is a diffeomorphism of class C^4 from $\mathbf{T}^1 \times [\nu, \infty)$ onto its image and has the intersection property. All this is proved in [13, Sections 2 and 4]. We intend to prove that S has invariant curves satisfying $v \to +\infty$ because, according to Proposition 4.2 in [13], this implies the boundedness of all solutions of (12).

We apply twice the proposition 6.1 in [15] to deduce that S has a lift with expansion

$$\begin{cases} \tau_1 = \tau_0 + T + \frac{1}{v_0} \int_0^T p(t+\tau_0)s(t)dt + R_1(\tau_0, v_0) \\ v_1 = v_0 + \int_0^T p(t+\tau_0)\dot{s}(t)dt + R_2(\tau_0, v_0) \end{cases}$$

where $T = \frac{\pi}{\sqrt{a}} + \frac{\pi}{\sqrt{b}}$ and $s(t)$ is the T-periodic function

$$s(t) = \begin{cases} \frac{\sin\sqrt{a}t}{\sqrt{a}}, & t \in [0, \frac{\pi}{\sqrt{a}}] \\ -\frac{\sin\sqrt{b}(t-\frac{\pi}{\sqrt{a}})}{\sqrt{b}}, & t \in (\frac{\pi}{\sqrt{a}}, T). \end{cases}$$

The remainders R_1 and R_2 satisfy

$$\sup_{\mathbf{R}\times[\nu,\infty)} \{v^{2+\alpha_2}|\partial^\alpha R_1(\tau,v)| + v^{1+\alpha_2}|\partial^\alpha R_2(\tau,v)|\} < \infty$$

for each multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2,\ \ 0 \le |\alpha| \le 4$.

Next we introduce a small parameter $\delta > 0$ by means of the change of variables

$$\frac{1}{v} = \delta r, \quad \tau = \theta.$$

In the region $A:\ \overline{\theta} \in \mathbf{T}^1, r \in [1,3]$, S becomes a mapping of the class (5) with

$$\omega = T,\ \ \ell_1(\theta,r) = r\int_0^T p(t+\theta)s(t)dt,\ \ \ell_2(\theta,r) = -r^2\int_0^T p(t+\theta)\dot{s}(t)dt.$$

The functions φ_1 and φ_2 come from R_1 and R_2 and satisfy

$$||\varphi_1(\cdot,\cdot;\delta)||_{C^4(A)} + ||\varphi_2(\cdot,\cdot;\delta)||_{C^4(A)} = O(\delta).$$

This implies that conditions (7) and (8) hold. Finally we notice that the conditions (9) and (10) are now equivalent to (13) and (14). To check the equivalence of (10) and (14) we observe that $\int_0^T s \neq 0$ and so

$$\int_0^{2\pi} \frac{\partial \ell_1}{\partial r}(\theta,r)d\theta = \int_0^{2\pi}\int_0^T p(t+\theta)s(t)dtd\theta = (\int_0^T s)(\int_0^{2\pi} p).$$

Thus, for small δ there exist invariant curves of this map which correspond to invariant curves of S lying in the region $v \in [\frac{1}{3\delta}, \frac{1}{\delta}]$.

3 Approximate Solutions of a Linear Difference Equation

Given $\omega \in \mathbf{R}$ and a function $F: A \to \mathbf{R}$, we consider the equation

$$X(\theta+\omega,r) - X(\theta,r) = F(\theta,r), \quad (\overline{\theta},r) \in A, \tag{15}$$

where the unknown is a function $X : A \to \mathbf{R}$.
Before studying the equation in a rigorous way it is convenient to perform some formal computations, without taking into account the regularity of F and X. By integrating the equation with respect to θ over a period we obtain the following necessary condition for the solvability of (15),

$$F^\star(r) = 0, \quad \forall r \in [a,b]. \tag{16}$$

Here $F^\star$ denotes the average of F with respect to θ, that is

$$F^\star(r) := \frac{1}{2\pi}\int_0^{2\pi} F(\theta,r)d\theta.$$

When $\omega \in 2\pi\mathbf{Q}$ it is easy to find additional necessary conditions for the solvability of (15). In fact, if (15) has a solution, then

$$\int_0^{2\pi} F(\theta,r)e^{-in\theta}d\theta = 0 \tag{17}$$

for each n for which $\frac{n\omega}{2\pi}$ is an integer.

When $\omega \notin 2\pi\mathbf{Q}$, the condition (16) becomes sufficient if we look for formal solutions. In fact, if F has the Fourier expansion

$$F(\theta,r) = \sum_{n\neq 0} F_n(r)e^{in\theta}, \quad F_{-n} = \overline{F_n},$$

then

$$X(\theta,r) = X_0(r) + \sum_{n\neq 0}\frac{F_n(r)}{e^{in\omega}-1}e^{in\theta},$$

where $X_0 = X_0(r)$ is an arbitrary function. In particular the solution is unique if we restrict to the class of solutions satisfying

$$X^\star(r) = 0 \quad \forall r \in [a,b].$$

The previous formula for X also reveals the existence of a problem of small divisors because

$$\inf_n |e^{in\omega} - 1| = 0.$$

Thus, the solution must be understood in a formal sense (or in the sense of distributions) and the regularity theory becomes delicate (see [12, 3, 4]).

Let $C_o^p(A)$, $0 \le p < \infty$, be the class of functions $F \in C^p(A)$ satisfying (16). It is a Banach space and we shall prove now that for a generic $F \in C_o^p(A)$ one cannot expect solutions in the same space $C_o^p(A)$. This is the well known phenomenon of loss of derivatives.

