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Multiplicity Results for the Supercritical Hénon Equation[*]

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Abstract

We study the Dirichlet problem for the Hénon equation $-\Delta u = |x|^\alpha u^{p-1}$ in the unit ball $B \subset \mathbf{R}^N$. For $N \geq 4$ and α large we prove the existence of positive nonradial solutions for a range of p's including supercritical values.

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1 Introduction

The equation

$$-\Delta u = |x|^\alpha u^{p-1},$$

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with $\alpha > 0$, has been introduced by M. Hénon [4] as a model to study spherically symmetric clusters of stars. From a mathematical point of view, it is a good example for interesting phenomena in critical point theory applied to nonlinear elliptic equations. Up to now researchers have devoted their work to positive solutions of the Dirichlet problem, that is to

$$\begin{cases} -\Delta u = |x|^{\alpha} u^{p-1} & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{1}$$

where $B = \{x \in \mathbf{R}^N \mid |x| < 1\}$, $\alpha > 0$ and $p > 2$.

It is not difficult to realize that the term $|x|^{\alpha}$ modifies the classical Pohozaev argument for non existence of solutions on star–shaped domains, so that the usual non existence results hold only for $p \geq 2^* + \frac{2\alpha}{N-2}$, where $2^* = \frac{2N}{N-2}$ is the critical exponent for the Sobolev embedding of H_0^1 into L^p. Hence, one can hope to prove existence of solutions of (1) when $p \in \left(2, 2^* + \frac{2\alpha}{N-2}\right)$, and indeed, in [7], W.M. Ni proved that (1) has a solution for each p in that range. This result is achieved by the application of the Mountain Pass Lemma in a space of radial functions, and the solution obtained is therefore radial. After Ni's result the work on problem (1) was not pursued, to our knowledge, until 2002, when Smets, Su and Willem (see [9]) proved that, in the subcritical case $p \in (2, 2^*)$, and at least for large α's, Ni's radial solution is not the ground state solution, that is the solution with minimal energy. So, in spite of the radiality of problem (1), an interesting symmetry–breaking phenomenon gives rise to multiple solutions. Of course in the subcritical case the existence of the ground state solution is not a difficult problem, because of the compactness of the embedding of $H_0^1(B)$ into $L^p(B)$, and the main difficulty, solved in [9], is to distinguish the ground state solution from the radial one. Further results on the subcritical Hénon equation (such as symmetry properties of solutions and blow–up profile of ground states) can be found in [1], [3] and [10].

The work of Smets, Su and Willem has also stimulated the recent paper [8], where a multiplicity result has been obtained in the critical case $p = 2^*$. In this case the non radial solution is obtained by minimizing a suitable functional on subspaces of $H_0^1(B)$ which are invariant under the action of some subgroups of $O(N)$, and of course two difficulties must be faced: to prove that the minimum is attained, which is not trivial because of the lack of compactness, and to distinguish the new solution from the radial one.

In the present paper we will prove some existence results of non radial solutions of (1) for ranges of p like $(2, p')$, where $2^* < p' < 2^* + \frac{2\alpha}{N-2}$, covering in this way subcritical, critical and supercritical growths. We look for solutions by minimizing the homogeneous functional

$$Q_{\alpha}(u) = \frac{\int_B |\nabla u|^2 dx}{\left(\int_B |x|^{\alpha} |u|^p dx\right)^{2/p}}. \tag{2}$$

It is well known that the positive minima of Q_{α}, after rescaling, give rise to solutions of (1). The functional Q_α is not well defined in $H_0^1(B)$, when $p > 2^*$, so we will restrict it to functions that exhibit a particular kind of symmetry: for $N \geq 4$ and $2 \leq N-l \leq l$, we write $x = (y,z) \in \mathbf{R}^l \times \mathbf{R}^{N-l}$ and we consider functions in $H_0^1(B)$ that are radial with respect to y and z, i.e. functions u such that $u(x) = u(y,z) = v(|y|,|z|)$. See Section 2 for precise notation and definitions. This technique has been used in the papers [5] and [6], where it is applied to problems on annuli. The fact that it works also in our case shows, as already pointed out in [8], that the effect of the weight $|x|^\alpha$ is comparable to the presence of a hole in B.

A first task is to prove that Q_α attains its infimum on the space of symmetric functions (in the sense introduced above) of $H_0^1(B)$. This a consequence of an embedding result, which implies some compactness for $p > 2^*$, that we prove in Section 2. After that, we have to distinguish these critical points one from another (when we consider different l's) and each from the radial one, which is always present. This last result is obtained in Section 3 with an argument inspired by [9] and [8], while an argument of Y.Y. Li (see [5]), that we recall in Section 4, implies that all the symmetric solutions that we find are different. The sum of all these arguments allows us to get different multiplicity results for different ranges of p's. We state and prove just two of these results, which in our opinion are the most interesting. To state these theorems, we define $[a]$ to be the integer part of a real number a.

