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Research article

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Blow-up criteria and instability of normalized standing waves for the fractional Schrödinger-Choquard equation

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Abstract: In this paper, we study blow-up criteria and instability of normalized standing waves for the fractional Schrödinger-Choquard equation

$$i\partial_t\psi - (-\Delta)^s\psi + (I_\alpha \star |\psi|^p)|\psi|^{p-2}\psi = 0.$$

By using localized virial estimates, we firstly establish general blow-up criteria for non-radial solutions in both L^2-critical and L^2-supercritical cases. Then, we show existence of normalized standing waves by using the profile decomposition theory in H^s. Combining these results, we study the strong instability of normalized standing waves. Our obtained results greatly improve earlier results.

Keywords: Fractional Schrödinger-Choquard equation; Blow-up criteria; Strong instability; Normalized standing waves

MSC: 35Q55; 35J10

1 Introduction

Over the past decade, there has been a great deal of interest in studying the fractional Schrödinger equation (NLS)

$$i\partial_t\psi = (-\Delta)^s\psi + f(\psi), \tag{1.1}$$

where $0 < s < 1$ and $f(\psi)$ is the nonlinearity. The fractional differential operator $(-\Delta)^s$ is defined by $(-\Delta)^s\psi = \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}(\psi)]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and inverse Fourier transform, respectively. The fractional NLS (1.1) was first deduced by Laskin in [29, 30] by extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. The fractional NLS also arises in the description of Bonson stars as well as in water wave dynamics (see e.g. [22]) and in the continuum limit of discrete models with long-range interactions (see e.g. [28]).

In this paper, we consider blow-up criteria and instability of normalized standing waves for the fractional nonlinear Schrödinger-Choquard equation

$$\begin{cases} i\partial_t\psi - (-\Delta)^s\psi + (I_\alpha \star |u|^p)|\psi|^{p-2}\psi = 0, & (t,x) \in [0,T^\star) \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1.2}$$

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where $\psi : [0, T^\star) \times \mathbb{R}^N \to \mathbb{C}$ is the complex valued function, $N \geq 1$, $\psi_0 \in H^s$, $0 < s < 1$, $0 < T^\star \leq \infty$, $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s}$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{A(\alpha)}{|x|^{N-\alpha}}, \quad A(\alpha) := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}$$

with $\alpha \in (0, N)$ and Γ is the Gamma function.

Equation (1.2) enjoys the scaling invariance. That is, if ψ is a solution of (1.2) with initial data ψ_0, then $\psi_\mu(t, x) := \mu^{\frac{\alpha+2s}{4s(p-1)}}\psi(\mu t, \mu^{\frac{1}{2s}}x)$ for all $\mu > 0$ is also a solution of (1.2) with initial data $\mu^{\frac{\alpha+2s}{4s(p-1)}}\psi_0(\mu^{\frac{1}{2s}}x)$. In particular, $\|\psi_\mu(t)\|_{\dot{H}^{s_c}} = \|\psi(\mu t)\|_{\dot{H}^{s_c}}$, where

$$s_c := \frac{N}{2} - \frac{\alpha + 2s}{2p - 2}. \tag{1.3}$$

Thus, s_c is referred as the critical Sobolev exponent of (1.2). If the initial data $\psi_0 \in H^s$, then equation (1.2) enjoys mass and energy conservation laws:

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad E(\psi(t)) = E(\psi_0),$$

where the energy E is defined by

$$E(\psi(t)) = \frac{1}{2}\|\psi(t)\|^2_{\dot{H}^s} - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha \star |\psi(t)|^p)(x)|\psi(t, x)|^p dx. \tag{1.4}$$

Before entering our main results, we firstly recall some known blow-up results for NLS. For the classical NLS, i.e., $s = 1$, when initial data $\psi_0 \in \Sigma := \{\psi_0 \in H^1 \text{ and } x\psi_0 \in L^2\}$, the following Variance-Virial Law holds

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|x|^2|\psi(t, x)|^2 dx = 2\mathrm{Im}\int_{\mathbb{R}^N}\bar{\psi}(t, x)x \cdot \nabla\psi(t, x)dx. \tag{1.5}$$

By using (1.5) and the virial identity, ones can prove existence of blow-up solutions for the classical NLS with negative energy $E(\psi_0) < 0$, see [7]. However, since identity (1.5) fails for $s < 1$, which readily checks by dimensional analysis, this argument cannot work. Rather, a possible generalization of the variance for the fractional NLS is given by the nonnegative quantity

$$\mathcal{V}^{(s)}[\psi(t)] := \int_{\mathbb{R}^N}\bar{\psi}(t, x)x \cdot (-\Delta)^{1-s}x\psi(t, x)dx = \|x(-\Delta)^{\frac{1-s}{2}}\psi(t)\|^2_{L^2}. \tag{1.6}$$

Let $\psi(t)$ be a sufficiently regular and spatially localized solution of equation $i\partial_t\psi = (-\Delta)^s\psi$, it follows that

$$\frac{1}{2}\frac{d}{dt}\mathcal{V}^{(s)}[\psi(t)] := 2\mathrm{Im}\int_{\mathbb{R}^N}\bar{\psi}(t, x)x \cdot \nabla\psi(t, x)dx. \tag{1.7}$$

This method has been successfully applied to prove the existence of radial blow-up solutions of (1.1) with focusing Hartree-type nonlinearities, i.e., $f(\psi) = -(|x|^{-\gamma} \star |\psi|^2)\psi$ with $\gamma \geq 1$, see [8, 9, 48]. But this method can not work due to the nontrivial error terms which seem very hard to control for the local nonlinearities $f(\psi) = -|\psi|^p\psi$, see [6]. In [6], Boulenger, Himmelsbach and Lenzmann applied the Balakrishman's formula

$$(-\Delta)^s = \frac{\sin \pi s}{\pi}\int_0^\infty m^{s-1}\frac{-\Delta}{-\Delta + m}dm, \tag{1.8}$$

and obtained the differential inequality

$$\frac{d}{dt}\left(\mathrm{Im}\int_{\mathbb{R}^N}\bar{\psi}(t)\nabla\varphi_R \cdot \nabla\psi(t)dx\right) \leq 4pNE(\psi_0) - 2\delta\|(-\Delta)^{\frac{s}{2}}\psi(t)\|^2_{L^2} + \circ_R(1)(1 + \|(-\Delta)^{\frac{s}{2}}\psi(t)\|^{p/s+}_{L^2}),$$

where $\delta = pN - 2s$. Based on this key estimate, they proved the existence of radial blow-up solutions by applying a standard comparison ODE argument.

For the fractional Schrödinger-Choquard equation (1.2), Saanouni in [39] proved the existence of radial blow-up solutions by using the method in [6]. In this paper, we will further study the existence of blow-up solutions of (1.2) for non-radial initial data by using the idea of Du, Wu and Zhang in [13]. The main difficulty is the appearance of the fractional order Laplacian $(-\Delta)^s$. When $s = 1$, the time derivative of the virial action can be easily obtained, that is

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \varphi(x)|\psi(t,x)|^2 dx = 2\mathrm{Im}\int_{\mathbb{R}^N} \bar{\psi}(t,x)\nabla\varphi(x)\cdot\nabla\psi(t,x)dx. \tag{1.9}$$

Using this identity, Du, Wu and Zhang in [13] derived an L^2-estimate in the exterior ball. Combining this L^2-estimate and the virial estimates, they established blow-up criteria for the classical NLS. When $s \in (\frac{1}{2}, 1)$, the identity (1.9) does not hold. However, by exploiting the ideas in [6, 12] and using the Balakrishman's formula (1.8), we can obtain the time derivative of the virial action, see Lemma 2.9. Thus, we can establish the blow-up criteria for (1.2).

Theorem 1.1. *Let $N \geq 1$, $s \in (\frac{1}{2}, 1)$, $1 + \frac{2s+\alpha}{N} \leq p < \frac{N+\alpha}{N-2s}$, $\psi_0 \in H^s$ and $\psi \in C([0, T^\star), H^s)$ be the corresponding solution of* (1.2). *Furthermore, we suppose either $E(\psi_0) < 0$, or, if $E(\psi_0) \geq 0$ and*

$$\begin{cases} E(\psi_0)^{s_c}\|\psi_0\|_{L^2}^{2(s-s_c)} < E(u)^{s_c}\|u\|_{L^2}^{2(s-s_c)}, \\ \|(-\Delta)^{s/2}\psi_0\|_{L^2}^{s_c}\|\psi_0\|_{L^2}^{s-s_c} > \|(-\Delta)^{s/2}u\|_{L^2}^{s_c}\|u\|_{L^2}^{s-s_c}, \end{cases} \tag{1.10}$$

where s_c is defined by (1.3) *and u is a ground state of the following elliptic equation*

$$(-\Delta)^s u + u - (I_\alpha \star |u|^p)|u|^{p-2}u = 0. \tag{1.11}$$

Then one of the following statements holds true:
- *$\psi(t)$ blows up in finite time, i.e. $T^\star < +\infty$;*
- *$\psi(t)$ blows up infinite time and there exists a time sequence $(t_n)_{n\geq 1}$ such that $t_n \to +\infty$ and*

$$\lim_{n\to\infty} \|(-\Delta)^{\frac{s}{2}}\psi(t_n)\|_{L^2} = \infty.$$

Remark 1. The uniqueness of ground state solutions to (1.11) is still unknown. However, it follows from the optimal constant in (2.2) that all ground states have the same L^2-norm. Moreover, we see from Pohozaev's identities (2.3) that all ground states have the same $\dot{H}^s$-norm and energy. Therefore, for different ground states, the quantities $E(u)^{s_c}\|u\|_{L^2}^{2(s-s_c)}$ and $\|(-\Delta)^{s/2}u\|_{L^2}^{s_c}\|u\|_{L^2}^{s-s_c}$ are same. These imply that the assumption (1.10) is reasonable.

Remark 2. When $p = 2$, similar blow-up criteria for (1.2) with radial solutions have been established in [8, 9, 25, 33, 37, 40, 41, 47, 48]. Here, we remove the assumption of radial solutions and extend these results to more general Choquard-type nonlinearity.

