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Uniqueness and Generic Non-degeneracy on Strips

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Abstract

In this paper, we prove the uniqueness of the positive decaying solution of $-\Delta u = u^p - \lambda u$ on $S, u = 0$ on ∂S where S is an infinite strip in R^n and $p > 1$ but close to 1. We can use this and real analyticity to prove that for "most" p in $(1, (n+2)/(n-2))$ our problem has a nondegenerate positive decaying solution.

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1 Introduction

In this paper, we prove two results for the problem

$$\begin{aligned} &-\Delta u = u^p - \lambda u \qquad \text{on } S = \widetilde{\Omega} \times \mathbb{R}^{n-k} \\ &u > 0 \text{ in } S \\ &u = 0 \text{ on } \partial S \\ &u \longrightarrow 0 \text{ as } |x| \longrightarrow \infty (x \in S). \end{aligned} \tag{1.1}$$

(More precisely if we write $x = (\widetilde{x}, x')$ with $\widetilde{x} \in \widetilde{\Omega} \subseteq \mathbb{R}^k, x' \in \mathbb{R}^{n-k}$, we require $u \longrightarrow 0$ uniformly in $\widetilde{x}$.) Here we assume $\widetilde{\Omega}$ is a bounded domain in $\mathbb{R}^k$ with smooth boundary.

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We prove two theorems. Firstly if $\lambda > -\lambda_1$ (where λ_1 denotes the first eigenvalue of $-\Delta_k$ on $\widetilde{\Omega}$ with Dirichlet boundary conditions), then the solution of (1.1) is unique up to translation in x' if p is close to 1 but $p > 1$. We then use this to prove that for fixed $\widetilde{\Omega}$ then for all p in $(1, (n+2)/(n-2)$, except for isolated points, (1.1) has a non-degenerate positive solution. Here we say a solution u is non-degenerate if the kernel of the linearized operator in $L^\infty(S)$ has kernel spanned by $\{\frac{\partial u}{\partial x_i}\}_{i=k+1,n}$. Note that we do not expect the solution to be unique. The second result follows by combining the first result with the techniques of [5]. Indeed the second result was proved in [5] for $n = 3$ and $\widetilde{\Omega}$ rather symmetric. (In this special case, the result there and the result here imply that the exceptional set of p is finite.) The difference here is that we prove uniqueness for p close to 1 while in [5], we prove uniqueness results for p close to $(n+2)/(n-2)$. Note that non-degeneracy here is much more difficult than for problems on $\mathbb{R}^n$ because the problem can not be reduced to an ordinary differential equation.

The uniqueness result is much more difficult to prove than the corresponding result on bounded domains (which was proved in [5]) because the asymptotic behaviour of solutions as p approaches 1 is much more complicated with rather different asymptotic behaviour in $\widetilde{x}$ and x'. More precisely as p approaches 1, solutions of (1.1) look like $bz_0((p-1)^{\frac{1}{2}}x')\phi_1(\widetilde{x})$ where $b > 0, \phi_1$ is the positive eigenfunction of $-\Delta_k$ on $\widetilde{\Omega}$ corresponding to λ_1 for Dirichlet boundary conditions and z_0 is the positive radial decaying solution of $-\Delta_{n-k}v = v$ log on $\mathbb{R}^{n-k}$. Here for *simplicity* we assume $\lambda_1 = 1$ and $\lambda = 0$. The log term comes about because $y^p - y = (p-1)y$ log $y + o(p-1)$ for p close to 1). Indeed, this asymptotic behaviour is the key step in proving uniqueness.

The techniques we develop to prove uniqueness seem likely to have a number of other uses. Indeed, we will show in a separate paper that they can be modified to solve an old problem of ours [6] on the behaviour of solutions near turning points of nonlinear elliptic problems on long domains.

It might be asked why we are interested in problems on strips. They arise in at least 3 ways. Firstly (and this was the immediate motivation for the present work), Shusen Yan and I [12] were considering the problem of multipeak positive solutions

$$-\Delta u = u^p - \lambda u$$

the "annular" domain on $\Omega_R = \{x' + t\nu(x') : R \leq t \leq R+1, x' \in \partial\Omega\}$ with R large and Dirichlet boundary conditions on $\partial\Omega_R$. Here Ω is a smooth bounded convex domain in $\mathbb{R}^n, 0 \in \Omega$ and $\nu(x')$ is the outward normal to Ω at $x' \in \partial\Omega$. It turns out that the natural limit problem is the strip problem with $\widetilde{\Omega}$ a ball and $k = 1$. Here we need to know there are non-degenerate solutions on strips (and uniqueness is unimportant). Our second theorem shows that this usually holds. As another application, the theory in our work [12] shows that strip problems are the natural limit problems for unstable solutions of $-\Delta u = u^p - \lambda u$ on long domains and the non-degeneracy problem arises there. Thirdly, the strip problem occurs as a natural

limiting problem when one looks for positive solutions for small ϵ

$$\Delta u = u^p \text{ on } \Omega_\epsilon$$
$$u = 0 \text{ on } \partial\Omega_\epsilon,$$

where the dumbell $\Omega_\epsilon = B_1(-\mu e_1) \cup B_1(\mu e_1) \cup \{(\widetilde{x}, x') : \widetilde{x} \in \epsilon\Omega, -\mu \leq x' \leq \mu\}$ with $\widetilde{x} \in \mathbb{R}^{n-1}, x' \in \mathbb{R}$ and $\mu > 1$. If we look for solutions concentrating in the joining strip, then once again the strip problem with $\widetilde{\Omega} = \Omega$ and $k = n-1$ is the natural limit problem [8], and we want non-degenerate solutions. In this case, Byeon [3] has managed to prove the result without assuming non-degeneracy (improving the result in [8]) but it seems that Wei and Liqun have managed to weaken the condition that the balls have the same radius at the expense of using the non-degeneracy again.

