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On the Solvability of Some Nonlinear Systems of Elliptic Equations

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Abstract

We study the solvability and the existence of multiple solutions of nonlinear systems of elliptic equations.

Key words. Nonlinear systems, Constrained variational problem, Bifurcation equation, p-Laplacian operator, Fibering method, Lagrange multipliers.

1 Introduction

Many authors are interested in the study of the solvability and in the existence of multiple solutions of nonlinear boundary value problems like the following one

$$\begin{aligned} -\Delta_p u &= f(x,u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

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where Δ_p denotes the p-Laplacian operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ by using, for example, the perturbation of Ljusternik–Schnirelmann theory [4] or the Mountain Pass Theorem [2] or the fibering method [8], [11]-[15] introduced by S. Pohozaev a few years ago.
This paper is focused on the study of the system

$$\begin{aligned} &-\Delta_p u_1 - \lambda_1 |u_1|^{p-2}u_1 = f_1(x,u_1,u_2) && \\ & && \text{in } \Omega \\ &-\Delta_p u_2 - \lambda_2 |u_2|^{p-2}u_2 = f_2(x,u_1,u_2) && \\ &u_1 = u_2 = 0 && \text{on } \partial\Omega \end{aligned}$$

where Ω is a bounded connected open set of R^N with locally Lipschitz boundary $\partial\Omega$, λ_i $(i = 1,2)$ are real parameters and functions $f_i(x,u_1,u_2)$ are chosen in the following ways:

$$f_i(x,u_1,u_2) = |a_1u_1 + a_2u_2|^{\gamma-2}(a_1u_1 + a_2u_2)a_i - b_i|u_i|^{r_i-2}u_i\,, \tag{1.1}$$

$$\begin{aligned} f_i(x,u_1,u_2) &= |a_1u_1 + a_2u_2|^{\gamma-2}(a_1u_1 + a_2u_2)a_i + \\ &\quad c_i|u_i|^{s-2}u_i - b_i|u_i|^{r_i-1}\,, \end{aligned} \tag{1.2}$$

$$f_i(x,u_1,u_2) = |a_1u_1^2 + a_2u_2^2|^{\frac{\gamma}{2}-1}a_iu_i - b_i|u_i|^{\gamma-2}u_i\,. \tag{1.3}$$

The functions a_i, b_i, c_i are essentially bounded in Ω; in all cases any exponent is greater than 1 and less than p^* [$p^* = Np/(N-p)$ if $N > p$, $+\infty$ otherwise], so, that the Rellich–Kondrashov Theorem [1] holds.

Let us add that the assumptions about the sign of coefficients and the relation between the exponents will be made for every case.
As far as we know, systems like these have not been studied yet. Our analysis starts with comparison between the parameters λ_i and the first eigenvalue of the problem

$$\begin{aligned} &-\Delta_p u = \lambda|u|^{p-2}u && \text{in } \Omega \\ &u = 0 && \text{on } \partial\Omega\,. \end{aligned}$$

We shall denote by λ_* the above mentioned eigenvalue and by u_* the eigenfunction corresponding to λ_* that is positive in Ω. Let us remark that

$$\lambda_* = \frac{\displaystyle\int_\Omega |\nabla u_*|^p dx}{\displaystyle\int_\Omega |u_*|^p dx} = \min_{v\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla v|^p\,dx}{\displaystyle\int_\Omega |v|^p\,dx}$$

and all eigenfunctions corresponding to λ_* are of the type cu_*, with c real constant [3], [5], [9].

The cases (1.1), (1.2) are studied in the same Section 2 due to the methodological affinity; in fact, when $\lambda_i < \lambda_*$ for $i = 1,2$, under suitable assumptions about the signs of coefficients and relations between exponents, by using prevalently the fibering approach we obtain that the system admits one solution with both components different from zero. In particular, choosing the norms in $L^\infty(\Omega)$ of coefficients a_i, b_i

in (1.1) sufficiently small we prove the existence of two different solutions with both components different from zero.
We reach the same conclusion by choosing f_i in (1.2), but in this case no restrictive assumption about the norms of coefficients is necessary.

In Section 3 our aim is to prove that there exists at least one weak solution of the system with f_i in (1.3) with both components different from zero. We reach this aim by studying a suitable constrained variational problem and using the Lagrange multipliers.
In Section 4 we study substantially the same system with $\lambda_1 = \lambda_2 = \lambda$; by using another constrained variational problem we obtain that when λ is different from λ_* and begins in a suitable right neighborhood of λ_* there exists another solution with both components different from zero.

On this subject we note that in some particular cases our hypotheses assure the existence of solutions like $(u_1, 0)$, $(0, u_2)$ with $u_i > 0$ due to some results of Drabek–Pohozaev [8]. It is necessary to point out that our method is quite similar to [8] even if our situation is more complicated. As for details of the results obtained in Sections 3 and 4 we refer to the conclusions, which we stated at the end of each of the Sections.

2 The first and the second case

Let us consider the following problem

$$\begin{aligned} -\Delta_p u_1 - \lambda_1 |u_1|^{p-2}u_1 &= |a_1u_1 + a_2u_2|^{\gamma-2}(a_1u_1 + a_2u_2)a_1 - b_1|u_1|^{r_1-2}u_1\,, \\ -\Delta_p u_2 - \lambda_2 |u_2|^{p-2}u_2 &= |a_1u_1 + a_2u_2|^{\gamma-2}(a_1u_1 + a_2u_2)a_2 - b_2|u_2|^{r_2-2}u_2 \end{aligned} \tag{2.1}$$

in Ω, and $u_1 = u_2 = 0$ on $\partial\Omega$.

Let us make the following assumptions:

$$a_1 \cdot a_2 > 0 \quad \text{a.e. in } \Omega \quad \text{or} \quad a_1 \cdot a_2 < 0 \quad \text{a.e. in } \Omega\,, \tag{2.2}$$

$$\lambda_i < \lambda_* \quad \text{for } i = 1, 2, \tag{2.3}$$

and set

$$\forall v_1, v_2 \in W_0^{1,p}(\Omega) \qquad H(v_1, v_2) = \int_\Omega (|\nabla v_1|^p + |\nabla v_2|^p - \lambda_1|v_1|^p - \lambda_2|v_2|^p)dx\,,$$

$$\forall v_1, v_2 \in W_0^{1,p}(\Omega) \qquad A(v_1, v_2) = \int_\Omega |a_1v_1 + a_2v_2|^\gamma dx\,,$$

$$\begin{aligned} S &= \{(v_1, v_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \,:\, H(v_1, v_2) < 1\}\,, \\ S' &= \{(v_1, v_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \,:\, H(v_1, v_2) = 1\}\,, \\ \Sigma &= \{(v_1, v_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \,:\, A(v_1, v_2) > 0\}\,. \end{aligned}$$

Pointing out that in the space $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ the functional $A(v_1, v_2)$ is weakly continuous and $H(v_1, v_2)$ is lower weakly semicontinuous in the same space, we state in advance the Propositions:

Proposition 2.1 *If (u_1, u_2) is a nontrivial weak solution to (2.1), under the assumption (2.2) $u_i \neq 0$ as $i = 1, 2$.*

Proof. By virtue of the definition of weak solution we get:

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla v_1 dx + \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \cdot \nabla v_2 dx - \lambda_1 \int_\Omega |u_1|^{p-2}u_1 v_1 dx +$$
$$- \lambda_2 \int_\Omega |u_2|^{p-2}u_2 v_2 dx =$$
$$= \int_\Omega |a_1 u_1 + a_2 u_2|^{\gamma-2}(a_1 u_1 + a_2 u_2)(a_1 v_1 + a_2 v_2)dx +$$
$$- \int_\Omega (b_1|u_1|^{r_1-2}u_1 v_1 + b_2|u_2|^{r_2-2}u_2 v_2)dx \qquad \forall\, v_1, v_2 \in W_0^{1,p}(\Omega)\,.$$

Arguing by contradiction if $u_2 = 0$ we have:

$$\int_\Omega |\nabla u_1||^{p-2}\nabla u_1 \cdot \nabla v_1 dx - \lambda_1 \int_\Omega |u_1|^{p-2}u_1 v_1 dx =$$
$$= \int_\Omega |a_1 u_1|^{\gamma-2} a_1 u_1 (a_1 v_1 + a_2 v_2)dx +$$
$$- \int_\Omega b_1|u_1|^{r_1-2}u_1 v_1 dx \qquad \forall\, v_1, v_2 \in W_0^{1,p}(\Omega)$$

from which, as $v_1 = 0$, $v_2 = u_1$ it follows that

$$\int_\Omega |a_1|^{\gamma-2} a_1 a_2 |u_1|^\gamma dx = 0$$

and from here, due to (2.2), it follows that $u_1 = 0$ a.e. in Ω. In the same way it is possible to check that $u_1 \neq 0$.

Proposition 2.2 *Under the assumption (2.3) there exist two positive constants c_{λ_1}, c_{λ_2} such that*

$$H(v_1, v_2) \geq c_{\lambda_1}\|v_1\|_{1,p}^p + c_{\lambda_2}\|v_2\|_{1,p}^p \qquad \forall\, v_1, v_2 \in W_0^{1,p}(\Omega)\,.$$

Proof. In the meanwhile as $\lambda_i > 0$, it results in:

$$\int_\Omega [|\nabla v_i|^p - \lambda_i |v_i|^p] dx \geq \int_\Omega \left[|\nabla v_i|^p - \frac{\lambda_i}{\lambda_*}|\nabla v_i|^p\right] dx = \frac{\lambda_* - \lambda_i}{\lambda_*}\|v_i\|_{1,p}^p\,;$$

on the other hand, as $\lambda_i \leq 0$, obviously it has:

$$\int_\Omega [|\nabla v_i|^p - \lambda_i |v_i|^p]\, dx \geq \|v_i\|_{1,p}^p\,.$$

The statement follows from here with

$$c_{\lambda_i} = \begin{cases} 1 & \text{if } \lambda_i \leq 0 \\ \dfrac{\lambda_* - \lambda_i}{\lambda_*} & \text{if } \lambda_i > 0\,. \end{cases}$$

In order to investigate the solvability of the system (2.1) we distinguish the cases

$$1 < \gamma \neq p < p^*\,, \qquad b_i = 0 \quad \text{as} \quad i = 1,2\,, \tag{2.4}$$

$$1 < \gamma < p\,, \qquad \gamma \leq r_i < p^*\,, \qquad b_i \geq 0 \ \text{ a.e. in } \Omega\,, \tag{2.5}$$

$$1 < r_1 = r_2 = r < p < \gamma < p^*\,, \qquad b_i < 0 \ \text{ a.e. in } \Omega\,. \tag{2.6}$$

As for the case (2.4) let us introduce the constrained variational problem

(P) *Find a maximizer* $(u_{10}, u_{20}) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *of the problem*

$$M = \sup\{A(v_1, v_2) \,:\, (v_1, v_2) \in S'\}$$

and let us note that it holds:

Theorem 2.1 *Under the assumptions (2.2), (2.3) the problem (P) admits at least one solution* (u_{10}, u_{20}). *In particular, if* $a_1 \cdot a_2 > 0$ *a.e. in* Ω, $u_{i0} \geq 0$, $i = 1,2$.