Based on blow-up criterion (1.10), we study the strong instability of normalized standing waves of (1.2). Firstly, we introduce some notations. Equation (1.2) enjoys a class of special solutions, which are called standing waves, namely solutions of the form $e^{i\omega t}u_\omega$, where $\omega \in \mathbb{R}$ is a frequency and $u_\omega \in H^s$ is a nontrivial solution to the elliptic equation

$$(-\Delta)^s u_\omega + \omega u_\omega - (I_\alpha \star |u_\omega|^p)|u_\omega|^{p-2}u_\omega = 0. \tag{1.12}$$

At this moment, our intention is reduced to study (1.12). To do this, there exist two substantially different choices in terms of the frequency ω. One is to fix the frequency $\omega \in \mathbb{R}$. In this situation, every solution to (1.12) corresponds to a critical point of the action functional $S_\omega(u)$ on H^s, where

$$S_\omega(u) := \frac{1}{2}\|u\|_{\dot{H}^s}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha \star |u|^p)(x)|u(x)|^p dx. \tag{1.13}$$

Alternatively, it is interesting to study solutions of (1.12) having prescribed L^2-norm. That is, for any given $c > 0$, ones study solutions of (1.12) satisfying the L^2-norm constraint

$$S(c) = \{u \in H^s : \ \|u\|_{L^2}^2 = c\}, \quad c > 0. \tag{1.14}$$

Physically, such solutions are called normalized solutions of (1.12), which formally corresponds to critical points of the energy functional $E(u)$ restricted on $S(c)$, where $E(u)$ is defined by (1.4). In particular, in this situation, the frequency $\omega \in \mathbb{R}$ is an unknown part, which corresponds to the associated Lagrange multiplier. Recently, these questions have received more attention, see [1–5, 26, 32, 42, 45].

In the L^2-subcritical case, i.e., $1 + \frac{\alpha}{N} < p < 1 + \frac{2s+\alpha}{N}$, the energy $E(u)$ is bounded from below on $S(c)$. Feng and Zhang in [21] studied existence of normalized ground states to (1.12) by using the profile decomposition theory in H^s. On the contrary, in the L^2-supercritical case, the energy $E(u)$ restricted on $S(c)$ becomes unbounded from below for any $c > 0$. For this reason, it is unlikely to obtain a solution to (1.12)-(1.14) by developing a global minimizing problem. Motivated by minimizing method on Pohozaev manifold, we try to construct a submanifold of $S(c)$, on which $E(u)$ is bounded from below and coercive, and then we look for minimizers of $E(u)$ on such a submanifold. Precisely, we introduce the following minimizing problem

$$m(c) := \inf_{u \in V(c)} E(u), \tag{1.15}$$

where the constraint $V(c)$ is defined by

$$V(c) := \{u \in S(c) : \ K(u) = 0\}, \tag{1.16}$$

and the functional $K(u)$ is defined by

$$K(u) := \partial_\lambda S_\omega(u^\lambda)|_{\lambda=1} = s\|u\|_{\dot{H}^s}^2 - \frac{\theta}{2p} \int\limits_{\mathbb{R}^N} (I_\alpha \star |u|^p)(x)|u(x)|^p \, dx, \tag{1.17}$$

where

$$\theta = Np - N - \alpha, \quad u^\lambda(x) := \lambda^{N/2} u(\lambda x). \tag{1.18}$$

Indeed, the identity $K(u) = 0$ is the Pohozaev identity related to (1.12). The constraint $V(c)$ is the so-called Pohozaev manifold related to (1.12)-(1.14). In the following theorem, we can prove the existence of minimizers of (1.15).

Theorem 1.2. *Let $1 + \frac{\alpha+2s}{N} < p < \frac{N+\alpha}{N-2s}$ and $c > 0$. Then there exists $u_c \in V(c)$ such that $E(u_c) = m(c)$.*

Remark. This theorem can be proved by using the method in [18]. Here, we will use the profile decomposition of bounded sequences in H^s to prove this theorem. The profile decomposition theory has been extensively applied to study existence of normalized standing waves in the L^2-subcritical case, see, e.g., [20, 21, 49]. Here, we successfully apply it to study existence of normalized standing waves in the L^2-supercritical case. Therefore, our approach is of particular interest.

Next, we denote the set of minimizers of E on $V(c)$ as

$$\mathcal{M}_c := \{u \in V(c) : \ E(u) = \inf_{v \in V(c)} E(v)\}. \tag{1.19}$$

In the following theorem, we can show any minimizer to (1.15) is a ground state to (1.12)-(1.14).

Theorem 1.3. *Let $1 + \frac{2s+\alpha}{N} < p < \frac{N+\alpha}{N-2s}$. Then for any $u_c \in \mathcal{M}_c$, there exists $\omega_c > 0$ such that $(u_c, \omega_c) \in H^s \times \mathbb{R}$ is a weak solution to problem (1.12). Furthermore, u_c is a ground state solution to problem (1.12) with $\omega = \omega_c$.*

Finally, we consider the strong instability of normalized standing waves. The usual strategy to study the strong instability of standing waves for the classical NLS (s=1) is to use the variational characterization of the ground states as minimizers of the action functional and obtain the key estimate $K(\psi(t)) \le 2(S_\omega(\psi_0) - S_\omega(u_\omega))$. Then, it follows from the virial identity that

$$\frac{d^2}{dt^2}\|x\psi(t)\|_{L^2}^2 = 8K(\psi(t)) \le 16(S_\omega(\psi_0) - S_\omega(u_\omega)) < 0,$$

where $K(\psi(t))$ is defined by (1.17) with $s = 1$. This implies that the solution $\psi(t)$ of (1.1) with $s = 1$ blows up in finite time. Thus, one can prove the strong instability of ground state standing waves, see [7, 11, 16, 17, 23, 24, 31, 34–36, 38, 43, 44].

Here, we only need to use the blow-up criterion (1.10) to study the strong instability of normalized standing waves.

Theorem 1.4. *Let $N \ge 1$, $s \in (\frac{1}{2}, 1)$, $1 + \frac{2s+\alpha}{N} < p < \frac{N+\alpha}{N-2s}$, $c > 0$. Then for any $u_c \in \mathcal{M}_c$, the standing wave $\psi(t,x) = e^{i\omega_c t}u_c(x)$ is strongly unstable in the following sense: there exists $\{\psi_{0,n}\} \subset H^s$ such that $\psi_{0,n} \to u_c$ in H^s as $n \to \infty$ and the corresponding solution ψ_n of (1.2) with initial data $\psi_{0,n}$ blows up in finite or infinite time for any $n \ge 1$.*

Remark 1. In previous results, in order to construct blow-up solutions around the ground state solution, one need to assume that the ground state solution u_ω is radial or $u_\omega \in \Sigma := \{v \in H^1 \text{ and } xv \in L^2\}$. Here, we remove these assumptions, so our result greatly improve some previous results.

Remark 2. When $p = 2$ and $N - \alpha = 2s$, i.e., in the L^2-critical case, Zhang and Zhu in [46] proved the strong instability of radial ground state standing waves of (1.2). Here, we remove this radial assumption and extend this result to the L^2-supercritical case and more general Choquard-type nonlinearity.

This paper is organized as follows: in Section 2, we will recall and prove some lemmas such as the local well-posedness theory of (1.2), a sharp Gagliardo-Nirenberg type inequality and the localized virial estimate related to (1.2). In section 3, we will establish blow-up criteria for (1.2). In section 4, we will prove the existence and strong instability of normalized standing waves.

Notations. In this paper, we use the following notations. For any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N);\ \int\limits_{\mathbb{R}^N} (1 + |\xi|^{2s})|\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \|u\|_{\dot{H}^s(\mathbb{R}^N)},$$

where up to a multiplicative constant

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)} = \left\{ \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right\}^{\frac{1}{2}}$$

is the so-called Gagliardo semi-norm of u. In this paper, we often use the abbreviations $L^r = L^r(\mathbb{R}^N)$, $H^s = H^s(\mathbb{R}^N)$. For $J \subset \mathbb{R}$ and $q, r \in [1, \infty]$, we define the mixed norm

$$\|u\|_{L^q(J,L^r)} := \left(\int\limits_J \left(\int\limits_{\mathbb{R}^N} |u(t,x)|^r dx \right)^{\frac{q}{r}} \right)^{\frac{1}{q}},$$

with the usual modification when either q or r are infinity. In the case $q = r$, we shall use $L^q(J \times \mathbb{R}^N)$ instead of $L^q(J, L^r)$.

2 Preliminaries

In this section, we recall some preliminary results that will be used later. Firstly, we recall the local well-posedness for the Cauchy problem (1.2). Hong and Sire in [27] first studied the local well-posedness of the fractional NLS in H^s by using Strichartz's estimates and the contraction mapping argument. Since Strichartz's estimates for non-radial data have a loss of derivatives, a weak local well-posedness holds in the energy space compared to the classical nonlinear Schrödinger equation, see [10, 27] for more details. One can remove the loss of derivatives in Strichartz's estimates by considering radially symmetric data. However, it needs a restriction on the validity of s, namely $\frac{N}{2N-1} \le s < 1$.

Proposition 2.1. *[15, Proposition 2.3][Non-radial H^s LWP] Let $s \in (0,1)\backslash\{1/2\}$, $2 \le p < \frac{N+\alpha}{N-2s}$ and $\max\{0, N-4s\} < \alpha < N$ be such that*

$$s > \begin{cases} \frac{1}{2} - \frac{2s}{\max\{2p-2,4\}} & \text{if } N = 1, \\ \frac{N}{2} - \frac{s}{p-1} & \text{if } N \ge 2. \end{cases} \tag{2.1}$$

Then for all $\psi_0 \in H^s$, there exist $T^\star \in (0, +\infty]$ and a unique solution $\psi \in C([0, T^\star), H^s) \cap L^q_{\mathrm{loc}}([0, T^\star), L^\infty)$, for some $q > \max\{2p-2, 4\}$ when $N = 1$ and some $q > 2p-2$ when $N \ge 2$. Moreover, the following properties hold:

- *If $T^\star < +\infty$, then $\|\psi(t)\|_{H^s} \to \infty$ as $t \uparrow T^\star$.*
- *The solution enjoys conservation of mass and energy, i.e. $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$ and $E(u(t)) = E(\psi_0)$ for all $t \in [0, T^\star)$, where $E(\psi(t))$ defined by (1.4).*

Remark. When $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s}$, it follows from the Hardy-Littlewood-Sobolev inequality that $\int_{\mathbb{R}^N} (I_\alpha \star |\psi|^p)(x)|\psi(x)|^p\,dx$ is well-defined for $\psi \in H^s$. Therefore, we guess that these results also hold for $1 + \frac{\alpha}{N} < p < 2$. However, we cannot prove these results since the nonlinearity $(I_\alpha \star |\psi|^p)|\psi|^{p-2}\psi$ is singular when $1 + \frac{\alpha}{N} < p < 2$, see [19].

