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A Simple Approach to Brouwer Degree Based on Differential Forms

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Abstract

We present Heinz' approach to Brouwer degree in a simpler, shorter and better motivated way. We link it to Kronecker index, use the language of differential forms at an elementary level, and prove the homotopy invariance using an unpublished result of Tartar.

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1 Introduction

Degree theory for continuous mappings from $\mathbb{R}$ to $\mathbb{R}$ can of course be traced to Bolzano's intermediate value theorem for continuous functions. Implicit in some of Gauss' proofs of the fundamental theorem of algebra, the concept of index or winding number around a closed curve, a forerunner of the notion of degree for continuous mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$, was first explicitely defined and developed by Cauchy in a series of memoirs published between 1831 and 1837. The first one, devoted to the special case of holomorphic functions from $\mathbb{C}$ to $\mathbb{C}$, was followed, in 1833 (published 1837), by a treatment of C^1 mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. See [21], p. 134-136 for an interesting discussion and references. As shown in [25], Cauchy's index was widely used by Poincaré in his early work on the qualitative theory of nonlinear differential equations [32].

Cauchy's index was generalized in 1869 by Kronecker [23] to C^1 mappings $f = (f_1, f_2, \ldots, f_n)$ from $\mathbb{R}^n$ into $\mathbb{R}^n$, on the compact $(n-1)$-dimensional manifold $F^{-1}(0)$ associated to a smooth mapping $F : \mathbb{R}^n \to \mathbb{R}$ and the regular value 0 of F. He defined the "characteristic of (F, f)", nowadays called the Kronecker's index of f on $F^{-1}(0)$, by the integral (written here in modern notations)

$$i_K[f, F^{-1}(0)] = \frac{1}{\mu_{n-1}} \int_{F^{-1}(0)} \|f\|^{-n} \sum_{j=1}^{n} (-1)^{j-1} f_j \, df_1 \wedge \ldots \wedge \widehat{df_j} \wedge \ldots df_n \quad (1.1)$$

where μ_{n-1} is the $(n-1)$-dimensional measure of the unit sphere S^{n-1} in $\mathbb{R}^n$, and $\widehat{df_j}$ means that the factor df_j is missing (see [21] and [34] for more details, as well as [25] for Poincaré's use of Kronecker's index in his qualitative study of nonlinear differential equations).

Influenced by Brouwer's construction (using simplicial methods) of a degree theory for continuous mappings between two oriented compact manifolds of the same finite dimension [9], Hadamard [17] extended in 1910 Kronecker's index to continuous mappings and more general $(n-1)$-dimensional compact manifolds (see [13] for the role of Hadamard in the genesis of Brouwer's ideas). After Brouwer's pioneering paper, degree theory was essentially developed using algebraic topological tools (see e.g. [2, 16]), with the exception of two papers of Nagumo [28, 29], who based his analytical construction of the (localized) Brouwer degree $d_B[f, D, z]$ of a continuous mapping $f : \overline{D} \to \mathbb{R}^n$, when $D \subset \mathbb{R}^n$ is open bounded and $z \notin f(\partial D)$, on approximating f by smooth mappings g and z by regular values u of g. For such approximations, one can define

$$d_B[g, D, u] = \sum_{x \in g^{-1}(u)} \text{sign } J_g(x),$$

with J_g the Jacobian of g. Nagumo's approach, which also uses the structure of compact 1-dimensional manifolds, has been adopted in the monographs [3, 6, 7, 20, 22, 27, 35, 39].

Inspired by de Rham cohomology theory based on differential forms [10], Heinz [19] proposed in 1959 another analytical approach to Brouwer degree. He first defines $d_B[f, D, z]$ for f smooth, $z \notin f(\partial D)$, by the integral

$$\int_D c(\|f(x) - z\|) J_f(x) \, dx, \quad (1.2)$$

where the continuous function c has support in $]0, \min_{\partial D} \|f(\cdot) - z\|[$, and then approximates a continuous f through smooth mappings. Heinz' approach, which avoids the explicit use of differential forms, and does not discuss the connection of integral (1.2) with Kronecker's one (1.1) when both are defined, has been adopted in many monographs [1, 5, 11, 12, 14, 24, 31, 33]. A variant of it, due to Lax, explicitely uses the language of differential forms, and can be found in the monographs [6, 7, 12, 30, 36, 40].