Theorem 1.1 *Assume* $N \geq 4$ *and* $p \in \left(2, \frac{2N-2}{N-3}\right)$. *Then, for all large* α, *problem (1) has at least* $[N/2]-1$ *different non radial solutions.*

Theorem 1.2 *Let* $p^*(N) = 2\frac{N+2}{N-2}$ *if* N *is even and* $p^*(N) = 2\frac{[N/2]+2}{[N/2]}$ *if* N *is odd. Assume* $N \geq 4$ *and* $p \in (2, p^*(N))$. *Then, for all large* α, *problem (1) has at least a non radial solution.*

Remark 1.3 Notice that $\frac{2N-2}{N-3}$ is the critical exponent for the Sobolev embedding of H_0^1 in dimension $N-1$, while $p^*(N)$ is the critical exponent for the dimension $N/2+1$ when N is even and $[N/2]+2$ when N is odd. For $N \geq 4$ there always results $p^*(N) \geq \frac{2N-2}{N-3} > 2^*$ (strict inequality if $N > 5$). We remark that Theorem 1.1 gives new results also for the subcritical and critical cases, while Theorem 1.2 only for the range $p \in (2^*, p^*(N))$.

This paper is organized as follows: after the Introduction we give in Section 2 an embedding result. Section 3 is devoted to the study of some energy levels, and in Section 4 we conclude the proof of Theorems 1.1 and 1.2.

We collect below the main notation used throughout the paper.

- $\mathbf{N}$ is the set of positive integers.
- For any $k \in \mathbf{N}$, $O(k)$ is the group of real orthogonal $k \times k$ matrices.
- For any $N, l \in \mathbf{N}$ such that $2 \leq N-l \leq l$ and for any $x = (x_1 \ldots, x_N) \in \mathbf{R}^N$ we write $x = (y,z)$, with $y = (x_1, \ldots, x_l) \in \mathbf{R}^l$ and $z = (x_{l+1}, \ldots, x_N) \in \mathbf{R}^{N-l}$.

- H^1, H_0^1 and L^p are the usual Sobolev and Lebesgue spaces, while $L^p(|x|^\alpha)$ denotes the space of functions that are in L^p with respect to the measure $|x|^\alpha dx$.
- We will use C to denote positive constants that can change from line to line.

2 An embedding result

We work in the space $H_0^1(B)$ or in some of its subspaces and we consider the functional Q_α defined by

$$Q_\alpha(u) = \frac{\int_B |\nabla u|^2 dx}{\left(\int_B |x|^\alpha |u|^p dx\right)^{2/p}}.$$

For a fixed $N \geq 4$ we consider $l \in \mathbf{N}$ such that $2 \leq N-l \leq l$ and we write $x=(y,z) \in \mathbf{R}^l \times \mathbf{R}^{N-l}$. We define

$$H_l = \{u \in H_0^1(B) \mid u(R_1 y, R_2 z) = u(y,z),\ \forall x=(y,z) \in B,\\ \forall (R_1,R_2) \in O(l) \times O(N-l)\}.$$

Notice that H_l is the subset of H_0^1 of functions which are radial in y and in z; therefore we will also write more simply

$$H_l = \{u \in H_0^1(B) \mid u(y,z) = v(|y|,|z|)\}.$$

The main result of this section is the following embedding result, which we believe to be interesting in its own. A similar result, without the weight $|x|^\alpha$, and with B replaced by an annulus has been proved in [5] and [6].

Theorem 2.1 *Let $N \geq 4$ and $2 \leq N-l \leq l$. Define $p_l = \frac{2l+2}{l-1}$ and assume that $\alpha > N+2$. Then there is a constant $C>0$ such that for all $u \in H_l$ there holds*

$$\left(\int_B |x|^\alpha |u|^{p_l} dx\right)^{1/p_l} \leq C \left(\int_B |\nabla u|^2 dx\right)^{1/2}.$$

Theorem 2.1 will be proved by studying some embedding properties on small annular domains; we will estimate the best embedding constant on such domains as their radii go to zero, and will obtain Theorem 2.1 by writing B as the sum of infinitely many annuli. A similar device has been used, for a different problem, in [2].

The main tool to prove Theorem 2.1 is the following lemma, which contains the technical estimate. In order to state it, we fix $\rho \in (0,1)$ and for each $j \in \mathbf{N}$ we define the annulus

$$\Omega_j = \{x \in \mathbf{R}^N \mid \rho^j < |x| < \rho^{j-1}\}$$

and the space

$$E_j = \{u \in H^1(\Omega_j) \mid u(y,z) = v(|y|,|z|),\ \forall x=(y,z) \in \Omega_j\}.$$

We also set $q_l = \frac{2N+(l-1)^2}{l+1}$.

Lemma 2.2 *Assume that N and l are as in Theorem 2.1. Then there is a constant $C > 0$ such that for all $j \in \mathbf{N}$ and all $u \in E_j$ there results*

$$\int_{\Omega_j} (|\nabla u|^2 + u^2)dx \geq C\rho^{jq_l} \left(\int_{\Omega_j} |u|^{p_l} dx \right)^{2/p_l} .$$

Proof. The fact that the embedding holds has already been established in [5] and [6]; the new part is the estimate of the embedding constant. We follow anyway (part of) the computations in [6].

In the proof we denote by C several positive numbers independent of j and u. We begin by defining the sets

$$D^j = \{(s,t) \in \mathbf{R}^2 \mid s \geq 0,\, t \geq 0,\, \rho^{2j} < s^2 + t^2 < \rho^{2(j-1)}\},$$

$$D_1 = D_1^j = \{(s,t) \in D^j \mid t > s\}$$

and

$$D_2 = D_2^j = \{(s,t) \in D^j \mid t < s\}.$$

For $u \in E_j$ let $v = v(s,t)$ be the function on D^j defined by $u(y,z) = v(|y|,|z|)$. We can write, by a standard change of variables,

$$\int_{\Omega_j} (|\nabla u|^2 + u^2)dx = C \int_{D^j} (|\nabla v|^2 + v^2)s^{l-1}t^{N-l-1}dsdt$$
$$= C \int_{D_1} (|\nabla v|^2 + v^2)s^{l-1}t^{N-l-1}dsdt + C \int_{D_2} (|\nabla v|^2 + v^2)s^{l-1}t^{N-l-1}dsdt.$$

Notice that C is produced by integration on the angular variables and does not depend on j (nor on u).