Proposition 2.2. *[15, Proposition 2.3][Radial H^s LWP] Let $N \ge 2$, $\frac{N}{2N-1} \le s < 1$, $2 \le p < \frac{N+\alpha}{N-2s}$ and $\max\{0, N-4s\} < \alpha < N$. Then for any $\psi_0 \in H^s$ radial, there exist $T^\star \in (0, +\infty]$ and a unique solution $\psi \in C([0, T^\star), H^s)$ to (1.2). Moreover, the following properties hold:*

- *$\psi \in L^a_{\mathrm{loc}}([0, T), W^{s,b})$ for any fractional admissible pair (a, b).*
- *If $T^\star < +\infty$, then $\|\psi(t)\|_{\dot H^s} \to \infty$ as $t \uparrow T^\star$.*
- *The solution enjoys conservation of mass and energy, i.e. $M(\psi(t)) = M(\psi_0)$ and $E(\psi(t)) = E(\psi_0)$ for all $t \in [0, T^\star)$.*

Next, we recall a sharp Gagliardo-Nirenberg type inequality established in [21].

Lemma 2.3. *[21, Theorem 2.3] Let $0 < s < 1$ and $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s}$. Then, for all $u \in H^s$,*

$$\int_{\mathbb{R}^N} (I_\alpha \star |u|^p)|u|^p\,dx \le C_{opt} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^{\frac{Np-N-\alpha}{s}} \|u\|_{L^2}^{\frac{N+\alpha-Np+2sp}{s}}, \tag{2.2}$$

where the optimal constant C_{opt} is given by

$$C_{opt} = \frac{2sp}{2sp - Np + N + \alpha} \left(\frac{2sp - Np + N + \alpha}{Np - N - \alpha} \right)^{\frac{Np-N-\alpha}{2s}} \|Q\|_{L^2}^{2-2p},$$

where Q is the ground state of the elliptic equation (1.11). In particular, in the L^2-critical case, i.e., $p = 1 + \frac{2s+\alpha}{N}$, $C_{opt} = p\|Q\|_{L^2}^{2-2p}$.

Moreover, the following Pohozaev's identities hold true:

$$\|Q\|_{\dot H^s}^2 = \frac{Np - N - \alpha}{2sp} \int_{\mathbb{R}^N} (I_\alpha \star |Q|^p)(x)|Q(x)|^p\,dx = \frac{Np - N - \alpha}{2sp - Np + N + \alpha} \|Q\|_{L^2}^2. \tag{2.3}$$

Next, we recall the profile decomposition of bounded sequences in H^s, which has been established in [48].

Lemma 2.4. *Let $N \geq 3$, $0 < s < 1$ and $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s}$. If $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in H^s, then there exist a subsequence of $\{u_n\}_{n=1}^{\infty}$ (still denoted by $\{u_n\}_{n=1}^{\infty}$), a family $\{x_n^j\}_{j=1}^{\infty}$ of sequences in $\mathbb{R}^N$ and a sequence $\{U^j\}_{j=1}^{\infty}$ in H^s such that*

(i) for every $k \neq j$, $|x_n^k - x_n^j| \to +\infty$, as $n \to \infty$;

(ii) for every $l \geq 1$ and every $x \in \mathbb{R}^N$, we have

$$u_n(x) = \sum_{j=1}^{l} U^j(x - x_n^j) + r_n^l, \tag{2.4}$$

with $\limsup_{n\to\infty} \|r_n^l\|_{L^q} \to 0$ *as $l \to \infty$ for every $q \in (2, \frac{2N}{N-2s})$. Moreover,*

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|U^j\|_{L^2}^2 + \|r_n^l\|_{L^2}^2 + \circ(1), \tag{2.5}$$

$$\|(-\Delta)^{s/2}u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|(-\Delta)^{s/2}U^j\|_{L^2}^2 + \|(-\Delta)^{s/2}r_n^l\|_{L^2}^2 + \circ(1), \tag{2.6}$$

$$\begin{aligned}&\int_{\mathbb{R}^N} I_\alpha \star \Big|\sum_{j=1}^{l} U^j(\cdot - x_n^j)\Big|^p \Big|\sum_{j=1}^{l} U^j(x - x_n^j)\Big|^p dx \\ &= \sum_{j=1}^{l} \int_{\mathbb{R}^N} I_\alpha \star |U^j(\cdot - x_n^j)|^p |U^j(x - x_n^j)|^p dx + \circ(1),\end{aligned} \tag{2.7}$$

where $\circ(1) = \circ_n(1) \to 0$ as $n \to \infty$.

Finally, we recall and prove some virial estimates related to (1.2) which is the main ingredient in the proof of Theorem 1.1.

Lemma 2.5 ([6]). *Let $N \geq 1$ and suppose $\varphi : \mathbb{R}^N \to \mathbb{R}$ is such that $\nabla\varphi \in W^{1,\infty}(\mathbb{R}^N)$. Then, for all $u \in H^{\frac{1}{2}}$, it holds that*

$$\left| \int_{\mathbb{R}^N} \overline{u}(x)\nabla\varphi(x) \cdot \nabla u(x)dx \right| \leq C\|\nabla\varphi\|_{W^{1,\infty}} \left(\|u\|_{\dot{H}^{\frac{1}{2}}}^2 + \|u\|_{L^2}\|u\|_{\dot{H}^{\frac{1}{2}}} \right),$$

for some constant $C > 0$ that depends only on N.

In order to study localized virial estimates for (1.2), we need to introduce the auxiliary function

$$u_m(x) := c_s \frac{1}{-\Delta + m} u(x) = c_s \mathcal{F}^{-1}\left(\frac{\hat{u}(\xi)}{|\xi|^2 + m} \right), \quad m > 0, \tag{2.8}$$

where

$$c_s := \sqrt{\frac{\sin \pi s}{\pi}}.$$

Lemma 2.6 ([6]). *Let $N \geq 1$, $s \in (0, 1)$ and suppose $\varphi : \mathbb{R}^N \to \mathbb{R}$ with $\Delta\varphi \in W^{2,\infty}(\mathbb{R}^N)$. Then, for all $u \in L^2$, it holds that*

$$\left| \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta^2\varphi)|u_m|^2 dx dm \right| \leq C\|\Delta^2\varphi\|_{L^\infty}^{s} \|\Delta\varphi\|_{L^\infty}^{1-s} \|u\|_{L^2}^2,$$

for some constant $C > 0$ that depends only on s and N.

We refer the reader to [6, Appendix A] for the proof of Lemma 2.5 and Lemma 2.6. Using the fact

$$\frac{\sin \pi s}{\pi} \int_0^\infty \frac{m^s}{(|\xi|^2 + m)^2} dm = s|\xi|^{2s-2},$$

the Plancherel's and Fubini's theorems imply

$$\begin{aligned} \int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla u_m|^2 dx dm &= \int_{\mathbb{R}^N} \left(\frac{\sin \pi s}{\pi} \int_0^\infty \frac{m^s dm}{(|\xi|^2 + m)^2} \right) |\xi|^2 |\hat{u}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^N} (s|\xi|^{2s-2}) |\xi|^2 |\hat{u}(\xi)|^2 d\xi = s\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2, \end{aligned} \tag{2.9}$$

for any $u \in \dot{H}^s$.

Lemma 2.7. *[12, Lemma 4.2] Let $N \geq 1$, $s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\nabla\varphi \in W^{1,\infty}$. Then for any $u \in L^2$, it holds that*

$$\left| \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta\varphi)|u_m|^2 dx dm \right| \leq C\|\Delta\varphi\|_{L^\infty}^{2s-1} \|\nabla\varphi\|_{L^\infty}^{2-2s} \|u\|_{L^2}^2,$$

for some constant $C > 0$ that depends only on s and N.

By the same argument as in Lemma 2.7 and using in addition Lemma 2.5, we obtain the following estimate.

Lemma 2.8. *Let $N \geq 1$, $s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\nabla\varphi \in W^{1,\infty}$. Then for any $u \in H^{1/2}$, it holds that*

$$\left| \int_0^\infty m^s \int_{\mathbb{R}^N} \overline{u}_m \nabla\varphi \cdot \nabla u_m dx dm \right| \leq C\|\nabla\varphi\|_{W^{1,\infty}} \|u\|_{H^{1/2}}^2,$$

for some constant $C > 0$ depending only on N.

Let $N \geq 1$, $1/2 < s < 1$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $\psi \in C([0, T^*), H^s)$ is a solution to (1.2). We define the localized virial action of ψ associated to φ by

$$\mathcal{V}_\varphi[\psi(t)] := \int_{\mathbb{R}^N} \varphi(x)|\psi(t, x)|^2 dx.$$

Lemma 2.9. *[12, Lemma 4.5][Virial identity] Let $N \geq 1, s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $\psi \in C([0, T^*), H^s)$ is a solution to (1.2). Then for any $t \in [0, T^*)$, it holds that*

$$\frac{d}{dt}\mathcal{V}_\varphi[\psi(t)] = -i \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta\varphi)|\psi_m(t)|^2 dx dm - 2i \int_0^\infty m^s \int_{\mathbb{R}^N} \overline{\psi}_m(t) \nabla\varphi \cdot \nabla\psi_m(t) dx dm,$$

where $\psi_m(t) = c_s(-\Delta + m)^{-1}\psi(t)$.

A direct consequence of Lemmas 2.7, Lemma 2.8 and 2.9 is the following estimate.

Corollary 2.10. *Let $N \geq 1, s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $\psi \in C([0, T^*), H^s)$ is a solution to (1.2). Then for any $t \in [0, T^*)$,*

$$\left| \frac{d}{dt}\mathcal{V}_\varphi[\psi(t)] \right| \leq C\|\nabla\varphi\|_{W^{1,\infty}} \|\psi(t)\|_{H^s}^2,$$

for some constant $C > 0$ depending only on s and N.

