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Comparison and Existence Results for Classes of Nonlinear Elliptic Equations with General Growth in the Gradient

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Abstract

In this paper we study the Dirichlet problem for a class of nonlinear elliptic equations in the form $A(u) = H(x, u, Du)$, where the principal term is a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$, and $H(x, u, Du)$ grows with respect to Du at most like $|Du|^q$, $p - 1 \leq q \leq p$. Comparison results are obtained between the rearrangement of a solution u of Dirichlet problem quoted above and the rearrangement of the solution of a problem whose data are radially symmetric.

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$. We consider the model problem

$$\begin{cases} -\Delta_p u = F(Du) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$ denotes the p-laplacian operator, $p > 1$, f is a bounded function and $F(Du)$ is a nonlinear term which grows like $|Du|^q$, with $p-1 \leq q \leq p$, that is:

$$|F(\xi)| \leq \theta|\xi|^q, \quad \forall \xi \in \mathbb{R}^n,$$

where θ is a positive constant.

Our aim is to prove that a solution to problem (1.1) can be compared, in terms of rearrangement, with the solution to a suitable "symmetrized" problem.

The first results in this direction can be found in [25], [29], [30], [33]. For example in [29], when $p = 2$ and $\theta = 0$, it has been proved that, if v is the solution of problem

$$\begin{cases} -\Delta v = f^{\#} & \text{in } \Omega^{\#} \\ u = 0 & \text{on } \partial\Omega^{\#}, \end{cases} \tag{1.2}$$

then:

$$u^*(s) \leq v^*(s), \quad \forall s \in [0, |\Omega|], \tag{1.3}$$

where $\Omega^{\#}$ denotes the ball centered at the origin such that $|\Omega| = |\Omega^{\#}|$ and $f^{\#}$, u^*, v^* are the decreasing rearrangements of f, u, v, respectively (see Section 2 for precise definitions).

Several similar results have been proved where a term, which depends on the gradient, appears in (1.1). For instance, (see [3], [9], [26], [31], [32]) when $q = p - 1$, inequality (1.3) can be obtained when v is the regular solution of problem

$$\begin{cases} -\Delta_p v = -\theta \frac{x}{|x|}|Dv|^{p-2}Dv + f^{\#} & \text{in } \Omega^{\#} \\ v = 0 & \text{on } \partial\Omega^{\#}. \end{cases} \tag{1.4}$$

When $p - 1 < q \leq p$, the results which can be found in the literature mainly concern the case $q = p$ (see, for example, [3], [15], [16], [19], [24], [26]). In the latter case $q = p$, inequality (1.3) is proved, where v is the solution of problem

$$\begin{cases} -\Delta_p v = \theta|Dv|^p + f^{\#} & \text{in } \Omega^{\#} \\ v = 0 & \text{on } \partial\Omega^{\#}. \end{cases} \tag{1.5}$$

The main difference with respect to the previous case consists in the fact that the comparison is proved for bounded solutions of (1.1) and under the assumption that a unique bounded radially symmetric solution of (1.5) exists, a requirement that is satisfied if a smallness assumption on $\|f\|_\infty$ is made. For instance in [26] such smallness assumption reads as:

$$\|f\|_\infty < \left(\frac{p-1}{\theta}\right)^{p-1} \lambda_p(\Omega^{\#}), \tag{1.6}$$

where $\lambda_p(\Omega^\#)$ is the first eigenvalue of the following Dirichlet problem

$$\begin{cases} -\Delta_p\psi = \lambda|\psi|^{p-2}\psi & \text{in } \Omega^\# \\ \psi = 0 & \text{on } \partial\Omega^\#. \end{cases} \tag{1.7}$$

Some properties of the first eigenvalue for problem (1.7) can be found, for example, in [1], [21], [23].

When $p = 2$ in (1.1), the case $1 < q < 2$ has been addressed in [3] and their approach does not seem to be extended to the case $p \neq 2$.

In the present paper we study solutions to (1.1) in the general case $p - 1 < q \le p$, $p > 1$. More precisely we consider the Dirichlet problem

$$\begin{cases} -\text{div}(a(x, u, Du)) = H(x, u, Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $a(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $H(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions, satisfying for some $p \in]1, +\infty[$, the following conditions

$$|a(x, s, \xi)| \le c(|\xi|^{p-1} + |s|^{p-1} + k(x)), \quad \text{a.e. } x \in \Omega, \ \forall(s, \xi) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.9}$$

$$a(x, s, \xi)\xi \ge |\xi|^p, \quad \text{a.e. } x \in \Omega, \ \forall(s, \xi) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.10}$$

$$|H(x, s, \xi)| \le \theta|\xi|^q + f(x), \quad \text{a.e. } x \in \Omega, \ \forall(s, \xi) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.11}$$