Proof. Let $\{(u_{1n}, u_{2n})\}$ a maximizing sequence of (P), so,

$$A(u_{1n}, u_{2n}) \to M\,, \qquad H(u_{1n}, u_{2n}) = 1\,;$$

moreover, because Proposition 2.2 holds, there exists $(u_{10}, u_{20}) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ such that, within a subsequence, we get

$$u_{in} \to u_{i0} \quad \text{weakly in} \quad W_0^{1,p}(\Omega)\,.$$

So, we have:

$$A(u_{10}, u_{20}) = M > 0\,, \qquad H(u_{10}, u_{20}) \leq 1\,.$$

Now, in order to prove that $H(u_{10}, u_{20}) = 1$, at first we note that $H(u_{10}, u_{20}) > 0$; in fact if on the contrary $H(u_{10}, u_{20}) = 0$, by virtue of Proposition 2.2 we have $u_{10} = u_{20} = 0$ and from here $A(u_{10}, u_{20}) = 0$. On the other hand, if $H(u_{10}, u_{20}) < 1$, denoting by $t_0 = [H(u_{10}, u_{20})]^{-\frac{1}{p}}$, it results in

$$A(t_0 u_{10}, t_0 u_{20}) = t_0^{\gamma} A(u_{10}, u_{20}) > M\,,$$

but this contradicts the definition of M. Now let us remark that

$$H(|u_{10}|, |u_{20}|) = H(u_{10}, u_{20}) = 1$$

and, due to the assumption $a_1 \cdot a_2 > 0$ a.e. in Ω, that

$$A(|u_{10}|, |u_{20}|) \geq A(u_{10}, u_{20})\,;$$

so, the statement holds with $|u_{i0}|$.

As a consequence of Theorem 2.1 we deduce the

Theorem 2.2 *In the case (2.4) under the assumptions (2.2), (2.3) the system (2.1) admits at least two solutions* (u_1, u_2) *and* $(-u_1, -u_2)$ *with* $u_i \neq 0$ *as* $i = 1, 2$. *In particular, if* $a_1 \cdot a_2 > 0$ *a.e. in* Ω, *it results in* $u_i \geq 0$, $i = 1, 2$.

Proof. Let (u_{10}, u_{20}) be a solution of (P); then there exists a Lagrange multiplier τ such that for any $v_1, v_2 \in W_0^{1,p}(\Omega)$:

$$\int_\Omega |a_1 u_{10} + a_2 u_{20}|^{\gamma-2}(a_1 u_{10} + a_2 u_{20})(a_1 v_1 + a_2 v_2)dx =$$
$$= \tau \left[\int_\Omega |\nabla u_{10}|^{p-2} \nabla u_{10} \cdot \nabla v_1 dx + \int_\Omega |\nabla u_{20}|^{p-2} \nabla u_{20} \cdot \nabla v_2 dx + \right.$$
$$\left. -\lambda_1 \int_\Omega |u_{10}|^{p-2} u_{10} v_1 dx - \lambda_2 \int_\Omega |u_{20}|^{p-2} u_{20} v_2 dx \right].$$

This equality, rewritten with $v_1 = u_{10}$ and $v_2 = u_{20}$, produces $\tau = M > 0$ and it is easy to verify that $(M^{1/(p-\gamma)} u_{10}, M^{1/(p-\gamma)} u_{20})$ is the solution of the system (2.1) with both the components different from zero, due to Proposition 2.1.

In order to study the cases (2.5), (2.6) it is not possible to use an argument like before; so we follow the "fibering approach". In particular, observing about (2.5) that the Euler functional related to the system (2.1) is

$$E(u_1, u_2) = \frac{1}{p} H(u_1, u_2) - \frac{1}{\gamma} A(u_1, u_2) + \frac{1}{r_1} \int_\Omega b_1 |u_1|^{r_1} dx + \frac{1}{r_2} \int_\Omega b_2 |u_2|^{r_2} dx,$$

we look for its critical points of the following type

$$(u_1, u_2) = (t v_1, t v_2) \quad \text{with} \quad t \in \mathbf{R} \backslash \{0\} \quad \text{and} \quad v_i \in W_0^{1,p}(\Omega)$$

under the constraint $H(v_1, v_2) = 1$. Then the reduced functional is

$$E(t, v_1, v_2) = E(t v_1, t v_2) = \frac{|t|^p}{p} - \frac{|t|^\gamma}{\gamma} A(v_1, v_2) + \frac{|t|^{r_1}}{r_1} \int_\Omega b_1 |v_1|^{r_1} dx + \frac{|t|^{r_2}}{r_2} \int_\Omega b_2 |v_2|^{r_2} dx$$

and the bifurcation equation :

$$|t|^{p-\gamma} + |t|^{r_1-\gamma} \int_\Omega b_1 |v_1|^{r_1} dx + |t|^{r_2-\gamma} \int_\Omega b_2 |v_2|^{r_2} dx = A(v_1, v_2), \qquad (2.7)$$

that, obviously, for any $(v_1, v_2) \in \Sigma$ admits two roots: $t^*(v_1, v_2) > 0$ and $-t^*(v_1, v_2)$ corresponding to the minimum of the function $E(t, v_1, v_2)$ in t; moreover $t^*(v_1, v_2)$ is weakly continuous in Σ and $t^*(v_1, v_2) \in C^1(\Sigma)$ in view of the implicit function theory. Then the functional

$$E^*(v_1, v_2) = E(t^*(v_1, v_2), v_1, v_2) < 0 \qquad \forall (v_1, v_2) \in \Sigma$$

is weakly continuous and C^1-class in Σ. Let us study the constrained variational problem

(P*) *Find a minimizer* $(u_{10}, u_{20}) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *of the problem*

$$m^* = \inf\{E^*(v_1, v_2) \,:\, (v_1, v_2) \in S' \cap \Sigma\}\,.$$

Let us point out that if $\{(v_{1n}, v_{2n})\}$ is a minimizing sequence of (P*) and (u_{10}, u_{20}) is its weak limit in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ (within a subsequence), taking into account the inequality

$$E^*(v_1, v_2) \geq \frac{(t^*(v_1, v_2))^p}{p} - \frac{(t^*(v_1, v_2))^\gamma}{\gamma} A(v_1, v_2)$$

and the equation (2.7), it is easy to obtain that $\{t^*(v_{1n}, v_{2n})\}$ is bounded and

$$(u_{10}, u_{20}) \in \Sigma\,. \tag{2.8}$$

From (2.8) we deduce that $H(u_{10}, u_{20}) > 0$; so, considering

$$\frac{d}{d\vartheta} E(t^*(\vartheta u_{10}, \vartheta u_{20})\vartheta u_{10}, t^*(\vartheta u_{10}, \vartheta u_{20})\vartheta u_{20}) = -\frac{1}{\vartheta}(t^*(\vartheta u_{10}, \vartheta u_{20}))^p < 0 \quad \forall\, \vartheta > 0\,,$$

(u_{10}, u_{20}) solves the problem (P*). Thus, there exists a Lagrange multiplier τ such that

$$\langle dE^*(u_{10}, u_{20}),\, (v_1, v_2)\rangle = \tau \langle dH(u_{10}, u_{20}),\, (v_1, v_2)\rangle \qquad \forall\, v_1, v_2 \in W_0^{1,p}(\Omega)$$

i.e., taking into account (2.7)

$$\begin{aligned}
&-(t^*(u_{10}, u_{20}))^\gamma \int_\Omega |a_1 u_{10} + a_2 u_{20}|^{\gamma-2}(a_1 u_{10} + a_2 u_{20})(a_1 v_1 + a_2 v_2)dx + \\
&+(t^*(u_{10}, u_{20}))^{r_1} \int_\Omega b_1 |u_{10}|^{r_1-2} u_{10} v_1 dx + (t^*(u_{10}, u_{20}))^{r_2} \int_\Omega b_2 |u_{20}|^{r_2-2} u_{20} v_2 dx \\
&= p\tau \left[\int_\Omega |\nabla u_{10}|^{p-2} \nabla u_{10} \cdot \nabla v_1 dx + \int_\Omega |\nabla u_{20}|^{p-2} \nabla u_{20} \cdot \nabla v_2 dx + \right. \\
&\left. -\lambda_1 \int_\Omega |u_{10}|^{p-2} u_{10} v_1 dx - \lambda_2 \int_\Omega |u_{20}|^{p-2} u_{20} v_2 dx \right] \qquad \forall\, v_1, v_2 \in W_0^{1,p}(\Omega)\,.
\end{aligned}$$

From here we deduce that $(t^*(u_{10}, u_{20})u_{10}, t^*(u_{10}, u_{20})u_{20})$ is the solution of (2.1). Then, by using also Proposition 2.1, we can state the following result

Theorem 2.3 *In the case (2.5) under the assumptions (2.2), (2.3) the system (2.1) admits at least two solutions* (u_1, u_2) *and* $(-u_1, -u_2)$ *with* $u_i \neq 0$, $i = 1, 2$.