We next define the localized Morawetz action of ψ associated to φ by

$$\mathcal{M}_\varphi[\psi(t)] := 2\,\mathrm{Im}\int_{\mathbb{R}^N} \bar{\psi}(t,x)\nabla\varphi(x)\cdot\nabla\psi(t,x)dx. \tag{2.10}$$

By Lemma 2.5, we obtain the bound

$$|\,\mathcal{M}_\varphi[\psi(t)]\,| \le C\,(\|\nabla\varphi\|_{L^\infty}, \|\Delta\varphi\|_{L^\infty})\,\|\psi(t)\|^2_{H^{\frac{1}{2}}}.$$

Hence the quantity $\mathcal{M}_\varphi[\psi(t)]$ is well-defined, since $\psi(t)\in H^s$ with some $s>\frac{1}{2}$ by assumption.

By a similar argument as that in [6, Lemma 2.1], we have the following time evolution of $\mathcal{M}_\varphi[u(t)]$.

Lemma 2.11 (Morawetz identity). *Let $N \ge 1$, $s\in(1/2,1)$ and $\varphi:\mathbb{R}^N\to\mathbb{R}$ be such that $\nabla\varphi\in W^{3,\infty}$. Assume that $\psi\in C([0,T^\star),H^s)$ is a solution to (1.2). Then for any $t\in[0,T^\star)$, it holds that*

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_\varphi[\psi(t)] = &\int_0^\infty m^s\int_{\mathbb{R}^N}\left\{4\overline{\partial_k\psi_m(t)}(\partial^2_{kl}\varphi)\partial_l\psi_m(t) - (\Delta^2\varphi)|\psi_m(t)|^2\right\}dxdm\\ &- \frac{2p-4}{p}\int_{\mathbb{R}^N}\Delta\varphi(I_\alpha\star|\psi(t)|^p)|\psi(t)|^p dx\\ &- \frac{2N-2\alpha}{p}A(\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi(t,x)|^p|\psi(t,y)|^p(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))}{|x-y|^{N-\alpha+2}}dxdy,\end{aligned} \tag{2.11}$$

where $\psi_m(t)=\psi_m(t,x)$ is defined in (2.8).

Proof. It follows from an integration by parts that

$$\begin{aligned}&\langle\psi(t),[-(I_\alpha\star|\psi(t)|^p)|\psi(t)|^{p-2}, i\Gamma_\varphi]\psi(t)\rangle\\ &= -\langle\psi(t),[(I_\alpha\star|\psi(t)|^p)|\psi(t)|^{p-2},\nabla\varphi\cdot\nabla+\nabla\cdot\nabla\varphi]\psi(t)\rangle\\ &= 2\int_{\mathbb{R}^N}\nabla\varphi\cdot\nabla(I_\alpha\star|\psi(t)|^p)|\psi(t)|^p dx + 2\int_{\mathbb{R}^N}(I_\alpha\star|\psi(t)|^p)|\psi(t)|^2\nabla\varphi\cdot\nabla(|\psi(t)|^{p-2})dx\\ &= -\frac{2p-4}{p}\int_{\mathbb{R}^N}\Delta\varphi(I_\alpha\star|\psi(t)|^p)|\psi(t)|^p dx\\ &\quad-\frac{2N-2\alpha}{p}A(\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi(t,x)|^p|\psi(t,y)|^p(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))}{|x-y|^{N-\alpha+2}}dxdy.\end{aligned}$$

Therefore, following the method used in [6], we prove Lemma 2.11. □

3 Blow-up criteria

In this section, we will prove Theorem 1.1. To this end, we will establish the following blow-up criterion for (1.2).

Lemma 3.1. *Let $N\ge 1$, $s\in(\frac{1}{2},1)$, $1+\frac{2s+\alpha}{N}\le p<\frac{N+\alpha}{N-2s}$. Assume that $\psi_0\in H^s$ and $\psi\in C([0,T^\star),H^s)$ is the corresponding solution of (1.2). If there exists $\delta>0$ such that*

$$\sup_{t\in[0,T^\star)} K(\psi(t)) \le -\delta < 0. \tag{3.1}$$

then one of the following statements holds true:

- *$\psi(t)$ blows up in finite time, i.e. $T^\star<+\infty$;*

- *$\psi(t)$ blows up infinite time and there exists a time sequence $(t_n)_{n\geq 1}$ such that $t_n \to +\infty$ and*

$$\lim_{n\to\infty} \|(-\Delta)^{\frac{s}{2}}\psi(t_n)\|_{L^2} = \infty. \tag{3.2}$$

Proof. If $T^\star < +\infty$, then the proof is completed. If $T^\star = +\infty$, then we show (3.2). Assume by contradiction that the solution $\psi(t)$ exists globally and there exists $C_0 > 0$ such that

$$C_0 := \sup_{t\in[0,+\infty)} \|(-\Delta)^{\frac{s}{2}}\psi(t)\|_{L^2} < \infty. \tag{3.3}$$

Combining this and the conservation of mass, we have

$$C_1 := \sup_{t\in[0,+\infty)} \|\psi(t)\|_{H^s} < \infty. \tag{3.4}$$

Next, we introduce a smooth function $\theta : [0,\infty) \to [0,1]$ and satisfy

$$\theta(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq 1/2, \\ 1 & \text{if } r \geq 1. \end{cases}$$

For $R > 1$, we define the radial function

$$\phi_R(x) = \phi_R(r) := \theta(r/R), \quad r = |x|.$$

After some simple calculations, we can obtain

$$\nabla\phi_R(x) = \frac{x}{rR}\theta'(r/R), \quad \Delta\phi_R(x) = \frac{1}{R^2}\theta''(r/R) + \frac{(N-1)}{rR}\theta'(r/R).$$

These imply

$$\|\nabla\phi_R\|_{W^{1,\infty}} \sim \|\nabla\phi_R\|_{L^\infty} + \|\Delta\phi_R\|_{L^\infty} \lesssim R^{-1}. \tag{3.5}$$

Thus, we can define the localized virial function

$$\mathcal{V}_{\phi_R}[\psi(t)] := \int_{\mathbb{R}^N} \phi_R(x)|\psi(t,x)|^2 dx.$$

It easily follows that

$$\begin{aligned} \mathcal{V}_{\phi_R}[\psi(t)] &= \mathcal{V}_{\phi_R}[\psi_0] + \int_0^t \frac{d}{d\tau}\mathcal{V}_{\phi_R}[\psi(\tau)]d\tau \\ &\leq \mathcal{V}_{\phi_R}[\psi_0] + \left(\sup_{\tau\in[0,t]}\left|\frac{d}{d\tau}\mathcal{V}_{\phi_R}[\psi(\tau)]\right|\right)t. \end{aligned}$$

Combining Corollary 2.10, (3.4) and (3.5), we can obtain

$$\sup_{\tau\in[0,t]}\left|\frac{d}{d\tau}\mathcal{V}_{\phi_R}[\psi(\tau)]\right| \lesssim \|\nabla\phi_R\|_{W^{1,\infty}} \sup_{\tau\in[0,t]}\|\psi(\tau)\|_{H^s}^2 \leq CC_1^2R^{-1},$$

for some constant $C > 0$ independent of R and C_1. We consequently obtain

$$\mathcal{V}_{\phi_R}[\psi(t)] \leq \mathcal{V}_{\phi_R}[\psi_0] + CC_1^2R^{-1}t,$$

for all $t \geq 0$. We infer from the definition of θ that

$$\mathcal{V}_{\phi_R}[\psi_0] = \int_{\mathbb{R}^N} \phi_R(x)|\psi_0(x)|^2 dx \leq \int_{|x|>R/2} |\psi_0(x)|^2 dx \to 0,$$

as $R \to \infty$. This implies that $\mathcal{V}_{\phi_R}[\psi_0] = o_R(1)$. In addition, it follows that

$$\int_{|x|\geq R} |\psi(t,x)|^2 dx \leq \mathcal{V}_{\phi_R}[\psi(t)].$$

Collecting the above estimates, we can obtain the following control about the L^2-norm of the solution $\psi(t)$ outside a large ball.

Lemma 3.2. *Let $\eta > 0$, $R > 1$ and C_1 be as in* (3.4). *Then there exists a constant $C > 0$ independent of R and C_1 such that for any $t \in [0, T_0]$ with $T_0 := \frac{\eta R}{CC_1^2}$,*

$$\int_{|x|\geq R} |\psi(t,x)|^2 dx \leq \eta + o_R(1).$$

Next, we introduce a radial function $\varphi(x) = \varphi(r)$ which satisfies

$$\varphi(r) = \begin{cases} \frac{r^2}{2} & \text{for } r \leq 1, \\ \text{const.} & \text{for } r \geq 10, \end{cases}$$

and $\varphi''(r) \leq 1$ for $r \geq 0$. For any $R > 0$, we define the rescaled function $\varphi_R : \mathbb{R}^N \to \mathbb{R}$ by

$$\varphi_R(x) := R^2 \varphi\left(\frac{x}{R}\right). \tag{3.6}$$

It easily follows that

$$1 - \varphi_R''(r) \geq 0, \quad 1 - \frac{\varphi_R'(r)}{r} \geq 0, \quad N - \Delta\varphi_R(x) \geq 0,$$

for all $r \geq 0$ and all $x \in \mathbb{R}^N$. It is easy to see that

$$\|\nabla^k \varphi_R\|_{L^\infty} \lesssim R^{2-k}, \quad k = 0, \cdots, 4,$$

and

$$\operatorname{supp}(\nabla^k \varphi_R) \subset \begin{cases} \{|x| \leq 10R\} & \text{for } k = 1, 2, \\ \{R \leq |x| \leq 10R\} & \text{for } k = 3, 4. \end{cases}$$

Applying Lemma 2.11, we can obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{M}_{\varphi_R}[\psi(t)] = & \int_0^\infty m^s \int_{\mathbb{R}^N} \left\{ 4\overline{\partial_k \psi_m(t)}(\partial_{kl}^2 \varphi_R)\partial_l \psi_m(t) - (\Delta^2 \varphi_R)|\psi_m(t)|^2 \right\} dxdm \\ & - \frac{2p-4}{p} \int_{\mathbb{R}^N} \Delta\varphi_R (I_\alpha \star |\psi(t)|^p)|\psi(t)|^p dx \\ & - \frac{2N-2\alpha}{p} A(\alpha) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi(t,x)|^p |\psi(t,y)|^p (x-y)\cdot(\nabla\varphi_R(x) - \nabla\varphi_R(y))}{|x-y|^{N-\alpha+2}} dxdy \end{aligned} \tag{3.7}$$

where $\psi_m(t) = \psi_m(t,x)$ is defined in (2.8). Due to $\operatorname{supp}(\Delta^2\varphi_R) \subset \{|x| \geq R\}$, we infer from Lemma 2.6 that

$$\begin{aligned} \left| \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta^2 \varphi_R)|\psi_m(t)|^2 dxdm \right| & \lesssim \|\Delta^2\varphi_R\|_{L^\infty}^s \|\Delta\varphi_R\|_{L^\infty}^{1-s} \|\psi(t)\|_{L^2(|x|\geq R)}^2 \\ & \lesssim R^{-2s} \|\psi(t)\|_{L^2(|x|\geq R)}^2. \end{aligned} \tag{3.8}$$