where $c > 0$, $\theta > 0$, $p - 1 < q \le p$, k and f are non negative functions such that $k \in L^{p'}(\Omega)$ and $f \in L^\infty(\Omega)$. We prove that for any bounded solution $u \in W_0^{1,p}(\Omega)$ of problem (1.8), inequality (1.3) holds true, where v is the bounded solution to the following problem

$$\begin{cases} -\Delta_p v = \theta|Dv|^q + f^\# & \text{in } \Omega^\# \\ v = 0 & \text{on } \partial\Omega^\# \end{cases} \tag{1.12}$$

under the additional assumption on f

$$\|f\|_\infty < \Big(\frac{\gamma - 1}{\theta}\Big)^{\gamma-1} \lambda_\gamma(\Omega^\#), \tag{1.13}$$

where $\gamma = \frac{q}{1-p+q} = \left(\frac{q}{p-1}\right)'$. In particular, condition (1.13) is used to prove that problem (1.12) admits a unique bounded radially symmetric solution. Actually, we not only prove inequality (1.3), but we also prove an estimate for the gradient of the solution, namely:

$$\int_\Omega \eta\big(|Du|^p\big)dx \le \int_{\Omega^\#} \eta\big(|Dv|^p\big)dx, \tag{1.14}$$

where η is any nondecreasing concave function on $[0, +\infty[$.

We finally mention that the estimates we have found can be used in order to prove the existence of a bounded solution to the problem (1.8). Indeed, using (1.3), which gives L^∞ bound on u, and (1.14), we can show that a suitable sequence of solutions to approximate problems converges to a solution of (1.8) (see, for example, [10], [18]).

2 Preliminaries and background information

In this section we recall the definition of decreasing rearrangement of a measurable function.

Let Ω be an open bounded subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. If one denotes by $|E|$ the Lebesgue measure of a set $E \subset \mathbb{R}^n$, one can define the distribution function μ_u of u as follows:

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|, \qquad t \geq 0.$$

The function μ_u is decreasing and right continuous; moreover, its generalized inverse function is the decreasing rearrangement u^* of u:

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) > s\}, \qquad s \in [0, |\Omega|].$$

The spherically symmetric decreasing rearrangement of u is defined by

$$u^\#(x) = u^*(\omega_n |x|^n), \qquad x \in \Omega^\#,$$

where $\Omega^\#$ is the ball centered at the origin having the same measure as Ω and ω_n is the measure of the unit ball in $\mathbb{R}^n$.

For an exhaustive treatment of the properties of rearrangements we refer to [2], [4], [20], [22], [27]; we just want to point out the Hardy-Littlewood inequality

$$\int_\Omega |f(x)g(x)|dx \leq \int_0^{|\Omega|} f^*(s)g^*(s)ds$$

where f and g are measurable functions in Ω.

Finally, let us conclude this section by recalling a technical lemma which will be useful in the following (see[23]).

Lemma 2.1 *If $p \geq 2$, then*

$$|\xi_2|^p \geq |\xi_1|^p + p|\xi_1|^{p-2}\xi_1(\xi_2 - \xi_1) + \frac{|\xi_2 - \xi_1|^p}{2^{p-1} - 1}$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$.

If $1 < p < 2$, then

$$|\xi_2|^p \geq |\xi_1|^p + p|\xi_1|^{p-2}\xi_1(\xi_2 - \xi_1) + c(p)\frac{|\xi_2 - \xi_1|^2}{(|\xi_1| + |\xi_2|)^{2-p}},$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$, where $c(p)$ is a positive constant depending only on p.

3 The radial case

In this section we consider the following Dirichlet problem with radially symmetric data

$$\begin{cases} -\Delta_p v = \theta |Dv|^q + f & \text{in } B \\ v = 0 & \text{on } \partial B, \end{cases} \tag{3.1}$$

where $B = B_R$ is a ball centered at the origin with radius R, θ is a positive constant, $f(x) = f(|x|)$ is a bounded radially decreasing function, with $f(x) \geq 0$, and $p-1 < q \leq p$, and, as usual, $p > 1$.

Theorem 3.1 *Let the following condition be satisfied:*

$$\|f\|_\infty < \left(\frac{\gamma - 1}{\theta}\right)^{\gamma-1} \lambda_\gamma(B), \tag{3.2}$$

where $\gamma = \frac{q}{1-p+q} = \left(\frac{q}{p-1}\right)'$ *and* $\lambda_\gamma(B)$ *is the first eigenvalue of the following Dirichlet problem*

$$\begin{cases} -\Delta_\gamma \psi = \lambda |\psi|^{\gamma-2} \psi & \text{in } B \\ \psi = 0 & \text{on } \partial B. \end{cases} \tag{3.3}$$

Then the problem (3.1) *admits at least one bounded radial solution. Such a solution is unique among the bounded and radial ones; moreover, it is such that:*

$$v(x) = v^{\#}(x). \tag{3.4}$$

Remark 3.1 Let us observe that, if $q = p$ then $\gamma = p$, so the result of Theorem 3.1 can be found in [26] and condition (3.2) reduces to that one given in [26]. However, in the case $q = p$, a stronger uniqueness result has been proved, that is, it has been shown that problem (3.1) admits a unique bounded solution (see [26]). In the case $p = 2$ general uniqueness results for problem (3.1) can be found, for example, in [5], [6], [7].