In D_1 we have $t > \rho^j/\sqrt{2}$, while in D_2 there results $s > \rho^j/\sqrt{2}$, so that

$$\int_{\Omega_j} (|\nabla u|^2 + u^2)dx \geq C\rho^{j(N-l-1)} \int_{D_1} (|\nabla v|^2 + v^2)s^{l-1}dsdt$$
$$+ C\rho^{j(l-1)} \int_{D_2} (|\nabla v|^2 + v^2)t^{N-l-1}dsdt.$$

We now define the sets

$$\widetilde{D_1} = \{(y,t) \in \mathbf{R}^{l+1} \mid (|y|,t) \in D_1\}, \qquad \widetilde{D_2} = \{(s,z) \in \mathbf{R}^{N-l+1} \mid (s,|z|) \in D_2\},$$

and the functions

$$u_1(y,t) = v(|y|,t), \qquad u_2(s,z) = v(s,|z|).$$

It is not difficult to see that $u_i \in H^1(\widetilde{D_i})$ so that, by the usual embeddings, $u_i \in L^{p_l}(\widetilde{D_i})$. Notice indeed that p_l is the critical exponent for the Sobolev embedding in dimension $l+1$ and that, by hypothesis, $l+1 \geq N-l+1$ so that the critical Sobolev exponent in dimension $N-l+1$, namely $\frac{2(N-l+1)}{N-l-1}$, satisfies

$$\frac{2(N-l+1)}{N-l-1} \geq p_l.$$

The crucial point now is to make explicit the dependence of the embedding constants on j. To this aim, we define

$$\overline{D_1} = \{(y,t) \in \mathbf{R}^{l+1} \mid 1 < |y|^2 + t^2 < \frac{1}{\rho^2},\, t > |y|\},$$

and

$$\overline{D_2} = \{(s,z) \in \mathbf{R}^{N-l+1} \mid 1 < s^2 + |z|^2 < \frac{1}{\rho^2},\, |z| < s\},$$

and we notice that $\widetilde{D_i} = \rho^j \overline{D_i}$, where of course $\overline{D_i}$ does not depend on j. Hence, setting $w_1(y,t) = u_1(\rho^j y, \rho^j t)$, by a simple rescaling we obtain

$$\int_{\widetilde{D_1}} \left(|\nabla u_1(y,t)|^2 + u_1(y,t)^2\right) dydt = \int_{\rho^j \overline{D_1}} \left(\frac{1}{\rho^{2j}} |\nabla w_1(\frac{y}{\rho^j}, \frac{t}{\rho^j})|^2 + w_1^2(\frac{y}{\rho^j}, \frac{t}{\rho^j})\right) dydt$$

$$\geq \int_{\overline{D_1}} \left(|\nabla w_1(y,t)|^2 + w_1^2(y,t)\right) \rho^{j(l+1)} dydt \geq C\rho^{j(l+1)} \left(\int_{\overline{D_1}} |w_1(y,t)|^{p_l} dydt\right)^{2/p_l}.$$

Here C is the embedding constant for the domain $\overline{D_1}$, so that it does not depend on j. We can now write

$$C\rho^{j(l+1)} \left(\int_{\overline{D_1}} |w_1(y,t)|^{p_l} dydt\right)^{2/p_l} = C\rho^{j(l+l)} \left(\int_{\overline{D_1}} |u_1(\rho^j y, \rho^j t)|^{p_l} dydt\right)^{2/p_l}$$

$$= C\rho^{j(l+1)} \left(\int_{\widetilde{D_1}} |u_1(y,t)|^{p_l} \frac{1}{\rho^{j(l+1)}} dydt\right)^{2/p_l} = C\rho^{j(l+1)(1-\frac{2}{p_l})} \left(\int_{\widetilde{D_1}} |u_1(y,t)|^{p_l} dydt\right)^{2/p_l}.$$

We have thus obtained

$$\int_{\widetilde{D_1}} \left(|\nabla u_1(y,t)|^2 + u_1(y,t)^2\right) dydt \geq C\rho^{j(l+1)(1-\frac{2}{p_l})} \left(\int_{\widetilde{D_1}} |u_1(y,t)|^{p_l} dydt\right)^{2/p_l},$$

where C does not depend on j and u. The same argument yields

$$\int_{\widetilde{D_2}} \left(|\nabla u_2(s,z)|^2 + u_2(s,z)^2\right) dsdz \geq C\rho^{j(N-l+1)(1-\frac{2}{p_l})} \left(\int_{\widetilde{D_2}} |u_2(s,z)|^{p_l} dsdz\right)^{2/p_l}.$$

From the previous computations we deduce that

$$\int_{\Omega_j} (|\nabla u|^2 + u^2) dx$$

$$\geq C\rho^{j(N-l-1)} \int_{\widetilde{D_1}} (|\nabla u_1|^2 + u_1^2) dydt + C\rho^{j(l-1)} \int_{\widetilde{D_2}} (|\nabla u_2|^2 + u_2^2) dsdz$$

$$\begin{aligned}&\geq C\rho^{j[N-l-1+(l+1)(1-2/p_l)]}\left(\int_{\widetilde{D_1}}|u_1|^{p_l}dydt\right)^{2/p_l}\\&+C\rho^{j[l-1+(N-l+1)(1-\frac{2}{p_l}]}\left(\int_{\widetilde{D_2}}|u_2|^{p_l}dsdz\right)^{2/p_l}.\end{aligned}$$