It seems likely that our techniques are useful for a number of other problems and for other nonlinearities. In general, it seems that analyticity has not been used sufficiently in the study of nonlinear equations.

2 Uniqueness and non-degeneracy for p close to 1

Theorem 2.1 *Assume that $\lambda > -\lambda_1$. Then there is a $\delta > 0$ such that if $1 < p < 1 + \delta$, (1.1) has a unique solution u (up to translation) and this solution is non-degenerate (that is the kernel in $L^\infty(S)$ is spanned by $\{\frac{\partial u}{\partial x_i}\}_{i=k+1,\cdots n}$.)*

The proof of this will take the whole section. First note that it is well known (cp [2] or [8]) that solutions decay exponentially in x' (uniformly in $\widetilde{x}$) and by a moving plane argument a translation of u is radial in x' and is decreasing in $|x'|$.

Thus it suffices to prove the uniqueness in the space of functions radial in x'. Indeed, by the arguments in the appendix to [12] (or [11]), it also suffices to prove the non-degeneracy in the more usual sense in the space of functions radial in x'. Asymptotically for large x', the solutions of (1.1) are of the form $g(|x'|)\phi_1(\widetilde{x})(1 + o(1))$ where $g(y) = y^{-\frac{1}{2}(n-k-1)} \exp(-(\lambda + \lambda_1)^{\frac{1}{2}} y)$ and the $o(1)$ is exponentially small in the C^1 norm for large $|x'|$ (cp [6] or [8]).

As one would expect, a key step in the proof of the theorem is to obtain the asymptotic behaviour of solutions. Before doing this note that by a transformation $\alpha^{2/(p-1)}u(\alpha x)$, we may assume that $\lambda_1(\Omega) = 1$. (Note that map $x \mapsto \alpha x$ maps a strip to another strip, that λ may change in this transformation but the condition $\lambda > -\lambda_1$ remains true.) We assume $\lambda_1 = 1$ henceforth. (Thus $\lambda > -1$).

Lemma 2.1 *Suppose $\epsilon > 0$. We can find $a, b > 0$ independent of ϵ and $\delta > 0$ such that, if $1 < p < 1 + \delta$, every solution u of (1.1) satisfies*

$$\|(1+\lambda)^{-1/(p-1)}u - bz_0((p-1)^{\frac{1}{2}}a^{\frac{1}{2}}|x'|)\phi_1(\widetilde{x})\|_\infty \leq \epsilon.$$

Proof Step 1. We first let $u = \|u\|_\infty v$. Hence

$$-\Delta v = (\|u\|_\infty)^{p-1}v^p - \lambda v \tag{2.1}$$

on $S, v = 0$ on $\partial S, \|v\|_\infty = 1$. We first prove that $(\|u\|_\infty)^{p-1} \longrightarrow 1 + \lambda$ as $p \longrightarrow 1$. We first use a blowing up argument to prove that $(\|u\|_\infty)^{p-1}$ is bounded. If not we may, by taking subsequences, assume that we have solutions u_i for $p = p_i$ such that $(\|u_i\|_\infty)^{p_i - 1} \longrightarrow \infty$ and $p_i \longrightarrow 1$ as $i \longrightarrow \infty$. In this case, a rather standard blowing-up argument (cp p 441 of [10]) shows that we have a non-trivial positive bounded solution $\tilde{v}$ of

$$-\Delta \tilde{v} = \tilde{v} \tag{2.2}$$

on $\mathbb{R}^n$ (or on a half space T with $\tilde{v} = 0$ on T). Whether the limit is a half space or all of $\mathbb{R}^n$ depends on how close the maximum of v is to ∂S. Note also that we have used here that $y^p \longrightarrow y$ uniformly on $[0,1]$ as $p \longrightarrow 1$. As in [5], it is well known that such a $\tilde{v}$ can not exist. Hence we see that $(\|u\|_\infty)^{p-1}$ is bounded. Hence we can choose a subsequence so that $(\|u_i\|_\infty)^{p_i - 1} \longrightarrow \alpha$ as $i \longrightarrow \infty$. We can than easily pass to the limit in (2.1) on compact sets of $\overline{S}$ and obtain a non-trivial bounded positive solution $\hat{v}$ of

$$-\Delta \hat{v} = (\alpha - \lambda)\, \hat{v} \text{ on } S, \hat{v} = 0 \text{ on } \partial S \tag{2.3}$$

with $\|\hat{v}\|_\infty = 1$. Here we have used that $\|v\|_\infty = 1$, that C^1 estimates ensure that v cannot take its maximum close to S and we have used again that $y^p \longrightarrow y$ uniformly on $[0,1]$ as $p \longrightarrow 1$. It is well known (cp [9]) and easy to prove that (2.3) can only have a non-trivial bounded positive solution if $\alpha - \lambda = \lambda_1(\Omega) = 1$, that is $\alpha = \lambda + 1$ and $\hat{v}(\tilde{x}, x') = \phi_1(\tilde{x})$ where ϕ_1 is normalized to have sup 1.