Remark 3.2 Theorem 3.1 is related to the case $p - 1 < q \leq p$. In the case $q = p - 1$ an analogous theorem can be found in [26]. In this paper the existence of a unique solution of problem (3.1) has been obtained without smallness conditions on f. Let us observe that, according to above, the term that appears on the right hand side of inequality (3.2) goes to $+\infty$ when q tends to $p - 1$. Indeed, it is known that (see [21])

$$\lim_{\gamma \to \infty} (\lambda_\gamma(B))^\gamma = \frac{1}{R},$$

then

$$\lim_{q \to p-1} \left(\frac{\gamma - 1}{\theta}\right)^{\gamma-1} \lambda_\gamma(B) = +\infty.$$

Remark 3.3 Let us observe that problems in the form (3.1) can admit unbounded radial solutions. Indeed, if we consider the problem

$$\begin{cases} -\Delta_p u = |Du|^q & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{3.5}$$

it obviously has $u = 0$ as a bounded radially symmetric solution in $W_0^{1,p}(B)$. On the other hand, it is not difficult to show that in the case $1 < p < n$ the unbounded function $u(x) = c\big(|x|^{-\frac{p-q}{1-p+q}} - R^{-\frac{p-q}{1-p+q}}\big)$, where c is a suitable positive constant depending on n, p, q and $\frac{p}{n} + p - 1 < q < p$, belongs to $W_0^{1,p}(B)$ and solves problem (3.5). A similar example has been given in [15] when $q = p$ (see also [7], [18]).

Proof of Theorem 3.1. According to the assumptions on p and q, we have $\gamma = \left(\frac{q}{p-1}\right)' > 1$. Let us set

$$g(x) = \Big(\frac{\theta}{\gamma - 1}\Big)^{\gamma-1} f(x),$$

and consider the following problem

$$\begin{cases} -\Delta_\gamma V & = g(V+1)^{\gamma-1} \quad \text{in } B \\ V = 0 & \qquad\qquad\qquad \text{on } \partial B. \end{cases} \tag{3.6}$$

It is known (see [8], [13], [14], [26]) that the problem (3.6), under the condition (3.2), has a unique nonnegative solution such that $V(x) = V^\#(x) = V(|x|)$. Furthermore, such a solution is bounded.

Let us consider the solution $V(x)$ of problem (3.6) and set

$$v(x) = \Big(\frac{\gamma - 1}{\theta}\Big)^{\frac{1}{1-p+q}} \int_{|x|}^{R} \Big(\frac{-V'(\rho)}{V(\rho)+1}\Big)^{\frac{1}{1-p+q}} d\rho.$$

Clearly $v(x) = v(|x|) = v^\#(x)$.

Let us prove that $v(x)$ is a bounded solution to (3.1). Indeed, setting $|x| = \rho$, we have that:

$$|Dv| = \Big(\frac{\gamma - 1}{\theta}\Big)^{\frac{1}{1-p+q}} \Big(\frac{-V'(\rho)}{V(\rho)+1}\Big)^{\frac{1}{1-p+q}},$$

and

$$-\Delta_p v = \Big(\frac{\gamma - 1}{\theta}\Big)^{\frac{p-1}{1-p+q}} \frac{-\Delta_\gamma V}{(V+1)^{\gamma-1}} + (\gamma - 1)\Big(\frac{-V'}{V+1}\Big)^{\gamma},$$

so a straightforward calculation gives that $v(x)$ solves (3.1).

As regards the boundedness of $v(x)$, let us observe that the solution $V(x)$ of (3.6) satisfies:

$$\frac{1}{\rho^{n-1}} \frac{d}{d\rho}\Big(\big(-V'(\rho)\big)^{\gamma-1}\rho^{n-1}\Big) = g(V(\rho)+1)^{\gamma-1}.$$

This implies:

$$\big(-V'(\rho)\big)^{\frac{1}{1-p+q}} = \Big(\frac{1}{\rho^{n-1}} \int_0^{\rho} g(\sigma)(V(\sigma)+1)^{\gamma-1}\sigma^{n-1} d\sigma\Big)^{\frac{1}{p-1}} \le K\rho^{\frac{1}{p-1}},$$

and, using the definition of $v(x)$, we immediately have that $v(x)$ is bounded.