Since φ_R is a radial function, applying

$$\partial_{jk}^2 = \left(\frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3}\right)\partial_r + \frac{x_j x_k}{r^2}\partial_r^2,$$

we can obtain

$$\int_0^\infty m^s \int_{\mathbb{R}^N} \overline{\partial_k \psi_m(t)}(\partial_{jk}^2 \varphi_R)\partial_l \psi_m(t)dxdm = \int_0^\infty m^s \int_{\mathbb{R}^N} \frac{\varphi_R'}{r}|\nabla \psi_m(t)|^2 dxdm + \int_0^\infty m^s \int_{\mathbb{R}^N} \left(\frac{\varphi_R''}{r^2} - \frac{\varphi_R'}{r^3}\right)|x \cdot \nabla \psi_m(t)|^2 dxdm.$$

It follows from (2.9) that

$$\int_0^\infty m^s \int_{\mathbb{R}^N} \frac{\varphi_R'}{r}|\nabla \psi_m(t)|^2 dxdm = s\|(-\Delta)^{s/2}\psi(t)\|_{L^2}^2 + \int_0^\infty m^s \int_{\mathbb{R}^N} \left(\frac{\varphi_R'}{r} - 1\right)|\nabla \psi_m(t)|^2 dxdm.$$

Since $\varphi_R'' \le 1$, we deduce from Cauchy-Schwarz inequality that

$$\int_0^\infty m^s \int_{\mathbb{R}^N} \left(\frac{\varphi_R'}{r} - 1\right)|\nabla \psi_m(t)|^2 dxdm + \int_0^\infty m^s \int_{\mathbb{R}^N} \left(\varphi_R'' - \frac{\varphi_R'}{r}\right)\frac{|x \cdot \nabla \psi_m(t)|^2}{r^2} dxdm \le 0.$$

Thus, we can obtain

$$4\int_0^\infty m^s \int_{\mathbb{R}^N} \overline{\partial_k \psi_m(t)}(\partial_{jk}^2 \varphi_R)\partial_l \psi_m(t)dxdm \le 4s\|(-\Delta)^{s/2}\psi(t)\|_{L^2}^2. \tag{3.9}$$

Next, we write

$$-\frac{2p-4}{p}\int_{\mathbb{R}^N} \Delta\varphi_R (I_\alpha \star |\psi(t)|^p)|\psi(t)|^p dx = -\frac{(2p-4)N}{p}\int_{\mathbb{R}^N} (I_\alpha \star |\psi(t)|^p)|\psi(t)|^p dx + \frac{2p-4}{p}\int_{\mathbb{R}^N} (N - \Delta\varphi_R)(I_\alpha \star |\psi(t)|^p)|\psi(t)|^p dx.$$

By the Hardy-Littlewood-Sobolev inequality and the conservation of mass, we can estimate as follows

$$\begin{aligned}\int_{\mathbb{R}^N} (N - \Delta\varphi_R)(I_\alpha \star |\psi(t)|^p)|\psi(t)|^p dx &\lesssim \int_{|x|\ge R} (I_\alpha \star |\psi(t)|^p)|\psi(t)|^p dx \\ &\lesssim \|I_\alpha \star |\psi(t)|^p\|_{L^{\frac{2N}{N-\alpha}}(|x|\ge R)} \||\psi(t)|^p\|_{L^{\frac{2N}{N+\alpha}}(|x|\ge R)} \\ &\lesssim \|\psi(t)\|^p_{L^{\frac{2Np}{N+\alpha}}} \|\psi(t)\|^p_{L^{\frac{2Np}{N+\alpha}}(|x|\ge R)} \\ &\lesssim \|\psi(t)\|_{H^s}^{\frac{Np-N-\alpha}{s}} \|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{N+\alpha-p(N-2s)}{2s}} \\ &\lesssim C_1^{\frac{Np-N-\alpha}{s}} \|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{N+\alpha-p(N-2s)}{2s}}.\end{aligned}$$

We consequently obtain

$$-\frac{2p-4}{p}\int_{\mathbb{R}^N} \Delta\varphi_R (I_\alpha \star |\psi(t)|^p)|\psi(t)|^p dx \le -\frac{(2p-4)N}{p}\int_{\mathbb{R}^N} (I_\alpha \star |\psi(t)|^p)|\psi(t)|^p dx + CC_1^{\frac{Np-N-\alpha}{s}} \|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{N+\alpha-p(N-2s)}{2s}}. \tag{3.10}$$

Denote the last term in (3.7) by $\mathcal{J}$. We can obtain

$$\mathcal{J} = -\frac{2N-2\alpha}{p}A(\alpha)\int_{\mathbb{R}^N} (I_\alpha \star |\psi(t)|^p)|\psi(t)|^p dx$$

$$+\frac{2N-2\alpha}{p}A(\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[|x-y|^2-(x-y)\cdot(\nabla\varphi_R(x)-\nabla\varphi_R(y))\right]\frac{|\psi(t,x)|^p|\psi(t,y)|^p}{|x-y|^{N-\alpha+2}}dxdy.$$

Note that

$$\operatorname{supp}(|x-y|^2-(x-y)\cdot(\nabla\varphi_R(x)-\nabla\varphi_R(y)))\subset\{|x|\geq R\}\cup\{|y|\geq R\}.$$

In the region $\{|x|\geq R\}$, it follows that

$$||x-y|^2-(x-y)\cdot(\nabla\varphi_R(x)-\nabla\varphi_R(y))|\lesssim|x-y|^2.$$

This implies that

$$\left|A(\alpha)\int_{|x|\geq R}\int_{\mathbb{R}^N}\left[|x-y|^2-(x-y)\cdot(\nabla\varphi_R(x)-\nabla\varphi_R(y))\right]\frac{|\psi(t,x)|^p|\psi(t,y)|^p}{|x-y|^{N-\alpha+2}}dxdy\right|$$
$$\lesssim\int_{|x|\geq R}(I_\alpha\star|\psi(t)|^p)|\psi(t)|^p dx.$$

We have a similar control in the region $\{|y|\geq R\}$. By a similar argument as above, we can obtain

$$\mathcal{I}\leq-\frac{2N-2\alpha}{p}\int_{\mathbb{R}^N}(I_\alpha\star|\psi(t)|^p)|\psi(t)|^p dx+C_1^{\frac{Np-N-\alpha}{s}}\|\psi(t)\|_{L^2(|x|\geq R)}^{\frac{N+\alpha-p(N-2s)}{2s}}. \tag{3.11}$$

Combining (3.8) – (3.11), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_{\varphi_R}[\psi(t)]&\leq 4s\|(-\Delta)^{s/2}\psi(t)\|_{L^2}^2-\frac{2Np-2N-2\alpha}{p}\int_{\mathbb{R}^N}(I_\alpha\star|\psi(t)|^p)|\psi(t)|^p dx\\&+CR^{-2s}\|\psi(t)\|_{L^2(|x|\geq R)}^2+CC_1^{\frac{2Np-N-\alpha}{s}}\|\psi(t)\|_{L^2(|x|\geq R)}^{\frac{N+\alpha-(N-2s)p}{2s}}.\end{aligned}\tag{3.12}$$

Applying Lemma 3.2, for any $\eta>0$ and any $R>1$, there exists $C>0$ independent of R and C_1 such that for any $t\in[0,T_0]$ with $T_0=\frac{\eta R}{CC_1^2}$,

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_{\varphi_R}[\psi(t)]&\leq 4K(\psi(t))+CR^{-2s}(\eta+o_R(1))^2+CC_1^{\frac{2Np-N-\alpha}{s}}(\eta+o_R(1))^{\frac{N+\alpha-(N-2s)p}{2s}}\\&\leq-4\delta+CR^{-2s}(\eta^2+o_R(1))+CC_1^{\frac{2Np-N-\alpha}{s}}(\eta^{\frac{N+\alpha-(N-2s)p}{2s}}+o_R(1)).\end{aligned}$$

We first choose $\eta>0$ small enough so that

$$CC_1^{\frac{2Np-N-\alpha}{s}}\eta^{\frac{N+\alpha-(N-2s)p}{2s}}\leq 2\delta.$$

We next choose $R>1$ large enough so that

$$\frac{d}{dt}\mathcal{M}_{\varphi_R}[\psi(t)]\leq-\delta<0, \tag{3.13}$$

for any $t\in[0,T_0]$ with $T_0=\frac{\eta R}{CC_1^2}$. Note that $\eta>0$ is fixed, so we can choose $R>1$ large enough so that T_0 is as large as we want. By (3.13), it follows that

$$\mathcal{M}_{\varphi_R}[\psi(t)]\leq-ct,$$

for all $t\in[t_0,T_0]$ with some sufficiently large $t_0\in[0,T_0]$. The constant $c>0$ depends only on δ. On the other hand, we deduce from Lemma 2.5 and the conservation of mass that for any $t\in[0,+\infty)$,

$$\begin{aligned}|\mathcal{M}_{\varphi_R}[\psi(t)]|&\lesssim C(\varphi_R)\left(\|\psi(t)\|_{\dot H^{1/2}}^2+\|\psi(t)\|_{L^2}\|\psi(t)\|_{\dot H^{1/2}}\right)\\&\lesssim C(\varphi_R)\left(\|\psi(t)\|_{\dot H^{1/2}}^2+\|\psi(t)\|_{L^2}^2\right)\end{aligned}$$

$$\lesssim C(\varphi_R)\left(\|\psi(t)\|^2_{\dot H^{1/2}}+1\right).$$

By interpolating between L^2 and $\dot H^s$, we get for any $t\in[t_0,T_0]$,

$$ct\le -\mathcal{M}_{\varphi_R}[\psi(t)]=|\mathcal{M}_{\varphi_R}[\psi(t)]|\lesssim C(\varphi_R)\left(\|(-\Delta)^{s/2}\psi(t)\|^{\frac{1}{s}}_{L^2}+1\right).$$

This implies that

$$\|(-\Delta)^{s/2}\psi(t)\|_{L^2}\ge Ct^s, \tag{3.14}$$

for all $t\in[t_1,T_0]$ with some sufficiently large $t_1\in[t_0,T_0]$. Taking t close to $T_0=\frac{\eta R}{CC_1^2}$, we see that $\|(-\Delta)^{s/2}\psi(t)\|_{L^2}\to\infty$ as $R\to\infty$. Taking $R>1$ sufficiently large, we have a contradiction with (3.4). The proof is complete.