The aim of this note is to present Heinz approach in a somewhat simpler, shorter and better motivated way, by linking it to the Kronecker index (Proposition 3.1), and using the language of differential forms, at the level of an advanced calculus treatise (for example [38]). The needed versions of Stokes formula are

$$\int_D d\lambda = \int_{\partial D} \lambda, \tag{1.3}$$

when the open bounded set $D \subset \mathbb{R}^n$ has a regular oriented boundary ∂D and λ is a differential $(n-1)$-form of class C^1 (to relate Kronecker and Heinz definitions), and

$$\int_D d\lambda = 0. \tag{1.4}$$

when the C^1 differential $(n-1)$-form λ has compact support in an arbitrary open set $D \subset \mathbb{R}^n$ (to justify Heinz definition and prove the homotopy invariance of degree). Furthermore, the homotopy invariance of the degree is proved here using an unpublished result of Tartar [37], quoted in the Introduction of [12] but not used later in the book, which is closely related to the theory of integral invariants. When written in terms of differential forms, this result (our Lemma 5.2) immediately follows from an elementary computation of exterior calculus given in Lemma 5.1. In the setting of Kronecker index, a similar approach was used recently by Hatziafratis and Tsarpalias [18], at the expense of much longer and tedious computations.

In another paper [26], we show how a special case of Lemma 5.2 provides, independently of any degree theory, very simple proofs of Brouwer's fixed point theorems on balls and spheres, of Poincaré-Brouwer's (hairy ball) theorem [8, 9, 32], and of the fundamental theorem of algebra in the complex and in the quaternionic fields, a subject which has called some attention in those last years.

2 An exact form

Let $0 < \varepsilon_0 < \mu_0$, $a \in C^k([0,+\infty[,\mathbb{R})$, $(k \geq 0)$ be such that

$$\text{supp } a \subset [\varepsilon_0, \mu_0],$$

and consider the differential n-form

$$\omega_a := a(\|y\|)\, dy_1 \wedge \ldots \wedge dy_n = a(\|y\|)\, \omega, \tag{2.1}$$

where

$$\omega := dy_1 \wedge \ldots \wedge dy_n \tag{2.2}$$

is the **volume n-form**. Let σ be the **solid angle $(n-1)$-form**

$$\sigma := \sum_{j=1}^{n} (-1)^{j-1} y_j\, dy_1 \wedge \ldots \wedge \widehat{dy_j} \wedge \ldots \wedge dy_n, \tag{2.3}$$

where $\widehat{dy_j}$ means that dy_j is missing. One has $\omega = d[(1/n)\sigma]$. The following lemma shows that ω_a is also exact on $\mathbb{R}^n$.

Lemma 2.1 *The equality*

$$\omega_a = d\sigma_A \tag{2.4}$$

with

$$\sigma_A := A(\|y\|)\,\sigma, \tag{2.5}$$

$A \in C^{k+1}([0,+\infty[,\mathbb{R})$ *and supp* $A \subset [\varepsilon_0,+\infty[$*, holds if and only if*

$$rA'(r) + nA(r) = a(r) \quad (r \geq \varepsilon_0), \tag{2.6}$$

i.e. if and only if A *is given by*

$$A(0) = 0, \quad A(r) = \frac{1}{r^n}\int_0^r a(s)s^{n-1}\,ds \quad (r > 0). \tag{2.7}$$

