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# Rotation Numbers and Global Bifurcation in Systems of Ordinary Differential Equations

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**Abstract**

Global bifurcation in boundary value problems for systems of ordinary differential equations is studied. Rotation numbers are associated with solutions and are shown to be invariant along bifurcating continua. This invariance is then used to analyze the global structure of the bifurcating continua, and to demonstrate the existence of infinitely many solutions to some boundary value problems.

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The purpose of this paper is to study global bifurcation of solutions to boundary value problems for systems of ordinary differential equations. For the most part the paper concerns two dimensional systems, but the results have extensions to systems of dimension three or more, as is explained in the final section. Consider

$$\frac{dw}{dt} = F(\mu, w, t),\ t \in [0, T] \tag{1}$$

$$Bw = 0 \tag{2}$$

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where $F = (F_1, F_2) \in C(\mathbb{R} \times \mathbb{R}^2 \times [0,T], \mathbb{R}^2)$, and $F(\mu, 0, t) = 0$ for all $\mu \in \mathbb{R}$ and $t \in [0,T]$. We will first study the case of $Bw := w(T) - w(0)$, the periodic case. Then we will further illustrate our method with other boundary conditions. Letting $w = (u, v)$ we will consider (1) with $Bw := (u(0), u(T))$ and $Bw := (u(0), v(T))$. Throughout we will assume that solutions to initial value problems are unique. In particular, if $w(t_0) = 0$ then $w(t) \equiv 0$. If $w = w(t)$ is a nontrivial $T$-periodic solution then the mapping $t \mapsto \frac{w(t)}{|w(t)|}$ defines a mapping from the circle $S^1$ into itself. If $\varphi$ denotes this mapping then the Brouwer degree $\deg(\varphi)$ is defined. It is the same as the rotation number of $w(t)$ (with respect to the origin). If $\theta = \tan^{-1}(\frac{v}{u})$ then

$$\frac{d\theta}{dt} = \frac{v'u - vu'}{u^2 + v^2}$$

and the rotation number of a non-trivial $T$-periodic solution is

$$\mathbf{rot}(w) = \frac{1}{2\pi} \int_0^T \frac{uv' - vu'}{u^2 + v^2}\, dt = \frac{1}{2\pi} \int_0^T \frac{F_2 u - vF_1}{u^2 + v^2}\, dt.$$

We will use the rotation number to distinguish solutions and branches of solutions. We then show that this idea may also be applied to other boundary value problems, such as the two already mentioned, by appropriately extending the solutions to a larger interval on which the extension is periodic and the rotation number is an integer. In the case of second order scalar Sturm-Liouville boundary value problems on an interval $[0,T]$, bifurcating branches can be distinguished by the number of nodal points in $]0,T[$ [6]. This can sometimes be accomplished in systems of special form [2], but not generally. The method used here can characterize and distinguish different bifurcation branches in cases in which the use of nodal points seems unavailable. Our method of proving the existence of bifurcation is based upon the Leray-Schauder degree and the change of the Leray-Schauder index, using an abstract operator formulation of our boundary value problems. Krasnosel'skii [5] first showed the existence of bifurcation at an eigenvalue of odd multiplicity and Rabinowitz [6] later showed global bifurcation from these eigenvalues and proved fundamental results on the global structure. The reader is referred to the fundamental paper [6] or the expositions in [7] or [1] for the fundamental ideas. The rotation numbers of solutions have been used before to analyze global solution structure for boundary value problems, see [3], [4].

In §1 we introduce our method and consider in detail the bifurcation of periodic solutions and show under general conditions the bifurcating continua do not intersect. In §2 we consider some other boundary conditions. In §3 we study the global behavior of the bifurcating continua, giving conditions for the curves to lie on one side or the other of the bifurcation point. In §4 we apply our bifurcation results to prove the existence of infinitely many non-trivial solutions to some systems. In §5 we explain how some our earlier results on two dimensional systems can be extended to systems of dimension $N > 2$.