Step 2. Next we write $v = B(x')\phi_i(\tilde{x}) + w$ where w is orthogonal to ϕ_1 for each x'. We prove that w is small in the C^1 norm if p is close to 1. This will show that the asymptotics of solutions are largely determined by B. Now

$$-\Delta v - v = \gamma^{p-1}(v^p - v) + (\gamma^{p-1} - (1 + \lambda))v \tag{2.4}$$

on S where $\gamma = \|u\|_\infty$. Since $\gamma^{p-1} \longrightarrow 1 + \lambda$ as $p \longrightarrow 1$ and $y^p \longrightarrow y$ uniformly on $[0,1]$ as $p \longrightarrow 1$, we deduce that $-\Delta v - v$ is uniformly small on S if p is close to 1.

Let T be the subspace of $L^\infty(S)$ of functions orthogonal to $\phi_1(\tilde{x})$ for each x'. (Thus $w \in T$). It is easy to see that T and its orthogonal complement are invariant under $-\Delta$. Hence $-\Delta w - w$ is uniformly small. We will prove below that there is a $c > 0$ such that

$$\| - \Delta \tilde{w} - \tilde{w}\|_\infty \geq c \|\tilde{w}\|_\infty \tag{2.5}$$

for all $w \in Z \equiv \{z \in T : z \in W^{2,p}_{loc}(S), z = 0 \text{ on } \partial S, \Delta z \in L^\infty(S)\}$.

Assuming this for the moment, we deduce that w is uniformly small on S. Hence by standard regularity theory, w is also uniformly small in C^1 on S if p is close to 1.

It remains to prove (2.5). Note that while we eventually only need the estimate for z radially in x', it is easier to prove the inequality in general. If (2.5) fails, there exists

$$w_i \in Z, \|w_i\|_\infty = 1 \text{ so that } - \Delta w_i - w_i \longrightarrow 0 \tag{2.6}$$

uniformly on S as $i \longrightarrow \infty$. By a translation in x', we may assume $w_i(\widetilde{x}_i, 0) \geq \frac{1}{2}$ for some $\widetilde{x}_i \in \widetilde{\Omega}$. By C^1 estimates for $w_i, \widetilde{x}_i$ is not close to $\partial\widetilde{\Omega}$. Thus we pass to the limit in (2.6) on compact sets of $\overline{S}$ and find $\overline{w} \in Z$ so that $-\Delta\overline{w} = \overline{w}$ and $\overline{w}(\widetilde{x}, 0) \geq \frac{1}{2}$ for some $\widetilde{x} \in \Omega$. By completeness, there is an eigenfunction ϕ_j of $-\Delta$ on $\widetilde{\Omega}$ for Dirichlet boundary conditions on $\widetilde{\Omega}$ such that $\widetilde{r}(0) \neq 0$ where $\widetilde{r}(x') = \int_\Omega \overline{w}(\widetilde{x}, x')\phi_j(\widetilde{x})d\widetilde{x}$. Note that ϕ_i corresponds to an eigenvalue λ_i greater than 1 since $\overline{w} \in Z$. Hence $-\Delta_{n-k}\widetilde{r} = (1 - \lambda_i)\widetilde{r}$ on $\mathbb{R}^{n-k}$ and r is bounded and non-trivial. Since $\lambda_i > 1$ it is well known and easy to see (for example by spherical harmonic expansions) that this is impossible and hence we have a contradiction. Thus (2.5) holds, as required. (In fact, it can be proved $-\Delta w - w$ is invertible on T).

Step 3. Next we obtain better estimates for $\|u\|_\infty$. We prove that $a^{-1/(p-1)}\|u\|_\infty$ is bounded where $a = 1 + \lambda$.

If $\frac{\gamma^{p-1}-a}{p-1} \longrightarrow \infty$ as $p \longrightarrow 1$ we will get a contradiction by a blowing up argument. We first obtain an equation for B (defined in the previous step). By a simple calculation from (2.1), we see that

$$-\Delta_k B = \mu^{-1}(\gamma^{p-1} \int_{\widetilde{\Omega}} (B\phi_1 + w)^p \phi_1 - aB\mu) \tag{2.7}$$

on $\mathbb{R}^k$ where $\mu = \int_{\widetilde{\Omega}} \phi_1^2$. Note that $\|B\|_\infty$ is close to 1 since $\|\phi_1\|_\infty = 1$ and $\|w\|_\infty$ is small. Next note that, by an easy application of the mean value theorem,

$$|y^p - y| \leq C(p-1)y|\log y|$$

on $[0, 2]$. (We differentiate $y^p - y$ in p). Thus the right hand side of (2.7) is