Finally, it is easy to show that if $v(x)$ is a bounded radially decreasing solution of (3.1), then the function

$$V(x) = \exp\Big[\Big(\frac{\theta}{\gamma-1}\Big)\int_{|x|}^{R}\big((-v(r))'\big)^{1-p+q}dr\Big] - 1,$$

is a solution of problem (3.6).

Since condition (3.2) is satisfied, the problem (3.6) has a unique bounded radially decreasing solution, and consequently also the problem (3.1) has a unique bounded radially decreasing solution. The theorem is so proved.

4 Comparison results

The present section is devoted to the comparison result between the decreasing rearrangement u^* of a solution $u \in W_0^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\text{div}(a(x,u,Du)) = H(x,u,Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

and the decreasing rearrangement v^* of the solution $v \in W_0^{1,p}(\Omega^\#)$ of the problem

$$\begin{cases} -\Delta_p v = \theta|Dv|^q + f^\# & \text{in } \Omega^\# \\ v = 0 & \text{on } \partial\Omega^\#. \end{cases} \tag{4.2}$$

Theorem 4.1 *Let $u \in W_0^{1,p}(\Omega)$ be a bounded solution of problem (4.1), under the assumptions (1.9)-(1.11). If f satisfies (3.2) and $v \in W_0^{1,p}(\Omega^\#)$ is the bounded radially symmetric solution of problem (4.2), we have:*

$$u^*(s) \le v^*(s), \qquad \forall s \in [0, |\Omega|]. \tag{4.3}$$

Moreover:

$$\int_\Omega \eta\big(|Du|^p\big)dx \le \int_{\Omega^\#} \eta\big(|Dv|^p\big)dx \tag{4.4}$$

for all functions η concave, nondecreasing on $[0,+\infty[$.

To prove Theorem 4.1 we need the following:

Lemma 4.1 *Under the hypotheses of Theorem 4.1, we have, a.e. in $(0,|\Omega|)$, that*

$$(-u^*(s))'(n\omega_n^{1/n}s^{1-1/n})^{\frac{p}{p-1}} \le \Big[\int_0^s \Big\{f^*(\sigma)\exp\Big(\int_\sigma^s \frac{\theta}{\big(n\omega_n^{1/n}\big)^{p-q}} \times \frac{\big((-u^*(\tau))'\big)^{1-p+q}}{\tau^{(1-1/n)(p-q)}}\,d\tau\Big)\Big\}\,d\sigma\Big]^{\frac{1}{p-1}}, \tag{4.5}$$

$$(-v^*(s))'(n\omega_n^{1/n}s^{1-1/n})^{\frac{p}{p-1}} = \left[\int_0^s \left\{ f^*(\sigma)\exp\Big(\int_\sigma^s \frac{\theta}{\big(n\omega_n^{1/n}\big)^{p-q}} \times \frac{\big((-v^*(\tau))'\big)^{1-p+q}}{\tau^{(1-1/n)(p-q)}}\,d\tau\Big)\right\} d\sigma\right]^{\frac{1}{p-1}}. \quad (4.6)$$

Proof. Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of problem (4.1). Using the test function:

$$\varphi(x) = \begin{cases} h \text{ sign } u & |u| > t+h \\ (|u|-t) \text{ sign } u & t < |u| \le t+h \\ 0 & |u| \le t, \end{cases}$$

where $h > 0$ and $t \in [0, \sup|u|[$, in a standard way (see, for example, [3], [29], [30]), we obtain:

$$-\frac{d}{dt}\int_{|u|>t} |Du|^p dx \le \theta \int_{|u|>t} |Du|^q dx + \int_{|u|>t} f(x)dx. \quad (4.7)$$

Now, let us estimate the term:

$$\int_{|u|>t} |Du|^q dx = \int_t^{+\infty} \Big(-\frac{d}{d\tau}\int_{|u|>\tau} |Du|^q dx\Big) d\tau. \quad (4.8)$$

Let us recall that, from Hölder inequality we have, for $0 < k < p$,

$$-\frac{d}{dt}\int_{|u|>t} |Du|^k dx \le \Big(-\frac{d}{dt}\int_{|u|>t} |Du|^p dx\Big)^{\frac{k}{p}} (-\mu_u'(t))^{1-\frac{k}{p}}. \quad (4.9)$$

So, using (4.9) with $k = q$, from (4.8) we get

$$\int_{|u|>t} |Du|^q dx \le \int_t^{+\infty} \Big[\Big(-\frac{d}{d\tau}\int_{|u|>\tau} |Du|^p dx\Big)^{\frac{q}{p}} (-\mu_u'(\tau))^{1-\frac{q}{p}}\Big] d\tau. \quad (4.10)$$