# 1 Bifurcation of periodic solutions

Let $\mathbb{R}$ denote the real numbers, $[0,T] = [0, 2\pi]$ and $\mathbb{Z}$ the integers. Let $|.|$ denote the Euclidean norm in $\mathbb{R}^n$, so if $w = (u,v)^T \in \mathbb{R}^2$ then $|w| = \sqrt{u^2 + v^2}$. If the function $F$ is

$C^1$ then we may write $F$ in the form

$$F(\mu, w, t) = A(\mu, t)w + g(\mu, w, t)$$

where $A$ is a $2 \times 2$ real matrix and $g(\mu, w, t) = o(|w|)$ as $|w| \to 0$, uniformly with respect to $\mu$ in compact sets and $t \in \mathbb{R}$. It suffices for our first result to assume that $A$ is independent of $t$ and, for $\mu$ in a neighborhood of some $k \in \mathbb{Z}$, we have $A(\mu) = \mu J + B(\mu)$ where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a $2 \times 2$ symplectic matrix and $B(\mu) = o(|\mu - k|)$ as $\mu \to k$. However, we are mainly interested in continuation along a bifurcation curve of the rotation value of solutions at a bifurcation point, which will imply the curves do not intersect. Thus for simplicity we will assume that the function $F$ has the form

$$F(\mu, w, t) = \mu J w + g(\mu, w, t)$$

where $g \in C(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})$, $g(\mu, w, t + 2\pi) = g(\mu, w, t)$ and $g(\mu, w, t) = o(|w|)$ as $|w| \to 0$. We shall study bifurcation of solutions to the periodic problem

$$\frac{dw}{dt} = \mu J w + g(\mu, w, t), \ t \in [0, 2\pi], \tag{3}$$

$$w(2\pi) - w(0) = 0 \tag{4}$$

Such strong assumptions on $g$ are not needed. We could assume that $g$ satisfies appropriate Carathéodory conditions. Sometimes we may find it convenient to use the $2\pi$-periodic extension of a solution. In any case, we will often call solutions to (3), (4) $2\pi$-periodic solutions. We will say that $(\mu_0, 0)$ is a bifurcation point for (3), (4) from the line of trivial solutions if every neighborhood of $(\mu_0, 0)$ in $\mathbb{R} \times C(\mathbb{R}, \mathbb{R}^2)$ contains some $(\mu, w)$ with $w$ a non-trivial $2\pi-$periodic solution of (3). If (3) is Hamiltonian then variational methods may be applied to the bifurcation problem. However, in that case one cannot generally show that there is a global continuum of solutions emanating from a bifurcation point. Thus outside a small neighborhood of a bifurcation point one can conclude little. For this reason we apply Leray-Schauder degree to prove the existence of bifurcating continua, and therefore cannot allow the linearized system at a bifurcation point to have an even dimensional set of solutions. The possible bifurcation points on the line of trivial solutions $\{(\mu, 0) : \mu \in \mathbb{R}\}$ are those with $\mu \in \mathbb{Z}$. If $\mu \in \mathbb{R}$ then a fundamental solution to the linear system

$$\frac{dw}{dt} = \mu J w \tag{5}$$

is

$$W(t) = \begin{pmatrix} \cos(\mu t) & -\sin(\mu t) \\ \sin(\mu t) & \cos(\mu t) \end{pmatrix}.$$

It follows that if $\mu = k \in \mathbb{Z}$ then the associated eigenspace of $2\pi$-periodic solutions to (5) is two dimensional. We will impose an added condition to obtain a one dimensional

eigenspace. Let $w = (u, v)^T$, then two possible conditions that would suffice are (i) require $u$ to be an odd function and $v$ even, or (ii) require $u$ to be even and $v$ to be odd. There is only a formal difference. In our first theorem, stated below, we will seek solutions $w = (u, v)^T$ of (3) that are $2\pi$-periodic with $u(-t) = -u(t)$ and $v(-t) = v(t)$. The associated eigenspace for$\mu = k \in \mathbb{Z}$ is spanned by $(-\sin(kt), \cos(kt))^T$.

We now describe the abstract framework to be used. Let $X$ be the real linear space of all continuous $2\pi$-periodic functions $w = (u, v)^T \in C_{2\pi}(\mathbb{R}, \mathbb{R}^2)$ with $u$ odd and $v$ even. Let $Y$ be the real linear space of all continuous $2\pi$-periodic functions $w = (u, v)^T \in C_{2\pi}(\mathbb{R}, \mathbb{R}^2)$ with $u$ even and $v$ odd. The spaces $X$ and $Y$ are Banach spaces, each with the norm

$$\|w\| := \max_{t \in \mathbb{R}} |w(t)|$$

for $w \in X$ or $w \in Y$. Let $D(L) = X \cap C^1(\mathbb{R}, \mathbb{R}^2)$ and let $c \in \mathbb{R} \backslash \mathbb{Z}$. We define $L : D(L) \subset X \to Y$ by

$$Lw := \frac{dw}{dt} + cJw$$

for $w \in D(L)$. It is not difficult to verify that $L$ is a bijection from $D(L)$ onto $Y$, and its inverse $L^{-1} : Y \to X$ is compact.