Lemma 2 *The set of functions $F \in C^p_o(A)$ for which (15) has a solution in $C^p(A)$ is of first category.*

Proof. When $\omega \in 2\pi\mathbf{Q}$ each equation of the kind (17) with $n \neq 0$ defines a subspace of $C^p_o(A)$ of codimension 2. Thus, the lemma is proved for this case. From now on we assume $\omega \notin 2\pi\mathbf{Q}$. Define the linear operator

$$L : C^p_o(A) \to C^p_o(A), \quad LX(\theta, r) = X(\theta + \omega, r) - X(\theta, r).$$

From the previous discussions we know that L is continuous and one-to-one. It follows from Banach theorem that either $ImL = X$ or ImL is of first category. We shall exclude the first possibility and this will prove the lemma.
Assume, by a contradiction argument, that $ImL = X$. The closed graph theorem implies that L^{-1} is also continuous and this is not compatible with the identities

$$L(e^{in\theta}) = (e^{in\omega} - 1)e^{in\theta}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Definition 3 *Given $F \in C^p_o(A)$ we say that (15) is approximately solvable (in class C^p) if for each $\epsilon > 0$ there exists $X_\epsilon \in C^p_o(A)$ such that the function*

$$\mathcal{R}_\epsilon(\theta, r) = X_\epsilon(\theta + \omega, r) - X_\epsilon(\theta, r) - F(\theta, r)$$

satisfies

$$||\mathcal{R}_\epsilon||_{C^p(A)} < \epsilon.$$

(It is not restrictive to assume in the definition that X_ϵ is C^∞ or even real analytic).

The next result is similar to lemma 3 in [8].

Proposition 4 *Assume $\omega \notin 2\pi\mathbf{Q}$. Given $F \in C^p(A)$, the equation (15) is approximately solvable if and only if the condition (16) holds.*

Proof. The necessity of (16) for approximate solvability follows from the identity

$$\mathcal{R}^\star_\epsilon(r) = -F^\star(r), \quad r \in [a, b],$$

that is valid for any $\epsilon > 0$. To prove the sufficiency of (16) we first find a trigonometric polynomial of the type

$$P(\theta, r) = \sum_{0<|n|\leq N} P_n(r)e^{in\theta}$$

with $N \geq 0$, $P_n \in C^\infty([a,b],\mathbf{C})$, $P_{-n} = \overline{P_n}$ and such that

$$||F - P||_{C^p(A)} < \epsilon.$$

The existence of this polynomial is guaranteed by the Stone-Weierstrass Theorem. The function

$$X_\epsilon(\theta, r) = \sum_{0<|n|\leq N} \frac{P_n(r)}{e^{in\omega} - 1} e^{in\theta}$$

solves (15) when F is replaced by P and so X_ϵ is an approximate solution of the original equation.

4 A Remark on the Intersection Property

In this section we consider a family of mappings

$$h_\delta : A \subset C \to C, \quad (\overline{\theta}, r) \mapsto (\overline{\theta_1}, r_1), \quad \delta \in [0,1],$$

that satisfies the intersection property for each δ. It is also assumed that the lift can be expressed in the form

$$\theta_1 = \theta + \omega + \delta F(\theta, r; \delta), \quad r_1 = r + \delta G(\theta, r; \delta)$$

where $\omega \in \mathbf{R}$ and $F, G : A \times [0,1] \to \mathbf{R}$ are continuous functions. The average of G with respect to θ is denoted by

$$G^\star(r;\delta) = \frac{1}{2\pi}\int_0^{2\pi} G(\theta, r; \delta) d\theta.$$

Proposition 5 *In the notations of this section assume that*

$$\omega \notin 2\pi\mathbf{Q}.$$

Then

$$G^\star(r;0) = 0 \quad \forall r \in [a,b].$$

Proof. By a contradiction argument assume that

$$G^\star(r_0;0) = \mu \neq 0 \tag{18}$$

for some $r_0 \in (a,b)$. Let us define

$$\tilde{G}(\theta, r; \delta) = G(\theta, r; \delta) - G^\star(r;\delta).$$

Since $\tilde{G}(\theta, r; 0)$ has zero average we can apply proposition 4 to find a function $X \in C^\infty(A)$ such that

$$\mathcal{R}(\theta, r) = X(\theta + \omega, r) - X(\theta, r) - \tilde{G}(\theta, r; 0)$$

satisfies

$$||\mathcal{R}||_{C^0(A)} < \frac{|\mu|}{2}. \tag{19}$$

For small δ the equation

$$r = r_0 + \delta X(\theta, r)$$

describes implicitly a Jordan curve Γ_δ that is close to $r = r_0$. Moreover it can be expressed in the form

$$r = \psi_\delta(\theta)$$

with $\psi_\delta \in C^\infty(\mathbf{T}^1)$, $a \leq \psi_\delta \leq b$, and

$$\lim_{\delta \to 0^+} \psi_\delta(\theta) = r_0$$

uniformly in $\theta \in \mathbf{R}$. In particular, Γ_δ belongs to $\mathcal{J}_A$. For small δ we shall prove that

$$\Gamma_\delta \cap h_\delta(\Gamma_\delta) = \emptyset. \tag{20}$$

This will be the required contradiction.
To prove (20) we first notice that $h_\delta(\Gamma_\delta)$ can be described by the parametric equations

$$\theta = \Theta + \omega + \delta F(\Theta, \psi_\delta(\Theta); \delta), \quad r = \psi_\delta(\Theta) + \delta G(\Theta, \psi_\delta(\Theta); \delta) \tag{21}$$

where $\Theta \in \mathbf{R}$. It will be sufficient to prove that

$$r \neq r_0 + \delta X(\theta, r)$$

when θ and r are given by (21). In fact, from the identity

$$\psi_\delta(\Theta) = r_0 + \delta X(\Theta, \psi_\delta(\Theta))$$

we deduce, for $\delta \to 0$,

$$\psi_\delta(\Theta) = r_0 + O(\delta), \quad X(\theta, r) = X(\Theta + \omega, r_0) + O(\delta)$$

and

$$r - r_0 - \delta X(\theta, r) = \psi_\delta(\Theta) + \delta G(\Theta, \psi_\delta(\Theta); \delta) - r_0 - \delta X(\Theta + \omega, r_0) + O(\delta^2) =$$

$$\delta\{X(\Theta, \psi_\delta(\Theta)) - X(\Theta + \omega, r_0)\} + \delta G(\Theta, \psi_\delta(\Theta); \delta) + O(\delta^2)$$
$$= -\delta R(\Theta, r_0) + \delta G^\star(r_0; 0) + o(\delta).$$