□

Applying Lemma 3.1, we can prove Theorem 1.1.

Proof of Theorem 1.1. We need only to check that (3.1) follows. In the L^2-critical case, i.e., $s_c=0$, we infer from (1.10) that $\|\psi_0\|_{L^2}<\|u\|_{L^2}$ and $\|\psi_0\|_{L^2}>\|u\|_{L^2}$, which is an contradiction. Thus, when $s_c=0$, we have $E(\psi_0)<0$. Applying the conservation of energy and $1+\frac{2s+\alpha}{N}\le p<\frac{N+\alpha}{N-2s}$, it follows that

$$\begin{aligned}K(\psi(t)) =&s\|\psi(t)\|^2_{\dot H^s}-\frac{\theta}{2p}\int_{\mathbb{R}^N}(I_\alpha\star|\psi(t)|^p)(x)|\psi(t,x)|^p\,dx\\ =&2sE(\psi(t))+\frac{2s-\theta}{2p}\int_{\mathbb{R}^N}(I_\alpha\star|\psi(t)|^p)(x)|\psi(t,x)|^p\,dx\le 2sE(\psi_0),\end{aligned}$$

for all $t\in[0,T^\star)$. Hence, (3.1) follows with $\delta=-2sE(\psi_0)$.

Next, we consider the case $E(\psi_0)\ge 0$. We deduce from the assumption (1.10) that

$$\begin{cases} E(\psi_0)\|\psi_0\|^{2\sigma}_{L^2}<E(u)\|u\|^{2\sigma}_{L^2},\\ \|(-\Delta)^{s/2}\psi_0\|_{L^2}\|\psi_0\|^{\sigma}_{L^2}>\|(-\Delta)^{s/2}u\|_{L^2}\|u\|^{\sigma}_{L^2},\end{cases} \tag{3.15}$$

where

$$\sigma:=\frac{s-s_c}{s_c}=\frac{2sp-\theta}{\theta-2s}.$$

Notice that the sharp constant in Gagliardo-Nirenberg inequality (2.2) is

$$C_{opt}=\frac{\int_{\mathbb{R}^N}(I_\alpha\star|u|^p)(x)|u(x)|^p\,dx}{\|u\|^{\frac{\theta}{s}}_{\dot H^s}\|u\|^{\frac{2sp-\theta}{s}}_{L^2}}. \tag{3.16}$$

By (2.3), we can rewrite C_{opt} as

$$C_{opt}=\frac{2sp}{\theta}\frac{1}{(\|u\|_{\dot H^s}\|u\|^{\sigma}_{L^2})^{\frac{\theta-2s}{s}}}. \tag{3.17}$$

By a direct calculation, we also have

$$E(u)\|u\|^{2\sigma}_{L^2}=\frac{\theta-2s}{2\theta}(\|u\|_{\dot H^s}\|u\|^{\sigma}_{L^2})^2. \tag{3.18}$$

Multiplying both sides of $E(\psi(t))$ by $\|\psi(t)\|^{2\sigma}_{L^2}$ and use the sharp Gagliardo-Nirenberg inequality (2.2), we obtain

$$\begin{aligned}E(\psi(t))\|\psi(t)\|^{2\sigma}_{L^2}&=\frac12\|\psi(t)\|^2_{\dot H^s}\|\psi(t)\|^{2\sigma}_{L^2}-\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha\star|\psi(t)|^p)(x)|\psi(t,x)|^p\,dx\|u(t)\|^{2\sigma}_{L^2}\\ &\ge\frac12(\|\psi(t)\|_{\dot H^s}\|\psi(t)\|^{\sigma}_{L^2})^2-\frac{C_{opt}}{2p}(\|\psi(t)\|_{\dot H^s}\|\psi(t)\|^{\sigma}_{L^2})^{\frac{Np-N-\alpha}{s}}\end{aligned}$$

$$= f(\|\psi(t)\|_{\dot{H}^s}\|\psi(t)\|_{L^2}^{\sigma}),$$

where $f(x) := \frac{1}{2}x^2 - \frac{C_{opt}}{2p}x^{\frac{\theta}{s}}$. It easily follows that f is increasing on $(0, x_0)$ and decreasing on (x_0, ∞), where

$$x_0 = \left(\frac{2sp}{C_{opt}\theta}\right)^{\frac{s}{\theta-2s}} = \|u\|_{\dot{H}^s}\|u\|_{L^2}^{\sigma},$$

where the last equality follows from (3.17). It follows from (3.17) and (3.18) that

$$f(\|u\|_{\dot{H}^s}\|u\|_{L^2}^{\sigma}) = E(u)\|u\|_{L^2}^{2\sigma}.$$

Thus the conservation of mass and energy together with the first condition in (1.10) imply

$$\begin{aligned} f(\|\psi(t)\|_{\dot{H}^s}\|\psi(t)\|_{L^2}^{\sigma}) &\le E(\psi(t))\|\psi(t)\|_{L^2}^{2\sigma} = E(\psi_0)\|\psi_0\|_{L^2}^{2\sigma} \\ &< E(u)\|u\|_{L^2}^{2\sigma} = f(\|u\|_{\dot{H}^s}\|u\|_{L^2}^{\sigma}), \end{aligned}$$

for all $t \in [0, T^\star)$. Using the second condition (1.10), the continuity argument shows that

$$\|\psi(t)\|_{\dot{H}^s}\|\psi(t)\|_{L^2}^{\sigma} > \|u\|_{\dot{H}^s}\|u\|_{L^2}^{\sigma} \tag{3.19}$$

for any $t \in [0, T^\star)$. On the other hand, since $E(\psi_0)\|\psi_0\|_{L^2}^{2\sigma} < E(u)\|u\|_{L^2}^{2\sigma}$, we pick $\eta > 0$ small enough so that

$$E(\psi_0)\|\psi_0\|_{L^2}^{2\sigma} \le (1-\eta)E(u)\|u\|_{L^2}^{2\sigma}.$$

Thus, by the conservation of energy, (3.18) and (3.19), we have

$$\begin{aligned} K(\psi(t))\|\psi(t)\|_{L^2}^{2\sigma} &= \theta E(\psi(t))\|\psi(t)\|_{L^2}^{2\sigma} - \frac{\theta-2s}{2}\|\psi(t)\|_{\dot{H}^s}^2\|\psi(t)\|_{L^2}^{2\sigma} \\ &= \theta E(\psi_0)\|\psi_0\|_{L^2}^{2\sigma} - \frac{\theta-2s}{2}(\|\psi(t)\|_{\dot{H}^s}\|\psi(t)\|_{L^2}^{\sigma})^2 \\ &\le \theta(1-\eta)E(u)\|u\|_{L^2}^{2\sigma} - \frac{\theta-2s}{2}(\|u\|_{\dot{H}^s}\|u\|_{L^2}^{\sigma})^2 \\ &= -\eta\theta E(u)\|u\|_{L^2}^{2\sigma}, \end{aligned}$$

for all $t \in [0, T^\star)$. This implies (3.1) with $\delta = \eta\theta E(u)\|u\|_{L^2}^{2\sigma}$. Thus, the solution $\psi(t)$ of (1.2) blows up in finite or infinite time. This completes the proof.

4 Existence and instability of normalized standing waves

In this section, we will prove the existence and instability of normalized standing waves of (1.2). Firstly, we prove Theorem 1.2.

Proof of Theorem 1.2. We first show $m(c) > 0$. By $K(v) = 0$ and the inequality (2.2), we have

$$s\|v\|_{\dot{H}^s}^2 = \frac{\theta}{2p}\int_{\mathbb{R}^N}(I_\alpha \star |v|^p)|v|^p\,dx \le C\|v\|_{\dot{H}^s}^{\frac{\theta}{s}}\|v\|_{L^2}^{\frac{2sp-\theta}{s}} \le C\|v\|_{\dot{H}^s}^{\frac{\theta}{s}},$$

where $\theta = Np - N - \alpha$, which implies that there exists $C_1 > 0$ such that $\|v\|_{\dot{H}^s} \ge C_1 > 0$. Thus, it follows from $K(v) = 0$ that

$$E(v) = \frac{1}{2s}K(v) + \frac{\theta-2s}{4sp}\int_{\mathbb{R}^N}(I_\alpha \star |v|^p)|v|^p\,dx = \frac{\theta-2s}{2\theta}\|v\|_{\dot{H}^s} \ge \frac{\theta-2s}{2\theta}C_1. \tag{4.1}$$

Taking the infimum over $v \in V(c)$, we have $m(c) > 0$.

Next, let $\{v_n\} \subseteq V(c)$ be a minimizing sequence of (1.15), i.e., $K(v_n) = 0$, $\|v_n\|_{L^2}^2 = c$ and $E(v_n) \to m(c)$ as $n \to \infty$. Thus, it follows from (4.1) that

$$\|v_n\|_{\dot{H}^s} = \frac{2\theta}{\theta-2s}E(v_n) \to \frac{2\theta m(c)}{\theta-2s},$$

which implies that $\{v_n\}$ is bounded in H^s.

