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A Note on Nonlinear Schrödinger Systems: Existence of A-symmetric Solutions

Antonio Ambrosetti*

SISSA, 2-4, Via Beirut, Trieste 34014 Italy
e-mail: ambr@sissa.it

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Abstract

The paper contains some results concerning the existence of a-symmetric solutions to a class of autonomous nonlinear Schrödinger systems, coupled through a small parameter.

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1 Introduction

This paper deals with some nonlinear Schrödinger (NLS) systems of the form

$$\begin{cases} -u'' + \lambda u &= f_u(u,v), \\ -v'' + \lambda v &= f_v(u,v), \end{cases} \tag{1.1}$$

where $\lambda > 0$, $u, v \in W^{1,2}(\mathbb{R})$, 'prime' stands for the derivative with respect to $x \in \mathbb{R}$ and subscripts denote partial derivatives. Systems like (1.1) arise in Nonlinear Optics and model e.g. the propagation of pulses in nonlinear dual-core directional coupler, see [1]. Most of the known results on (1.1) deal with the existence of explicit solutions in the presence of specific nonlinearities $f(u,v)$ or are based on numerical arguments. More general and rigorous studies have been carried out very recently, only. See [4, 5, 6, 8, 10]. In particular, it has been proved in [6] that a class of (1.1) possesses multi-bump solutions, a feature

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previously observed in [2] using numerical calculation. It is worth pointing out that it is somewhat surprising to find multi-bump solutions in an autonomous problem. Actually the existence of multi-bumps is a property that is typical of single NLS *non-autonomous* equations, namely NLS equations with *potentials depending on* $x \in \mathbb{R}$.

The aim of the present paper is to discuss a further class of autonomous systems like (1.1) which share another feature that is typical of NLS equations with a potential, namely the break of symmetry. Roughly, in an autonomous system the coupling term can have the same effect that a potential $V(x)$ has on a single NLS equation (see e.g. [3]), giving rise to an a-symmetric solution. More precisely, we will consider a class of autonomous NLS systems (2.1), coupled through a small parameter. We will show that such a system possesses a family of symmetric solutions from which branches off a second family of a-symmetric solutions.

2 Existence of symmetric solutions

We will deal with the system (1.1) when f has the form $f(u,v) = \frac{1}{4}\left(u^4 + v^4\right) + \varepsilon g(u,v)$, where

$$g(u,v) = \left(u^2v^2 - uv\right),$$

so that (1.1) becomes

$$\begin{cases} -u'' + \lambda u &= u^3 + \varepsilon(2uv^2 - v), \\ -v'' + \lambda v &= v^3 + \varepsilon(2u^2v - u). \end{cases} \tag{2.1}$$

For $u, v \in W^{1,2}(\mathbb{R})$, let

$$I_\lambda(u) = \tfrac{1}{2}\int_{\mathbb{R}} (|u'|^2 + \lambda u^2)dx - \tfrac{1}{4}\int_{\mathbb{R}} u^4 dx,$$

and set

$$\Phi_{\lambda,\varepsilon}(u,v) = I_\lambda(u) + I_\lambda(v) - \varepsilon \int_{\mathbb{R}} g(u,v)dx.$$

Solutions $(u,v) \in W^{1,2}(\mathbb{R}) \times W^{1,2}(\mathbb{R})$ of (2.1) are the critical points of $\Phi_{\lambda,\varepsilon}$. Similarly, we can consider the sub-space $W_e^{1,2}(\mathbb{R})$ of the functions $u \in W^{1,2}(\mathbb{R})$ which are even, $u(-x) = u(x)$. Critical points of $\Phi_{\lambda,\varepsilon}$ on $W_e^{1,2}(\mathbb{R}) \times W_e^{1,2}(\mathbb{R})$ are solutions (u,v) of (2.1) such that both the two components u and v are even. These solutions will be called *symmetric solutions*.