Thus, from (18) and (19),

$$|r - r_0 - \delta X(\theta, r)| \geq \delta \frac{|\mu|}{2} + o(\delta) > 0$$

if δ is small.

To conclude this section we shall present an example showing that the condition $\omega \notin 2\pi\mathbf{Q}$ is essential in the previous result. We need some preliminary discussions on homeomorphisms of the cylinder

$$\mathcal{H} : C \to C, \quad (\overline{\theta}, r) \mapsto (\overline{\theta_1}, r_1)$$

that satisfy

$$r_1 \to \pm\infty, \qquad \text{as } r \to \pm\infty.$$

Lemma 6 *Assume that* Γ *is a Jordan curve in* $\mathcal{J}$ *such that*

$$\Gamma \cap \mathcal{H}(\Gamma) = \emptyset,$$

where $\mathcal{H}$ *is in the previous conditions. Then*

$$\Gamma \cap \mathcal{H}^n(\Gamma) = \emptyset, \quad n = 2, 3, \ldots$$

Proof. We employ the notation $\Gamma_n = \mathcal{H}^n(\Gamma)$. The complement of Γ, $C - \Gamma$, is divided in two components $C_+(\Gamma)$ and $C_-(\Gamma)$ with

$$\mathbf{T}^1 \times [r_0, +\infty) \subset C_+(\Gamma), \quad \mathbf{T}^1 \times (-\infty, -r_0] \subset C_-(\Gamma)$$

for some $r_0 > 0$. The conditions imposed on $\mathcal{H}$ imply that

$$\mathcal{H}(C_+(\Gamma)) = C_+(\Gamma_1), \quad \mathcal{H}(C_-(\Gamma)) = C_-(\Gamma_1).$$

Since Γ_1 does not intersect Γ, it will lie in one of the two components of $C - \Gamma$. For instance, let us assume that $\Gamma_1 \subset C_+(\Gamma)$. Then $\Gamma_{k+1} \subset C_+(\Gamma_k)$ for each $k = 1, 2, \ldots$ In consequence, $C_+(\Gamma_{k+1}) \subset C_+(\Gamma_k)$ and we find the chain of inclusions

$$\Gamma_n \subset C_+(\Gamma_{n-1}) \subset \ldots \subset C_+(\Gamma).$$

This finishes the proof.

The homeomorphism $\mathcal{H}$ has the intersection property if

$$\mathcal{H}(\Gamma) \cap \Gamma \neq \emptyset \quad \forall \Gamma \in \mathcal{J}.$$

The previous lemma implies that $\mathcal{H}$ has the intersection property as soon as $\mathcal{H}^n$ has this property for some $n \geq 2$.

Example. It is inspired by the discussions on the intersection property that can be found in [5]. Consider the family of homeomorphisms of C given by

$$h_\delta : \quad \theta_1 = \theta + \omega, \;\; r_1 = r + \delta\beta(\theta)$$

where $\omega \in \mathbf{R}$ and $\beta \in C(\mathbf{T}^1)$.
When $\omega \notin 2\pi\mathbf{Q}$ we can apply Proposition 5 to deduce that the intersection property of h_δ implies

$$\int_0^{2\pi} \beta(\theta) d\theta = 0.$$

The converse is also valid. This follows from a standard argument based on the exactness of the differential form $r_1 d\theta_1 - r d\theta$.

Let us now consider the case

$$\omega = 2\pi \frac{p}{q}$$

where $\frac{p}{q}$ is a fraction in reduced form. The iterate h_δ^q is defined by

$$\theta_q = \theta + 2\pi p, \;\; r_q = r + \delta q \beta^{\#}(\theta),$$

where

$$\beta^{\#}(\theta) = \frac{1}{q} \sum_{k=0}^{q-1} \beta(\theta + k\omega).$$

This kind of discrete average has already appeared in the first example of section 2.

Let us assume that $\beta^{\#}$ vanishes at some point $\theta^\star$. Then every point of the type $(\overline{\theta^\star}, r)$, $-\infty < r < \infty$, is fixed for h_δ^q. This fact implies that h_δ^q has the intersection property. From the previous lemma we deduce that also h_δ has the intersection property. This shows that Proposition 5 is not valid when $\omega \in 2\pi\mathbf{Q}$ because there are functions β with nonzero average and such that $\beta^{\#}$ vanishes at some points. For instance, $\beta(\theta) = \frac{1}{2} + \sin q\theta$.

5 Proof of Theorem 1

We shall assume

$$\int_0^{2\pi} \frac{\partial \ell_1}{\partial r}(\theta, r)d\theta > 0 \quad \forall r \in [a, b]. \tag{22}$$

The case of a negative average is reduced to this positive case via the change of variables $(\theta, r) \mapsto (\theta, -r)$.

The proof is divided in four steps. In what follows Δ_i, $i = 1, 2, \ldots$ will denote positive constants satisfying $\Delta_i \geq \Delta_{i+1}$.