Now, applying the profile decomposition in H^s, there exists a subsequence, still denoted by $\{v_n\}$, a family $\{x_n^j\}_{j=1}^{\infty}$ of sequences in $\mathbb{R}^N$ and a sequence $\{U^j\}_{j=1}^{\infty}$ in H^s such that for every $l \geq 1$ and every $x \in \mathbb{R}^N$, we have

$$v_n(x) = \sum_{j=1}^{l} U^j(x - x_n^j) + r_n^l, \tag{4.2}$$

and (2.5)-(2.7) hold. Moreover, we deduce from (2.5)-(2.6) that

$$\begin{aligned} 0 = K(v_n) =& s\|v_n\|_{\dot{H}^s}^2 - \frac{\theta}{2p}\int_{\mathbb{R}^N}(I_\alpha \star |v_n|^p)|v_n|^p dx \\ =& s\sum_{j=1}^{l}\|U^j\|_{\dot{H}^s}^2 + s\|v_n^l\|_{\dot{H}^s}^2 - \frac{\theta}{2p}\int_{\mathbb{R}^N}(I_\alpha \star |v_n|^p)|v_n|^p dx + \circ_n(1) \\ =& \sum_{j=1}^{l}K(U^j) + \frac{\theta}{2p}\sum_{j=1}^{l}\int_{\mathbb{R}^N}(I_\alpha \star |U^j|^p)|U^j|^p dx + s\|v_n^l\|_{\dot{H}^s}^2 \\ &- \frac{\theta}{2p}\int_{\mathbb{R}^N}(I_\alpha \star |v_n|^p)|v_n|^p dx + \circ_n(1), \end{aligned} \tag{4.3}$$

where $\circ_n(1) \to 0$ as $n \to \infty$. Since $K(v_n) = 0$,

$$\int_{\mathbb{R}^N}(I_\alpha \star |v_n|^p)|v_n|^p dx = \frac{4sp}{\theta - 2s}E(v_n) \to \frac{4spm(c)}{\theta - 2s},\ as\ n \to \infty$$

and $s\|v_n^l\|_{\dot{H}^s}^2 \geq 0$ for all $n \geq 1$, we infer that

$$\sum_{j=1}^{l}K(U^j) + \frac{\theta}{2p}\sum_{j=1}^{l}\int_{\mathbb{R}^N}(I_\alpha \star |U^j|^p)|U^j|^p dx - \frac{\theta}{2p}\frac{4spm(c)}{\theta - 2s} \leq 0, \tag{4.4}$$

or equivalently,

$$s\sum_{j=1}^{l}\|U^j\|_{\dot{H}^s}^2 - \frac{2sm(c)\theta}{\theta - 2s} \leq 0. \tag{4.5}$$

On the other hand, we see from (2.7) that

$$\frac{4spm(c)}{\theta - 2s} = \lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha \star |v_n|^p)|v_n|^p dx = \sum_{j=1}^{\infty}\int_{\mathbb{R}^N}(I_\alpha \star |U^j|^p)|U^j|^p dx. \tag{4.6}$$

Combining (4.4)-(4.6), we obtain

$$\sum_{j=1}^{\infty}K(U^j) \leq 0,\ \ \sum_{j=1}^{\infty}\|U^j\|_{\dot{H}^s}^2 \leq \frac{2m(c)\theta}{\theta - 2s}. \tag{4.7}$$

We claim that $K(U^j) = 0$ for all $j \geq 1$. Indeed, suppose that there exists $j_0 \geq 1$ such that $K(U^{j_0}) < 0$. Notice that

$$K(\lambda U^{j_0}) = \lambda^2 s\|U^{j_0}\|_{\dot{H}^s}^2 - \frac{\theta}{2p}\lambda^{2p}\int_{\mathbb{R}^N}(I_\alpha \star |U^{j_0}|^p)|U^{j_0}|^p dx > 0, \tag{4.8}$$

for sufficiently small $\lambda > 0$. There exists $\lambda_0 \in (0, 1)$ such that $K(\lambda_0 U^{j_0}) = 0$. Let $V = \lambda_0 U^{j_0}$, then we have

$$K(V) = 0,\ \ \int_{\mathbb{R}^N}|V(x)|^2 dx < c. \tag{4.9}$$

let $V_\mu = \mu^{\frac{2s+\alpha}{2p-2}} V(\mu x)$,

$$K(V_\mu) = \mu^{\frac{2s+\alpha}{p-1}+2s-N} K(V) = 0. \tag{4.10}$$

Since $p > \frac{N+\alpha+2s}{N}$ and

$$\int_{\mathbb{R}^N} |V_\mu(x)|^2 dx = \mu^{\frac{2s-\theta}{p-1}} \int_{\mathbb{R}^N} |V(x)|^2 dx = c. \tag{4.11}$$

there exists $\mu_0 \in (0,1)$ such that $\|V_{\mu_0}\|_{L^2}^2 = c$. We consequently estimate as follows:

$$\begin{aligned} m(c) \le E(V_{\mu_0}) &= \frac{\theta-2s}{4sp} \int_{\mathbb{R}^N} (I_\alpha \star |V_{\mu_0}|^p)|V_{\mu_0}|^p dx \\ &= \frac{\theta-2s}{4sp} \mu_0^{\frac{2s+\alpha}{p-1}+2s-N} \lambda_0^{2p} \int_{\mathbb{R}^N} (I_\alpha \star |U^{j_0}|^p)|U^{j_0}|^p dx \\ &< \frac{\theta-2s}{4sp} \int_{\mathbb{R}^N} (I_\alpha \star |U^{j_0}|^p)|U^{j_0}|^p dx \\ &\le \frac{\theta-2s}{4sp} \frac{4spm(c)}{\theta-2s} = m(c), \end{aligned} \tag{4.12}$$

which is a contraction. Finally, we claim that there exists only one term $U^j \neq 0$. Indeed, if there exist two terms $U^{j_1} \neq 0$ and $U^{j_2} \neq 0$, it follows from (4.7) that $K(U^{j_1}) = 0$, $K(U^{j_2}) = 0$ and

$$\int_{\mathbb{R}^N} (I_\alpha \star |U^{j_1}|^p)|U^{j_1}|^p dx < \frac{4spm(c)}{\theta-2s} \ and \int_{\mathbb{R}^N} (I_\alpha \star |U^{j_2}|^p)|U^{j_2}|^p dx < \frac{4spm(c)}{\theta-2s}. \tag{4.13}$$

Next, we set

$$U_\mu^{j_1} = \mu^{\frac{2s+\alpha}{2p-2}} U^{j_1}(\mu x), \quad U_\mu^{j_2} = \mu^{\frac{2s+\alpha}{2p-2}} U^{j_2}(\mu x). \tag{4.14}$$

It follows from that $K(U_\mu^{j_1}) = K(U^{j_1}) = 0$, and $K(U_\mu^{j_2}) = K(U^{j_2}) = 0$ for all $\mu > 0$. By $\int_{\mathbb{R}^N} |U^{j_1}|^2 dx < c$ and $\int_{\mathbb{R}^N} |U^{j_2}|^2 dx < c$, we obtain that there exist $\mu_1, \mu_2 \in (0,1)$ such that

$$\int_{\mathbb{R}^N} |U_{\mu_1}^{j_1}|^2 dx = c, \quad \int_{\mathbb{R}^N} |U_{\mu_2}^{j_2}|^2 dx = c. \tag{4.15}$$

Thus, we can estimate as follows:

$$\begin{aligned} m(c) \le E(U_{\mu_1}^{j_1}) &= \frac{\theta-2s}{4sp} \int_{\mathbb{R}^N} (I_\alpha \star |U_{\mu_1}^{j_1}|^p)|U_{\mu_1}^{j_1}|^p dx \\ &= \frac{\theta-2s}{4sp} \mu_1^{\frac{2s+\alpha}{2p-2}} \int_{\mathbb{R}^N} (I_\alpha \star |U^{j_1}|^p)|U^{j_1}|^p dx \\ &\le \frac{\theta-2s}{4sp} \frac{4spm(c)}{\theta-2s} = m(c), \end{aligned}$$

which is a contradiction. Therefore, there exists only one term $U^{j_0} \neq 0$ in the decomposition (4.2) and $K(U^{j_0}) = 0$, which together with (2.5) implies that the infimum of the variational problem (1.15) is attained at U^{j_0}. This completes the proof.

Proof of Theorem 1.3. With Theorem 1.2 in hand, ones can prove Theorem 1.3 by a similar argument as Theorem 1.3 in [18]. So we omit it.

Proof of Theorem 1.4. Let $u_c \in \mathcal{M}_c$, a direct computation shows

$$E(u_c^\lambda) = \frac{1}{2}\lambda^{2s}\|u_c\|_{\dot{H}^s}^2 - \frac{\lambda^\theta}{2p} \int_{\mathbb{R}^N} (I_\alpha \star |u_c|^p)(x)|u_c(x)|^p dx,$$

and

$$\partial_\lambda E(u_c^\lambda) = s\lambda^{2s-1}\|u_c\|_{\dot H^s}^2 - \frac{\theta\lambda^{\theta-1}}{2p}\int_{\mathbb{R}^N}(I_\alpha \star |u_c|^p)(x)|u_c(x)|^p\,dx = \frac{K(u_c^\lambda)}{\lambda}.$$

It is easy to see that the equation $\partial_\lambda E(u_c^\lambda) = 0$ has a unique non-zero solution

$$\left(\frac{2sp\|u_c\|_{\dot H^s}^2}{\theta\int_{\mathbb{R}^N}(I_\alpha \star |u_c|^p)(x)|u_c(x)|^p\,dx}\right)^{\frac{1}{\theta-2s}} = 1.$$

The last inequality comes from the fact that $K(u_c) = 0$, which follows from Pohozaev's identities (2.3). We thus obtain

$$\begin{cases} \partial_\lambda E(u_c^\lambda) > 0 \ \ if\ \lambda \in (0,1), \\ \partial_\lambda E(u_c^\lambda) < 0 \ \ if\ \lambda \in (1,\infty). \end{cases}$$

This implies that for any $\lambda > 1$

$$E(u_c^\lambda) < E(u_c). \tag{4.16}$$

Let $\lambda_n > 1$ such that $\lim_{n\to\infty}\lambda_n = 1$. We take the initial data

$$\psi_{0,n}(x) = u_c^{\lambda_n}(x) = \lambda_n^{\frac{N}{2}} u_c(\lambda_n x).$$

By Brezis-Lieb's lemma, we have $\psi_{0,n} \to u_c$ in H^s as $n\to\infty$. We deduce from (4.16) that

$$E(\psi_{0,n}) < E(u_c),$$

and

$$\|(-\Delta)^{s/2}\psi_{0,n}\|_{L^2} = \lambda_n^s\|(-\Delta)^{s/2}u_c\|_{L^2} > \|(-\Delta)^{s/2}u_c\|_{L^2}.$$