Since $N-l\leq l$, it is easy to check that

$$l-1+(N-l+1)(1-2/p_l)\geq N-l-1+(l+1)(1-2/p_l),$$

so that

$$\rho^{j[N-l-1+(l+1)(1-2/p_l)]}\geq\rho^{j[l-1+(N-l+1)(1-2/p_l)]}.$$

Moreover, by definition of p_l, we see that

$$l-1+(N-l+1)(1-2/p_l)=\frac{2N+(l-1)^2}{l+1}=q_l,$$

so that we obtain

$$\begin{aligned}\int_{\Omega_j}(|\nabla u|^2+u^2)dx&\geq C\rho^{jq_l}\left[\left(\int_{\widetilde{D_1}}|u_1|^{p_l}dydt\right)^{2/p_l}+\left(\int_{\widetilde{D_2}}|u_2|^{p_l}dsdz\right)^{2/p_l}\right]\\&\geq C\rho^{jq_l}\left[\int_{\widetilde{D_1}}|u_1|^{p_l}dydt+\int_{\widetilde{D_2}}|u_2|^{p_l}dsdz\right]^{2/p_l}.\end{aligned}\qquad(3)$$

It is now easy to see that

$$\int_{\widetilde{D_1}}|u_1|^{p_l}dydt=C\int_{D_1}|v(s,t)|^{p_l}s^{l-1}dsdt\geq C\int_{D_1}|v(s,t)|^{p_l}s^{l-1}t^{N-l-1}dsdt\qquad(4)$$

and, likewise,

$$\int_{\widetilde{D_2}}|u_2|^{p_l}dsdz\geq C\int_{D_2}|v(s,t)|^{p_l}s^{l-1}t^{N-l-1}dsdt.\qquad(5)$$

Collecting the estimates in (3), (4) and (5) we can write

$$\begin{aligned}\int_{\Omega_j}(|\nabla u|^2+u^2)dx&\geq C\rho^{jq_l}\left(\int_{D^j}|v(s,t)|^{p_l}s^{l-1}t^{N-l-1}dsdt\right)^{2/p_l}\\&=C\rho^{jq_l}\left(\int_{\Omega_j}|u|^{p_l}dx\right)^{2/p_l},\end{aligned}$$

and the proof is complete. ■

We can now finish the proof of the main result of this section.

Proof of Theorem 2.1. Fix $\rho \in (0,1)$ and define Ω_j as in the previous lemma. For every $u \in H_l$ we have, by Lemma 2.2,

$$\int_B |x|^\alpha |u|^{p_l} dx = \sum_{j=1}^{\infty} \int_{\Omega_j} |x|^\alpha |u|^{p_l} dx \leq \sum_{j=1}^{\infty} \rho^{(j-1)\alpha} \int_{\Omega_j} |u|^{p_l} dx$$

$$\leq \sum_{j=1}^{\infty} \rho^{(j-1)\alpha} C \rho^{-j\frac{q_l p_l}{2}} \left(\int_{\Omega_j} (|\nabla u|^2 + u^2) dx \right)^{p_l/2}$$

$$\leq C \sum_{j=1}^{\infty} \rho^{j(\alpha - \frac{q_l p_l}{2})} \rho^{-\alpha} \left(\int_B (|\nabla u|^2 + u^2) dx \right)^{p_l/2} \leq C \left(\int_B |\nabla u|^2 dx \right)^{p_l/2} \sum_{j=1}^{\infty} \rho^{j(\alpha - \frac{q_l p_l}{2})}.$$

Now by definition we have $\frac{q_l p_l}{2} = \frac{2N+(l-1)^2}{l-1}$; since $2 \leq N - l \leq l$, it is easy to see that $\frac{2N+(l-1)^2}{l-1} \leq N+2$, so that $\alpha - q_l p_l/2 > 0$. As $\rho \in (0,1)$, we therefore see that $\sum_{j=1}^{\infty} \rho^{j(\alpha - q_l p_l/2)} < +\infty$, and this concludes the proof. ■

By standard interpolation arguments we derive from Theorem 2.1 the following result, which gives the embedding and compactness properties that we need.

Corollary 2.3 *Under the same assumptions of Theorem 2.1,*

i) if $p \in [1, p_l]$, there exists a constant $C > 0$ such that

$$\left(\int_B |x|^\alpha |u|^p dx \right)^{2/p} \leq C \int_B |\nabla u|^2 dx$$

for all $u \in H_l$.

ii) If $p \in [1, p_l)$ and $\{u_k\}$ is a sequence in H_l such that $u_k \to 0$ weakly in H_l, then

$$\int_B |x|^\alpha |u_k|^p dx \to 0.$$

In other words, H_l embeds in $L^p(|x|^\alpha)$ continuously if $p \leq p_l$ and compactly if $p < p_l$.

We now define

$$m_{\alpha,l} = \inf_{H_l \setminus \{0\}} Q_\alpha$$

and we notice that from Corollary 2.3 for $N \geq 4$, $2 \leq N - l \leq l$ and $\alpha > N+2$, we have

$$m_{\alpha,l} > 0. \tag{6}$$

In the next section we study the asymptotic behavior of $m_{\alpha,l}$ as $\alpha \to \infty$.