1. Preliminaries
Define

$$a_1 = a + \frac{1}{8}, \quad a_2 = a + \frac{1}{4}, \quad b_2 = b - \frac{1}{4}, \quad b_1 = b - \frac{1}{8}$$

so that $b_1 - a_1 \geq \frac{7}{4}$ and $b_2 - a_2 \geq \frac{3}{2}$. We shall make use of the regions

$$A_1 = \mathbf{T}^1 \times [a_1, b_1], \quad A_2 = \mathbf{T}^1 \times [a_2, b_2].$$

They satisfy

$$A_2 \subset A_1 \subset A \subset C.$$

From (5) we find $\Delta_1 > 0$ such that

$$f_\delta(A_1) \subset A \quad \text{if } \delta < \Delta_1. \tag{23}$$

Next we present two preliminary results that follow from the Chain Rule.

Lemma 7 *Let $\Phi = (\Phi^1, \Phi^2) : A_2 \to A_1$ be a mapping of class C^p, $p \geq 1$, such that*

$$||\partial^\alpha \Phi^i||_{C^0(A_2)} \leq 7, \quad 1 \leq |\alpha| \leq p, \quad i = 1, 2.$$

Then there exists a constant C_p, independent of Φ, such that

$$||F \circ \Phi||_{C^p(A_2)} \leq C_p ||F||_{C^p(A_1)}$$

for every function $F \in C^p(A_1)$.

Remark. The notation $\Phi = (\Phi^1, \Phi^2)$ is understood in the following sense,

$$\Phi^i = \Pi_i \circ \tilde{\Phi},$$

where $\tilde{\Phi} : A_2 \to \mathbf{R} \times [a_1, b_1]$ is a lift of Φ and $\Pi_i : \mathbf{R}^2 \to \mathbf{R}$ is the projection onto the corresponding axis.

Lemma 8 *Given $p \geq 1$, $F \in C^{p+1}(A)$ and $R > 0$ there exists $\gamma = \gamma(p, F, R) > 0$ such that*

$$||F \circ f_1 - F \circ f_2||_{C^p(A_1)} \leq \gamma ||f_1 - f_2||_{C^p(A_1)} \tag{24}$$

for any functions $f_1, f_2 : A_1 \to A$ of class C^p and satisfying

$$||f_1||_{C^p}, ||f_2||_{C^p} \leq R.$$

Remark. If F only belongs to $C^p(A)$, then (24) must be replaced by

$$||F \circ f_1 - F \circ f_2||_{C^p(A_1)} \leq \Omega(||f_1 - f_2||_{C^p(A_1)}),$$

where $\Omega : [0, \infty) \to \mathbf{R}$ is a modulus of continuity that only depends on F and R. In particular, Ω is decreasing and $\lim_{r \to 0^+} \Omega(r) = 0$.

Later in the proof we shall apply the Small Twist Theorem as stated in the Appendix. The region $\mathcal{A}$ will be A_2 and the twist function

$$\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} \ell_1(\theta, r) d\theta. \tag{25}$$

From now on, $\epsilon_0 > 0$ will be the constant given by Theorem 9 for this choice. It only depends on ℓ_1.

To conclude the preliminary remarks we notice that the functions φ_i satisfy

$$\lim_{\delta \to 0^+} \{||\varphi_1(\cdot, \cdot; \delta)||_{C^4(A)} + ||\varphi_2(\cdot, \cdot; \delta)||_{C^4(A)}\} = 0. \tag{26}$$

This is a consequence of (7) and (8).

2. Scheme of the proof

We shall find $\Delta_2 > 0$ and a smooth change of variables

$$\Theta = \Theta(\theta, r; \delta), \quad R = R(\theta, r; \delta)$$

defined for $\delta \leq \Delta_2$ such that

$$\Psi_\delta : A \subset C \to C, \ \ (\overline{\theta}, r) \to (\overline{\Theta}, R)$$

satisfies

- Ψ_δ is a diffeomorphism of class C^∞ from A onto $\Psi_\delta(A)$.
- $A_2 \subset \Psi_\delta(A_1)$.

The inverse of Ψ_δ will be denoted by Φ_δ. In view of (23) we can define

$$g_\delta = \Psi_\delta \circ f_\delta \circ \Phi_\delta : A_2 \subset C \to C, \quad (\overline{\Theta}, R) \to (\overline{\Theta_1}, R_1),$$

for each $\delta \in [0, \Delta_2]$. The mapping g_δ has the intersection property in A_2 and will be expressed in the form

$$\begin{cases} \Theta_1 = \Theta + \omega + \delta\alpha(R) + \delta\phi_1(\Theta, R; \delta) \\ R_1 = R + \delta\phi_2(\Theta, R; \delta) \end{cases} \tag{27}$$

where α is given by (25) and $\phi_1, \phi_2 \in C^{4,0}(A_2 \times [0, \Delta_2])$. These remainders will satisfy

$$||\phi_1(\cdot,\cdot;\delta)||_{C^4(A_2)} + ||\phi_2(\cdot,\cdot;\delta)||_{C^4(A_2)} \leq \tag{28}$$

$$C_4\{||\varphi_1(\cdot,\cdot;\delta)||_{C^4(A)} + ||\varphi_2(\cdot,\cdot;\delta)||_{C^4(A)}\} + \frac{\epsilon_0}{2} + \Omega_1(\delta),$$

where C_4 is the constant given by Lemma 7 and $\Omega_1 : [0, \Delta_2] \to \mathbf{R}$ is an increasing function satisfying $\lim_{\delta \to 0^+} \Omega_1(\delta) = 0$. In view of (26) we can find $\Delta_3 > 0$ such that

$$||\varphi_1(\cdot,\cdot;\delta)||_{C^4(A)} + ||\varphi_2(\cdot,\cdot;\delta)||_{C^4(A)} < \frac{\epsilon_0}{4C_4}$$

and

$$\Omega_1(\delta) < \frac{\epsilon_0}{4}$$

if $\delta \in [0, \Delta_3]$. Then Theorem 9 is applicable and we deduce that g_δ has invariant curves. Undoing the change of variables we obtain invariant curves of f_δ.