Let $\mathcal{S}_0$ denote the set of all nontrivial solutions $(\mu, w)$ of (3), (4) and let $\mathcal{S}$ denote the closure of $\mathcal{S}_0$. If $\psi \in C(S^1, S^1)$ we denote the rotation number of $\psi$ by $\mathbf{rot}(\psi)$. We state our first result.

**Theorem 1** *Let $g = (g_1, g_2)^T$ be continuous. Assume $g_1(\mu, -u, v, -t) = g_1(\mu, u, v, t)$ and $g_2(\mu, -u, v, -t) = -g_2(\mu, u, v, t)$. Then*

*(C1) For each $k \in \mathbb{Z}$, $(k, 0)$ is a bifurcation point of (3), (4). That is, in every neighborhood of $(k, 0) \in \mathbb{R} \times X$ there is a solution $(\mu, w) \in \mathbb{R} \times D(L)$, $w \neq 0$, of (3).*

*(C2) For each $k \in \mathbb{Z}$ let $\mathcal{C}_k \subset \mathbb{R} \times X$ denote the component of $\mathcal{S}$ which meets $(k, 0)$. Each $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times X$ and if $(\mu, w) \in \mathcal{C}_k$, $w \neq 0$, then $\mathbf{rot}(w)) = k$. If $j \neq k$ then $\mathcal{C}_j \cap \mathcal{C}_k = \varnothing$.*

*Proof.* The problem

$$\frac{dw}{dt} = \mu Jw + g(\mu, w, t),$$
$$w(t + 2\pi) = w(t),\ w \in X$$

is equivalent to

$$\frac{dw}{dt} + cJw = (\mu + c)Jw + g(\mu, w, t), \tag{6}$$
$$w(t + 2\pi) = w(t),\ w \in X.$$

We wish to write this as an abstract bifurcation problem. Let $G(\lambda, w)(t) := g(\lambda - c, w(t), t)$ for $w \in X$ and $t \in \mathbb{R}$. The operator $G$ maps $\mathbb{R} \times X$ into $Y$. It is easy to see that it is continuous and maps bounded sets in $\mathbb{R} \times X$ into bounded sets in $Y$. Let $\widetilde{J}$ denote the bounded linear operator mapping $X$ into $Y$ defined by the action of the matrix

$J$, so $(\widetilde{J}w)(t) := J(w(t))$ for $w \in X$ and $t \in \mathbb{R}$. Letting $\lambda = \mu + c$, we may rewrite (6) in abstract form as

$$Lw = \lambda \widetilde{J}w + G(\lambda, w). \tag{7}$$

Let $K := L^{-1}$. We may rewrite (7) equivalently as

$$w = \lambda K\widetilde{J}w + KG(\lambda, w). \tag{8}$$

The bounded operator $\widetilde{J}$ maps $X$ into $Y$ and $K$ maps $Y$ into $X$ compactly, so $K\widetilde{J}$ is a compact linear map of $X$ into itself. Similarly, $G$ is a continuous and bounded map from $X$ into $Y$, so $KG$ is a continuous map that takes bounded sets in $X$ into relatively compact ones; i.e., $KG$ is a completely continuous map of $X$ into itself. The Leray-Schauder index of $I - \lambda K\widetilde{J}$ is defined (and is $\pm 1$) whenever $\mu = \lambda - c \notin \mathbb{Z}$. Moreover the index changes sign when $\lambda - c$ crosses an integer , since for $k \in \mathbb{Z}$ the null space of $I - (k + c)K\widetilde{J}$ is one (and, hence, odd) dimensional ([1], [5], [6]). Now $KG(\lambda, w) = o(\|w\|)$ as $\|w\| \to 0$, uniformly with respect to $\lambda$ in compact sets. It follows by application of the Rabinowitz global bifurcation theorem [6] that each $(k + c, 0)$, $k \in \mathbb{Z}$ is a point of bifurcation from the line of trivial solutions, and there is a continuum $\widetilde{\mathcal{C}}_k \subset \mathbb{R} \times X$ of solutions $(\lambda, w)$ to (8) meeting $(k + c, 0)$ and such that either $\widetilde{\mathcal{C}}_k$ is unbounded or else it meets another bifurcation point $(j + c, 0)$, $j \neq k$. Now if $(\lambda, w) \in \widetilde{\mathcal{C}}_k$ then $(\lambda, w)$ solves (7) so

$$\frac{dw}{dt} + cJw = \lambda Jw + g(\lambda - c, w, t)$$

or, since $\lambda = \mu + c$,

$$\frac{dw}{dt} = \mu Jw + g(\mu, w, t)$$

and $(\mu, w)$ is a solution of the latter equation. Thus $\mathcal{C}_k = \{(\mu, w) : (\mu + c, w) \in \widetilde{\mathcal{C}}_k$ is a continuum of solutions to (3), (4) meeting $(k, 0)$, and $\mathcal{C}_k$ is either unbounded in $\mathbb{R} \times X$ or else $\mathcal{C}_k$ meets another bifurcation point $(j, 0)$, $j \neq k$.