On the other hand, inequality (4.9), with $k = 1$, Fleming-Rishel coarea formula (see [17]) and the isoperimetric inequality give:

$$\begin{aligned}\Big(-\frac{d}{dt}\int_{|u|>t} |Du|^p dx\Big)^{\frac{p-q}{p}} &\ge \left(\Big(-\frac{d}{dt}\int_{|u|>t} |Du| dx\Big)(-\mu_u'(t))^{-1+\frac{1}{p}}\right)^{p-q} \\ &\ge \big(n\omega_n^{1/n}(\mu_u(y))^{1-\frac{1}{n}}\big)^{p-q}(-\mu_u'(t))^{-\frac{(p-1)(p-q)}{p}}. \quad (4.11)\end{aligned}$$

Thus, from (4.10) and (4.11) we obtain

$$\int\limits_{|u|>t} |Du|^q dx \le \frac{1}{\left(n\omega_n^{1/n}\right)^{p-q}} \times \int\limits_t^{+\infty} \Big(-\frac{d}{d\tau} \int\limits_{|u|>\tau} |Du|^p dx \Big) \left(\frac{-\mu_u'(\tau)}{(\mu_u(\tau))^{1-\frac{1}{n}}} \right)^{p-q} d\tau. \quad (4.12)$$

From (4.7), (4.12) and the Hardy-Littlewood inequality we have

$$-\frac{d}{dt} \int\limits_{|u|>t} |Du|^p dx \le \int\limits_0^{\mu_u(t)} f^*(s) ds + \frac{\theta}{\left(n\omega_n^{1/n}\right)^{p-q}} \int\limits_t^{+\infty} \Big(-\frac{d}{d\tau} \int\limits_{|u|>\tau} |Du|^p dx \Big) \left(\frac{-\mu_u'(\tau)}{(\mu_u(\tau))^{1-\frac{1}{n}}} \right)^{p-q} d\tau.$$

According to Gronwall's Lemma, we have

$$-\frac{d}{dt} \int\limits_{|u|>t} |Du|^p dx \le \int\limits_0^{\mu_u(t)} f^*(s) ds + \int\limits_t^{+\infty} \frac{\theta}{\left(n\omega_n^{1/n}\right)^{p-q}} \left(\frac{-\mu_u'(\tau)}{(\mu_u(\tau))^{1-\frac{1}{n}}} \right)^{p-q} \times \left(\int\limits_0^{\mu_u(\tau)} f^*(s)\, ds \right) \exp\left(\int\limits_t^{\tau} \frac{\theta}{\left(n\omega_n^{1/n}\right)^{p-q}} \left(\frac{-\mu_u'(r)}{(\mu_u(r))^{1-\frac{1}{n}}} \right)^{p-q} dr \right) d\tau, \quad (4.13)$$

and, using again the Fleming-Rishel coarea formula, the isoperimetric inequality and inequality (4.9), with $k=1$, we obtain for a.e. $t>0$,

$$\frac{\left(n\omega_n^{1/n} (\mu_u(t))^{1-1/n}\right)^p}{(-\mu_u'(t))^{p-1}} \le \int\limits_0^{\mu_u(t)} f^*(s) ds + \int\limits_t^{+\infty} \frac{\theta}{\left(n\omega_n^{1/n}\right)^{p-q}} \left(\frac{-\mu_u'(\tau)}{(\mu_u(\tau))^{1-\frac{1}{n}}} \right)^{p-q} \times \left(\int\limits_0^{\mu_u(\tau)} f^*(s)\, ds \right) \exp\left(\int\limits_t^{\tau} \frac{\theta}{\left(n\omega_n^{1/n}\right)^{p-q}} \left(\frac{-\mu_u'(r)}{(\mu_u(r))^{1-\frac{1}{n}}} \right)^{p-q} dr \right) d\tau.$$

Integrating by parts, we have, for a.e. $t>0$,

$$\frac{\left(n\omega_n^{1/n} (\mu_u(t))^{1-1/n}\right)^{\frac{p}{p-1}}}{-\mu_u'(t)} \le \Bigg[\int\limits_t^{+\infty} f^*(\mu_u(\tau))(-\mu_u'(\tau)) \times \exp\left(\int\limits_t^{\tau} \frac{\theta}{\left(n\omega_n^{1/n}\right)^{p-q}} \left(\frac{-\mu_u'(r)}{(\mu_u(r))^{1-\frac{1}{n}}} \right)^{p-q} dr \right) d\tau \Bigg]^{\frac{1}{p-1}}. \quad (4.14)$$

Let us observe that arguing, for example, as in [11] and [28] (see also [12]), we have

$$\int_t^\tau \Big(\frac{-\mu_u'(r)}{(\mu_u(r))^{1-\frac{1}{n}}}\Big)^{p-q} dr = \frac{1}{n^{1-p+q}\omega_n^{\frac{n-p+q}{n}}} \int_{\tau>u^\#(x)>t} \frac{|Du^\#|^{1-p+q}}{|x|^{n-1}} dx$$
$$= \int_{\mu_u(\tau)}^{\mu_u(t)} \frac{\big((-u^*(\rho))'\big)^{1-p+q}}{\rho^{(1-\frac{1}{n})(p-q)}} d\rho.$$