$$\mu^{-1}(\gamma^{p-1} - a)\mu B + 0(p-1) = (\gamma^{p-1} - a)(B + o(1))$$

since we are assuming $\frac{\gamma^{p-1}-a}{p-1} \longrightarrow \infty$ as $p \longrightarrow 1$. Hence by a standard blowing up argument (using the rescaling $X' = (\gamma^{p-1} - a)^{\frac{1}{2}}x'$ on $\mathbb{R}^k$) we obtain a bounded positive radial non-trivial solution of $-\Delta_k\widetilde{B} = \widetilde{B}$ on $\mathbb{R}^k$. Since we see much as before that there is no such solution, it follows that $\frac{\gamma^{p-1}-a}{p-1}$ is bounded, i.e. $\frac{\frac{\gamma^{p-1}}{a}-1}{p-1} = \frac{e^{(p-1)\log(a^{-1/(p-1)}\gamma)}-1}{p-1}$ is bounded. Since $e^x \geq 1 + x$ for $x \geq 0$, it follows that $a^{-1/(p-1)}\gamma$ is bounded as required. Moreover,

$$\frac{\gamma^{p-1}-a}{p-1} = a \log(\gamma a^{-1/(p-1)}) + o(1). \tag{2.8}$$

We now use a more careful argument to determine the limit of γ and also to obtain a limiting equation for B (rescaled). We return to the right hand of (2.7). Since $y \log y$ is continuous on $[0, 1]$, we can easily use the mean value theorem to prove that $\frac{y^p - y}{p-1} \longrightarrow y \log y$ uniformly for $y \in [0, 2]$ as $p \longrightarrow 1$. Thus we can write the right

hand side of (2.7) as

$$(p-1)[\tfrac{\gamma^{p-1}-a}{p-1}\ B\ +\mu^{-1}\gamma^{p-1}\int_{\widetilde{\Omega}}(v\log v\phi_1+o(1))]$$
$$=(p-1)[\tfrac{\gamma^{p-1}-a}{p-1}\ B\ +\mu^{-1}a\int_{\widetilde{\Omega}}B\phi_1^2\log(B\phi_1)+o(1)].$$

Thus, if we rescale B by a factor of $(p-1)^{\frac{1}{2}}$ in x' and noting that the right hand side of (2.7) is bounded by $K(p-1)$, we can pass to the limit and deduce B rescaled converges on compact sets to $\widetilde{B}$ where $\widetilde{B}$ is bounded and non-negative on $\mathbb{R}^n$, $\|\widetilde{B}\|_\infty=1$ and

$$-\Delta_k\widetilde{B}=\widetilde{\tau}\widetilde{B}+\mu^{-1}a\widetilde{B}\int_\Omega\phi_1^2\log\phi_1+a\widetilde{B}\log\widetilde{B}.$$

Here $\widetilde{\tau}$ is a limit of $\frac{\gamma^{p-1}-a}{p-1}$ and we know that $\lim\|B\|_\infty=\|\widetilde{B}\|_\infty$ because both B and $\widetilde{B}$ are radial and decreasing in x'. (Note that B is radial because v is radial). Provided we know that $\widetilde{B}(r)\longrightarrow 0$ as $r\longrightarrow\infty$, we can also deduce uniform convergence of the B. After a little rescaling in the space variables, we can write our equation as

$$-\Delta_k\widetilde{B}=(\tfrac{\widetilde{\tau}}{a}-\hat{\mu})\widetilde{B}+\widetilde{B}\ \log\ \widetilde{B}. \tag{2.9}$$

$\|\widetilde{B}\|_\infty=1$ where $\hat{\mu}=-\mu^{-1}\int_{\widetilde{\Omega}}\phi_1^2\ \log\ \phi_1>0$. Here we are interested in non-negative decreasing radial solutions of (2.9). One possible solution is the constant solution $\widetilde{B}\equiv 1$. This is only possible of $\widetilde{\tau}=a\hat{\mu}$. We show this case cannot occur. If this case occurs, then since B (rescaled) converges to $\widetilde{B}$ uniformly on compact sets, B is uniformly close to 1 on compact sets. Thus if we choose $\widetilde{r}$ so that $B(\widetilde{r})=\frac{1}{2}$, $\widetilde{r}\longrightarrow\infty$ as $p\longrightarrow 1$. Hence if we use a blow up for the rescaled equation for B with origin at $\widetilde{r}$, we obtain in the limit a non-negative bounded decreasing solution of

$$-y''=y\ \log\ y$$

y is decreasing, $\|y\|_\infty\leq 1, y(0)=\frac{1}{2}$. (Note that $\widetilde{\tau}=a\hat{\mu}$ and that the term $\frac{k-1}{r}\ \frac{\partial B}{\partial r}$ (where $r=|X'|$) vanishes in the limit because neighbourhoods of $\widetilde{r}$ correspond to large r). Hence $y''\geq 0$. It is easy to see that y bounded and y convex and not constant leads to a contradiction. Thus this case does not occur.