If $\varepsilon = 0$, (2.1) is decoupled and has the solutions $(\pm U_\lambda, 0)$ and $(0, \pm U_\lambda)$ as well as $(\pm U_\lambda, \pm U_\lambda)$, where

$$U_\lambda(x) = \frac{\sqrt{2\lambda}}{\cosh(\sqrt{\lambda}\, x)}$$

denotes the even positive solution of

$$-u'' + \lambda u = u^3,\ u \in W^{1,2}(\mathbb{R}). \tag{2.2}$$

Of course, for all $\xi \in \mathbb{R}$, any translated function $U_\lambda(\cdot + \xi)$ is also a solution of (2.2) in $W^{1,2}(\mathbb{R})$. On the other hand, U_λ is the unique solution of (2.2) in $W_e^{1,2}(\mathbb{R})$, see [9]. In the sequel we will work near the pair (U_λ, U_λ) or, in the next section, near $(U_\lambda(\cdot + \xi), U_\lambda)$. It is clear that the same arguments could be carried out for any of the other pairs $(U_\lambda, -U_\lambda)$, $(-U_\lambda, -U_\lambda)$, etc..

In the following remark we collect some well known facts (see e.g. Lemmas 3.2 and 8.6 of [7]) on the spectral properties of $I''_\lambda(U_\lambda)$, namely on the eigenvalues of the linearized equation

$$-\phi'' + \lambda\phi - 3U_\lambda^2(\cdot + \xi)\phi = \omega\phi, \qquad \phi \in W^{1,2}(\mathbb{R}). \tag{2.3}$$

Remark 2.1 (a) The linearized problem (2.3) has one negative eigenvalue, corresponding to the fact that $U_\lambda(\cdot + \xi)$ is a Mountain Pass critical point of I_λ on $W^{1,2}(\mathbb{R})$. The same holds for $U_\lambda \in W_e^{1,2}(\mathbb{R})$ which is a Mountain-Pass critical point of I_λ on $W_e^{1,2}(\mathbb{R})$.

(b) $\omega = 0$ is an eigenvalue of (2.3) and $Ker[I''_\lambda(U_\lambda(\cdot+\xi))]$ is spanned by $U'_\lambda(\cdot+\xi)$.

(b') Since U'_λ is odd function, then it follows from (b) that $Ker[I''_\lambda(U_\lambda)] = \{0\}$, provided we work in the space of even functions $W_e^{1,2}(\mathbb{R})$;

(c) the rest of the spectrum of $I''_\lambda[U_\lambda(\cdot + \xi)]$ is positive.

We are now in position to prove the following result concerning the existence of symmetric solutions to (2.1), near to $(\pm U_\lambda, \pm U_\lambda)$.

Theorem 2.1 *Given $\Lambda > 0$ there exists $\varepsilon_0 > 0$ such that for all $\lambda \in (0, \Lambda]$ and $\varepsilon \in (0, \varepsilon_0]$, (2.1) has a unique symmetric solution $(u_{\lambda,\varepsilon}, v_{\lambda,\varepsilon}) \in W_e^{1,2}(\mathbb{R}) \times W_e^{1,2}(\mathbb{R})$ such that $u_{\lambda,\varepsilon} \to U_\lambda$, $v_{\lambda,\varepsilon} \to U_\lambda$, as $\varepsilon \to 0$.*

Proof. We follow the arguments of Theorem 6.4 in [5]. From the preceding Remark one infers that (U_λ, U_λ) is a non-degenerate critical point of the unperturbed functional $\Phi_{\lambda,0}$ on $W_e^{1,2}(\mathbb{R}) \times W_e^{1,2}(\mathbb{R})$. Then a straight application of the Local Inversion theorem yields, for ε small enough, the existence of $(u_{\lambda,\varepsilon}, v_{\lambda,\varepsilon}) \in W_e^{1,2}(\mathbb{R}) \times W_e^{1,2}(\mathbb{R})$ such that $\nabla\Phi_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}, v_{\lambda,\varepsilon}) = 0$, with $(u_{\lambda,\varepsilon}, v_{\lambda,\varepsilon}) \to (U_\lambda, U_\lambda)$ as $\varepsilon \to 0$.