Consequently, using the properties of rearrangements, from (4.14) we have, a.e. in $[0,|\Omega|]$, that:

$$(-u^*(s))'(n\omega_n^{1/n}s^{1-1/n})^{\frac{p}{p-1}} \le \Big[\int_0^s \Big\{f^*(\sigma)\exp\Big(\int_\sigma^s \frac{\theta}{\big(n\omega_n^{1/n}\big)^{p-q}} \times \frac{\big((-u^*(\rho))'\big)^{1-p+q}}{\rho^{(1-\frac{1}{n})(p-q)}} d\rho\Big)\Big\} d\sigma\Big]^{\frac{1}{p-1}}.$$

So the condition (4.5) is proved.

As regards equality (4.6), let us remember that in Section 3 it has been proved that, under assumption (3.2), the problem (3.1) admits a unique positive, radially symmetric solution $v \in W_0^{1,p}(\Omega^\#) \cap L^\infty(\Omega^\#)$ such that $v(x) = v^\#(x)$. So, obviously, the arguments leading to (4.5) proceed in the same way except that the equalities now replace the inequalities. Thus, instead of (4.5) we have the differential equality (4.6). So the lemma is proved.

Proof of Theorem 4.1. Let us set:

$$U(s) = \exp\Big(\alpha \int_s^{s_0} \frac{\big((-u^*(\tau))'\big)^{1-p+q}}{\tau^{(1-1/n)(p-q)}} d\tau\Big) - 1, \tag{4.15}$$

where $\alpha > 0$ is a constant to be fixed later, $s_0 \in]0,|\Omega|]$ and $s \in [0, s_0]$.

Let us observe that $U(s_0) = 0$, $U(s)$ is decreasing in $[0, s_0]$, and

$$\exp\Big(\int_s^{s_0} \frac{\big((-u^*(\tau))'\big)^{1-p+q}}{\tau^{(1-1/n)(p-q)}} d\tau\Big) = (U(s)+1)^{1/\alpha}. \tag{4.16}$$

Moreover:

$$U'(s) = -\alpha \frac{\big((-u^*(s))'\big)^{1-p+q}}{s^{(1-1/n)(p-q)}}(U(s)+1), \qquad \forall s \in [0, s_0]. \tag{4.17}$$

Then:

$$(-u^*(s))' = \Big[-\frac{U'(s)s^{(1-1/n)(p-q)}}{\alpha(U(s)+1)}\Big]^{\frac{1}{1-p+q}}. \tag{4.18}$$

Setting:

$$\beta = \frac{\theta}{(n\omega_n^{1/n})^{p-q}},$$

from (4.16), $\forall \sigma \le s$, we have

$$\exp\Big(\beta \int_\sigma^s \frac{\big((-u^*(\tau))'\big)^{1-p+q}}{\tau^{(1-1/n)(p-q)}}\, d\tau\Big) = \Big(U(\sigma)+1\Big)^{\frac{\beta}{\alpha}} \Big(U(s)+1\Big)^{-\frac{\beta}{\alpha}}.$$

Using the above equality in (4.5) and bearing in mind (4.18), we have, a.e. in $[0, s_0]$, that

$$\frac{\big(-U'(s)\big(U(s)+1\big)^{\frac{\beta}{\alpha}\frac{1-p+q}{p-1}-1}\big)^{\frac{1}{1-p+q}}}{(n\omega_n^{1/n})^{-\frac{p}{p-1}}\big(\alpha s^{(\frac{1}{n}-1)\frac{q}{(p-1)}}\big)^{\frac{1}{1-p+q}}} \le \left[\int_0^s f^*(\sigma)\big(U(\sigma)+1\big)^{\frac{\beta}{\alpha}} d\sigma\right]^{\frac{1}{p-1}}. \tag{4.19}$$

Recalling the definition of β , if we choose

$$\alpha = \frac{\theta(1-p+q)}{(p-1)\big(n\omega_n^{1/n}\big)^{p-q}} > 0,$$

we have:

$$\frac{1}{1-p+q} - \frac{\beta}{\alpha}\,\frac{1}{p-1} = 0.$$

Setting, as in Theorem 3.1, $\gamma = \left(\frac{q}{p-1}\right)'$ and observing that $\frac{\beta}{\alpha} = \gamma - 1$, from (4.19) it follows, a.e. in $[0, s_0]$, that

$$\big(-U'(s)\big)^{\gamma-1} s^{(1-\frac{1}{n})\gamma}(n\omega_n^{\frac{1}{n}})^{\gamma}\Big(\frac{\gamma-1}{\theta}\Big)^{\gamma-1} \le \int_0^s f^*(\sigma)\big(U(\sigma)+1\big)^{\frac{p-1}{1-p+q}} d\sigma,$$

so, a.e. in $[0, s_0]$, we have:

$$-U'(s) \le \frac{\theta\, s^{-(1-\frac{1}{n})\gamma'}}{(\gamma-1)(n\omega_n^{1/n})^{\gamma'}}\left[\int_0^s f^*(\sigma)\Big(U(\sigma)+1\Big)^{\gamma-1} d\sigma\right]^{\frac{\gamma'}{\gamma}}. \tag{4.20}$$