Hence $\widetilde{B}(r)\longrightarrow 0$ as $r\longrightarrow\infty$ (which case we will return to in a moment) or $\widetilde{B}(r)\longrightarrow C>0$ as $r\longrightarrow\infty$ where $\frac{\widetilde{\tau}}{a}-\hat{\mu}+\log C=0$. A simple computation shows that C is a simple zero of $g(y)=(\frac{\widetilde{\tau}}{a}-\hat{\mu}+\log y)$ and $g(y)>0$ for $y>C$. Then $\widetilde{B}(r)-C$ is a positive decaying solution of $\Delta z=g(z+C)$ on R^k and well known standard results ensure that this case does not occur (cp Proposition 3 in [10]). Hence $\widetilde{B}(r)\longrightarrow 0$ as $r\longrightarrow\infty$. Then, by (2.9), $\widetilde{C}\widetilde{B}(r)$ is a positive decaying solution of

$$-\Delta z=z\ \log\ z \tag{2.10}$$

if

$$\log \widetilde{C} = a^{-1}\widetilde{\tau} - \hat{\mu}. \tag{2.11}$$

Then $\|\widetilde{C}\widetilde{B}(r)\|_\infty = \widetilde{C}$. Now by a slight modification of the main theorems in [17] (or [7]) this equation has a unique positive decaying radial solution z_0. We need to make a few remarks here. The proof of uniqueness is by shooting from the left hand end ($r = 0$). Note that existence and uniqueness for our problem is fine as long as the solutions stay positive and that it is easy to check that positive solution cannot join up with the trivial solution for $r > 0$. Note that the non-degeneracy part of the uniqueness proof is discussed later in this section. (There we prove a non-degeneracy result on infinite intervals. We also need a similar non-degeneracy result on finite intervals. This is proved by essentially the same argument.) Moreover $\widetilde{C}$ (and hence, by (2.11), $\widetilde{\tau}$) is determined by the equation $\|z_0\|_\infty = \widetilde{C}$ (and $\widetilde{B}(r) = (\widetilde{C})^{-1}z_0(r)$). Hence, by (2.9), we have shown that asymptotically $a^{-1/(p-1)}u$ is

$$e^{\hat{\mu}} z_0(a^{\frac{1}{2}}(p-1)^{\frac{1}{2}}|x'|)\phi_1(\widetilde{x}) .$$

Hence we have determined the asymptotics of u which completes the proof of Lemma 2.1.

Remark 2.1 The moving plane techniques of Ni and Li [18] can be easily modified to show that any positive decaying solution of

$$-\Delta B = B \log B$$

in R^k is radial (up to translation).

Proof of Theorem 2.1 The uniqueness and non-degeneracy proofs are very similar. We start with uniqueness. Suppose by way of contribution that u_1 and u_2 are distinct positive decaying solutions on S for p close to 1. Since our nonlinearity is convex, it is easy to see that $u_1 - u_2$ changes sign in S. Let $z = u_1 - u_2$. Then z is a changing sign decaying solution on S of

$$\begin{aligned} -\Delta z &= (p\widetilde{u}^{p-1} - \lambda)z \text{ on } S \\ z &= 0 \text{ on } \partial S, \end{aligned} \tag{2.12}$$

where $\widetilde{u}(x)$ is between $u_1(x)$ and $u_2(x)$ for each x. We will normalize z so $\|z\|_\infty = 1$. We will write $z = \widetilde{\gamma}(x')\phi(\widetilde{x}) + w_1$ where $w_1 \in Z$. Our main difficulties occur because u^{p-1} does not converge well to 1 as $|x'| \longrightarrow \infty$. Note that, by the equation, z is bounded in C^1 and thus is uniformly small close to ∂S. We first prove that w_1 is C^1 small if p is close to 1. It suffices to prove that w_1 is uniformly small because we can use regularity properties of the Laplacian. Note that $\widetilde{u}^{p-1}$ is uniformly bounded by Lemma 2.1.

Before proving these results, we need some extra information about solutions. Assume that $K_1 \subset \widetilde{\Omega}$ is compact in $\widetilde{\Omega}$ and K_2 is compact in $\mathbb{R}^{n-k}$. Then

$\sup_{T(x_0)} u \,/\, \inf_{T(x_0)} u \leq C_3$ where C_3 is independent of the solution and of x_0. Here $T(x_0) = K_1 \times (x_0 + K_2)$. This follows from the Harnack inequality (cp [15]) applied to a translate of u and since $-\Delta u = (u^{p-1} - \lambda)u$ (and by Lemma 2.1). Thus

$$(\sup_{T(x_0)} u / inf_{T(x_0)} u)^{\frac{1}{p-1}} \leq C_3^{1/(p-1)}$$

and hence we see that u^{p-1} is nearly constant on such a set $T(x_0)$ uniformly in x_0. There is a slight problem here in that we need these estimate for $\tilde{u}$ rather than u_1 or u_2 (where $\tilde{u}(x)$ is between $u_1(x)$ and $u_2(x)$). If $u_1(x)$ and $u_2(x)$ are of comparable magnitude at one point of $T(x_0)$ and hence all points of $T(x_0)$), we can argue as before. On the other hand if $u_1(x)/u_2(x) \leq s$ on $T(x_0)$ where s is small, then by the original construction of $\tilde{u}$

$$p(\tilde{u}(x))^{p-1} = \frac{u_2(x)^p - u_1(x)^p}{u_2(x) - u_1(x)} = u_2(x)^{p-1} \frac{1 - s(x)^p}{1 - s(x)}, \tag{2.13}$$