In the case (2.6), by following the same approach, the reduced functional is

$$E(t, v_1, v_2) = \frac{|t|^p}{p} - \frac{|t|^\gamma}{\gamma} A(v_1, v_2) + \frac{|t|^r}{r} \int_\Omega (b_1 |v_1|^r + b_2 |v_2|^r)\, dx$$

from which we obtain the bifurcation equation

$$|t|^{p-r} - |t|^{\gamma-r} A(v_1, v_2) = -\int_\Omega (b_1 |v_1|^r + b_2 |v_2|^r) dx\,.$$

Denoting by $\tilde{c}_{p\gamma}$ and $\tilde{c}_{pr}$ the Sobolev constants such that

$$\|v\|_\gamma \leq \tilde{c}_{p\gamma}\|v\|_{1,p}\,, \qquad \|v\|_r \leq \tilde{c}_{pr}\|v\|_{1,p} \qquad \forall\, v \in W_0^{1,p}(\Omega)$$

and observing that Proposition 2.2 holds, for any $(v_1, v_2) \in S \cup S'$ we get:

$$A(v_1, v_2) \leq \left[2\tilde{c}_{p\gamma}(c_{\lambda_1}^{-1} + c_{\lambda_2}^{-1})^{1/p} \max\{\|a_1\|_\infty\,,\, \|a_2\|_\infty\}\right]^\gamma\,,$$

$$-\int_\Omega (b_1|v_1|^r + b_2|v_2|^r)dx \leq 2\tilde{c}_{pr}^r(c_{\lambda_1}^{-1} + c_{\lambda_2}^{-1})^{r/p} \max\{\|b_1\|_\infty\,,\, \|b_2\|_\infty\}\,.$$

Introducing the function

$$\varphi(t) = |t|^{p-r} - |t|^{\gamma-r}[2\tilde{c}_{p\gamma}(c_{\lambda_1}^{-1} + c_{\lambda_2}^{-1})^{1/p} \max\{\|a_1\|_\infty\,,\, \|a_2\|_\infty\}]^\gamma$$

we have

$$\varphi(t) \leq |t|^{p-r} - |t|^{\gamma-r} A(v_1, v_2) \qquad \forall\,(v_1, v_2) \in S \cup S'\,;$$

then, if

$$2\tilde{c}_{pr}^r(c_{\lambda_1}^{-1} + c_{\lambda_2}^{-1})^{r/p} \max\{\|b_1\|_\infty\,,\, \|b_2\|_\infty\} < \max \varphi(t)$$

or if

$$\max\{\|b_1\|_\infty\,,\, \|b_2\|_\infty\}[\max\{\|a_1\|_\infty\,,\, \|a_2\|_\infty\}]^{\gamma\frac{p-r}{\gamma-p}} <$$

$$< \frac{\gamma-p}{\gamma-r}\,\frac{1}{2\tilde{c}_{pr}^r(c_{\lambda_1}^{-1} + c_{\lambda_2}^{-1})^{r/p}} \left[\frac{p-r}{\gamma-r}\,\frac{1}{(2\tilde{c}_{p\gamma}(c_{\lambda_1}^{-1} + c_{\lambda_2}^{-1})^{1/p})^\gamma}\right]^{\frac{p-r}{\gamma-p}} \tag{2.9}$$

it results in

- as $(v_1, v_2) \in (S \cup S') \cap \Sigma$ the bifurcation equation has four different roots: $t_1(v_1, v_2)$ and $-t_1(v_1, v_2)$ corresponds to the local minimum of $E(t, v_1, v_2)$ in t, $t_2(v_1, v_2) > t_1(v_1, v_2)$ and $-t_2(v_1, v_2)$ corresponds to the maximum of $E(t, v_1, v_2)$ in t;
- as $(v_1, v_2) \in (S \cup S'\backslash\{(0,0)\})\backslash\Sigma$ the bifurcation equation has two roots: $t_1(v_1, v_2)$ and the opposite one.

Moreover we can note about these roots that

$$0 < t_1(v_1, v_2) \leq t' \;\; \forall\,(v_1, v_2) \in S \cup S'\backslash\{(0,0)\}\,, \quad t_2(v_1, v_2) \geq t'' \;\; \forall\,(v_1, v_2) \in (S \cup S') \cap \Sigma\,,$$

where $t' < t''$ are the positive roots of the equation

$$\varphi(t) = 2\tilde{c}_{pr}^r(c_{\lambda_1}^{-1} + c_{\lambda_2}^{-1})^{r/p} \max\{\|b_1\|_\infty\,,\, \|b_2\|_\infty\}\,,$$

and let us point out that $t_1(v_1, v_2)$ [resp. $t_2(v_1, v_2)$] is weakly continuous in $S \cup S'\backslash\{(0,0)\}$ [resp. $(S \cup S') \cap \Sigma$], $t_1(v_1, v_2) \in C^1(S\backslash\{(0,0)\})$ [resp. $t_2(v_1, v_2) \in C^1(S \cap \Sigma)$], there exists a C^1-extension of $t_1(v_1, v_2)$ [resp. $t_2(v_1, v_2)$] in a neighborhood of

every point of S' [resp. $S' \cap \Sigma$].
Thus, setting

$$E_i(v_1, v_2) = E(t_i(v_1, v_2), v_1, v_2)\,,$$

with $E_1(v_1, v_2)$ [resp. $E_2(v_1, v_2)$] weakly continuous and C^1-class in $S \cup S' \backslash \{0,0)\}$ [resp. $(S \cup S') \cap \Sigma$], let us consider the constrained variational problems

(P_1) *Find a minimizer* $(u_{10}^1, u_{20}^1) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *of the problem*

$$m_1 = \inf\{E_1(v_1, v_2) \,:\, (v_1, v_2) \in S'\}\,,$$

(P_2) *Find a minimizer* $(u_{10}^2, u_{20}^2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *of the problem*

$$m_2 = \inf\{E_2(v_1, v_2) \,:\, (v_1, v_2) \in S' \cap \Sigma\}\,.$$

It is easy to prove the following statements:

α_1) If $\{(v_{1n}, v_{2n})\}$ is a minimizing sequence of (P_1) and (u_{10}^1, u_{20}^1) is its weak limit in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ (within a subsequence), then $(u_{10}^1, u_{20}^1) \neq (0,0)$.

α_2) If $\{(v_{1n}, v_{2n})\}$ is a minimizing sequence of (P_2) and (u_{10}^2, u_{20}^2) is its weak limit in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ (within a subsequence), then $\{t_2(v_{1n}, v_{2n})\}$ is bounded and $(u_{10}^2, u_{20}^2) \in \Sigma$.

By α_1) and α_2) we deduce as in the problem (P^*) that (u_{10}^i, u_{20}^i) is the solution of (P_i), $i = 1, 2$. Finally, by using suitably the Lagrange multipliers we check ([10]; Th.3.3) that $(t_i(u_{10}^i, u_{20}^i)u_{10}^i, t_i(u_{10}^i, u_{20}^i)u_{20}^i)$ solves (2.1). Obviously the two solutions are different; so, holding the Proposition 2.1, we state the following

Theorem 2.4 *In the case (2.6) under the assumptions (2.2), (2.3),(2.9) the system (2.1) admits at least four different solutions* (u_1, u_2), $(-u_1, -u_2)$, (w_1, w_2), $(-w_1, -w_2)$ *with all components different from zero.*

Let us continue studying the system

$$\begin{aligned}
&-\Delta_p u_1 - \lambda_1 |u_1|^{p-2}u_1 = \\
&= |a_1u_1 + a_2u_2|^{\gamma-2}(a_1u_1 + a_2u_2)a_1 + c_1|u_1|^{s-2}u_1 - b_1|u_1|^{r_1-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega \\
&-\Delta_p u_2 - \lambda_2 |u_2|^{p-2}u_2 = \\
&= |a_1u_1 + a_2u_2|^{\gamma-2}(a_1u_1 + a_2u_2)a_2 + c_2|u_2|^{s-2}u_2 - b_2|u_2|^{r_2-1} \\
&u_1 = u_2 = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega
\end{aligned} \tag{2.10}$$

under assumptions (2.2), (2.3) and, moreover,

$$c_i > 0 \quad \text{a.e. in } \Omega\,, \qquad 1 < p < r_i < s < p^*\,, \qquad p < \gamma \leq s\,. \tag{2.11}$$

Obviously, Proposition 2.1 holds also for the system (2.10). Noting that the Euler functional related to (2.10) is

$$\begin{aligned}
E(u_1, u_2) \;=\; & \frac{1}{p}H(u_1, u_2) - \frac{1}{s}\int_\Omega (c_1|u_1|^s + c_2|u_2|^s)dx - \frac{1}{\gamma}A(u_1, u_2) + \\
& + \frac{1}{r_1}\int_\Omega b_1|u_1|^{r_1-1}u_1 dx + \frac{1}{r_2}\int_\Omega b_2|u_2|^{r_2-1}u_2 dx
\end{aligned}$$

and following the fibering method with the constraint $H(v_1, v_2) = 1$ like before, we obtain the reduced functional:

$$E(t, v_1, v_2) = \frac{|t|^p}{p} - \frac{|t|^s}{s}\int_\Omega (c_1|v_1|^s + c_2|v_2|^s)dx - \frac{|t|^\gamma}{\gamma}A(v_1, v_2) + \\ + \frac{|t|^{r_1-1}t}{r_1}\int_\Omega b_1|v_1|^{r_1-1}v_1 dx + \frac{|t|^{r_2-1}t}{r_2}\int_\Omega b_2|v_2|^{r_2-1}v_2 dx\,.$$

By using a geometrical analysis of bifurcation equation:

$$|t|^{s-p}\int_\Omega (c_1|v_1|^s + c_2|v_2|^s)dx + |t|^{\gamma-p}A(v_1, v_2) - t|t|^{r_1-p-1}\int_\Omega b_1|v_1|^{r_1-1}v_1 dx + \\ - t|t|^{r_2-p-1}\int_\Omega b_2|v_2|^{r_2-1}v_2 dx = 1$$

we give that for every $(v_1, v_2) \in [W_0^{1,p}(\Omega)]^2\backslash\{(0,0)\}$ it admits two roots: $t_1(v_1, v_2) > 0$, $t_2(v_1, v_2) < 0$ and denoting by $E_i(v_1, v_2) = E(t_i(v_1, v_2), v_1, v_2)$, we obtain

$$E_1(v_1, v_2) = \max_{t>0} E(t, v_1, v_2)\,, \quad E_2(v_1, v_2) = \max_{t<0} E(t, v_1, v_2)\,. \tag{2.12}$$

Moreover, about these roots we add that $t_i(v_1, v_2)$ are weakly continuous and C^1-class in $[W_0^{1,p}(\Omega)]^2\backslash\{(0,0)\}$; so these properties are in possession of $E_i(v_1, v_2)$ too. Now we formulate the constrained variational problems:

(Q_i) *Find a minimizer* $(u_{10}^i, u_{20}^i) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *of the problem*

$$m_i = \inf\{E_i(v_1, v_2) : (v_1, v_2) \in S'\} \qquad i = 1, 2\,.$$

In order to study the problem (Q_i) it is easy to show that there exist two real positive constants, k_1 and k_2, independent on v_1, v_2 such that

$$E(t, v_1, v_2) \geq \frac{|t|^p}{p} - k_1\frac{|t|^s}{s}\int_\Omega (|v_1|^s + |v_2|^s)dx - k_2 \equiv \psi(t, v_1, v_2);$$

so, from (2.12) we have

$$E_i(v_1, v_2) \geq \max_{t\in R}\psi(t, v_1, v_2)$$

i.e.