Using (4.6) instead of (4.5), obviously the arguments leading to (4.20) proceed in the same way except that the equalities now replace the inequalities. So, instead of (4.20) we have, a.e. in $[0, s_0]$, that:

$$-V'(s) = \frac{\theta\, s^{-(1-\frac{1}{n})\gamma'}}{(\gamma-1)(n\omega_n^{1/n})^{\gamma'}}\left[\int_0^s f^*(\sigma)\Big(V(\sigma)+1\Big)^{\gamma-1} d\sigma\right]^{\frac{\gamma'}{\gamma}}, \tag{4.21}$$

where:

$$V(s) = \exp\left(\frac{\theta(1-p+q)}{(p-1)\big(n\omega_n^{1/n}\big)^{p-q}} \int_s^{s_0} \frac{\big((-v^*(\tau))'\big)^{1-p+q}}{\tau^{(1-1/n)(p-q)}}\, d\tau\right) - 1.$$

Let us set:

$$W(s)=\int_0^s f^*(\sigma)\Big(U(\sigma)+1\Big)^{\gamma-1}d\sigma,\quad \forall s\in[0,s_0],$$

and

$$Z(s)=\int_0^s f^*(\sigma)\Big(V(\sigma)+1\Big)^{\gamma-1}d\sigma,\quad \forall s\in[0,s_0].$$

Using arguments similar to those in Theorem 5.1 of [26], we can prove that

$$W(s)\le Z(s),\quad \forall s\in[0,s_0]. \tag{4.22}$$

Indeed, if (4.22) is not satisfied, then there exists $\bar{s}\in]0,s_0]$ such that

$$W(\bar{s})-Z(\bar{s})=\max_{s\in[0,s_0]}(W(s)-Z(s))>0.$$

Here we analyze the simplest case when $\bar{s}=s_0$. For the case $\bar{s}<s_0$ one can refer to the proof of the quoted theorem in [26].

Let us put

$$s_1=\inf\{s\in[0,s_0]:W(t)>Z(t),\forall t\in[s,s_0]\}.$$

Observing that W/Z and Z/W are bounded in $[s_1,s_0]$, we can use the following test functions:

$$\varphi_1(s)=\frac{(W(s))^{\gamma'}-(Z(s))^{\gamma'}}{(W(s))^{\gamma'-1}},\quad \varphi_2(s)=\frac{(W(s))^{\gamma'}-(Z(s))^{\gamma'}}{(Z(s))^{\gamma'-1}}.$$

Inserting φ_1 and φ_2 in the relations (4.20) and (4.21), respectively, and integrating between s_1 and s_0, we have:

$$\int_{s_1}^{s_0}[-U'(s)\varphi_1(s)+V'(s)\varphi_2(s)]ds\le 0. \tag{4.23}$$

Integrating by parts the first member of (4.23) and bearing in mind that $\varphi_1(s_1)=\varphi_2(s_1)=0$ and $U(s_0)=V(s_0)=0$, we have

$$\int_{s_1}^{s_0}[(U(s)+1)\varphi_1'(s)-(V(s)+1)\varphi_2'(s)]ds\le\varphi_1(s_0)-\varphi_2(s_0). \tag{4.24}$$

But:

$$\varphi_1(s_0)-\varphi_2(s_0)=\big((W(s_0))^{\gamma'}-(Z(s_0))^{\gamma'}\big)\big(\frac{1}{(W(s_0))^{\gamma'-1}}-\frac{1}{(Z(s_0))^{\gamma'-1}}\big)<0,$$

so, from (4.24) it follows that:

$$\int_{s_1}^{s_0}[(U(s)+1)\varphi_1'(s)-(V(s)+1)\varphi_2'(s)]ds<0.$$

Then, using the definitions of φ_1 and φ_2, we get:

$$\int_{s_1}^{s_0} f^*\Big[W^{\gamma'}\left(\frac{(U+1)^\gamma}{W^{\gamma'}}-\gamma'\frac{(U+1)^{\gamma-1}}{W}\frac{(V+1)}{Z^{\gamma'-1}}+(\gamma'-1)\frac{(V+1)^\gamma}{Z^{\gamma'}}\right)$$
$$+Z^{\gamma'}\left(\frac{(V+1)^\gamma}{Z^{\gamma'}}-\gamma'\frac{(V+1)^{\gamma-1}}{Z}\frac{(U+1)}{W^{\gamma'-1}}+(\gamma'-1)\frac{(U+1)^\gamma}{W^{\gamma'}}\right)\Big]ds<0. \quad (4.25)$$

On the other hand, from Lemma 2.1, the first member of (4.25) is greater than or equal to 0; so we have a contradiction.