3 Existence of a-symmetric solutions

We are going to show that, in addition to the symmetric solutions found above, there exist solutions of (2.1) which have the first component a-symmetric. For this reason, we will work in the space $X := W^{1,2}(\mathbb{R}) \times W_e^{1,2}(\mathbb{R})$. In the sequel we will use the abstract tools carried out in [7] to find critical points of perturbed functional like $\Phi_{\lambda,\varepsilon}$. Though we assume that the reader is familiar with these topics, let us sketch, using the present notation, the specific result we need. Setting $G(u, v) = \int_{\mathbb{R}} g(u, v)dx$, let us write $\Phi_{\lambda,\varepsilon}(u, v) = \Phi_{\lambda,0}(u, v) - \varepsilon G(u, v)$. The unperturbed functional $\Phi_{\lambda,0}$ has on X a one-dimensional manifold of critical points, namely

$$Z_\lambda = \{z_{\lambda,\xi}(x) = (U_\lambda(x + \xi), U_\lambda(x)) : \xi \in \mathbb{R}\}.$$

Let $T_{z_{\lambda,\xi}} Z_\lambda$ denote the tangent space to $Z_\lambda \subset X$ at $z_{\lambda,\xi}$.

Lemma 3.1 *Z_λ is a non-degenerate critical manifold of $\Phi_{\lambda,0}$ on X, in the sense that $Ker[\Phi''_{\lambda,0}(z_{\lambda,\xi})] = T_{z_{\lambda,\xi}} Z_\lambda$.*

Proof. One has that $(\phi, \psi) \in X$ belongs to $Ker[\Phi''_{\lambda,0}(z_{\lambda,\xi})]$ iff

$$\begin{cases} I''_\lambda(U_\lambda(\cdot+\xi))[\phi] &= 0, \quad \phi \in W^{1,2}(\mathbb{R}), \\ I''_\lambda(U_\lambda)[\psi] &= 0, \quad \psi \in W^{1,2}_e(\mathbb{R}). \end{cases}$$

From Remark 2.1-(b') it follows that $\psi = 0$. Using Remark 2.1-(b) we infer that $\phi = U'_\lambda(\cdot+\xi)$. Hence $Ker[\Phi''_{\lambda,0}(z_{\lambda,\xi})] = \{(U'_\lambda(\cdot+\xi), 0)\} = T_{z_{\lambda,\xi}} Z_\lambda$.

The preceding Lemma enables us to use Theorem 2.16 of [7] which asserts that if Z_λ is non-degerate and if ξ^* is a strict isolated local maximum or minimum of $\Gamma_\lambda = G_{|Z_\lambda}$, then $\Phi_{\lambda,\varepsilon}$ has a critical point near z_{ξ^*}, for $\varepsilon \ll 1$. In our case, this result can be formulated as follows:

Lemma 3.2 *Any strict isolated local maximum or minimum ξ^* of*

$$\Gamma_\lambda(\xi) = \int_{\mathbb{R}} \left[U^2_\lambda(x+\xi)U^2_\lambda(x) - U_\lambda(x+\xi)U_\lambda(x)\right] dx$$

gives rise to a solution $(u^*_{\lambda,\varepsilon}, v^*_{\lambda,\varepsilon}) \in X$ of (2.1), provided ε is sufficiently small. Moreover, $u^*_{\lambda,\varepsilon} \to U_\lambda(\cdot+\xi^*)$ and $v^*_{\lambda,\varepsilon} \to U_\lambda$, as $\varepsilon \to 0$.