Proof. Let $A \in C^{k+1}([0,+\infty[,\mathbb{R})$, with supp $A \subset [\varepsilon_0,+\infty[$. σ_A is of class C^{k+1} on $\mathbb{R}^n$ and $\sigma_A = 0$ on $B[0;\varepsilon_0]$. For $y \neq 0$, we have

$$\begin{aligned} d\sigma_A &= d[A(\|y\|)] \wedge \sigma + A(\|y\|)\,d\sigma = A'(\|y\|)\sum_{k=1}^{n}\frac{y_k}{\|y\|}\,dy_k \wedge \sigma + nA(\|y\|)\,\omega \\ &= [\|y\|A'(\|y\|) + nA(\|y\|)]\,\omega. \end{aligned}$$

Consequently, $\omega_a = d\sigma_A$ is and only if, for all $r \in [\varepsilon_0,+\infty[$, A satisfies (2.6). For $r > 0$, this is equivalent to

$$\frac{d}{dr}[r^n A(r)] = r^{n-1}a(r),$$

i.e. to A given by formula (2.7). Clearly, supp $A \subset [\varepsilon_0,+\infty[$. ∎

Corollary 2.1 *If we assume furthermore that*

$$\int_0^{+\infty} a(r)r^{n-1}\,dr = 0, \tag{2.8}$$

then supp $A \subset [\varepsilon_0,\mu_0]$.

Proof. For $r > \mu_0$, we have, by formula (2.7),

$$A(r) = \frac{1}{r^n}\int_0^r a(s)s^{n-1}\,ds = \frac{1}{r^n}\int_0^{\mu_0} a(s)s^{n-1}\,ds = \frac{1}{r^n}\int_0^{+\infty} a(s)s^{n-1}\,ds = 0.$$

∎

3 The Kronecker index

Let $D \subset \mathbb{R}^n$ be a bounded open set with oriented smooth boundary ∂D and $f \in C^2(\overline{D}, \mathbb{R}^n)$ be such that $0 \notin f(\partial D)$. Then

$$\mu := \min_{\partial D} \|f\| > 0. \tag{3.1}$$

Definition 1 The **Kronecker index** $i_K[f, \partial D]$ is defined by

$$i_K[f, \partial D] = \frac{1}{\mu_{n-1}} \int_{\partial D} f^* \left[\|y\|^{-n} \sigma \right], \tag{3.2}$$

where μ_{n-1} denotes the $(n-1)$-dimensional measure of the unit sphere

$$S^{n-1} = \{y \in \mathbb{R}^n : \|y\| = 1\}.$$

Recall that

$$f^* \left[\|y\|^{-n} \sigma \right] = \|f\|^{-n} \left[\sum_{j=1}^{n} (-1)^{j-1} f_j \, df_1 \wedge \ldots \wedge \widehat{df_j} \wedge \ldots \wedge df_n \right].$$

For $n = 2$, formula (3.2) reduces to a line integral, already defined by Cauchy, and also named the **winding number** or the **Poincaré index** of f around ∂D

$$i_K[f, \partial D] = \frac{1}{2\pi} \int_{\partial D} \frac{f_1 \, df_2 - f_2 \, df_1}{f_1^2 + f_2^2}.$$

If ∂D has the parametric representation $\varphi : [0, 2\pi] \to \mathbb{R}^2$, then

$$i_K[f, \partial D] = \frac{1}{2\pi} \int_0^{2\pi} \frac{(f_1 \circ \varphi)(s)\,(f_2 \circ \varphi)'(s) - (f_2 \circ \varphi)(s)\,(f_1 \circ \varphi)'(s)}{[(f_1 \circ \varphi)(s)]^2 + [(f_2 \circ \varphi)(s)]^2} \, ds.$$

The Kronecker integral in the right-hand member of (3.2) is associated to the differential $(n-1)$-form σ_A with $A(r) = r^{-n}$, which is not defined at 0 and does not vanish near 0. However, in (3.2), one only has to consider the values of $f(x)$ near ∂D, i.e. values of $f(x)$ for which $\|f(x)\| \geq \mu$. Hence, let

$$0 < \varepsilon_0 < \mu_0 < \mu,$$

and define $B \in C^1([0, +\infty[, \mathbb{R})$ by

$$\begin{array}{rcll} B(r) & = & 0 & \text{if} \quad r \in [0, \varepsilon_0] \\ B(r) & > & 0 & \text{if} \quad r \in]\varepsilon_0, \mu_0[\\ B(r) & = & r^{-n} & \text{if} \quad r \in [\mu_0, +\infty[. \end{array} \tag{3.3}$$