where $s(x) = \frac{u_1(x)}{u_2(x)}$. Since $1 - s^p$ converges uniformly to $1 - s$ on $[0,1]$ as $p \longrightarrow 1$, we see that $\frac{1-s^p}{1-s}$ converges uniformly to 1 on $[0, \frac{1}{2}]$. Thus, by (2.13), we see that in this case, $(1-\epsilon)(u_2(x))^{p-1} \leq p((\tilde{u}(x))^{p-1} \leq (1+\epsilon)(u_2(x))^{p-1}$ and we can argue as before. Hence in all case $p\tilde{u}^{p-1}$ is nearly constant on $T(x_0)$. Let $I - P$ be the projection on $L^\infty(S)$ defined by

$$(I - P)v = \mu^{-1}\phi_1(\tilde{x}) \int_{\tilde{\Omega}} v(\tilde{x}, x')\phi_1(\tilde{x})d\tilde{x}.$$

It is easy to see that P is the natural projection onto T. Then our equation for w_1 is

$$-\Delta w_1 = P((p\tilde{u}^{p-1} - \lambda)w_1) + P((p\tilde{u}^{p-1} - \lambda)\tilde{\gamma}\phi_1). \tag{2.14}$$

We next show that $P((p\tilde{u}^{p-1} - \lambda)\tilde{\gamma}\phi_1) = \tilde{\gamma}P(p\tilde{u}^{p-1} - \lambda)\phi_1)$ is uniformly small if p is close to 1. If $K_1 \subset \tilde{\Omega}$ is compact, then by the result above $p\tilde{u}^{p-1} - \lambda$ is uniformly close to a constant C on K_1 (uniform in x_0 but where the constant may depend on x_0) and hence $(p\tilde{u}^{p-1} - \lambda)\phi_1$ is uniformly close to $C\phi_1$. Note that $\tilde{u}^{p-1}$ is uniformly bounded and ϕ_1 vanishes on $\partial\tilde{\Omega}$. Hence $P((p\tilde{u}^{p-1} - \lambda)\phi_1)$ is uniformly small (since $P\phi_1 = 0$). Since $\tilde{\gamma}$ is bounded, this proves our claim.

Now suppose there exist $x_i \in S$ such that $|w_1^i(x_i)| \geq \alpha > 0$ for p_i arbitrarily close to 1 (where w_1^i is w_1 for $p = p_i$). We shift origin to x_i and pass to the limit in (2.14) and find a subsequence of $w_1^i(x - x_i)$ converge uniformly on compact subsets of S to a bounded solution $\tilde{w} \in Z$ of

$$\begin{aligned} -\Delta\tilde{w} &= (C - \lambda)\tilde{w} \text{ on } S \\ \tilde{w} &= 0 \text{ on } \partial S, \end{aligned} \tag{2.15}$$

where C is constant, $C \leq 1 + \lambda$. We need to explain the limit of the term $P((p_i\tilde{u}^{p_i-1} - \lambda)w_1^i)$. Firstly note that by (2.14) the w_1^i are uniformly bounded

in C^1 and thus uniformly small near ∂S. Hence the boundary condition is preserved. Moreover, much as in the previous paragraph, we can deduce $(p_i(\widetilde{u}(x - x_i))^{p_i-1} - \lambda)w_1^i$ is uniformly close to Cw_1^i on sets $\widetilde{\Omega} \times K_2$ for some C (possibly depending on i). Hence $P(p^i(\widetilde{u}(x - x_i))^{p_i-1} - \lambda)w_1^i)$ is uniformly close to CPw_1^i on the same sets. Thus we can pass to the limit and obtain (2.15). Note that since $|w_1^i|$ achieves maximum away from the boundary, $\widetilde{w}$ is non-trivial. Note that the estimates of Lemma 2.1 ensure that $\|\widetilde{u}\|_\infty \leq C_1(1+\lambda)^{p-1}$ and hence we obtain our estimate for C. Since $\widetilde{w} \in Z$, we can easily use eigenfunction expansions as before to show (2.15) has no non-trivial solution in Z. Hence we have a contradiction and thus $w_1 \longrightarrow 0$ uniformly on S as $p \longrightarrow 1$. As usual, we can then use regularity properties of the Laplacian to deduce that $w_1 \longrightarrow 0$ uniformly on S in C^1 as $p \longrightarrow 1$.

This implies that $|w_1(x)) \leq \epsilon|\widetilde{\gamma}(x')|\phi_1(\widetilde{x})$ whenever $\widetilde{\gamma}(x')$ is not small (and p is close to 1).

Much as in the proof of Lemma 2.1, we now obtain an equation for $\widetilde{\gamma}$. We have that

$$-\Delta_{n-k}\widetilde{\gamma} = \mu^{-1}\int_\Omega (p\widetilde{w}^{p-1} - a)z\phi_1 d\widetilde{x}. \tag{2.16}$$

Let

$$\begin{aligned} R = \tfrac{p\widetilde{w}^{p-1}-a}{p-1} &= \widetilde{w}^{p-1} + \tfrac{\widetilde{w}^{p-1}-a}{p-1} \\ &= \widetilde{w}^{p-1} + a(p-1)^{-1}[\exp\ ((p-1)\log(\widetilde{u}/a^{\frac{1}{p-1}})) - 1]. \end{aligned} \tag{2.17}$$

By Lemma 2.1, it follows easily that R is bounded above (uniformly in p).