3. The change of variables

Proposition 5 implies that the function ℓ_2 satisfies

$$\int_0^{2\pi} \ell_2(\theta, r)d\theta = 0 \quad \forall r \in [a, b].$$

Also the function $\tilde{\ell_1}(\theta, r) = \ell_1(\theta, r) - \alpha(r)$ has zero average with respect to θ. This allows us to apply Proposition 4 to deduce that (15) is approximately solvable (in class C^4) when $F = -\tilde{\ell_1}$ or $F = -\ell_2$. In this way we can find functions $X_1, X_2 \in C_o^\infty(A)$ such that

$$||\mathcal{R}_i||_{C^4(A)} < \frac{\epsilon_0}{4C_4}, \quad i = 1, 2,$$

where $\mathcal{R}_i$ is defined as

$$\mathcal{R}_1(\theta, r) = X_1(\theta+\omega, r) - X_1(\theta, r) + \tilde{\ell}_1(\theta, r)$$

and

$$\mathcal{R}_2(\theta, r) = X_2(\theta+\omega, r) - X_2(\theta, r) + \ell_2(\theta, r).$$

Here C_4 is given by Lemma 7. At this point it is convenient to notice that the functions X_1, X_2 only depend on ω, ℓ_1 and ℓ_2.

The change of variables is defined by

$$\begin{cases} \Theta = \theta + \delta X_1(\theta, r) \\ R = r + \delta X_2(\theta, r). \end{cases}$$

It is clear that we can find $\Delta_2 > 0$ as indicated in Step 2. Moreover we can assume that Δ_2 is so small that the inverse of Ψ_δ satisfies

$$||\partial^\alpha \Phi_\delta^i||_{C(A_2)} \le 7, \quad 1 \le |\alpha| \le 4, \ i = 1, 2, \ \delta \in [0, \Delta_2].$$

4. The map g_δ

The way we defined the change of variables and a computation lead to

$$\Theta_1 = \Theta + \omega + \delta\alpha(R) + \delta\{\varphi_1(\theta, r; \delta) + \mathcal{R}_1(\theta, r) + D_1(\theta, r; \delta) + \alpha(r) - \alpha(R)\}$$

$$R_1 = R + \delta\{\varphi_2(\theta, r; \delta) + \mathcal{R}_2(\theta, r) + D_2(\theta, r; \delta)\}$$

where

$$D_i(\theta, r; \delta) = X_i(\theta_1, r_1) - X_i(\theta+\omega, r).$$

Since X_i is of class C^∞, it is possible to apply Lemma 8 with $p = 4$, $F = X_i$, to find the estimate

$$||D_1(\cdot,\cdot;\delta)||_{C^4(A_1)} + ||D_2(\cdot,\cdot;\delta)||_{C^4(A_1)} \le$$

$$\gamma\delta\{||\ell_1||_{C^4(A)} + ||\ell_2||_{C^4(A)} + ||\varphi_1(\cdot,\cdot;\delta)||_{C^4(A)} + ||\varphi_2(\cdot,\cdot;\delta)||_{C^4(A)}\} \le k^\star \delta$$

where

$$k^\star := \gamma\{||\ell_1||_{C^4(A)} + ||\ell_2||_{C^4(A)} + ||\varphi_1||_{C^{4,0}(A\times[0,1])} + ||\varphi_2||_{C^{4,0}(A\times[0,1])}\}.$$

Notice that $f_1(\theta, r) = (\theta+\omega, r)$ and $f_2(\theta, r) = (\theta_1, r_1)$ are bounded in $C^4(A)$ by a number R that depends on ω, $||\ell_1||_{C^4(A)}$, $||\ell_2||_{C^4(A)}$ and $||\varphi_1||_{C^{4,0}(A\times[0,1])}$, $||\varphi_2||_{C^{4,0}(A\times[0,1])}$.

The function

$$\beta_\delta : A_1 \to \mathbf{R}, \quad \beta_\delta(\theta, r) = \alpha(r) - \alpha(R(\theta, r))$$

is only of class C^4 and we shall apply the remark after Lemma 8. In this way we can get an estimate of the form

$$||\beta_\delta||_{C^4(A_1)} \le \Omega_2(\delta),$$

where Ω_2 is an appropriate modulus of continuity that depends on α and X_2.

The map g_δ can be rewritten in the form (27) with

$$\phi_1(\cdot,\cdot;\delta) = \{\varphi_1(\cdot,\cdot;\delta) + \mathcal{R}_1 + D_1(\cdot,\cdot;\delta) + \beta_\delta\} \circ \Phi_\delta$$

$$\phi_2(\cdot,\cdot;\delta) = \{\varphi_2(\cdot,\cdot;\delta) + \mathcal{R}_2 + D_2(\cdot,\cdot;\delta)\} \circ \Phi_\delta.$$

Lemma 7 implies

$$||\phi_1(\cdot,\cdot;\delta)||_{C^4(A_2)} + ||\phi_2(\cdot,\cdot;\delta)||_{C^4(A_2)} \le$$

$$C_4\{||\varphi_1(\cdot,\cdot;\delta)||_{C^4(A)} + ||\varphi_2(\cdot,\cdot;\delta)||_{C^4(A)} + \frac{\epsilon_0}{2C_4} + k^\star\delta + \Omega_2(\delta)\}.$$

Thus the estimate (28) holds with $\Omega_1(\delta) = k^\star\delta + \Omega_2(\delta)$. We can now go back to Step 2 and finish the proof.