$$E_i(v_1, v_2) \geq \left[k_1\int_\Omega (|v_1|^s + |v_2|^s)dx\right]^{-\frac{p}{s-p}}\left(\frac{1}{p} - \frac{1}{s}\right) - k_2\,.$$

Therefore, if $\{(v_{1n}, v_{2n})\}$ is a minimizing sequence of (Q_i) and (u_{10}^i, u_{20}^i) is its weak limit in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ (within a subsequence), we obtain $(u_{10}^i, u_{20}^i) \neq (0,0)$; consequently

$$H(u_{10}^i, u_{20}^i) > 0\ , \qquad E_i(u_{10}^i, u_{20}^i) = m_i\,.$$

Now following the procedure used in the above situations we obtain that (u_{10}^i, u_{20}^i) solves the problem (Q_i) and for $i = 1, 2$, $(t_i(u_{10}^i, u_{20}^i)u_{10}^i,\ t_i(u_{10}^i, u_{20}^i)u_{20}^i)$ are different solutions of (2.10). In conclusion we state the following result

Theorem 2.5 *Under the assumptions (2.2), (2.3), (2.11) the system (2.10) admits at least two different solutions with all components different from zero.*

3 The third case

This section is focused on the study of the existence of nontrivial weak solutions of the system

$$\begin{aligned} -\Delta_p u_1 - \lambda_1 |u_1|^{p-2}u_1 &= |a_1u_1^2 + a_2u_2^2|^{\frac{\gamma}{2}-1}a_1u_1 - b_1|u_1|^{\gamma-2}u_1 \\ -\Delta_p u_2 - \lambda_2 |u_2|^{p-2}u_2 &= |a_1u_1^2 + a_2u_2^2|^{\frac{\gamma}{2}-1}a_2u_2 - b_2|u_2|^{\gamma-2}u_2 \end{aligned} \quad \text{in } \Omega \tag{3.1}$$

$$u_1 = u_2 = 0 \qquad \text{on } \partial\Omega$$

under assumption

$$1 < p \neq \gamma < p^*, \qquad \gamma > 2. \tag{3.2}$$

We note that from some results obtained by Drabek–Pohozaev [8] it is possible to deduce assumptions that the system (3.1) admits nontrivial solutions like $(u_1, 0)$, $(0, u_2)$ with $u_i > 0$. So, it is interesting to find solutions (u_1, u_2) with both components different from zero. Let us set

$$\forall\, v_1, v_2 \in W_o^{1,p}(\Omega) \qquad A(v_1, v_2) = \int_\Omega |a_1v_1^2 + a_2v_2^2|^{\frac{\gamma}{2}-1}(a_1v_1^2 + a_2v_2^2)dx + - \int_\Omega (b_1|v_1|^\gamma + b_2|v_2|^\gamma)dx\,,$$

$$\forall\, \lambda \in R \quad \text{and} \quad \forall\, v \in W_0^{1,p}(\Omega) \qquad H_\lambda(v) = \int_\Omega (|\nabla v|^p - \lambda|v|^p)dx\,,$$

$$S_{\lambda_1\lambda_2} = \{(v_1, v_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \,:\, H_{\lambda_1}(v_1) + H_{\lambda_2}(v_2) = 1\}\,,$$

$$\Sigma = \{(v_1, v_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \,:\, A(v_1, v_2) > 0\}\,.$$

Obviously, the functional $A(v_1, v_2)$ is weakly continuous in the space $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ and the functional $H_{\lambda_1}(v_1) + H_{\lambda_2}(v_2)$ is lower weakly semicontinuous in the same space.

For obtaining some results of existence, the following assumption, i.e.,

$$\begin{aligned} &\text{there exist } c_0 > 0 \text{ and a measurable set } \Omega^+ \subset \Omega \text{ with } |\Omega^+| > 0 \text{ such that} \\ &\qquad a_1 + c_0^2 a_2 > 0 \quad \text{and} \quad (a_1 + c_0^2 a_2)^{\gamma/2} > b_1 + c_0^\gamma b_2 \quad \text{in} \quad \Omega^+ \end{aligned} \tag{3.3}$$

plays a fundamental role. By virtue of this condition we can prove

Proposition 3.1 *Under the assumptions (3.2), (3.3) there exists $v_i \in C_0^\infty(\Omega)\backslash\{0\}$ with $v_i \geq 0$ such that $(v_1, v_2) \in S_{\lambda_1\lambda_2} \cap \Sigma$.*

Proof. Let us denote by K a compact set included in Ω^+ with $|K| > 0$, by v_0 the characteristic function of K and set

$$J(x) = \begin{cases} \sigma e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}, \qquad J_\varepsilon(x) = \varepsilon^{-N} J\left(\frac{x}{\varepsilon}\right) \qquad (\varepsilon > 0)$$

where σ is a positive constant such that

$$\int_{R^N} J(x)dx = 1\,.$$

Let $\{\varepsilon_n\}$ be an infinitesimal numerical sequence, with $0 < \varepsilon_n < \operatorname{dist}(K, \partial\Omega)$, and $v_n = J_{\varepsilon_n} * v_0$. Let us note that $v_n \in C_0^\infty(\Omega)$ and

$$v_n \to v_0 \qquad \text{strongly in } L^s(\Omega) \qquad \forall\, s \in [1, +\infty[\,. \tag{3.4}$$

By virtue of (3.4)

$$\lim_n A(v_n, c_0 v_n) = \int_K [a_1 + c_0^2 a_2]^{\frac{\gamma}{2}} dx - \int_K [b_1 + c_0^\gamma b_2]dx > 0\,,$$

so, a positive integer ν exists such that

$$A(v_n, c_0 v_n) > 0 \qquad \forall\, n > \nu\,. \tag{3.5}$$

Again (3.4) and the relation

$$v_0 \notin W_0^{1,p}(\Omega)$$

assure that

$$\sup_n \int_\Omega |\nabla v_n|^p dx = +\infty\,,$$

so, within a subsequence $\lim_n \int_\Omega |\nabla v_n|^p dx = +\infty$.
Consequently, there is $\nu' \geq \nu$ such that

$$(1 + c_0^p) \int_\Omega |\nabla v_n|^p dx > (\lambda_1 + \lambda_2 c_0^p) \int_\Omega |v_n|^p dx \qquad \forall\, n > \nu'\,.$$

By choosing $n > \nu'$ and setting

$$t_n = \left[(1 + c_0^p) \int_\Omega |\nabla v_n|^p dx - (\lambda_1 + \lambda_2 c_0^p) \int_\Omega |v_n|^p dx\right]^{\frac{1}{p}}\,, \qquad w_n = \frac{v_n}{t_n}$$

we get:

$$H_{\lambda_1}(w_n) + H_{\lambda_2}(c_0 w_n) = 1$$

and due to (3.5)

$$A(w_n, c_0 w_n) > 0\,.$$

The proved Proposition makes it possible to consider the constrained variational problem

$(\mathrm{P}_{\lambda_1\lambda_2})$ *Find a maximizer* $(u_{10}, u_{20}) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *of the problem*

$$M_{\lambda_1\lambda_2} = \sup\{A(v_1, v_2) \,:\, (v_1, v_2) \in S_{\lambda_1\lambda_2} \cap \Sigma\}\,.$$

Since $M_{\lambda_1\lambda_2}$ is a positive number, by using the Lagrange multipliers it is easy to verify the following

Proposition 3.2 *Under the assumptions (3.2), (3.3) if* (u_{10}, u_{20}) *is a solution of the problem* $(P_{\lambda_1\lambda_2})$, *then* $(M_{\lambda_1\lambda_2}^{\frac{1}{p-\gamma}} u_{10}, M_{\lambda_1\lambda_2}^{\frac{1}{p-\gamma}} u_{20})$ *is a weak solution of the system (3.1).*

Again, taking into account that of $M_{\lambda_1\lambda_2}$ is positive, it is obvious that the subsequent result holds.

Proposition 3.3 *Under the assumptions (3.2), (3.3) if*

$$|a_1|^{\frac{\gamma}{2}-1} a_1 \leq b_1 \quad a.e.\ on\ \ \Omega\,, \tag{3.6}$$

$$|a_2|^{\frac{\gamma}{2}-1} a_2 \leq b_2 \quad a.e.\ on\ \ \Omega\,, \tag{3.7}$$

the components of every solution of $(P_{\lambda_1\lambda_2})$ *are both different from zero.*

Remark 3.1 Let us point out that if $b_i \neq 0$ for $i = 1, 2$ it is easy to find examples for which the assumptions (3.3), (3.6) and (3.7) hold. If at least one of these functions is equal to zero a.e. in Ω the conditions (3.6) and (3.7) are not compatible with (3.3).

In order to study the problem $(\mathrm{P}_{\lambda_1\lambda_2})$, we analyze separately the following cases

$$\lambda_1 < \lambda_*\,, \quad \lambda_2 < \lambda_*\,, \tag{3.8}$$

$$\lambda_1 \geq \lambda_*\,, \quad \lambda_2 < \lambda_*\,, \tag{3.9}$$

$$\lambda_1 < \lambda_*\,, \quad \lambda_2 \geq \lambda_*\,, \tag{3.10}$$

$$\lambda_1 \geq \lambda_*\,, \quad \lambda_2 \geq \lambda_*\,. \tag{3.11}$$

At first let us examine the case (3.8).

Theorem 3.1 *In the case (3.8) under the assumptions (3.2), (3.3) the problem* $(P_{\lambda_1\lambda_2})$ *admits at least one solution* (u_{10}, u_{20}) *with* $u_{i0} \geq 0$, $i = 1, 2$. *In the above mentioned hypotheses (3.6), (3.7) it results in* $u_{i0} \neq 0$, $i = 1, 2$.