Thus we have proved inequality (4.22). Then, from (4.20) and (4.21) we have:

$$-U'(s)\le -V'(s), \qquad \text{a.e. } s\in[0,s_0],$$

thus, integrating from s to s_0 and recalling that $U(s_0)=V(s_0)=0$, we obtain

$$U(s)\le V(s), \qquad \forall s\in[0,s_0].$$

Consequently, recalling the definitions of $U(s)$ and $V(s)$ we have, $\forall s\in[0,s_0]$,

$$\int_s^{s_0}\frac{1}{\tau^{(1-1/n)(p-q)}}\big((-u^*(\tau))'\big)^{1-p+q}d\tau\le\int_s^{s_0}\frac{1}{\tau^{(1-1/n)(p-q)}}\big((-v^*(\tau))'\big)^{1-p+q}d\tau.$$

Then, since s_0 is an arbitrary point of $]0,|\Omega|]$, (4.5) and (4.6) imply

$$(-u^*(s))'\le(-v^*(s))', \qquad \text{a.e. } s\in[0,|\Omega|],$$

thus, integrating from 0 to $|\Omega|$, we obtain

$$u^*(s)\le v^*(s), \qquad \forall s\in[0,|\Omega|].$$

The inequality (4.3) is so proved.

Now, let us show the inequality (4.4). Let us recall that:

$$\int_\Omega \eta\big(|Du|^p\big)dx=\int_0^{+\infty}\Big(-\frac{d}{dt}\int_{|u|>t}\eta\big(|Du|^p\big)dx\Big)dt. \quad (4.26)$$

By standard arguments, using the Jensen integral inequality we obtain that:

$$-\frac{d}{dt}\int_{|u|>t}\eta\big(|Du|^p\big)dx\le-\mu_u'(t)\eta\Big(\frac{1}{-\mu_u'(t)}\big(-\frac{d}{dt}\int_{|u|>t}|Du|^p dx\big)\Big). \quad (4.27)$$

From (4.13), integrating by parts, we have:

$$-\frac{d}{dt}\int_{|u|>t}|Du|^p dx \;\le\; \int_t^{+\infty} f^*(\mu_u(\tau))(-\mu_u'(\tau))$$
$$\times\exp\left(\int_t^\tau\frac{\theta}{\big(n\omega_n^{1/n}\big)^{p-q}}\left(\frac{-\mu_u'(r)}{(\mu_u(r))^{1-\frac{1}{n}}}\right)^{p-q}dr\right)d\tau. \quad (4.28)$$

From (4.26), (4.27) and (4.28), using the properties of rearrangements, a.e. in $[0,|\Omega|]$ we have :

$$\int_\Omega \eta\big(|Du|^p\big)dx \le \int_0^{|\Omega|} \eta\Bigg[(-u^*(s))' \times \int_0^s f^*(\sigma)\exp\Bigg(\int_\sigma^s \frac{\theta}{\big(n\omega_n^{1/n}\big)^{p-q}} \frac{\big((-u^*(\rho))'\big)^{1-p+q}}{\rho^{(1-1/n)(p-q)}}d\rho\Bigg)d\sigma\Bigg]ds.$$

Consequently, since $(-u^*(s))' \le (-v^*(s))'$ a.e. in $[0,|\Omega|]$, from (4.6) we have

$$\int_\Omega \eta\big(|Du|^p\big)dx \le \int_0^{|\Omega|} \eta\Big(((-v^*(s))')^p(n\omega_n^{1/n}s^{(1-1/n)})^p\Big)ds = \int_{\Omega^\#} \eta\big(|Dv|^p\big)dx.$$

The theorem is so proved.

We finally point out that the result given in Theorem 4.1 provides a priori estimates on bounded solutions to problem (4.1) which can be used in order to prove the existence of a solution for the same problem. Indeed, using well known approximation techniques which can be found, for example, in [10], one can easily prove, as a consequence of Theorem 4.1, the following result (see also [18]).

Theorem 4.2 *Let us suppose that* (1.9)-(1.11), (3.2) *hold true and that:*

$$[a(x,s,\xi)-a(x,s,\xi')](\xi-\xi')>0, \quad a.e.\ x\in\Omega,\ \forall s\in\mathbb{R},\ \forall\xi,\xi'\in\mathbb{R}^n,\ \xi\ne\xi'.$$

Then there exists at least one bounded solution to problem (4.1).

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