In order to apply such a result, let us study the behavior of $\Gamma_\lambda(\xi)$. Let us recall the following elementary formulas

$$\int_{\mathbb{R}} \frac{1}{\cosh^p x} dx = \begin{cases} 2 & \text{if } p = 2, \\ \frac{4}{3} & \text{if } p = 4, \\ \frac{16}{15} & \text{if } p = 6, \end{cases}$$

which imply

$$\int_{\mathbb{R}} U^p_\lambda(x) dx = \begin{cases} 4\lambda^{1/2} & \text{if } p = 2, \\ \frac{16}{3}\lambda^{3/2} & \text{if } p = 4, \\ \frac{8\cdot 16}{15}\lambda^{5/2} & \text{if } p = 6. \end{cases} \tag{3.1}$$

First, a direct application of the Dominated convergence theorem yields

$$\lim_{|\xi|\to\infty} \Gamma_\lambda(\xi) = 0. \tag{3.2}$$

Moreover, there holds

$$\Gamma_\lambda(\xi) < 0, \qquad \forall\, |\xi| \gg 1. \tag{3.3}$$

Next, from (3.1) we immediately get

$$\Gamma_\lambda(0) = \int_{\mathbb{R}} \left[U_\lambda^4(x) - U_\lambda^2(x)\right] dx = 4\sqrt{\lambda}\,(\tfrac{4}{3}\lambda - 1), \tag{3.4}$$

and $\Gamma'_\lambda(0) = 0$. Moreover, using also (2.2), one has

$$\begin{aligned}\Gamma''_\lambda(0) &= \int_{\mathbb{R}} \Big[2U_\lambda^2(x)U_\lambda^{'2}(x) + 2U_\lambda^3(x)U''_\lambda(x) - U_\lambda(x)U''_\lambda(x)\Big]\, dx \\ &= \int_{\mathbb{R}} \Big[2U_\lambda^2(x)U_\lambda^{'2}(x) + \big(2U_\lambda^3(x) - U_\lambda(x)\big)\,\big(\lambda U_\lambda(x) - U_\lambda^3(x)\big)\Big]\, dx.\end{aligned}$$

With elementary calculation we find

$$\begin{aligned}\int_{\mathbb{R}} \Big[2U_\lambda^2(x)U_\lambda^{'2}(x)\Big]\, dx &= 8\,\lambda^{5/2} \int_{\mathbb{R}} \frac{\sinh^2 x}{\cosh^6 x} dx \\ &= 8\,\lambda^{5/2} \int_{\mathbb{R}} \left[\frac{1}{\cosh^4 x} - \frac{1}{\cosh^6 x}\right] dx = \frac{32}{15}\,\lambda^{5/2}.\end{aligned}$$

Furthermore, using again (3.1) we get

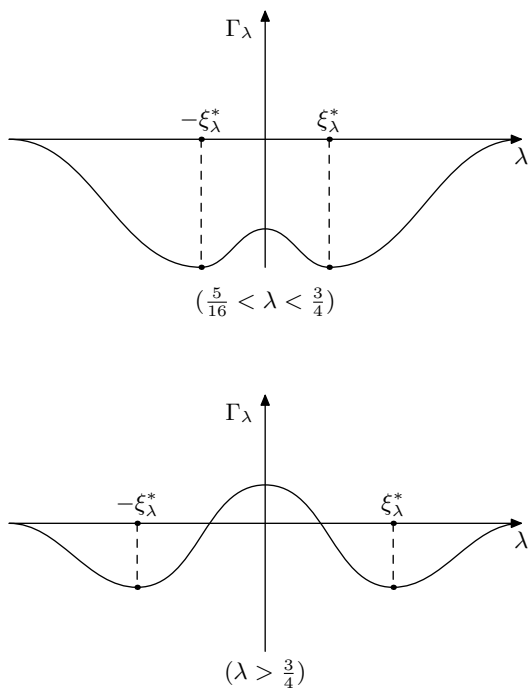

Figure 1: Graph of $\Gamma_\lambda(\xi)$ for $\lambda > \frac{5}{16}$.

$$\int_{\mathbb{R}} \big(2U_\lambda^3(x) - U_\lambda(x)\big)\,\big(\lambda U_\lambda(x) - U_\lambda^3(x)\big)\, dx = \frac{4}{3}\lambda^{3/2}(1 - \frac{24}{5}\lambda).$$

In conclusion, one has that $\Gamma''_\lambda(0) = \frac{4}{3}\lambda^{3/2}(1 - \frac{16}{5}\lambda)$.