We now consider when $\widetilde{\gamma}$ can be close to a positive maximum (or a negative minimum). Note that since $\|z\| = 1$ and $\|w_1\|_\infty$ is small, $|\widetilde{\gamma}|$ must be close to 1 at such a point.

If $\frac{\widetilde{u}}{a^{\frac{1}{p-1}}} \leq e^{-T}$ on $\widetilde{\Omega}$ where T is large positive when γ has such a maximum,

$$\tfrac{e^{(p-1)\ \log\ (\frac{\widetilde{u}}{a^{\frac{1}{p-1}}})}-1}{p-1} \leq \tfrac{e^{-(p-1)T}-1}{p-1} \longrightarrow -T \text{ as } p \longrightarrow 1.$$

Hence we we see this term is large negative on $\widetilde{\Omega}$ in this case. Thus $R(\widetilde{x}, x')$ is large negative on $\widetilde{\Omega}$. On the other hand since $|\widetilde{\gamma}(x')|$ is close to 1 and w_1 is C^1 small

$$(1+\epsilon)|\widetilde{\gamma}(x')|\phi_1(\widetilde{x}) \geq |\widetilde{\gamma}(x')\phi_1(\widetilde{x}) + w_1(x)| \geq (1-\epsilon)|\widetilde{\gamma}(x')|\phi_1(\widetilde{x})$$

on $\widetilde{\Omega}$ and $\widetilde{\gamma}(x')\phi_1(\widetilde{x}) + w_1(x)$ and $\widetilde{\gamma}(x')$ have the same sign on $\widetilde{\Omega}$. Hence by (2.16) and (2.17) see that $-\Delta_{n-k}\widetilde{\gamma}(x') = (p-1)\tau\widetilde{\gamma}(x')$ where τ is large negative. Thus $\widetilde{\gamma}$ can not have a local maximum or minimum at such x'.

By this and Lemma 2.1, we see that $|\gamma|$ can only have its maximum at points x' where $\widetilde{u}/a^{\frac{1}{p-1}}$ is not small and hence at points where $(p-1)^{\frac{1}{2}}|x'|$ is not large. We now want to work in the rescaled variable $X' = a^{\frac{1}{2}}(p-1)^{\frac{1}{2}}x'$. Note that this is a

slightly different rescaling to earlier. By what we have proved, $|\tilde{\gamma}|$ has its maximum at a bounded X' and on bounded X' sets $\tilde{u}(x', \tilde{x})$ is not small unless $\tilde{x}$ is close to the boundary.

On bounded X' sets, w is C^1 small and B is uniformly close to $\tilde{B}$ (with the notation of the proof of Lemma 2.1) and hence for these X' we see that

$$a^{-\frac{1}{p-1}} u_1 = (B\phi_1 + w) \geq C\phi_1$$

and hence $\log (a^{-1/(p-1)} u_1) \geq \log \phi_1 - \tilde{C}_1$.

Since a similar inequality also holds for u_2, these inequalities also hold for $\tilde{u}$. Hence, for these X',

$$\left| \frac{\exp\ ((p-1)\ \log (\tilde{u}/a^{1/(p-1)})-1}{p-1}\ \phi_1 \right| \leq K\phi_1 \left| \log \left(\frac{\tilde{u}}{a^{1/(p-1)}}\right) \right|$$

$$\text{(since } \frac{e^x-1}{x} \leq 1 \text{ if } x \leq 0)$$

$$\leq K\phi_1(\tilde{x})(|\ \log\ \phi_1(\tilde{x})| + \tilde{C}_1)$$

which is small near the boundary and is uniformly bounded. Hence we easily see that from the dominated convergence theorem that as $p \longrightarrow 1$

$$\mu^{-1} \int_\Omega R(\tilde{x}, x')\phi_1(\tilde{x})(\tilde{\gamma}\phi_1(\tilde{x}) + w_1)$$

$$\longrightarrow \mu^{-1} a \int_{\tilde{\Omega}} \tilde{\gamma}(x')(\phi_1(\tilde{x}))^2 d\tilde{x} + \mu^{-1} a \int_{\tilde{\Omega}} \log\ (e^{\hat{\mu}} z_0 (a^{\frac{1}{2}}(p-1)^{\frac{1}{2}} x')\phi_1)\tilde{\gamma}(x')(\phi_1(\tilde{x}))^2 d\tilde{x}$$

on bounded X' intervals. Hence the limit equation for γ in the X' variables is

$$-\Delta_{n-k}\tilde{\gamma} = \tilde{\gamma} + \mu^{-1}\tilde{\gamma} \int_{\tilde{\Omega}} \log\ (e^{\hat{\mu}} z_0(X')\phi_1(\tilde{x}))(\phi_1(\tilde{x}))^2 d\tilde{x}.$$