3 Level estimates

In this section we establish the level estimates necessary to separate the level $m_{\alpha,l}$ from the level of radial solutions, which are always present. We start by recalling Ni's result ([7]), which provides the existence of such solutions.

Let $H^1_{0,r}$ be the space of radial functions in $H^1_0(B)$; the following result is taken from [7] and is restated according to our notation.

Theorem 3.1 *i) Q_α is well defined in $H^1_{0,r}\setminus\{0\}$ when $p \in [2, 2^* + \frac{2\alpha}{N-2}]$.*
ii) Let

$$c_\alpha = \inf_{H^1_{0,r}\setminus\{0\}} Q_\alpha.$$

Then $c_\alpha > 0$ and for $p \in [2, 2^* + \frac{2\alpha}{N-2})$ it is attained in $H^1_{0,r} \setminus \{0\}$.

It is obvious that $m_{\alpha,l} \leq c_\alpha$ whenever $m_{\alpha,l}$ is defined. We want to prove that, at least for large α, there results $m_{\alpha,l} < c_\alpha$. Once this is done, we will know that all functions at level $m_{\alpha,l}$ are not radial.

To obtain the inequality $m_{\alpha,l} < c_\alpha$ we will estimate c_α from below and $m_{\alpha,l}$ from above. We begin with the estimates on c_α. This is essentially the same estimate obtained in [9]; the proof in that paper however works only for $p \leq 2^*$, while we are mainly interested in supercritical growths. Therefore we give an extension of the estimate in [9] which holds up to the exponent $p_l = \frac{2l+2}{l-1}$ defined in Theorem 2.1 of the preceding section.

Lemma 3.2 *Let $\sigma \in (0, 1]$ and, for $u : B \to \mathbf{R}$, define $(T_\sigma u)(x) = u(|x|^{\sigma-1}x)$. Then $T_\sigma(H^1_{0,r}) \subset H^1_{0,r}$ and for $p \in [2, 2^* + \frac{2\alpha}{N-2}]$ there results*

$$c_\alpha = \inf_{v\in T_\sigma(H^1_{0,r}\setminus\{0\})} \frac{1}{\sigma^{1+2/p}} \frac{\int_B |\nabla v|^2 |x|^{(\sigma-1)(N-2)} dx}{\left(\int_B |v|^p |x|^{\sigma(\alpha+N)-N} dx\right)^{2/p}}.$$

Proof. For the first statement we write $v = T_\sigma u$ as $v(\rho) = u(\rho^\sigma)$, with a slight abuse of notation, and we compute, with obvious changes of variables,

$$\int_B |\nabla v|^2 dx = C \int_0^1 |v'(\rho)|^2 \rho^{N-1} d\rho = C\sigma^2 \int_0^1 |u'(\rho^\sigma)|^2 \rho^{2\sigma-2} \rho^{N-1} d\rho$$

$$= C\sigma \int_0^1 |u'(r)|^2 r^{(1-\frac{1}{\sigma})(2-N)} r^{N-1} dr \leq C\sigma \int_0^1 |u'(r)|^2 r^{N-1} dr = \sigma \int_B |\nabla u|^2 dx,$$

since $\sigma \leq 1$.

To prove the second statement we let $u = T_\sigma^{-1} v$ (namely $u(\rho) = v(\rho^{1/\sigma})$) in the expression of $Q_\alpha(u)$. Again by a direct change of variables we obtain

$$Q_\alpha(u) = \frac{1}{\sigma^{1+2/p}} \frac{\int_B |\nabla v|^2 |x|^{(\sigma-1)(N-2)} dx}{\left(\int_B |v|^p |x|^{\sigma(\alpha+N)-N} dx\right)^{2/p}}.$$

Recalling the definition of c_α the result immediately follows. ■

We can now prove the estimate from below on c_α.

Proposition 3.3 *Assume that $N \geq 4$, $2 \leq N - l \leq l$ and $p \in \left(2, \frac{2l+2}{l-1}\right)$. Then there is a constant $C > 0$ such that, for all large α,*

$$c_\alpha \geq C\,\alpha^{1+\frac{2}{p}}.$$

Proof. In view of Lemma 3.2 we estimate

$$\inf_{v\in T_\sigma(H^1_{0,r}\setminus\{0\})} \frac{1}{\sigma^{1+2/p}} \frac{\int_B |\nabla v|^2 |x|^{(\sigma-1)(N-2)} dx}{\left(\int_B |v|^p |x|^{\sigma(\alpha+N)-N} dx\right)^{2/p}},$$

where we choose $\sigma = \frac{4N}{\alpha+N}$. Notice that $\sigma \leq 1$ if α is large enough. With this choice of σ we see that

$$\int_B |v|^p |x|^{\sigma(\alpha+N)-N} dx = \int_B |v|^p |x|^{3N} dx.$$

Since $\sigma \leq 1$, we also have

$$\int_B |\nabla v|^2 |x|^{(\sigma-1)(N-2)} dx \geq \int_B |\nabla v|^2 dx,$$

so that for all $v \in T_\sigma(H^1_{0,r}) \setminus \{0\}$,

$$\frac{1}{\sigma^{1+2/p}} \frac{\int_B |\nabla v|^2 |x|^{(\sigma-1)(N-2)} dx}{\left(\int_B |v|^p |x|^{\sigma(\alpha+N)-N} dx\right)^{2/p}} \geq \frac{1}{\sigma^{1+2/p}} \frac{\int_B |\nabla v|^2 dx}{\left(\int_B |v|^p |x|^{3N} dx\right)^{2/p}}.$$