Collecting the preceding informations, we infer that for any $\lambda > \frac{5}{16}$ there exists $\xi^*_\lambda > 0$ such that Γ_λ has a strict isolated minimum at $\pm\xi^*_\lambda$. Then, according to Lemma 3.2, we can state the following existence result.

Theorem 3.1 *Given $\Lambda > 0$ there exists $\varepsilon_1 > 0$ such that for all $\lambda \in (\frac{5}{16}, \Lambda]$ and $\varepsilon \in (0, \varepsilon_1]$, (2.1) has an a-symmetric solution $(u^*_{\lambda,\varepsilon}, v^*_{\lambda,\varepsilon}) \in X$, resp. $(\widetilde{u}^*_{\lambda,\varepsilon}, \widetilde{v}^*_{\lambda,\varepsilon}) \in X$. Moreover, one has that*

(i) $u^*_{\lambda,\varepsilon} \to U_\lambda(\cdot + \xi^*_\lambda)$, resp. $\widetilde{u}^*_{\lambda,\varepsilon} \to U_\lambda(\cdot - \xi^*_\lambda)$ as $\varepsilon \to 0$;

(ii) $v^*_{\lambda,\varepsilon} \to U_\lambda$ and $\widetilde{v}^*_{\lambda,\varepsilon} \to U_\lambda$ as $\varepsilon \to 0$.

Remark 3.1 For all $\lambda > 0$, the reduced functional Γ_λ has a stationary point at $\xi = 0$, which is a minimum for $\lambda < \frac{5}{16}$, and a maximum for $\lambda > \frac{5}{16}$. This $\xi = 0$ gives rise to the symmetric solution $(u_{\lambda,\varepsilon}, v_{\lambda,\varepsilon}) \sim (U_\lambda, U_\lambda)$, found in Theorem 2.1. Moreover, we can evaluate the Morse index of these solutions. Precisely, since each U_λ has Morse index 1, see Remark 2.1, then the unperturbed solution (U_λ, U_λ) has Morse index equal to 2 on X. Using Theorem 2.24 of [7], it follows that the Morse index of $(u_{\lambda,\varepsilon}, v_{\lambda,\varepsilon})$ is 2 when $\xi = 0$ is a minimum, namely for $\lambda < \frac{5}{16}$. On the other hand, when $\xi = 0$ is a maximum, namely for $\lambda > \frac{5}{16}$ the Morse index of $(u_{\lambda,\varepsilon}, v_{\lambda,\varepsilon})$ is 3. This change of index implies that $\lambda = \frac{5}{16}$ is a bifurcation point for the family $C_\varepsilon = \{(u_{\lambda,\varepsilon}, v_{\lambda,\varepsilon}) : \lambda \in (0, \Lambda]\}$. The solutions branching off from C_ε are nothing but the a-symmetric solutions $(u^*_{\lambda,\varepsilon}, v^*_{\lambda,\varepsilon})$ (jointly with $(\widetilde{u}^*_{\lambda,\varepsilon}, \widetilde{v}^*_{\lambda,\varepsilon})$) found in Theorem 3.1. Since these solutions correspond to a minimum of Γ_λ, another application of Theorem 2.24 of [7] shows that these a-symmetric solutions have Morse index 2.

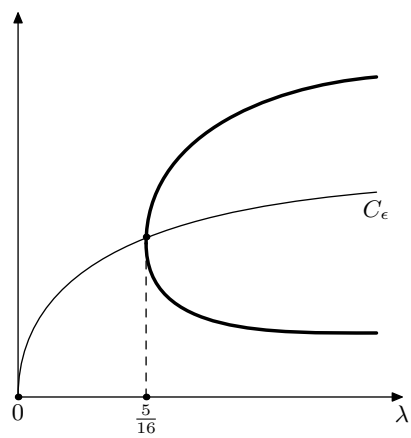

Figure 2: Bifurcation diagram of symmetric and a-symmetric solutions (the latter ones correspond to the bold line).

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