There is one point here. From our previous estimates for R, we see that in the X' variables $\Delta_{n-k}\tilde{\gamma}$ is bounded and this we can choose a subsequence so $\tilde{\gamma}$ converges uniformly on compact sets. Hence we see that $\tilde{\gamma}$ is non-trivial, $|\tilde{\gamma}| \leq 1$ always and $-\Delta_{n-k}\tilde{\gamma} = \tilde{\gamma} + \mu^{-1}\tilde{\gamma} \int_\Omega \log(e^{\hat{\mu}} z_0(X')\phi_1(\tilde{x}))(\phi_1(\tilde{x}))^2 d\tilde{x}$. We expanding the log term and using the definition of $\hat{\mu}$, this becomes

$$-\Delta_{n-k}\tilde{\gamma} = (1 + \ \log\ z_0)\tilde{\gamma}$$

which is the linearization of our equation for z at z_0. Thus, we see that it suffices to consider the equation

$$-\Delta_{n-k} h \ = \ (1 + \ \log\ z_0)h, \tag{2.18}$$

where $h \longrightarrow 0$ as $|X'| \longrightarrow \infty$. However, we know a little more. u_1 and u_2 are radial in x' and hence z is radial in x' and hence in the limit we see that h is radial in X'. Thus we have an ordinary differential equation. Since $\log z_0(|X'|) \longrightarrow -\infty$ as $|X'| \longrightarrow \infty$, standard comparison principles for linear ordinary differential equations

(cp [4] or [16]) implies that $z_0(X')$ decays faster than $\exp(-k|X'|)$ for any $k > 0$ and a similar decay holds for h. It is then easy to justify scalar multiplying (2.18) by z_0 and the equation for z_0 by h, integrating by parts and subtracting to ensure $\int_{R^{n-k}} z_0 h dX' = 0$. Hence h must change sign. Now the function $f(y) = y \log y$ is easily seen to satisfy that $\Theta(y) = yf'(y)/f(y) = 1 + (\log y)^{-1}$ satisfies $\Theta(y) < 1$ if $0 < y < 1$ and $\Theta(y) > 1$ and strictly decreasing for $y > 1$. We can then argue as in the proof of Theorem 1 in [7] (or in [17]) to deduce that h cannot have exactly one positive zero. The argument is actually rather simpler than that in [7]. Note that the non lipschitzean nature of the nonlinearity at zero does not affect the argument in [7]. It remains to prove that h has at most one positive zero. There are many ways to prove this. Probably the easiest is to use that z is the limit of mountain positive solutions of

$$-\Delta_{n-k} u = f_n(u)$$

on large balls when $f_n(y)$ is $y \log y$ but smoothed at zero to be C^1. (Recall that at a mountain pass point the linearization has at most 1 negative value and hence the radial solution of the linearized equation has at most 1 positive zero (cp [14], Corollary XIII.7.54). An alternative argument is to use the ideas in [17].

Lastly, the non degeneracy is proved by similar arguments to the uniqueness. In fact it is a little simpler. Note that we only prove non-degeneracy in the space of functions radial in x'.

This completes the proof of Theorem 2.1.

3 Generic non-degeneracy

Theorem 3.1 *Assume that the conditions of Theorem 2.1 hold except we only require* $1 < p < (n+2)/(n-2) (1 < p < \infty$ *if* $n = 2)$. *Then for all* p *in* $(1, (n+2)/(n-2))$ *except for isolated points, (1.1) has a non-degenerate positive solution.*

Remark 3.1 Here as in §2, we consider positive solutions which decay in x' as $|x| \longrightarrow \infty$ uniformly in $\widetilde{x}$. As there, the solutions are necessarily radial in $\widetilde{x}$. When we speak of non-degeneracy, we mean non-degeneracy in the space of functions radial in $\widetilde{x}$ and converging uniformly to zero as $|x'| \longrightarrow \infty$ uniformly in $\widetilde{x}$ (though it could be proved in larger spaces).

Proof This is almost identical to the proof of Theorem 4 in [5] except where we combine Theorem 1 there with Theorem 2.1 here (instead of Lemma 2 there). Thus we use the uniqueness for p close to 1 rather than p close to $(n+2)/(n-2)$. There are a few minor changes. Firstly, we work with spaces which no longer have symmetrics in $\widetilde{x}$ but otherwise use the same weighted spaces $\widetilde{E}$ as these. Finally, note that the arguments in the proof of Theorem 4 in [5] are still value if $\lambda > -\lambda_1$. (We noted there they were valid if $\lambda \geq 0$).

Remarks 3.2

1. The positive solution is not always unique. Indeed, we suspect that this can also be proved (and more information obtained) by using similar ideas to those in [11], combined with Du's fixed point index formula in [13] which generalizes a theorem of the author. In the case where $\lambda = 0$ there is a variational proof similar to but more complicated than the one sketched in [10]. This example implies that the δ in Theorem 2.1 cannot be chosen independent of $\widetilde{\Omega}$. It is possible to prove there are sometimes degenerate solutions e.g. if $\widetilde{\Omega}$ is a suitable annulus.

2. If $n = 3$ and $\widetilde{\Omega}$ has the symmetry properties in [5], we could combine our ideas with the ideas in §4 of [5] to prove that the exceptional set is finite in this case.

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