Proof. Let us note that $S_{\lambda_1\lambda_2}$ is bounded in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ (Prop. 2.2). If $\{(u_{1n}, u_{2n})\}$ is a maximizing sequence of the problem $(\mathrm{P}_{\lambda_1\lambda_2})$, then there exist $v_1, v_2 \in W_0^{1,p}(\Omega)$ such that within a subsequence:

$$u_{in} \to v_i \quad \text{weakly in} \quad W_0^{1,p}(\Omega)\,,$$

so

$$\begin{aligned} &A(v_1, v_2) = M_{\lambda_1\lambda_2}\,, \\ &H_{\lambda_1}(v_1) + H_{\lambda_2}(v_2) \leq 1\,; \end{aligned} \tag{3.12}$$

moreover

$$H_{\lambda_1}(v_1) + H_{\lambda_2}(v_2) > 0$$

since $\lambda_i < \lambda_*$ and due to (3.12) at least one of the functions v_1 and v_2 is not equal to zero. Let us add that $H_{\lambda_1}(v_1)+H_{\lambda_2}(v_2) = 1$; in fact on the contrary by assuming that $t_0 = (H_{\lambda_1}(v_1) + H_{\lambda_2}(v_2))^{-\frac{1}{p}} > 1$ we obtain the contradiction

$$(t_0 v_1, t_0 v_2) \in S_{\lambda_1\lambda_2}\,, \qquad A(t_0 v_1, t_0 v_2) = t_0^{\gamma} M_{\lambda_1\lambda_2} > M_{\lambda_1\lambda_2}\,.$$

Then, noting that

$$H_{\lambda_1}(|v_1|) + H_{\lambda_2}(|v_2|) = 1\,, \qquad A(|v_1|, |v_2|) = M_{\lambda_1\lambda_2}\,,$$

taking into account Proposition 3.3, the statement of the Theorem holds for $u_{i0} = |v_i|$.

Let us note that the condition (3.3) holds, in particular, if

$$\begin{aligned} &\text{there exists a measurable set } \Omega^+ \subset \Omega \text{ with } |\Omega^+| > 0 \text{ such that} \\ &a_i > 0 \quad (i = 1,2) \quad \text{and } \min\{a_1^{\gamma/2}, a_2^{\gamma/2}\} > \max\{b_1, b_2\} \text{ in } \Omega^+\,. \end{aligned} \tag{3.13}$$

We want to remark now that (3.13) implies that (3.6) and (3.7) are not valid; in this case owing to $\lambda_i < \lambda_*$ as $i = 1,2$ [8] assure the solvability of the constrained variational problems

$$M_{\lambda_i} = \max\left\{\int_\Omega (|a_i|^{\frac{\gamma}{2}-1}a_i - b_i)|v|^\gamma dx \,:\, H_{\lambda_i}(v) = 1\right\} > 0 \qquad i = 1,2\,. \tag{3.14}$$

Therefore, it could be $M_{\lambda_1\lambda_2} = M_{\lambda_1}$ or $M_{\lambda_1\lambda_2} = M_{\lambda_2}$ i.e. it could happen that the solution of the problem $(\mathrm{P}_{\lambda_1\lambda_2})$, which Theorem 3.1 refers to, gives one of the components equal to zero. A sufficient condition that excludes this possibility is stated by

Theorem 3.2 *In the case (3.8) under the assumptions (3.2), (3.13) the components of every solution of the problem $(P_{\lambda_1\lambda_2})$ are both different from zero when $\gamma > p > 2$.*

Proof. Let (u_{10}, u_{20}) be a solution of $(P_{\lambda_1\lambda_2})$. As it is possible to assume $u_{i0} \geq 0$, let us verify that $u_{20} \neq 0$. Arguing by contradiction, let u_{20} be equal to zero, i.e.

$$M_{\lambda_1\lambda_2} = A(u_{10}, 0) \qquad \text{with } (u_{10}, 0) \in S_{\lambda_1\lambda_2}\,. \tag{3.15}$$

The relation (3.15) implies that

$$\int_\Omega \left(|a_1|^{\frac{\gamma}{2}-1}a_1 - b_1\right) u_{10}^{\gamma} dx = \max\left\{\int_\Omega \left(|a_1|^{\frac{\gamma}{2}-1}a_1 - b_1\right)|v|^\gamma dx \,:\, H_{\lambda_1}(v) = 1\right\}\,,$$

so, $\hat{u} = M_{\lambda_1\lambda_2}^{\frac{1}{p-\gamma}}$ is a non-negative weak solution of the problem

$$-\Delta_p \hat{u} - \lambda_1 |\hat{u}|^{p-2}\hat{u} = \left(|a_1|^{\frac{\gamma}{2}-1}a_1 - b_1\right)|\hat{u}|^{\gamma-2}\hat{u} \quad \text{in } \Omega$$

$$\hat{u} = 0 \qquad \text{on } \partial\Omega .$$

From the above mentioned it follows [8] that $\hat{u} > 0$ a.e. in Ω, so,

$$u_{10} > 0 \quad \text{a.e. in } \Omega . \tag{3.16}$$

(3.16) and the assumption (3.13) imply that there exists a compact set K such that

$$K \subseteq \Omega^+ \cap \{x \in \Omega : u_{10}(x) > 0\} , \quad |K| > 0 .$$

By using the characteristic function of K and arguing like in the proof of Proposition 3.1, it is possible to find a non-negative function $u \in C_0^\infty(\Omega)$ such that $H_{\lambda_2}(u) = 1$ and

$$\int_\Omega |a_1|^{\frac{\gamma}{2}-1} a_2 u_{10}^{\gamma-2} u^2 dx > 0 . \tag{3.17}$$

By introducing the function

$$F(s) = A(s^{1/p} u_{10}, (1-s)^{1/p} u) \qquad \forall\, s \in [0,1] ,$$

let us note that due to (3.15) it follows that

$$F(1) = M_{\lambda_1\lambda_2} ;$$

moreover,

$$\lim_{s \to 1^-} F'(s) = -\infty \tag{3.18}$$

by (3.17) and the assumption $p > 2$. The existence of $s_0 \in]0,1[$ follows from relation (3.18), so that $F(s_0) > F(1)$. Then

$$(s_0^{1/p} u_{10}, (1-s_0)^{1/p} u) \in S_{\lambda_1\lambda_2} , \qquad A(s_0^{1/p} u_{10}, (1-s_0)^{1/p} u) = F(s_0) > M_{\lambda_1\lambda_2}$$

and this is impossible. Arguing in the same way it is possible to obtain that $u_{10} \neq 0$.

As for the case (3.9) we shall obtain the existence at least of one solution of the problem $(\mathrm{P}_{\lambda_1\lambda_2})$, by using some preliminary results, the proof of which requires the following assumption

$$\int_\Omega \left(|a_1|^{\frac{\gamma}{2}-1} a_1 - b_1 \right) u_*^\gamma dx < 0 , \tag{3.19}$$

that is compatible with the hypothesis (3.3).

Proposition 3.4 *For $\lambda_2 < \lambda_*$ under the assumptions (3.2), (3.3), (3.19) there exists $\delta_1' > 0$ such that for every $\lambda_1 \in [\lambda_*, \lambda_* + \delta_1'[$ $S_{\lambda_1\lambda_2} \cap \Sigma$ is bounded in the space $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.*

Proof. Let us argue by contradiction and denote by $\{\delta_{1k}\}$ an infinitesimal sequence of positive numbers. For every positive integer k there exist $\lambda_{1k} \in [\lambda_*, \lambda_* + \delta_{1k}[$ and a sequence $\{(u_{1n}^k, u_{2n}^k)\} \subseteq S_{\lambda_{1k}\lambda_2} \cap \Sigma$ such that

$$\lim_n \int_\Omega (|\nabla u_{1n}^k|^p + |\nabla u_{2n}^k|^p) dx = +\infty . \tag{3.20}$$

The relation (3.20) makes it possible to obtain a strictly increasing sequence of positive integers $\{n_k\}$, so that

$$\lim_k \int_\Omega (|\nabla u^k_{1n_k}|^p + |\nabla u^k_{2n_k}|^p)dx = +\infty\,.$$

Noting that

$$A(u^k_{1n_k}\,,\, u^k_{2n_k}) > 0\,, \tag{3.21}$$

$$H_{\lambda_{1k}}(u^k_{1n_k}) + H_{\lambda_2}(u^k_{2n_k}) = 1\,, \tag{3.22}$$

and setting

$$t_k = \left(\int_\Omega (|\nabla u^k_{1n_k}|^p + |\nabla u^k_{2n_k}|^p)dx\right)^{1/p}\,, \qquad v_{ik} = \frac{u^k_{in_k}}{t_k}\,,$$

we get

$$\int_\Omega (|\nabla v_{1k}|^p + |\nabla v_{2k}|^p)dx = 1\,, \tag{3.23}$$

and due to (3.22)

$$\lambda_{1k}\int_\Omega |v_{1k}|^p dx + \lambda_2 \int_\Omega |v_{2k}|^p dx = 1 - \frac{1}{t_k^p}\,. \tag{3.24}$$

The equality (3.23) implies that there exist $v_1, v_2 \in W^{1,p}_0(\Omega)$ such that within a subsequence

$$v_{ik} \to v_i \qquad \text{weakly in } \; W^{1,p}_0(\Omega)\,,$$

then from the relations (3.23), (3.24), by passing to the limit as $k \to +\infty$ we obtain

$$\begin{aligned} &\int_\Omega (|\nabla v_1|^p + |\nabla v_2|^p)dx \le 1\,,\\ &\lambda_* \int_\Omega |v_1|^p dx + \lambda_2 \int_\Omega |v_2|^p dx = 1\,. \end{aligned} \tag{3.25}$$

Since $\lambda_2 < \lambda_*$, we must have $v_2 = 0$. In fact, on the contrary with (3.25) it follows

$$\int_\Omega (|\nabla v_1|^p + |\nabla v_2|^p)dx < \lambda_* \int_\Omega (|v_1|^p + |v_2|^p)dx\,,$$

and this inequality contradicts the variational characterization of λ_*. Then the same characterization implies that

$$\lambda_* \int_\Omega |v_1|^p dx = \int_\Omega |\nabla v_1|^p dx \qquad \text{with} \quad v_1 \neq 0\,,$$

from which we deduce the existence of a real constant $c_1 \neq 0$ such that $v_1 = c_1 u_*$. Thus we have

$$\lim_k A(v_{1k}, v_{2k}) = A(v_1, 0) = |c_1|^\gamma \int_\Omega \left(|a_1|^{\frac{\gamma}{2}-1}a_1 - b_1\right) u_*^\gamma dx < 0$$

due to (3.19), and from this it follows that

$$\lim_k A(u^k_{1n_k}, u^k_{2n_k}) = \lim_k t_k^\gamma A(v_{1k}, v_{2k}) = -\infty\,,$$

but this contradicts (3.21).