6 Appendix

Let us consider the mapping $f : \mathcal{A} \subset C \to C$ defined in the region $\mathcal{A} = \mathbf{T}^1 \times [a_\star, b_\star]$ with $b_\star - a_\star \ge \frac{3}{2}$. The lift of f is

$$\begin{cases} \theta_1 = \theta + \omega + \lambda(\alpha(r) + F(\theta,r)) \\ r_1 = r + \lambda G(\theta,r) \end{cases} \qquad (29)$$

where

$$\alpha \in C^4[a_\star, b_\star] \qquad (30)$$

and

$$F, G \in C^4(\mathcal{A}). \qquad (31)$$

The number $\omega \in \mathbf{R}$ is arbitrary and

$$\lambda \in (0,1]$$

is a parameter.

Next result resembles theorem 3 in the original paper of Moser [12].

Theorem 9 *Let $c_0 > 1$ be a given constant and let f be a mapping satisfying the intersection property and the previous conditions (29), (30) and (31). Then there exists $\epsilon_0 > 0$ (depending only on c_0) such that f has an invariant curve if the conditions below hold*

$$c_0^{-1} \leq \alpha'(r) \leq c_0 \;\; \forall r \in [a_\star, b_\star], \quad ||\alpha||_{C^4[a_\star,b_\star]} \leq c_0, \tag{32}$$

$$||F||_{C^4(\mathcal{A})} + ||G||_{C^4(\mathcal{A})} \leq \epsilon_0. \tag{33}$$

Remarks 1. The conclusion of the theorem can be improved. The proof will show the existence of an uncountable collection of invariant curves in $\mathcal{A}$. Moreover, these curves can be chosen so that they are graphs of functions ψ that belong to the Sobolev space $H^3(\mathbf{T}^1)$ and satisfy

$$a_\star \leq \psi(\theta) \leq b_\star \quad \forall \theta \in \mathbf{R}.$$

The rotation number of $r = \psi(\theta)$ is in the interval $[\omega + \lambda\alpha(a_\star), \omega + \lambda\alpha(b_\star)]$.
2. In the original statement of Moser in [12], the functions F and G were small in the C^0-norm while they were controlled in a certain C^k-norm. This C^k-norm was not necessarily small. This can also be achieved with the previous theorem if one uses the C^5-norm. In fact (31) and (33) are implied by the conditions

$$F, G \in C^5(\mathcal{A}),$$

$$||F||_{C^0(\mathcal{A})} + ||G||_{C^0(\mathcal{A})} \leq \epsilon_\star, \quad ||F||_{C^5(\mathcal{A})} + ||G||_{C^5(\mathcal{A})} \leq c_0,$$

where $\epsilon_\star$ only depends on ϵ_0 and c_0. This implication is proved using the following inequality: given $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$||H||_{C^4(\mathcal{A})} \leq \epsilon ||H||_{C^5(\mathcal{A})} + C_\epsilon ||H||_{C^0(\mathcal{A})} \quad \forall H \in C^5(\mathcal{A}).$$

The theorem will be obtained from another invariant curve theorem that follows along the lines of the work of Herman in [3] and [4]. To state this second result it is convenient to consider the fixed cylinder

$$\mathcal{A}_1 = \mathbf{T}^1 \times [-\frac{1}{4}, \frac{1}{4}]$$

with coordinates (θ, ρ), $|\rho| \leq \frac{1}{4}$.
Let $g : \mathcal{A}_1 \subset C \to C$ be a mapping with lift

$$\begin{cases} \theta_1 = \theta + \beta + \lambda(\alpha_1(\rho) + F_1(\theta, \rho)) \\ \rho_1 = \rho + \lambda G_1(\theta, \rho) \end{cases} \tag{34}$$

where $\alpha_1 \in C^4[-\frac{1}{4}, \frac{1}{4}]$ and $F_1, G_1 \in C^4(\mathcal{A}_1)$. The function α_1 satisfies

$$\alpha_1(0) = 0, \quad c_0^{-1} \leq \alpha_1'(r) \leq c_0 \ \forall r \in [-\frac{1}{4}, \frac{1}{4}], \quad ||\alpha_1||_{C^4} \leq c_0,$$

for a fixed positive constant $c_0 > 1$. We also assume that β is a number of constant type with Markov constant $\gamma > 0$ (see [3]). Finally, λ is a parameter in the interval

$$b^{-1}\gamma \leq \lambda \leq b\gamma, \tag{35}$$

where $b > 1$ is a fixed number.

Theorem 10 *Let $g : \mathcal{A}_1 \subset C \to C$ be a mapping satisfying the intersection property and assume that it is in the previous conditions. Then there exists $\epsilon_1 > 0$ (depending only on c_0 and b) such that g has an invariant curve of rotation number β if*

$$||F_1||_{C^4(\mathcal{A}_1)} + ||G_1||_{C^4(\mathcal{A}_1)} \leq \epsilon_1.$$

Sketch of the proof of theorem 10. The reader will be familiar with the methods and ideas of [3] and [4], especially with chapters IV and VII. We conjugate g by $H(\theta, \rho) = (\theta, \lambda\rho)$ to obtain

$$g_\lambda : \mathcal{A}_\lambda \subset C \to C, \ \ g_\lambda(\theta, \rho) = (\theta + \beta + \lambda\alpha_1(\frac{\rho}{\lambda}) + \lambda F_1(\theta, \frac{\rho}{\lambda}), \rho + \lambda^2 G_1(\theta, \frac{\rho}{\lambda}))$$

where $\mathcal{A}_\lambda = \mathbf{T}^1 \times [-\frac{\lambda}{4}, \frac{\lambda}{4}]$. Next we consider the functional equation

$$\psi \circ d_\lambda(\theta) = \psi(\theta) + \lambda^2 G_1(\theta, \frac{\psi(\theta)}{\lambda}) \tag{36}$$

with

$$d_\lambda(\theta) = \theta + \beta + \lambda\alpha_1(\frac{\psi(\theta)}{\lambda}) + \lambda F_1(\theta, \frac{\psi(\theta)}{\lambda}).$$