On the other hand, let $u_c(x) = \omega_c^{\frac{\alpha+2s}{4s(p-1)}} u(\omega_c^{\frac{1}{2s}}x)$ in (1.12), then u satisfies equation (1.11). In particular, by some basic calculations, we have

$$E(u_c)^{s_c}\|u_c\|_{L^2}^{2(s-s_c)} = E(u)^{s_c}\|u\|_{L^2}^{2(s-s_c)}, \tag{4.17}$$

and

$$\|(-\Delta)^{\frac{s}{2}}u_c\|_{L^2}^{s_c}\|u_c\|_{L^2}^{s-s_c} = \|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{s_c}\|u\|_{L^2}^{s-s_c}. \tag{4.18}$$

Thus, by (4.17), (4.18) and $\|\psi_{0,n}\|_{L^2} = \|u_c\|_{L^2}$, we have

$$E(\psi_{0,n})^{s_c}\|\psi_{0,n}\|_{L^2}^{2(s-s_c)} < E(u_c)^{s_c}\|u_c\|_{L^2}^{2(s-s_c)} = E(u)^{s_c}\|u\|_{L^2}^{2(s-s_c)},$$

and

$$\|(-\Delta)^{s/2}\psi_{0,n}\|_{L^2}^{s_c}\|\psi_{0,n}\|_{L^2}^{s-s_c} > \|(-\Delta)^{s/2}u_c\|_{L^2}^{s_c}\|u_c\|_{L^2}^{s-s_c} = \|(-\Delta)^{s/2}u\|_{L^2}^{s_c}\|u\|_{L^2}^{s-s_c},$$

where $s_c = \frac{N}{2} - \frac{\alpha+2s}{2p-2}$. Applying Theorem 1.1, the solution ψ_n of (1.2) and initial data $\psi_{0,n}$ blows up in finite or infinite time. This completes the proof.

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References

[1] T. Bartsch, L. Jeanjean, *Normalized solutions for nonlinear Schrödinger systems*, Proc. Roy. Soc. Edinburgh Sect. A 148 (2018) 225–242.

[2] T. Bartsch, L. Jeanjean, N. Soave, *Normalized solutions for a system of coupled cubic Schrödinger equations on* $\mathbb{R}^3$, J. Math. Pures Appl. 106 (2016) 583–614.

[3] T. Bartsch, N. Soave, *A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems*, J. Funct. Anal. 272 (2017) 4998–5037.

[4] T. Bartsch, N. Soave, *Multiple normalized solutions for a competing system of Schrödinger equations*, Calc. Var. Partial Differential Equations 58 (2019) 24 pp.

[5] J. Bellazzini, L. Jeanjean, T. Luo, *Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations*, Proc. Lond. Math. Soc. 107 (2013) 303–339.

[6] T. Boulenger, D. Himmelsbach, E. Lenzmann, *Blowup for fractional NLS*, J. Funct. Anal. 271 (2016) 2569–2603.

[7] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics vol. 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[8] Y. Cho, H. Hajaiej, G. Hwang, T. Ozawa, *On the Cauchy problem of fractional Schrödinger equations with Hartree type nonlimearity*, Funkcial. Ekvac. 56 (2013) 193–224.

[9] Y. Cho, G. Hwang, S. Kwon, S. Lee, *On finite time blow-up for the mass-critical Hartree equations*, Proc. Roy. Soc. Edinburgh Sect. A 145 (2015) 467–479.

[10] V. D. Dinh, *Well-posedness of nonlinear fractional Schrödinger and wave equations in Sobolev spaces*, Int. J. Appl. Math. 31 (2018) 483–525.

[11] V. D. Dinh, *On instability of standing waves for the mass-supercritical fractional nonlinear Schrödinger equation*, Z. Angew. Math. Phys. 70 (2019) 17 pp.

[12] V. D. Dinh, *Blow-up criteria for fractional nonlinear Schrödinger equations*, Nonlinear Anal. Real World Appl. 48 (2019) 117–140.

[13] D. Du, Y. Wu, K. Zhang, *On blow-up criterion for the nonlinear Schrödinger equation*, Discrete Contin. Dyn. Syst. 36 (2016) 3639–3650.

[14] B. Feng, *On the blow-up solutions for the fractional nonlinear Schrödinger equation with combined power-type nonlinearities*, Comm. Pure Appl. Anal. 17 (2018) 1785–1804.

[15] B. Feng, R. Chen, J. Ren, *Existence of stable standing waves for the fractional Schrödinger equations with combined power-type and Choquard-type nonlinearities*, J. Math. Phys. 60 (2019) 051512 12 pp.

[16] B. Feng, R. Chen, Q. Wang, *Instability of standing waves for the nonlinear Schrödinger-Poisson equation in the* L^2*-critical case*, J. Dynam. Differential Equations (2019) https://doi.org/10.1007/s10884-019-09779-6.

[17] B. Feng, J. Liu, H. Niu, B. Zhang, *Strong instability of standing waves for a fourth-order nonlinear Schrödinger equation with the mixed dispersion*, Nonlinear Anal. 196 (2020) 111791.

[18] B. Feng, J. Ren, Q. Wang, *Normalized ground states for the fractional nonlinear Schrödinger equations*, arXiv:1907.03433, 2019.

[19] B. Feng, X. Yuan, *On the Cauchy problem for the Schrödinger-Hartree equation*, Evol. Equ. Control Theory 4 (2015) 431–445.

[20] B. Feng, H. Zhang, *Stability of standing waves for the fractional Schrödinger-Choquard equation*, Comput. Math. Appl. 75 (2018) 2499–2507.

[21] B. Feng, H. Zhang, *Stability of standing waves for the fractional Schrödinger-Hartree equation*, J. Math. Anal. Appl. 460 (2018) 352–364.

[22] J. Fröhlich, G. Jonsson, E. Lenzmann, *Boson stars as solitary waves*, Comm. Math. Phys. 274 (2007) 1–30.

[23] R. Fukuizumi, *Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential*, Discrete Contin. Dyn. Sys. 7 (2001) 525–544.

[24] R. Fukuizumi, M. Ohta, *Instability of standing waves for nonlinear Schrödinger equations with potentials*, Differential Integral Equations 16 (2003) 691–706.

[25] Q. Guo, S. Zhu, *Sharp threshold of blow-up and scattering for the fractional Hartree equation*, J. Differential Equations 264 (2018) 2802–2832.

[26] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal. 28 (1997) 1633-1659.

[27] Y. Hong, Y. Sire, *On fractional Schrödinger equations in Sobolev spaces*, Comm. Pure Appl. Anal. 14 (2015) 2265–2282.

[28] K. Kirkpatrick, E. Lenzmann, G. Staffilani, *On the continuum limit for discrete NLS with long-range lattice interactions*, Comm. Math. Phys. 317 (2013) 563–591.

[29] N. Laskin, *Fractional Quantum Mechanics and Lèvy Path Integrals*, Phys. Lett. A 268 (2000) 298–304.

[30] N. Laskin, *Fractional Schrödinger equations*, Physics Review E 66 (2002) 056108.

[31] S. Le Coz, *A note on Berestycki-Cazenave's classical instability result for nonlinear Schrödinger equations*, Adv. Nonlinear Stud. 8 (2008) 455–463.

[32] X. Luo, *Normalized standing waves for the Hartree equations*, J. Differential Equations 267 (2019) 4493–4524.

[33] A. Mohammed, V. Rădulescu, A. Vitolo, *Blow-up solutions for fully nonlinear equations: existence, asymptotic estimates and uniqueness*, Adv. Nonlinear Anal. 9 (2020) 39–64.

[34] C. Ning, M. Ohta, Y. Wu, *Instability of solitary wave solutions for derivative nonlinear Schrödinger equation in endpoint case*, J. Differential Equations 262 (2017) 1671–1689.

[35] M. Ohta, *Strong instability of standing waves for nonlinear Schrödinger equations with harmonic potential*, Funkcial. Ekvac. 61 (2018) 135–143.
[36] M. Ohta, *Strong instability of standing waves for nonlinear Schrödinger equations with a partial confinement*, Comm. Pure Appl. Anal. 17 (2018) 1671–1680.
[37] N. Pan, P. Pucci, R. Xu, B. Zhang, *Degenerate Kirchhoff-type wave problems involving the fractional Laplacian with nonlinear damping and source terms*, J. Evol. Equ. 19 (2019) 615–643.
[38] T. Saanouni, *Strong instability of standing waves for the fractional Choquard equation*, J. Math. Phys. 59 (2018) 081509.
[39] T. Saanouni, *A note on the fractional Schrödinger equation of Choquard type*, J. Math. Anal. Appl. 470 (2019) 1004–1029.
[40] M. Xiang, V. Rădulescu, B. Zhang, *Nonlocal Kirchhoff diffusion problems: Local existence and blow-up of solutions*, Nonlinearity 31 (2018) 3228–3250.
[41] M. Xiang, V. Rădulescu, B. Zhang, *A critical fractional Choquard-Kirchhoff problem with magnetic field*, Commun. Contemp. Math. 21 (2019) 1850004 36 pp.
[42] M. Xiang, B. Zhang, V. Rădulescu, *Superlinear Schrödinger-Kirchhoff type problems involving the fractional p–Laplacian and critical exponent*, Adv. Nonlinear Anal. 9 (2020) 690–709.
[43] J. Zhang, *Cross-constrained variational problem and nonlinear Schrödinger equation*, in Foundations of Computational Mathematics, World Scientific Publishing, River Edge, NJ (2002) 457–469.
[44] J. Zhang, *Sharp threshold for blowup and global existence in nonlinear Schrödinger equations under a harmonic potential*, Comm. Partial Differential Equations 30 (2005) 1429–1443.
[45] B. Zhang, V. Rădulescu, L. Wang, *Existence results for Kirchhoff-type superlinear problems involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A 149 (2019) 1061–1081.
[46] J. Zhang, S. Zhu, *Stability of standing waves for the nonlinear fractional Schrödinger equation*, J. Dynam. Differential Equations 29 (2017) 1017–1030.
[47] J. Zhang, S. Zhu, *Sharp energy criteria and singularity of blow-up solutions for the Davey-Stewartson system*, Commun. Math. Sci. 17 (2019) 653–667.
[48] S. Zhu, *On the blow-up solutions for the nonlinear fractional Schrödinger equation*, J. Differential Equations 261 (2016) 1506–1531.
[49] S. Zhu, *Existence of stable standing waves for the fractional Schrödinger equations with combined nonlinearities*, J. Evol. Equ. 17 (2017) 1003–1021.