The previous statement assures only that for every $\lambda_1 \in [\lambda_*, \lambda_* + \delta_1'[$ the set

$$\{(u_{10}, u_{20}) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \,:\, A(u_{10}, u_{20}) = M_{\lambda_1\lambda_2}\}$$

is not empty and, therefore, Proposition 3.4 is completed by the

Proposition 3.5 *For $\lambda_2 < \lambda_*$ under the assumptions (3.2), (3.3), (3.19) there exists a positive number $\delta_1 \le \delta_1'$ such that for every $\lambda_1 \in [\lambda_*, \lambda_* + \delta_1[$*

$$A(u_{10}, u_{20}) = M_{\lambda_1\lambda_2} \Rightarrow H_{\lambda_1}(u_{10}) + H_{\lambda_2}(u_{20}) > 0\,.$$

Proof. Arguing by contradiction, letting $\{\delta_{1k}\}$ be an infinitesimal sequence of positive numbers with $\delta_{1k} \le \delta_1'$ there exists $\lambda_{1k} \in [\lambda_*, \lambda_* + \delta_{1k}[$ such that

$$\int_\Omega (|\nabla u^k_{10}|^p + |\nabla u^k_{20}|^p)dx \le \lambda_{1k} \int_\Omega |u^k_{10}|^p dx + \lambda_2 \int_\Omega |u^k_{20}|^p dx$$

at least for one pair $(u^k_{10}, u^k_{20}) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ such that

$$A(u^k_{10}, u^k_{20}) = M_{\lambda_{1k}\lambda_2}\,. \tag{3.26}$$

Let us note that due to (3.26)

$$\int_\Omega (|\nabla u^k_{10}|^p + |\nabla u^k_{20}|^p)dx > 0\,,$$

by setting

$$t_k = \left[\int_\Omega (|\nabla u^k_{10}|^p + |\nabla u^k_{20}|^p)dx\right]^{1/p}, \qquad v_{ik} = \frac{u^k_{i0}}{t_k}\,,$$

we have

$$\begin{aligned} &A(v_{1k}, v_{2k}) > 0\,,\\ &\int_\Omega (|\nabla v_{1k}|^p + |\nabla v_{2k}|^p)dx = 1\,,\\ &\lambda_{1k} \int_\Omega |v_{1k}|^p dx + \lambda_2 \int_\Omega |v_{2k}|^p dx \ge 1\,; \end{aligned} \tag{3.27}$$

moreover, such $v_1, v_2 \in W_0^{1,p}(\Omega)$ exist that within a subsequence

$$v_{ik} \to v_i \qquad \text{weakly in} \quad W_0^{1,p}(\Omega)\,.$$

So, passing to the limit as $k \to +\infty$, we obtain from the relations (3.27)

$$A(v_1, v_2) \geq 0\,, \tag{3.28}$$

$$\int_\Omega (|\nabla v_1|^p + |\nabla v_2|^2) dx \leq 1\,, \tag{3.29}$$

$$\lambda_* \int_\Omega |v_1|^p dx + \lambda_2 \int_\Omega |v_2|^p dx \geq 1\,. \tag{3.30}$$

The inequalities (3.29), (3.30) imply that $v_2 = 0$ and then

$$\lambda_* \int_\Omega |v_1|^p dx = \int_\Omega |\nabla v_1|^p dx \qquad \text{for} \quad v_1 \neq 0\,;$$

thus, a real constant $c_1 \neq 0$ exists such that $v_1 = c_1 u_*$. Consequently, (3.28) can be rewritten

$$|c_1|^\gamma \int_\Omega \left(|a_1|^{\frac{\gamma}{2}-1} a_1 - b_1 \right) u_*^\gamma dx \geq 0$$

and this inequality contradicts (3.19).

Proposition 3.5 produces

Theorem 3.3 *In the case (3.9), under the assumptions (3.2), (3.3), (3.19) for every $\lambda_2 < \lambda_*$ there exists $\delta_1 > 0$ such that for each $\lambda_1 \in [\lambda_*, \lambda_* + \delta_1[$ the problem $(P_{\lambda_1\lambda_2})$ admits at least one solution (u_{10}, u_{20}) with $u_{i0} \geq 0$ as $i = 1, 2$. With the above mentioned hypotheses, (3.6), (3.7) we obtain $u_{i0} \neq 0$, $i = 1, 2$.*

As it has already been pointed out in the Remark 3.1, if conditions (3.3), (3.6) and (3.7) simultaneously hold as in the Theorem 3.3, it is not possible that at least one of the functions b_i is equal to zero a.e. in Ω. Unlike the above case (3.8) when at least one of parameters λ_i is greater than λ_* we can formulate another sufficient condition in order to obtain that both components of each solution of $(\mathrm{P}_{\lambda_1\lambda_2})$ are different from zero, even if it results in $b_i = 0$ a.e. in Ω.

Theorem 3.4 *In the case (3.9) under the assumptions (3.2), (3.3), (3.6), (3.19) for every $\lambda_2 < \lambda_*$ there exists $\delta_1 > 0$ such that for each $\lambda_1 \in]\lambda_*, \lambda_* + \delta_1[$ every solution of the problem $(P_{\lambda_1\lambda_2})$ has components both different from zero when $p < 2$.*

Proof. Let (u_{10}, u_{20}) be a solution of the problem $(\mathrm{P}_{\lambda_1\lambda_2})$. The assumption (3.6) excludes $u_{20} = 0$. Now we have to check that also $u_{10} \neq 0$. Arguing by contradiction, let $u_{10} = 0$; so

$$\int_\Omega \left(|a_2|^{\frac{\gamma}{2}-1} a_2 - b_2 \right) |u_{20}|^\gamma dx = M_{\lambda_1\lambda_2}\,. \tag{3.31}$$

Owing to the relation

$$\lambda_1 > \lambda_* \Rightarrow H_{\lambda_1}(u_*) < H_{\lambda_*}(u_*) = 0\,,$$

by setting

$$\tilde{u} = u_* |H_{\lambda_1}(u_*)|^{-\frac{1}{p}}$$

we get

$$H_{\lambda_1}(\tilde{u}) = -1\,.$$

Let us consider the function

$$G(s) = A((s-1)^{1/p}\tilde{u},\, s^{1/p}u_{20}) \qquad \forall\, s \geq 1\,;$$

by (3.31) it results in

$$G(1) = M_{\lambda_1\lambda_2}\,,$$

and, moreover,

$$G'(1) = \frac{\gamma}{p} M_{\lambda_1\lambda_2} > 0\,.$$

So, there exists $s_0 > 1$ such that $G(s_0) > G(1)$ and then

$$H_{\lambda_1}((s_0-1)^{1/p}\tilde{u}) + H_{\lambda_2}(s_0^{1/p}u_{20}) = 1\,, \qquad A((s_0-1)^{1/p}\tilde{u},\, s_0^{1/p}u_{20}) > M_{\lambda_1\lambda_2}$$

but this is impossible.

Remark 3.2 We think it is correct to point out that it is possible to find examples of functions a_i, b_i that verify conditions (3.3) and (3.6) [resp. (3.7)], but do not satisfy (3.7) [resp. (3.6)].

If (3.13) holds, the inequality (3.19) and the fact that λ_1 belongs to a suitable right neighborhood of λ_* assure [8] that for $i = 1$ the problem (3.14) is solvable; on the other hand the problem (3.14) is solvable also as $i = 2$ because $\lambda_2 < \lambda_*$. Then some considerations like the ones related to the Theorem 3.2 lead us to obtaining a condition in order to get that both components of the solution are different from zero; its proof is absolutely analogous to the proof of the Theorem 3.2 and therefore we can state:

Theorem 3.5 *In the case (3.9) under the assumptions (3.2), (3.13), (3.19) for every $\lambda_2 < \lambda_*$ there exists $\delta_1 > 0$ such that for each $\lambda_1 \in [\lambda_*, \lambda_* + \delta_1[$ every solution of the problem $(P_{\lambda_1\lambda_2})$ has components both different from zero when $\gamma > p > 2$.*

In the case (3.10) replacing the assumption (3.19) by

$$\int_\Omega \left(|a_2|^{\frac{\gamma}{2}-1}a_2 - b_2\right) u_*^\gamma dx < 0 \tag{3.32}$$

and the condition (3.6) of the Theorem 3.4 by (3.7) we obtain some statements that are absolutely similar to the results about the case (3.9).

On passing to the case (3.11) let us substitute for the hypotheses (3.19) and (3.32) the following more restrictive one

$$\max_{\vartheta\in[0,\pi/2]}\int_\Omega \Big[|a_1\cos^2\vartheta + a_2\sin^2\vartheta|^{\frac{\gamma}{2}-1}(a_1\cos^2\vartheta + a_2\sin^2\vartheta)+ \\ -(b_1\cos^\gamma\vartheta + b_2\sin^\gamma\vartheta)\Big]u_*^\gamma dx < 0\,. \tag{3.33}$$

It is important to note that the conditions (3.3) and (3.33) are compatible; also (3.3), (3.33), (3.6) and (3.7) are compatible. Now we state two Propositions that can be proved by using arguments similar to the Propositions 3.4 and 3.5 and we shortly give the proof of the first of them in order to point out the role of the assumption (3.33).

Proposition 3.6 *Under the assumptions (3.2), (3.3), (3.33) there exist* $\delta_1', \delta_2' > 0$ *such that for every* $(\lambda_1,\lambda_2) \in [\lambda_*, \lambda_* + \delta_1'[\times[\lambda_*, \lambda_* + \delta_2'[$, $S_{\lambda_1\lambda_2}\cap\Sigma$ *is bounded in the space* $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$.

Proof. Arguing by contradiction makes it possible to find a sequence $\{(v_{1k}, v_{2k})\}$ and $v_1, v_2 \in W_0^{1,p}(\Omega)$ such that

$$A(v_{1k}, v_{2k}) > 0\,, \tag{3.34}$$

$$v_{ik} \to v_i \quad \text{weakly in} \quad W_0^{1,p}(\Omega) \tag{3.35}$$

$$\int_\Omega (|\nabla v_1|^p + |\nabla v_2|^p)dx \le 1\,, \qquad \lambda_*\int_\Omega (|v_1|^p + |v_2|^p)dx = 1\,. \tag{3.36}$$

The relations (3.36) (differently from (3.25)) imply that

$$\lambda_*\int_\Omega |v_i|^p dx = \int_\Omega |\nabla v_i|^p dx\,,$$

so, there exist two real constants c_1, c_2 such that both are not equal to zero, and such that $v_i = c_i u_*$. Thus, choosing $\bar\vartheta \in [0,\pi/2]$ such that

$$\cos\bar\vartheta = |c_1|(c_1^2+c_2^2)^{-1/2}\,, \qquad \sin\bar\vartheta = |c_2|(c_1^2+c_2^2)^{-1/2}\,,$$

and taking into account (3.33), (3.35) we get:

$$\lim_k A(v_{1k}, v_{2k}) = \\ (c_1^2+c_2^2)^{\gamma/2}\left[\int_\Omega |a_1\cos^2\bar\vartheta + a_2\sin^2\bar\vartheta|^{\frac{\gamma}{2}-1}(a_1\cos^2\bar\vartheta + a_2\sin^2\bar\vartheta)u_*^\gamma dx + \right. \\ \left. -\int_\Omega (b_1\cos^\gamma\bar\vartheta + b_2\sin^\gamma\bar\vartheta)u_*^\gamma dx\right] < 0$$

but this contradicts (3.34).