The graphs of the continuous solutions of (36) will be invariant curves of g_λ. To be precise, if ψ is a solution of (36) then the curve $\Gamma = \{\rho = \psi(\theta)\}$ satisfies $g_\lambda(\Gamma) \subset \Gamma$. If we know that d_λ is a homeomorphism then we can say that $g_\lambda(\Gamma) = \Gamma$. Our aim will be to prove the existence of solutions of (36).

From now on, x and y are two positive parameters that will be made suitably small in the course of the proof. To start with we can assume

$$||F_1||_{C^4(\mathcal{A}_1)} + ||G_1||_{C^4(\mathcal{A}_1)} \leq y < c_0^{-1}.$$

Let us consider the class of functions ψ in $H^3(\mathbf{T}^1)$ that satisfy

$$|\overline{\psi}| \leq \frac{1}{8b}\gamma, \quad ||D^3\psi||_{L^2(\mathbf{T}^1)} \leq \gamma x, \tag{37}$$

where $\overline{\psi} = \frac{1}{2\pi}\int_0^{2\pi} \psi(\theta)d\theta$. Then, given a function of this class,

$$||\psi||_{L^\infty(\mathbf{T}^1)} \leq |\overline{\psi}| + \pi||D\psi||_{L^\infty(\mathbf{T}^1)} \leq \frac{1}{8b}\gamma + \pi k_1 \gamma x,$$

where $k_1 > 0$ is a constant such that

$$||p||_{L^\infty(\mathbf{T}^1)} \leq k_1||D^2p||_{L^2(\mathbf{T}^1)} \quad \forall p \in H^2(\mathbf{T}^1), \ \int_{\mathbf{T}^1} p = 0.$$

Assuming $b\pi k_1 x < \frac{1}{8}$ and using (35), we obtain

$$\frac{|\psi(\theta)|}{\lambda} \leq \frac{1}{4} \quad \forall \theta \in \mathbf{R}.$$

This estimate allows us to define, for each ψ satisfying (37),

$$d_\lambda = Id + \beta + \lambda\alpha_1 \circ \frac{\psi}{\lambda} + \lambda F_1 \circ \mathcal{G}_\lambda,$$

where $\mathcal{G}_\lambda(\theta) = (\theta, \frac{\psi(\theta)}{\lambda})$. The derivative of d_λ is given by

$$Dd_\lambda = 1 + \lambda(\alpha_1' \circ \frac{\psi}{\lambda})\frac{D\psi}{\lambda} + \lambda\{F_{1\theta} \circ \mathcal{G}_\lambda + (F_{1\rho} \circ \mathcal{G}_\lambda)\frac{D\psi}{\lambda}\}$$

and we make x and y small enough so that

$$\frac{1}{2} \leq Dd_\lambda(\theta) \leq \frac{3}{2} \quad \forall \theta \in \mathbf{R}.$$

This inequality implies that d_λ is the lift of a diffeomorphism of $\mathbf{T}^1$ and so the rotation number $\mu(d_\lambda)$ is well defined.

Let us denote by K the set of functions $\psi \in H^3(\mathbf{T}^1)$ that satisfy (37) and are such that $\mu(d_\lambda) = \beta$. We shall look at K as a metric subspace of $C^2(\mathbf{T}^1)$. The set K depends upon x, λ, α_1 and F_1. It can be proved that if x and y are small enough then K is compact and homeomorphic to a convex subset of $C^2(\mathbf{T}^1)$. Actually, the projection $\Pi : \psi \to \psi - \overline{\psi}$ defines a homeomorphism between K and

$$\mathcal{K}_x = \{\psi \in H^3(\mathbf{T}^1) : \ \overline{\psi_1} = 0, \ ||D^3\psi_1||_{L^2(\mathbf{T}^1)} \leq \gamma x\}.$$

This fact is proven in Chapter VII of [4] in the case $\alpha(r) = r$ and $F_1 = 0$. In the general case the proof is similar and we just prove

$$\Pi(K) = \mathcal{K}_x.$$

By restricting the size of x we can guarantee that each function ψ_1 in $\mathcal{K}_x$ satisfies

$$\frac{|\psi_1(\theta)|}{\lambda} \leq \frac{1}{16b^2} \quad \forall \theta \in \mathbf{R}.$$

We also impose

$$y < \min\{\alpha_1(\frac{1}{16b^2}), -\alpha_1(-\frac{1}{16b^2})\}.$$

The monotonicity of the rotation number implies that for each $\psi_1 \in \mathcal{K}_x$ there exists $c \in [-\frac{1}{8b^2}, \frac{1}{8b^2}]$ such that the diffeomorphism of $\mathbf{T}^1$

$$\theta \to \theta + \beta + \lambda\alpha_1(\frac{\psi_1(\theta) + \lambda c}{\lambda}) + \lambda F_1(\theta, \frac{\psi_1(\theta) + \lambda c}{\lambda})$$

has rotation number β. For this value of c the function $\psi = \psi_1 + \lambda c$ belongs to K.