Notice that, for $r > \mu_0$, we have

$$rB'(r) + nB(r) = 0. \tag{3.4}$$

Proposition 3.1 *If B is given by (3.3), the function $b \in C([0,+\infty[,\mathbb{R})$ defined by*

$$b(r) = rB'(r) + nB(r) \tag{3.5}$$

is such that

$$\text{supp } b \subset [\varepsilon_0, \mu_0], \qquad \int_0^{+\infty} b(r) r^{n-1}\, dr = 1,$$

and

$$i_K[f, \partial D] = \frac{1}{\mu_{n-1}} \int_D f^* \omega_b = \frac{1}{\mu_{n-1}} \int_D b(\|f(x)\|) J_f(x)\, dx. \tag{3.6}$$

Proof. For $x \in \partial D$, we have $\|f(x)\| \geq \mu > \mu_0$, and hence $B(\|f(x)\|) = \|f(x)\|^{-n}$. Using Lemma 2.1 and Stokes formula (1.3), we get

$$\begin{aligned} i_K[f, \partial D] &= \frac{1}{\mu_{n-1}} \int_{\partial D} f^* \left[\|y\|^{-n} \sigma \right] = \frac{1}{\mu_{n-1}} \int_{\partial D} f^* \sigma_B \\ &= \frac{1}{\mu_{n-1}} \int_D d[f^* \sigma_B] = \frac{1}{\mu_{n-1}} \int_D f^*[d\sigma_B] = \frac{1}{\mu_{n-1}} \int_D f^* \omega_b. \end{aligned}$$

It is clear from its definition and (3.4) that $b(r) = 0$ for $r \in [0, \varepsilon_0[\cup]\mu_0, +\infty[$, and that

$$\int_0^{+\infty} b(r) r^{n-1}\, dr = \int_0^{+\infty} [r^n B'(r) + n r^{n-1} B(r)]\, dr = \int_0^{+\infty} \frac{d}{dr}[r^n B(r)]\, dr = 1.$$

■

Remark 3.1 In the construction above, the functions b and B can be taken as smooth as needed.

4 The Brouwer degree for smooth mappings as a volume integral

Let now D be open and bounded in $\mathbb{R}^n$ and $f \in C^2(\overline{D}, \mathbb{R}^n)$ be such that $0 \notin f(\partial D)$. We again define $\mu > 0$ by (3.1), and take $0 < \varepsilon_0 < \mu_0 < \mu$. The following definition is due to Heinz [19].

Definition 2 The **Brouwer degree** $d_B[f, D]$ is defined by

$$d_B[f, D] = \int_D f^* \omega_c = \int_D c(\|f(x)\|) J_f(x)\, dx, \tag{4.1}$$

where $c \in C([0,+\infty[,\mathbb{R})$ is such that supp $c \subset [\varepsilon_0, \mu_0]$ and

$$\int_{\mathbb{R}^n} c(\|x\|)\, dx = 1 \quad \left(i.e. \int_0^{+\infty} c(r) r^{n-1}\, dr = \frac{1}{\mu_{n-1}} \right). \tag{4.2}$$

To justify this definition, it suffices to notice that if $\widetilde{c} \in C([0, +\infty[, \mathbb{R})$ satisfies the same conditions as c, then $a = c - \widetilde{c}$ is such that

$$\int_0^{+\infty} a(r) r^{n-1}\, dr = 0.$$

Hence, by Corollary 2.1, supp $A \subset [\varepsilon_0, \mu_0]$, with A given by (2.7), so that,

$$\int_D f^* \omega_c - \int_D f^* \omega_{\widetilde{c}} = \int_D f^* \omega_a = \int_D f^*[d\sigma_A] = \int_D d[f^* \sigma_A] = 0,$$

using (1.4), and the fact that $\|f(x)\| > \mu_0$ in a neighbourhood of ∂D. ∎

Remark 4.1 It follows immediately from Definition 2 that $d_B[f, \emptyset] = 0$.