Proposition 3.7 *Under the assumptions (3.2), (3.3), (3.33) there exist* δ_1 *and* δ_2 *with* $0 < \delta_i \leq \delta_i'$ *such that for every* $(\lambda_1, \lambda_2) \in [\lambda_*, \lambda_* + \delta_1[\times[\lambda_*, \lambda_* + \delta_2[$

$$A(u_{10}, u_{20}) = M_{\lambda_1\lambda_2} \Rightarrow H_{\lambda_1}(u_{10}) + H_{\lambda_2}(u_{20}) > 0\,.$$

From here follows

Theorem 3.6 *In the case (3.11) under the assumptions (3.2), (3.3), (3.33) there exist* $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that for every* $(\lambda_1, \lambda_2) \in [\lambda_*, \lambda_* + \delta_1[\times[\lambda_*, \lambda_* + \delta_2[$ *the problem* $(P_{\lambda_1\lambda_2})$ *admits at least one solution* (u_{10}, u_{20}) *with* $u_{i0} \geq 0$. *With the above mentioned hypotheses (3.6), (3.7) it follows that* $u_{i0} \neq 0$ *as* $i = 1, 2$.

Let us add the following

Theorem 3.7 *In the case (3.11) under the assumptions (3.2), (3.3), (3.33) there exist* $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that for each* $(\lambda_1, \lambda_2) \in]\lambda_*, \lambda_* + \delta_1[\times]\lambda_*, \lambda_* + \delta_2[$ *every solution of the problem* $(P_{\lambda_1\lambda_2})$ *has both components different from zero when* $p < 2$.

Since $\lambda_i > \lambda_*$ as $i = 1, 2$, we can deduce this statement following the argument used in the Theorem 3.4 in order to prove that $u_{10} \neq 0$.

The assumption (3.33) implies that (3.19) and (3.32) are true, so, if (3.13) holds, as λ_1 and λ_2 belong to a suitable right neighborhood of λ_*, the solution of the problems (3.14) exists [8] and then analogously to the above cases, by arguing as in the Theorem 3.2, we state the

Theorem 3.8 *In the case (3.11) under the assumptions (3.2), (3.13), (3.33) there exist* $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that for each* $(\lambda_1, \lambda_2) \in [\lambda_*, \lambda_* + \delta_1[\times[\lambda_*, \lambda_* + \delta_2[$ *every solution of the problem* $(P_{\lambda_1\lambda_2})$ *has both components different from zero when* $\gamma > p > 2$.

In view of the obtained results and bearing [8] in mind we can draw the following

Conclusion

In the case (3.8) under the assumption (3.2):

- if (3.3) holds there exists at least one nontrivial solution of (3.1), (u_1, u_2), with $u_i \geq 0$; moreover, if (3.6), (3.7) are valid too, then $u_i \neq 0$ as $i = 1, 2$;
- if (3.13) holds and $\gamma > p$, the system (3.1) admits at least two solutions of the type $(u_1, 0)$, $(0, u_2)$ with $u_i > 0$; moreover, when $p > 2$ there exists at least another solution (u_1, u_2) with $u_i \geq 0$ and $u_i \neq 0$ as $i = 1, 2$.

In the case (3.9) [resp. (3.10)] under the assumption (3.2):

- if (3.3), (3.19) [resp. (3.32)] hold, then for every $\lambda_2 < \lambda_*$ [resp. $\lambda_1 < \lambda_*$] and for every λ_1 [resp. λ_2] belonging to a suitable right neighborhood of λ_* (depending on λ_2 [resp. λ_1]) the system (3.1) admits at least one nontrivial solution (u_1, u_2) with $u_i \geq 0$; moreover, if (3.6), (3.7) are valid or (3.6) [resp. (3.7)] holds, λ_1 [resp. λ_2] is different from λ_* and $p < 2$ it results in $u_i \neq 0$ as $i = 1, 2$;

- if (3.13), (3.19) [resp. (3.32)] hold and $\gamma > p$, then for every $\lambda_2 < \lambda_*$ [resp. $\lambda_1 < \lambda_*$] and for every λ_1 [resp. λ_2] belonging to a suitable right neighborhood of λ_* (depending on λ_2 [resp. λ_1]) the system (3.1) admits at least two solutions of the type $(u_1, 0)$, $(0, u_2)$ with $u_i > 0$, and at least another solution with both non-negative and different from zero components when $p > 2$.

In the case (3.11) under the assumption (3.2):

- if (3.3), (3.33) hold for every λ_1 and λ_2 belonging to a suitable right neighborhood of λ_* the system (3.1) admits at least one nontrivial solution (u_1, u_2) with $u_i \geq 0$; moreover, if (3.6), (3.7) are valid or $p < 2$ and $\lambda_i \neq \lambda_*$ $(i = 1, 2)$ it results in $u_i \neq 0$ as $i = 1, 2$;
- if (3.13), (3.33) hold and $\gamma > p$, for every λ_1 and λ_2 belonging to a suitable right neighborhood of λ_* the system (3.1) has solutions like $(u_1, 0)$, $(0, u_2)$ with $u_i > 0$, and when $p > 2$ it admits at least another solution with both non-negative and different from zero components.

4 Further existence result in the third case

Our investigation continues for the case (3.11) when at least one of λ_i is distinct from λ_*; in fact in this case it is possible to obtain the existence of another solutions of (3.1). In order to obtain this purpose, setting

$$S = \{(v_1, v_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) : A(v_1, v_2) = -1\}$$
$$\Xi = \{(v_1, v_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) : H_{\lambda_1}(v_1) + H_{\lambda_2}(v_2) < 0\},$$

let us suppose that (3.33) is true and consider the constrained variational problem

$(Q_{\lambda_1\lambda_2})$ *Find a minimizer* $(w_{10}, w_{20}) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *of the problem*

$$m_{\lambda_1\lambda_2} = \inf\{H_{\lambda_1}(v_1) + H_{\lambda_2}(v_2) : (v_1, v_2) \in S \cap \Xi\} < 0\,.$$

It is important to note that the problem $(Q_{\lambda_1\lambda_2})$ is well posed by virtue of the following

Proposition 4.1 *Under the assumptions (3.2), (3.33) there exists* (v_1, v_2) *in* $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, *with* $v_i \neq 0$ *as* $i = 1, 2$, *such that* $A(v_1, v_2) = -1$ *and* $H_{\lambda_1}(v_1) + H_{\lambda_2}(v_2) < 0$.

Proof. The relation (3.33) implies that

$$\int_\Omega \left[|a_1 + a_2|^{\frac{\gamma}{2}-1}(a_1 + a_2) - (b_1 + b_2)\right] u_*^\gamma dx < 0$$

so, denoting by

$$\sigma = \left| \int_\Omega \left[|a_1 + a_2|^{\frac{\gamma}{2}-1}(a_1 + a_2) - (b_1 + b_2)\right] u_*^\gamma dx \right|^{-1/\gamma}$$

we get $A(\sigma u_*, \sigma u_*) = -1$ and

$$\begin{aligned} H_{\lambda_1}(\sigma u_*) + H_{\lambda_2}(\sigma u_*) &= \sigma^p \left[\int_\Omega (|\nabla u_*|^p - \lambda_1 u_*^p) dx + \int_\Omega (|\nabla u_*|^p - \lambda_2 u_*^p) dx \right] = \\ &= \sigma^p \left[(\lambda_* - \lambda_1) \int_\Omega u_*^p dx + (\lambda_* - \lambda_2) \int_\Omega u_*^p dx \right] < 0 . \end{aligned}$$

By using Lagrange multipliers again, we note that

Proposition 4.2 *Under the assumptions (3.2), (3.33) if* (w_{10}, w_{20}) *is a solution of the problem* $(Q_{\lambda_1,\lambda_2})$ *then* $(|m_{\lambda_1\lambda_2}|^{\frac{1}{\gamma-p}} w_{10}, |m_{\lambda_1\lambda_2}|^{\frac{1}{\gamma-p}} w_{20})$ *is a weak solution of the system (3.1).*

Then, let us state the following

Proposition 4.3 *Under the assumptions (3.2), (3.33) there exist* $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that for every* $(\lambda_1, \lambda_2) \in [\lambda_*, \lambda_* + \delta_1[\times]\lambda_*, \lambda_* + \delta_2[$ *or* $(\lambda_1, \lambda_2) \in]\lambda_*, \lambda_* + \delta_1[\times [\lambda_*, \lambda_* + \delta_2[$, *the set* $S \cap \Xi$ *is bounded in the space* $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

Proof. Arguing by contradiction let $\{\delta_{ik}\}$ an infinitesimal sequence of positive numbers. For every positive integer k there exist $\lambda_{1k} \in [\lambda_*, \lambda_* + \delta_{1k}[$, $\lambda_{2k} \in]\lambda_*, \lambda_* + \delta_{2k}[$ and a sequence $\{(u_{1n}^k, u_{2n}^k)\}$ such that

$$A(u_{1n}^k, u_{2n}^k) = -1 , \tag{4.1}$$

$$H_{\lambda_{1k}}(u_{1n}^k) + H_{\lambda_{2k}}(u_{2n}^k) < 0 , \tag{4.2}$$

$$\lim_n \int_\Omega (|\nabla u_{1n}^k|^p + |\nabla u_{2n}^k|^p) dx = +\infty . \tag{4.3}$$

The relation (4.3) makes it possible to obtain a strictly increasing sequence of positive integers $\{n_k\}$, so that

$$\lim_k \int_\Omega (|\nabla u_{1n_k}^k|^p + |\nabla u_{2n_k}^k|^p) dx = +\infty .$$