The Schauder's fixed point theorem implies that $\mathcal{K}_x$ has the fixed point property. Since this property is topological, also K has it.
After taking two derivatives in (36) we arrive at the second order functional differential equation

$$D^2\psi \circ d_\lambda - a_\lambda D^2\psi = B_\lambda \tag{38}$$

where

$$a_\lambda = \frac{1}{(Dd_\lambda)^2}\{1 + \lambda(G_{1\rho} \circ \mathcal{G}_\lambda) - (D\psi \circ d_\lambda)(\alpha_1' \circ \frac{\psi}{\lambda} + F_{1\rho} \circ \mathcal{G}_\lambda)\}$$

$$B_\lambda = \frac{1}{(Dd_\lambda)^2}[\lambda^2\{G_{1\theta\theta} \circ \mathcal{G}_\lambda + 2(G_{1\theta\rho} \circ \mathcal{G}_\lambda)\frac{D\psi}{\lambda} + (G_{1\rho\rho} \circ \mathcal{G}_\lambda)(\frac{D\psi}{\lambda})^2\}$$

$$-\lambda(D\psi \circ d_\lambda)\{F_{1\theta\theta} \circ \mathcal{G}_\lambda + 2(F_{1\theta\rho} \circ \mathcal{G}_\lambda)\frac{D\psi}{\lambda} + (F_{1\rho\rho} \circ \mathcal{G}_\lambda)(\frac{D\psi}{\lambda})^2 + (\alpha_1'' \circ \frac{\psi}{\lambda})(\frac{D\psi}{\lambda})^2\}].$$

Given $\psi \in K$, the corresponding functions d_λ, a_λ, B_λ satisfy estimates of the kind

$$||D^2 \log Dd_\lambda||_{L^2(\mathbf{T}^1)} \leq k_2(x + y)\gamma, \quad ||D^2 a_\lambda||_{L^2(\mathbf{T}^1)} \leq k_3(x + y)\gamma$$

$$||D^2 B_\lambda||_{L^2(\mathbf{T}^1)} \leq k_4(x^2 + y)\gamma^2,$$

where k_2, k_3 and k_4 are certain numbers that depend on c_0 and b. These estimates are important because they will allow us to use the theory of linear difference equations developed in chapter VII of [4].
Given $\psi \in K$, the functions d_λ, a_λ and B_λ are defined according to the previous rules. They depend on ψ and $D\psi$. Once these functions have been constructed we can consider the linear difference equation

$$D^2\psi^\star \circ d_\lambda - a_\lambda D^2\psi^\star = B_\lambda + \nu, \tag{39}$$

where the unknown $(\psi^\star, \nu)$ is searched in $H_0^3(\mathbf{T}^1) \times \mathbf{R}$. It is possible to prove that there exists a unique solution of (39) for x and y smaller than certain numbers. These numbers can be computed. Moreover,

$$||D^3\psi^\star||_{L^2} \leq \frac{k_5}{\gamma}||D^2B_\lambda||_{L^2}.$$

By restricting once again the size of x and y one can assume that $\psi^\star$ satisfies the estimate

$$||D^3\psi^\star||_{L^2} \leq \gamma x.$$

In this way we can define the map

$$K \to K, \quad \psi \to \Phi(\psi^\star)$$

where Φ is the homeomorphism from $\mathcal{K}_x$ onto K which was previously defined. It is not hard to prove that the possible fixed points of this mapping are solutions of (38). On the other hand, the uniqueness for (39) implies that this mapping is continuous with respect to the C^2-topology. Thus, there exists at least one solution of (38). The solution of (38) is also a solution of (36) because g has the intersection property.

Remarks The previous proof can be simplified if one starts with the change of variables $R = \alpha(\rho)$. In this way α becomes the identity. However, it seems to me that this simplification forces additional regularity conditions on α. Probably the previous theorem can also be proved using Holder spaces as in [3]. However I find more natural (and beautiful) the approach based on Sobolev spaces in [4].

Proof of theorem 9. Let us fix $\lambda \in (0,1]$. Since the length of $[a_\star, b_\star]$ is at least $3/2$, the interval with center $\frac{a_\star+b_\star}{2}$ and radius $3/4$ is included in $[a_\star, b_\star]$. Consider the interval

$$I_\lambda = [\omega + \lambda\alpha(\frac{a_\star + b_\star}{2} - \frac{1}{2}), \omega + \lambda\alpha(\frac{a_\star + b_\star}{2} + \frac{1}{2})].$$

In view of (32), the length of I_λ is at least λc_0^{-1}. Thus, we can apply the results in section 3.5 of [3] and lemma 4.4 in [13] to find a number of constant type $\beta \in I_\lambda$ with Markov constant γ satisfying

$$\frac{\lambda c_0^{-1}}{16} \leq \gamma \leq \frac{\lambda c_0^{-1}}{4}. \tag{40}$$

The strict monotonicity of α implies that we can find a unique $\xi \in [\frac{a_\star+b_\star}{2} - \frac{1}{2}, \frac{a_\star+b_\star}{2} + \frac{1}{2}]$ such that

$$\beta = \omega + \lambda\alpha(\xi).$$

The interval $[\xi - \frac{1}{4}, \xi + \frac{1}{4}]$ is contained in $[a_\star, b_\star]$ and we consider the translation

$$r = \rho + \xi, \quad \rho \in [-\frac{1}{4}, \frac{1}{4}].$$

Then (29) takes the form (34) with

$$\alpha_1(\rho) = \alpha(\rho+\xi) - \alpha(\xi), \quad F_1(\theta,\rho) = F(\theta,\rho+\xi), \quad G_1(\theta,\rho) = G(\theta,\rho+\xi).$$

Theorem 10 can be applied and the size of ϵ_1 is independent of λ. This is so because the constant b in condition (35) can be controlled by c_0 through (40). In consequence, the size of ϵ_0 only depends on c_0.

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