Remark 4.2 Proposition 3.6 implies that, if ∂D is smooth enough, one has

$$i_K[f, \partial D] = d_B[f, D]. \tag{4.3}$$

Proposition 4.1 *If the open set $D' \subset D$ is such that $0 \notin f(\overline{D} \setminus D')$, then*

$$d_B[f, D] = d_B[f, D'].$$

In particular, if $0 \notin f(\overline{D})$, then $d_B[f, D] = 0$.

Proof. By assumption, $\mu \geq \mu' := \min_{\overline{D} \setminus D'} \|f\| > 0$. If we take $0 < \varepsilon_0 < \mu_0 < \mu'$ in the function c defining $d_B[f, D]$, we have $c(\|f(x)\|) = 0$ if $x \in \overline{D} \setminus D'$, and

$$d_B[f, D] = \int_D c(\|f(x)\|) J_f(x)\, dx = \int_{D'} c(\|f(x)\|) J_f(x)\, dx = d_B[f, D'].$$

∎

It is convenient to introduce a new notation.

Definition 3 If $D \subset \mathbb{R}^n$ is open and bounded, $f \in C^2(\overline{D}, \mathbb{R}^n)$ and $z \notin f(\partial D)$, the **Brouwer degree** $d_B[f, D, z]$ is defined by

$$d_B[f, D, z] = d_B[f(\cdot) - z, D].$$

Of course, $d_B[f, D] = d_B[f, D, 0]$.

Definition 4 $z \in \mathbb{R}^n$ is a **regular value** for f if $J_f(x) \neq 0$ when $x \in f^{-1}(z)$.

So any z with $f^{-1}(z)$ empty is a regular value.

Proposition 4.2 *If $z \notin f(\partial D)$ is a regular value for f, then*

$$d_B[f, D, z] = \sum_{x \in f^{-1}(z)} \mathit{sign}\ J_f(x) = N_+ - N_-, \tag{4.4}$$

where N_+ (resp. N_-) denotes the number of elements of $f^{-1}(z)$ with positive (resp. negative) Jacobian.

Proof. If $f^{-1}(z) = \emptyset$, the result is trivial. If $z \notin f(\partial D)$ is a regular value for f and $f^{-1}(z) \neq \emptyset$, then $f^{-1}(z) \subset D$ is compact, and is discrete by the inverse function theorem. Hence it is finite, namely $f^{-1}(z) = \{x^1, \ldots, x^m\}$. By the inverse function theorem again, there exists, for each $1 \leq j \leq m$, an open neighbourhood $U_j \subset D$ of x^j such that f is a diffeomorphism on U_j, and $U_j \cap U_k = \emptyset$ if $j \neq k$. Thus the sets $f(U_j)$ are open neighbourhoods of z, as well as $N = \cap_{j=1}^m f(U_j)$, so that $N \supset B[z; r]$ for some $r > 0$. As $f(x) \neq z$ for all $x \in K := \overline{D} \setminus \cup_{j=1}^m U_j$, Proposition 4.1 gives

$$\begin{aligned} d_B[f, D, z] &= d_B[f, \cup_{j=1}^m U_j, z] = \int_{\cup_{j=1}^m U_j} c(\|f(x) - z\|) J_f(x)\, dx \\ &= \sum_{j=1}^m \int_{U_j} c(\|f(x) - z\|) J_f(x)\, dx = \sum_{j=1}^m d_B[f, U_j, z], \end{aligned}$$

where we have taken

$$\mu_0 < \min\{r, \min_{\overline{D} \setminus \cup_{j=1}^m U_j} \|f(\cdot) - z\|\}$$

in the definition of c. The change of variables formula in an integral gives

$$\begin{aligned} d[f, U_j, z] &= \int_{U_j} c(\|f(x) - z\|) J_f(x)\, dx \\ &= \text{sign } J_f(x^j) \int_{U^j} c(\|f(x) - z\|) |J_f(x)|\, dx \\ &= \text{sign } J_f(x^j) \int_{f(U^j) - z} c(\|y\|)\, dy. \end{aligned}$$