As $T_\sigma(H^1_{0,r}) \subset H^1_{0,r}$, we obtain

$$\begin{aligned} &\inf_{v\in T_\sigma(H^1_{0,r}\setminus\{0\})} \frac{1}{\sigma^{1+2/p}} \frac{\int_B |\nabla v|^2 |x|^{(\sigma-1)(N-2)} dx}{\left(\int_B |v|^p |x|^{\sigma(\alpha+N)-N} dx\right)^{2/p}} \\ &\geq \frac{1}{\sigma^{1+2/p}} \inf_{v\in H^1_{0,r}\setminus\{0\}} \frac{\int_B |\nabla v|^2 dx}{\left(\int_B |v|^p |x|^{3N} dx\right)^{2/p}} = \frac{1}{\sigma^{1+2/p}}\, c_{3N}. \end{aligned}$$

The last equality holds, by Theorem 3.1 with $\alpha = 3N$, provided that

$$p < \frac{2N}{N-2} + \frac{2(3N)}{N-2} = \frac{8N}{N-2}.$$

But this is indeed the case, since the assumptions $p < 2\frac{l+1}{l-1}$ and $2 \leq N - l \leq l$ immediately imply that $p < 2\frac{N+2}{N-2}$, which is always smaller than $\frac{8N}{N-2}$.

We remark that c_{3N} does not depend on α. Hence, from Lemma 3.2 and the above computations we obtain that for all large α,

$$c_\alpha \geq C \frac{1}{\left(\frac{4N}{\alpha+N}\right)^{1+2/p}} \geq \tilde{C} \alpha^{1+\frac{2}{p}},$$

where $\tilde{C}$ does not depend on α. ■

We now turn to the estimates from above for the levels $m_{\alpha,l}$.

Proposition 3.4 *Under the same assumptions of Proposition 3.3, there exists a constant C such that, for all large α,*

$$m_{\alpha,l} \leq C\alpha^{(l+1-N)(1-\frac{2}{p})+\frac{4}{p}}.$$

Proof. Let

$$D = \{(s,t) \in \mathbf{R}^2 \mid s,t \geq 0,\, 0 \leq s^2+t^2 < 1\}.$$

For every $u \in H_l(B)$ we have

$$\int_B |x|^\alpha |u|^p dx = C\int_D (s^2+t^2)^{\alpha/2}|u(s,t)|^p s^{l-1}t^{N-l-1}dsdt$$

and

$$\int_B |\nabla u|^2 dx = C\int_D |\nabla u(s,t)|^2 s^{l-1}t^{N-l-1}dsdt,$$

where we have written, for $x = (y,z) \in \mathbf{R}^l \times \mathbf{R}^{N-l}$ and with a little abuse of notation,

$$u(x) = u(|y|,|z|) = u(s,t).$$

We now study $Q_\alpha(u)$ by introducing polar coordinates in $\mathbf{R}^2$. Defining

$$s = \rho\cos\theta, \qquad t = \rho\sin\theta,$$

$$A = \{(\rho,\theta) \in \mathbf{R}^2 \mid 0 \leq \rho < 1,\, 0 \leq \theta \leq \pi/2\},$$

and

$$v(\rho,\theta) = u(\rho\cos\theta, \rho\sin\theta)$$

we can write

$$\int_D (s^2+t^2)^{\alpha/2}|u(s,t)|^p s^{l-1}t^{N-l-1}dsdt = \int_A |v(\rho,\theta)|^p \rho^{\alpha+N-1}H(\theta)d\rho d\theta$$

and

$$\int_D |\nabla u(s,t)|^2 s^{l-1}t^{N-l-1}dsdt = \int_A \big(v_\rho(\rho,\theta)^2 + \frac{1}{\rho^2}v_\theta(\rho,\theta)^2\big)\rho^{N-1}H(\theta)d\rho d\theta,$$

where $H(\theta) = (\cos\theta)^{l-1}(\sin\theta)^{N-l-1}$.

Hence, for every $u \in H_l(B)$, we obtain

$$Q_\alpha(u) = \frac{\int_B |\nabla u|^2 dx}{\left(\int_B |x|^\alpha |u|^p dx\right)^{2/p}} = C\,\frac{\int_A \left(v_\rho^2 + \frac{1}{\rho^2}v_\theta^2\right)\rho^{N-1}H(\theta)d\rho d\theta}{\left(\int_A |v|^p\, \rho^{\alpha+N-1}H(\theta)d\rho d\theta\right)^{2/p}}, \tag{7}$$

where $v(\rho,\theta) = u(\rho\cos\theta, \rho\sin\theta)$.

Since we want an estimate from above for the infimum $m_{\alpha,l}$, it is enough to estimate Q_α at some particular $u \in H_l(B)$. To this aim we define

$$\widetilde{A} = (1/4, 3/4) \times (\theta_1, \theta_2),$$

with $0 < \theta_1 < \theta_2 < \pi/2$ and we pick a function $\psi \in C_0^\infty(\widetilde{A})$. For every small $\varepsilon > 0$ (to be fixed later) we define

$$v^\varepsilon(\rho,\theta) = \psi(\rho^{1/\varepsilon}, \theta/\varepsilon)$$

and we notice that, setting

$$\widetilde{A}_\varepsilon = \left\{(\rho,\theta) \in \mathbf{R}^2 \mid (1/4)^\varepsilon < \rho < (3/4)^\varepsilon,\ \varepsilon\theta_1 < \theta < \varepsilon\theta_2\right\},$$

the function v^ε is in $C_0^\infty(\widetilde{A}_\varepsilon)$.