Setting

$$t_k = \left(\int_\Omega (|\nabla u_{1n_k}^k|^p + |\nabla u_{2n_k}^k|^p) dx \right)^{1/p} , \qquad v_{ik} = \frac{u_{in_k}^k}{t_k} ,$$

taking into account (4.1), (4.2) we get:

$$\int_\Omega \left[|a_1 v_{1k}^2 + a_2 v_{2k}^2|^{\frac{\gamma}{2}-1} (a_1 v_{1k}^2 + a_2 v_{2k}^2) - (b_1 |v_{1k}|^\gamma + b_2 |v_{2k}|^\gamma) \right] dx = -\frac{1}{t_k^\gamma} , \tag{4.4}$$

$$\int_\Omega (|\nabla v_{1k}|^p + |\nabla v_{2k}|^p) dx < \lambda_{1k} \int_\Omega |v_{1k}|^p dx + \lambda_{2k} \int_\Omega |v_{2k}|^p dx ; \tag{4.5}$$

moreover, there exist $v_1, v_2 \in W_0^{1,p}(\Omega)$ such that within a subsequence

$$v_{ik} \to v_i \quad \text{weakly in} \quad W_0^{1,p}(\Omega)\,.$$

From (4.4), (4.5), passing to the limit as $k \to +\infty$, we obtain

$$\int_\Omega \left[|a_1 v_1^2 + a_2 v_2^2|^{\frac{\gamma}{2}-1}(a_1 v_1^2 + a_2 v_2^2) - (b_1|v_1|^\gamma + b_2|v_2|^\gamma)\right] dx = 0\,, \quad (4.6)$$

$$\int_\Omega (|\nabla v_1|^p + |\nabla v_2|^p)dx \le \lambda_* \left[\int_\Omega |v_1|^p dx + \int_\Omega |v_2|^p dx\right]\,. \quad (4.7)$$

Owing to the variational characterization of λ_*, the inequality (4.7) implies that

$$\lambda_* \int_\Omega |v_i|^p dx = \int_\Omega |\nabla v_i|^p dx \quad \text{as} \quad i = 1, 2.$$

Let us add that at least one of the two functions v_1, v_2 is non-zero since

$$\lambda_* \left[\int_\Omega |v_1|^p dx + \int_\Omega |v_2|^p dx\right] \ge 1$$

that follows from (4.5). Then there exist two real constants c_1, c_2, not both equal zero, such that $v_i = c_i u_*$; consequently (4.6) can be rewritten as:

$$\int_\Omega \left[|a_1 \cos^2\bar{\vartheta} + a_2 \sin^2\bar{\vartheta}|^{\frac{\gamma}{2}-1}(a_1\cos^2\bar{\vartheta} + a_2\sin^2\bar{\vartheta}) - (b_1\cos^\gamma\bar{\vartheta} + b_2\sin^\gamma\bar{\vartheta})\right] u_*^\gamma dx = 0$$

with $\bar{\vartheta} \in [0, \pi/2]$ such that $\cos\bar{\vartheta} = |c_1|(c_1^2 + c_2^2)^{-1/2}$, $\sin\bar{\vartheta} = |c_2|(c_1^2 + c_2^2)^{-1/2}$, and this equality contradicts (3.33).
From Proposition 4.3 we have

Theorem 4.1 *Under the assumptions (3.2), (3.33) there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for every $(\lambda_1, \lambda_2) \in [\lambda_*, \lambda_* + \delta_1[\times]\lambda_*, \lambda_* + \delta_2[$ or $(\lambda_1, \lambda_2) \in]\lambda_*, \lambda_* + \delta_1[\times[\lambda_*, \lambda_* + \delta_2[$ the problem $(Q_{\lambda_1\lambda_2})$ admits at least one solution (w_{10}, w_{20}) with $w_{i0} \ge 0$.*

Let us note that the conditions (3.19), (3.32) (which are valid if (3.33) holds) assure that for $\lambda_i \ne \lambda_*$ belonging in a suitable right neighborhood of λ_* the constrained variational problems

$$m_{\lambda_i} = \inf\left\{H_{\lambda_i}(v) \,:\, \int_\Omega \left(|a_i|^{\frac{\gamma}{2}-1}a_i - b_i\right)|v|^\gamma dx = -1\right\} \qquad i = 1, 2$$

are solvable [8][1)] . So, in this case it is possible to have $m_{\lambda_1\lambda_2} = m_{\lambda_1}$ or $m_{\lambda_1\lambda_2} = m_{\lambda_2}$. Then we have to solve the question of the existence of solutions with both

[1)] In the Proposition (6.5) of [8] the condition (f.1) is mentioned like assumption, but it is not necessary in order to obtain this result.

components different from zero for the problem $(Q_{\lambda_1\lambda_2})$. In order to obtain a similar result let us set $\delta = \min\{\delta_1, \delta_2\}$ and suppose

$$a_i > 0 \quad \text{and} \quad b_i > 0 \quad \text{a.e. in} \quad \Omega \quad \text{as} \quad i = 1,2\,, \tag{4.8}$$

$$a_i^{\gamma/2} < b_i \quad \text{a.e. in} \quad \Omega \quad \text{as} \quad i = 1,2\,, \tag{4.9}$$

$$\sup \frac{b_1}{a_1 a_2^{\frac{\gamma}{2}-1}} < +\infty\,, \tag{4.10}$$

$$\sup \frac{b_2}{a_2 a_1^{\frac{\gamma}{2}-1}} < +\infty\,. \tag{4.11}$$

Let us note that the conditions (3.33) and (4.8)-(4.11) can hold simultaneously, so we state the following result

Theorem 4.2 *Under the assumptions (3.2), (3.33), (4.8)-(4.11), for each* $\lambda \in]\lambda_*, \lambda_* + \delta[$ *every solution of the problem* $(Q_{\lambda\lambda})$ *gives both the components different from zero.*

Proof. By choosing $\lambda \in]\lambda_*, \lambda_* + \delta[$ let (w_{10}, w_{20}) be solution of the problem $(Q_{\lambda\lambda})$. We have to prove that $w_{10} \neq 0$. Arguing by contradiction, let w_{10} be equal to zero, so

$$\int_\Omega [a_2^{\gamma/2} - b_2]|w_{20}|^\gamma dx = -1\,, \tag{4.12}$$

$$H_\lambda(w_{20}) = m_{\lambda\lambda}\,. \tag{4.13}$$

Let us consider the following function

$$F(s,t) = \int_\Omega \left[(a_1 s^2 + a_2 t^2)^{\gamma/2} - (b_1 s^\gamma + b_2 t^\gamma)\right] |w_{20}|^\gamma dx \qquad \forall\, s,t \geq 0\,;$$

by fixing $t_0 \in]0,1[$ by virtue of (4.12) we have

$$F(0,t_0) = -t_0^\gamma > -1 \tag{4.14}$$

and due to (4.9) for $i = 1$:

$$\begin{aligned} &\lim_{s\to+\infty} F(s,t_0) = \\ &= \lim_{s\to+\infty} s^\gamma \int_\Omega \left[(a_1 + a_2(t_0/s)^2)^{\gamma/2} - b_1\right] |w_{20}|^\gamma dx - t_0^\gamma \int_\Omega b_2 |w_{20}|^\gamma dx \\ &= -\infty\,. \end{aligned} \tag{4.15}$$

From (4.14),(4.15) we deduce the existence of $s(t_0) > 0$ such that

$$F(s(t_0), t_0) = -1\,. \tag{4.16}$$

Let us note that

$$F_s(s,t_0) = \gamma \int_\Omega \left[(a_1 s^2 + a_2 t_0^2)^{\frac{\gamma}{2}-1} a_1 s - b_1 s^{\gamma-1}\right] |w_{20}|^\gamma dx \quad \forall\, s \geq 0$$

and due to (4.8) the inequality

$$(a_1 s^2 + a_2 t_0^2)^{\frac{\gamma}{2}-1} a_1 s - b_1 s^{\gamma-1} > 0 \quad \text{a.e. in } \Omega \qquad \forall\, s \in]0, (1-t_0^p)^{1/p}]$$

holds if

$$a_2^{\frac{\gamma}{2}-1} a_1 t_0^{\gamma-2} > b_1 (1-t_0^p)^{\frac{\gamma-2}{p}} \qquad \text{a.e. in } \Omega\,.$$

Let us add that by virtue of the assumption (4.10) it is possible to choose $t_0 \in]0,1[$ such that

$$\sup \frac{b_1}{a_1 a_2^{\frac{\gamma}{2}-1}} < \left(\frac{t_0}{(1-t_0^p)^{1/p}} \right)^{\gamma-2}.$$

Consequently,

$$F_s(s, t_0) > 0 \qquad \forall\, s \in]0, (1-t_0^p)^{1/p}]\,;$$

so, taking into account (4.14) and (4.16), it must be that $s(t_0) > (1-t_0^p)^{1/p}$. In conclusion

$$A(s(t_0)w_{20}, t_0 w_{20}) = F(s(t_0), t_0) = -1$$

and due to (4.13) it results in

$$H_\lambda(s(t_0)w_{20}) + H_\lambda(t_0 w_{20}) = (s(t_0))^p H_\lambda(w_{20}) + t_0^p H_\lambda(w_{20}) = ((s(t_0))^p + t_0^p) m_{\lambda\lambda} < m_{\lambda\lambda}\,,$$

but this is impossible.
By using (4.9) with $i = 2$ and (4.11) arguing as before we prove that also $w_{20} \neq 0$.
Finally we come to the following

Conclusion
In the case (3.11) under the assumptions (3.2), (3.33), for each λ_1, λ_2 belonging to a suitable right neighborhood of λ_* we have:

- The system (3.1) admits at least one solution like $(w_1, 0)$ if $\lambda_1 \neq \lambda_*$ and at least one solution like $(0, w_2)$ if $\lambda_2 \neq \lambda_*$ with $w_i \geq 0$ and $w_i \neq 0$ (these solutions are different from ones obtained in the previous Section under the assumption that the condition (3.13) holds);
- If (4.8)-(4.11) are valid too and if $\lambda_1 = \lambda_2 \neq \lambda_*$ then the system (3.1) admits another solution (w_1, w_2) with $w_i \geq 0$ and $w_i \neq 0$ (this solution is different from the one obtained in the previous Section under the hypothesis that the condition (3.3) holds).

Remark 4.1 We clearly note that the conditions (3.3), (3.33) and (4.8)-(4.11) are compatible.

We conclude this paper by pointing out an open problem: It would be interesting to study the question of solvability of the system

$$\begin{aligned} -\Delta_p u_1 - \lambda_1 |u_1|^{p-2} u_1 &= |a_1 u_1^2 + a_2 u_2^2|^{\frac{\gamma}{2}-1} a_1 u_1 - b_1 |u_1|^{r_1-2} u_1 \\ & \qquad \text{in } \Omega \\ -\Delta_p u_2 - \lambda_2 |u_2|^{p-2} u_2 &= |a_1 u_1^2 + a_2 u_2^2|^{\frac{\gamma}{2}-1} a_2 u_2 - b_2 |u_2|^{r_2-2} u_2 \\ u_1 = u_2 &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

with $1 < r_i < p^*$ and at least one of these exponents different from γ.

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