Now, for $y \notin f(U^j) - z$, we have $\|y\| \geq r \geq \mu_0$, and hence $c(\|y\|) = 0$. Thus

$$\int_{f(U^j) - z} c(\|y\|)\, dy = \int_{\mathbb{R}^n} c(\|y\|)\, dy = 1,$$

which gives (4.4). ■

Remark 4.3 Proposition 4.2 explains the name of "algebraic number of zeros" sometimes given to the degree.

Remark 4.4 For each regular value $z \notin f(\partial D)$, $d_B[f, D, z]$ is an integer.

Corollary 4.1 *If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism, then*

$$\begin{aligned} d_B[A, D, z] &= \textit{sign det } A \quad \textit{if} \quad z \in A(D) \\ &= 0 \quad \textit{if} \quad z \notin A(\overline{D}). \end{aligned}$$

5 Another exact form and homotopy invariance

For $D \subset \mathbb{R}^n$ open and bounded, let $w \in C^1(\mathbb{R}^n, \mathbb{R})$, $F \in C^2(\overline{D} \times [0,1], \mathbb{R}^n)$, and

$$dF_j = \sum_{j=1}^{n} \partial_j F(\cdot, \lambda)\, dx_j.$$

The following lemma shows that $\partial_\lambda \{F(\cdot,\lambda)^*[w(y)\,\omega]\}$ is an exact form.

Lemma 5.1 *For each* $\lambda \in [0,1]$, *we have*

$$\begin{aligned} & \partial_\lambda \left[(w \circ F)\, dF_1 \wedge \ldots \wedge dF_n\right] \\ = \;& d\left[(w \circ F)\left(\sum_{j=1}^{n}(-1)^{j-1}\partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n\right)\right]. \end{aligned} \tag{5.1}$$

Proof. We first notice that

$$\partial_\lambda dF_j = \partial_\lambda \left[\sum_{k=1}^{n} \partial_k F_j\, dx_k\right] = \sum_{k=1}^{n} \partial_\lambda \partial_k F_j\, dx_k = \sum_{k=1}^{n} \partial_k \partial_\lambda F_j\, dx_k = d[\partial_\lambda F_j].$$

Now

$$\begin{aligned} & \partial_\lambda \left[(w \circ F)\, dF_1 \wedge \ldots \wedge dF_n\right] \\ = \;& \partial_\lambda[(w \circ F)]\, dF_1 \wedge \ldots \wedge dF_n + (w \circ F)\, \partial_\lambda\left(dF_1 \wedge \ldots \wedge dF_n\right) \\ = \;& \left[\sum_{j=1}^{n}(\partial_j w \circ F)\, \partial_\lambda F_j\right] dF_1 \wedge \ldots \wedge dF_n \\ + \;& (w \circ F)\left(\sum_{j=1}^{n} dF_1 \wedge \ldots \wedge \partial_\lambda dF_j \wedge \ldots \wedge dF_n\right) \\ = \;& \sum_{j=1}^{n}(-1)^{j-1}(\partial_j w \circ F)\, dF_j \wedge \partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \\ + \;& (w \circ F)\left[\sum_{j=1}^{n}(-1)^{j-1} d(\partial_\lambda F_j) \wedge dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n\right] \\ = \;& \sum_{j=1}^{n}(-1)^{j-1}\left[\sum_{k=1}^{n}(\partial_k w \circ F)\, dF_k\right] \wedge \partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \\ + \;& (w \circ F)\left[\sum_{j=1}^{n}(-1)^{j-1} d\left(\partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= d(w\circ F)\wedge\left(\sum_{j=1}^{n}(-1)^{j-1}\partial_\lambda F_j\,dF_1\wedge\ldots\wedge\widehat{dF_j}\wedge\ldots\wedge dF_n\right)\\
&+ (w\circ F)\,d\left(\sum_{j=1}^{n}(-1)^{j-1}\partial_\lambda F_j\,dF_1\wedge\ldots\wedge\widehat{dF_j}\wedge\ldots\wedge dF_n\right)\\
&= d\left[(w\circ F)\left(\sum_{j=1}^{n}(-1)^{j-1}\partial_\lambda F_j\,dF_1\wedge\ldots\wedge\widehat{dF_j}\wedge\ldots\wedge dF_n\right)\right].
\end{aligned}
$$