We evaluate the right–hand–side of (7) at $v = v^\varepsilon$. This corresponds of course to evaluating Q_α at u^ε, defined by $u^\varepsilon(\rho\cos\theta, \rho\sin\theta) = v^\varepsilon(\rho,\theta)$; clearly this u^ε, when interpreted as a function of $x = (y,z) \in \mathbf{R}^l \times \mathbf{R}^{N-l}$ via $u^\varepsilon(x) = u^\varepsilon(|y|,|z|)$, belongs to $C_0^\infty(B) \cap H_l(B)$.

As far as the denominator is concerned, we have

$$\begin{aligned}\int_A |v^\varepsilon(\rho,\theta)|^p \rho^{\alpha+N-1} H(\theta)\,d\rho d\theta &= \int_{\widetilde{A}_\varepsilon} |v^\varepsilon(\rho,\theta)|^p \rho^{\alpha+N-1} H(\theta)\,d\rho d\theta\\ &= \int_{\widetilde{A}_\varepsilon} |\psi(\rho^{1/\varepsilon},\theta/\varepsilon)|^p \rho^{\alpha+N-1} H(\theta)\,d\rho d\theta.\end{aligned}$$

Introducing the new variables

$$r = \rho^{1/\varepsilon}, \quad \varphi = \theta/\varepsilon,$$

the last integral becomes

$$\begin{aligned}&\int_{\widetilde{A}_\varepsilon} |\psi(\rho^{1/\varepsilon},\theta/\varepsilon)|^p\, \rho^{\alpha+N-1} H(\theta)\,d\rho d\theta\\ &= \int_{\widetilde{A}} |\psi(r,\varphi)|^p\, r^{\varepsilon(\alpha+N-1)} H(\varepsilon\varphi)\varepsilon^2 r^{\varepsilon-1}\,dr d\varphi = \varepsilon^2 \int_{\widetilde{A}} |\psi|^p\, r^{\varepsilon(\alpha+N)-1} H(\varepsilon\varphi)\,dr d\varphi.\end{aligned}$$

We now choose $\varepsilon = \frac{N}{\alpha+N}$, so that we obtain the estimate

$$\int_A |v^\varepsilon|^p\, \rho^{\alpha+N-1} H(\theta)\,d\rho d\theta = \varepsilon^2 \int_{\widetilde{A}} |\psi|^p\, r^{N-1} H(\varepsilon\varphi)\,dr d\varphi. \tag{8}$$

We pass to the study the integral involving the gradient.

As $v^\varepsilon(\rho,\theta) = \psi(\rho^{1/\varepsilon},\theta/\varepsilon)$, we have

$$v^\varepsilon_\rho(\rho,\theta) = \frac{1}{\varepsilon}\psi_1(\rho^{1/\varepsilon},\theta/\varepsilon)\rho^{\frac{1}{\varepsilon}-1} \quad \text{and} \quad v^\varepsilon_\theta(\rho,\theta) = \frac{1}{\varepsilon}\psi_2(\rho^{1/\varepsilon},\theta/\varepsilon),$$

where ψ_i $(i = 1,2)$ are the two partial derivatives of ψ.

With standard changes of variables we obtain

$$\int_A \left((v^\varepsilon_\rho)^2 + \frac{1}{\rho^2}(v^\varepsilon_\theta)^2\right) \rho^{N-1} H(\theta)\,d\rho d\theta$$

$$
\begin{aligned}
&= \int_{\widetilde{A}_\varepsilon} \left(\frac{1}{\varepsilon^2 \rho^{2-\frac{2}{\varepsilon}}} \psi_1^2(\rho^{1/\varepsilon}, \theta/\varepsilon) + \frac{1}{\varepsilon^2 \rho^2} \psi_2^2(\rho^{1/\varepsilon}, \theta/\varepsilon) \right) \rho^{N-1} H(\theta) d\rho d\theta \\
&= \int_{\widetilde{A}} \left(\frac{1}{\varepsilon^2 r^{2\varepsilon-2}} \psi_1^2(r,\varphi) + \frac{1}{\varepsilon^2 r^{2\varepsilon}} \psi_2^2(r,\varphi) \right) r^{\varepsilon(N-1)} H(\varepsilon\varphi) \varepsilon^2 r^{\varepsilon-1} dr d\varphi \\
&= \int_{\widetilde{A}} \left(\psi_1^2 + \frac{1}{r^2} \psi_2^2 \right) r^{(\varepsilon-1)(N-2)} r^{N-1} H(\varepsilon\varphi) dr d\varphi.
\end{aligned}
$$

Substituting this and (8) in (7), we arrive at

$$
Q_\alpha(u^\varepsilon) = C \, \frac{\int_{\widetilde{A}} \left(\psi_1^2 + \frac{1}{r^2} \psi_2^2 \right) r^{(\varepsilon-1)(N-2)} r^{N-1} H(\varepsilon\varphi) dr d\varphi}{\varepsilon^{4/p} \left(\int_{\widetilde{A}} |\psi|^p \, r^{N-1} H(\varepsilon\varphi) dr d\varphi \right)^{2/p}}. \tag{9}
$$