■

Define now

$$
\begin{aligned}
I_w[F(\cdot,\lambda)] &:= \int_D (w\circ F)\,dF_1\wedge\ldots\wedge dF_n\\
&= \int_D F(\cdot,\lambda)^*[w(y)\,dy_1\wedge\ldots\wedge dy_n] \qquad (5.2)\\
&= \int_D w[F(x,\lambda)]\,J_{F(\cdot,\lambda)}(x,\lambda)\,dx.
\end{aligned}
$$

The following result, close in spirit to Poincaré's theory of integral invariants, was stated, and proved in a special case, by Tartar [12, 37], without using differential forms.

Lemma 5.2 *If $w\in C^1(\mathbb{R}^n,\mathbb{R})$ and $F\in C^2(\overline{D}\times[0,1])$ are such that*

$$F(\partial D\times[0,1])\cap \ \mathit{supp}\ w=\emptyset, \qquad (5.3)$$

then $I_w[F(\cdot,\lambda)]$ is independent of λ on $[0,1]$.

Proof. Using Lemma 5.1, Assumption (5.3) and Stokes formula (1.4), we get

$$
\begin{aligned}
&\frac{d}{d\lambda}I_w[F(\cdot,\lambda)] = \int_D \partial_\lambda[(w\circ F)\,dF_1\wedge\ldots\wedge dF_n]\\
&= \int_D d\left[(w\circ F)\left(\sum_{j=1}^{n}(-1)^{j-1}\partial_\lambda F_j\,dF_1\wedge\ldots\wedge\widehat{dF_j}\wedge\ldots\wedge dF_n\right)\right]=0.
\end{aligned}
$$

■

Corollary 5.1 *If $F\in C^2(\overline{D}\times[0,1],\mathbb{R}^n)$ is such that $0\notin F(\partial D\times[0,1])$, then $d_B[F(\cdot,\lambda),D]$ is independent of λ over $[0,1]$.*

Proof. By assumption,

$$\mu := \inf_{\partial D\times[0,1]}\|F\|>0,$$

and, if $0<\varepsilon_0<\mu_0<\mu$, the corresponding function c in Definition 2 (which can always be chosen of class C^1) is such that $w(y)=c(\|y\|)$ has the property (5.3). Thus, by Lemma 5.2, $d_B[F(\cdot,\lambda),D]=I_w[F(\cdot,\lambda)]$ is independent of λ in $[0,1]$. ■

Corollary 5.2 *If $f,\ g \in C^2(\overline{D}, \mathbb{R}^n)$ are such that*

$$\nu := \max_{\partial D} \|f - g\| < \mu := \min\{\min_{\partial D} \|f\|, \min_{\partial D} \|g\|\}, \tag{5.4}$$

then $d_B[f, D] = d_B[g, D]$.

Proof. Let us define $F : \overline{D} \times [0,1] \to \mathbb{R}^n$ by $F(x, \lambda) = (1-\lambda)f(x) + \lambda g(x)$. Then we have, for all $(x, \lambda) \in \partial D \times [0,1]$,

$$\|F(x,\lambda)\| \geq \|f(x)\| - \|g(x) - f(x)\| \geq \mu - \nu > 0.$$

The result follows from Corollary 5.1. ■

Corollary 5.3 *For each $f \in C^2(\overline{D}, \mathbb{R}^n)$ and $z \notin f(\partial D)$, $d_B[f, D, z]$ is an integer.*