Recalling that $2 \le N - l \le l$ and that $\theta_1 < \varphi < \theta_2$, we see from the definition of H that there are positive constants C_1 and C_2 such that

$$
C_1 \varepsilon^{N-l-1} \le H(\varepsilon\varphi) \le C_2 \varepsilon^{N-l-1},
$$

for all small ε. Hence (9) becomes

$$
\begin{aligned}
Q_\alpha(u^\varepsilon) &\le C \, \frac{\varepsilon^{N-l-1} \int_{\widetilde{A}} \left(\psi_1^2 + \frac{1}{r^2} \psi_2^2 \right) r^{(\varepsilon-1)(N-2)} r^{N-1} dr d\varphi}{\varepsilon^{(N-l-1)\frac{2}{p}+\frac{4}{p}} \left(\int_{\widetilde{A}} |\psi|^p \, r^{N-1} dr d\varphi \right)^{2/p}} \\
&\le C \varepsilon^{(N-l-1)(1-\frac{2}{p})-\frac{4}{p}} \, \frac{\int_{\widetilde{A}} \left(\psi_1^2 + \frac{1}{r^2} \psi_2^2 \right) r^{N-1} dr d\varphi}{\left(\int_{\widetilde{A}} |\psi|^p \, r^{N-1} dr d\varphi \right)^{2/p}}.
\end{aligned} \tag{10}
$$

The last inequality comes from the fact that in $\widetilde{A}$ we have $1/4 \le r \le 3/4$, so that $r^{(\varepsilon-1)(N-2)}$ is uniformly bounded for all $\varepsilon > 0$.

Now the ratio in the last term of (10) is a constant independent of α and ε, and therefore

$$
Q_\alpha(u^\varepsilon) \le C \varepsilon^{(N-l-1)(1-\frac{2}{p})-\frac{4}{p}},
$$

which implies

$$
m_{\alpha,l} \le C \varepsilon^{(N-l-1)(1-\frac{2}{p})-\frac{4}{p}}.
$$

Recalling that we had chosen $\varepsilon = \frac{N}{\alpha+N}$, we finally obtain

$$
m_{\alpha,l} \le C \left(\frac{N}{\alpha+N} \right)^{(N-l-1)(1-\frac{2}{p})-\frac{4}{p}} \le \tilde{C} \alpha^{(l+1-N)(1-\frac{2}{p})+\frac{4}{p}},
$$

where $\tilde{C}$ does not depend on α. ■

We can now state the main result of this section.

Proposition 3.5 *Assume that $N \ge 4$, $2 \le N - l \le l$ and $p \in \left(2, \frac{2l+2}{l-1}\right)$. Then, for all large α,*

$$
m_{\alpha,l} < c_\alpha.
$$

Proof. As a consequence of Propositions 3.3 and 3.4, the claim is true if

$$(l+1-N)(1-\frac{2}{p})+\frac{4}{p}<1+\frac{2}{p}. \tag{11}$$

It is immediate to see that (11) holds whenever $p>2$. ■

4 Proofs of the main theorems

Collecting the results of the preceding sections, it is now easy to prove Theorems 1.1 and 1.2. Before giving some details, we recall a result by Y.Y. Li, Lemma 3.3 of [5], which we will need to conclude the proofs.

Lemma 4.1 *Let* $N \geq 4$, $2 \leq l_1 \leq N-l_1$ *and* $2 \leq l_2 \leq N-l_2$, *with* $l_1 \neq l_2$. *Then* $H_{l_1} \cap H_{l_2} = H^1_{0,r}$.

Proof of Theorem 1.1. Let $l \in \mathbf{N}$ be such that $2 \leq N-l \leq l$. For $p \in \left(2, \frac{2l+2}{l-1}\right)$ and $\alpha > N+2$, consider the number

$$m_{\alpha,l} = \inf_{H_l \setminus \{0\}} Q_\alpha .$$

By (6), $m_{\alpha,l}>0$, and by Corollary 2.3, $m_{\alpha,l}$ is attained at some nonzero $u_l \in H_l$.

Since Q_α is invariant under the action of the group $O(l)\times O(N-l)$ used to define H_l, critical points of Q_α restricted to H_l are free critical points of Q_α. Therefore u_l is a critical point of Q_α and, by standard arguments, a suitable multiple of u_l is a classical solution of problem (1). Notice that there are exactly $[N/2]-1$ integers l that satisfy $2 \leq N-l \leq l$. If we want that $p \in \left(2, \frac{2l+2}{l-1}\right)$ for all such l's, we must take $p \in \left(2, \frac{2N-2}{N-3}\right)$, since we have $\frac{2N-2}{N-3} \leq \frac{2l+2}{l-1}$ for all the admissible l's, when $N \geq 4$.

By Proposition 3.5, for all large α, u_l is not radial, because

$$Q_\alpha(u_l) = m_{\alpha,l} < c_\alpha = \inf_{H^1_{0,r}\setminus\{0\}} Q_\alpha .$$

Finally, by Lemma 4.1, $u_{l_1} \neq u_{l_2}$ if $l_1 \neq l_2$, otherwise they would both coincide with a radial function.

We have therefore found $[N/2]-1$ different nonradial solutions of (1). ■

Proof of Theorem 1.2. We repeat the first part of the above argument with $l=N/2$ if N is even and $l=[N/2]+1$ if N is odd, which means that $p \in (2, p^*(N))$. We obtain in this way a function $u_l \in H_l$ such that $Q_\alpha(u_l) = m_{\alpha,l}$. The same level estimates used in the proof of Theorem 1.1 allow us to conclude that u_l is not radial. As above, a multiple of u_l solves problem (1). ■

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