Proof. By Sard's theorem, there exists a regular value v such that

$$\|v - z\| < \mu = \min_{\partial D} \|f(\cdot) - z\|.$$

Hence,

$$\max_{\overline{D}} \|f(\cdot) - z - [f(\cdot) - u]\| = \|z - u\| < \mu,$$

and, using Corollary 5.2 and Proposition 4.2 we get

$$d_B[f, D, z] = d_B[f, D, u] \in \mathbb{Z}.$$

■

6 The Brouwer degree for continuous mappings

The contents of this section is classical, and only given for completeness. Let $D \subset \mathbb{R}^n$ be open and bounded, $f \in C(\overline{D}, \mathbb{R}^n)$ and $z \notin f(\partial D)$, so that

$$\mu := \min_{\partial D} \|f(\cdot) - z\| > 0.$$

Using Weierstrass' approximation theorem, let $(f_k)_k$ be a sequence of mappings $f_k \in C^2(\overline{D}, \mathbb{R}^n)$ such that $\max_{\overline{D}} \|f_k - f\| < \mu/4$ for all $k \in \mathbb{N}$. Then

$$\min_{\partial D} \|f_k(\cdot) - z\| > \mu/2, \quad \max_{\overline{D}} \|f_k - f_j\| < \mu/2 \quad (j, k \in \mathbb{N}),$$

and hence, by Proposition 5.2,

$$d_B[f_k, D, z] = d_B[f_j, D, z] \quad (j, k \in \mathbb{N}).$$

Definition 5 If $D \subset \mathbb{R}^n$ is open and bounded, $f \in C(\overline{D}, \mathbb{R}^n)$ and $z \notin f(\partial D)$, the **Brouwer degree** $d_B[f, D, z]$ is defined, for an arbitrary $k \in \mathbb{N}$, by

$$d_B[f, D, z] = d_B[f_k, D, z]. \tag{6.1}$$

The definition is justified by the fact that the right-side of (6.1) does not depend upon the choice of the approximating sequence $(f_k)_k$: if $(g_k)_k$ is another sequence satisfying the same properties as $(f_k)_k$, then

$$\max_{\partial D} \|f_k - g_k\| \leq \max_{\partial D} \|f_k - f\| + \max_{\partial D} \|f - g_k\| < \mu,$$

and we can apply Corollary 5.2. The basic properties of Brouwer degree easily extend to the continuous case, and the standard proofs are omitted.

Theorem 6.1 *The following properties hold.*

1. **Excision**. *If the open subset $D' \subset D$ is such that $z \notin f(\overline{D} \setminus D')$, then*
$$d_B[f, D, z] = d_B[f, D', z].$$
2. **Existence**. *If $z \notin f(\overline{D})$, then $d_B[f, D, z] = 0$. Equivalently, if $d_B[f, D, z] \neq 0$, there exists at least one $x \in D$ such that $f(x) = z$.*
3. **Homotopy invariance**. *If $F \in C(\overline{D} \times [0, 1])$ and $z \notin F(\partial D \times [0, 1])$, then $d_B[F(\cdot, \lambda), D, z]$ is independent of λ on $[0, 1]$.*
4. **Rouché's property**. *If $f,\ g \in C(\overline{D}, \mathbb{R})$, $z \notin f(\partial D \times [0, 1])$, $\max_{\partial D} \|f - g\| < \mu$, then $d_B[f, D] = d_B[g, D]$.*
5. **Additivity**. *Suppose that there exists a sequence $(D_j)_j$ of open and mutually disjoint subsets of D. If $z \notin f(\overline{D} \setminus \cup_{j=1}^{\infty} D_j)$, then $d_B[f, D_j, z] = 0$ for all but finitely many j, and $d_B[f, D, z] = \sum_j d_B[f, D_j, z]$.*

Recall finally that, as shown independently by Führer [15] and Amann-Weiss [4], the Brouwer degree is uniquely determined, as an integer-valued function, by its properties of homotopy invariance, additivity (for two sets) and **normalization,** namely Corollary 4.1 with $A = I$.

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