It remains to show the rotation properties of solutions and the non-intersection of $\mathcal{C}_k$ with $\mathcal{C}_j$ if $j \neq k$. Let $\{(\mu_n, w_n)\}$ be a sequence in $\mathcal{C}_k$ with $(\mu_n, w_n) \to (k, 0)$ as $n \to \infty$ and each $w_n \neq 0$. Let $y_n = w_n / \|w_n\|$. Then with $\lambda_n = \mu_n + c$ we have

$$y_n = \lambda_n K J y_n + KG(\lambda_n, w_n)/\|w_n\|$$

and we see there is a subsequence of $\{y_n\}$ convergent in $X$ to some $y^*$, $\|y^*\| = 1$, and $y^* = (k + c)KJy^*$. Hence

$$\frac{dy^*}{dt} = kJy^*$$

and it follows that $y^*(t) = \pm(-\sin(kt), \cos(kt))$. Take the plus sign (it makes no difference). Then $t \mapsto y^*(t)$ defines a map from $\mathbb{R}/(2\pi\mathbb{Z}) = S^1$ into itself, $t \mapsto (-\sin(kt), \cos(kt))$. If $k > 0$ we see that as $t$ runs from 0 to $2\pi$, i.e., once around the circle counterclockwise, $(-\sin(kt), \cos(kt))$ runs around the circle counterclockwise clockwise $k$ times, and the rotation number is $k$. If $k < 0$, $(-\sin(kt), \cos(kt))$ runs around the circle $|k|$ times in the clockwise direction, and the rotation number is $-|k| = k$. Now , for any $(\mu, u) \in C_k$ with

$u \neq 0$ the mapping $t \mapsto u(t)/|u(t)|$ also defines a mapping of $S^1$ into itself. If $(\mu, u)$ is sufficiently close to $(k, 0)$, then the rotation number of that map must also be $k$. It may be helpful to recall that the rotation number is the same as the topological degree of the map $\psi(w) \in C(S^1, S^1)$ given by $t \mapsto \frac{w(t)}{|w(t)|}$, a self map of the circle $S^1 = [0, 2\pi]/\{0.2\pi\}$. That is,

$$\deg(\psi(w)) = \mathbf{rot}(\psi(w)),$$

a fact we will use repeatedly. Now let

$$U = \{w \in X : w(t) \neq 0 \text{ for } t \in \mathbb{R}\}.$$

The set $U$ is open in $X$. For each $w \in U$ associate a mapping $\psi(w) \in C(S^1, S^1)$ given by $\psi(w)(t) := w(t)/|w(t)|$. The mapping $\psi : U \to C(S^1, S^1)$ is clearly continuous. Let deg denote the topological degree of a selfmap of $S^1$, and let $\Phi : U \to \mathbb{Z}$ be defined by $\Phi(w) := \deg(\psi(w))$. The mapping $\Phi$ is a continuous map from $U$ into the integers $\mathbb{Z}$. Thus the sets $U_k := \Phi^{-1}(k)$, $k \in \mathbb{Z}$, are disjoint and open in $X$. Now let $P$ be the continuous projection from $\mathbb{R} \times X$ into $X$ given by $P(\mu, w) := w$, so that $P\mathcal{C}_k$ is connected in $X$. We have seen there is a number $\delta > 0$ such that if $(\mu, u) \in \mathcal{C}_k \backslash \{(k, 0)\}$ and $|k - \mu| + \|u\| < \delta$ then $\mathbf{rot}(\psi(u)) = \deg(\psi(u)) = k$. Let $B(k, \delta) = \{(\mu, w) \in \mathbb{R} \times X : |k - \mu| + \|w\| < \delta\}$. Each $(\mu, w) \in \mathcal{C}_k \backslash \{(k, 0)\}$ must be in a component of $\mathcal{C}_k \backslash \{(k, 0)\}$ that meets $B(k, \delta)$. Call this component $\mathcal{C}_k(\mu, w)$, and notice that $P\,\mathcal{C}_k(\mu, w) \subset U$. It follows that $P\mathcal{C}_k(\mu, w) \subset U_j$ for some $j \in \mathbb{Z}$. But $\mathcal{C}_k(\mu, w)$ meets $B(k, \delta)$, so it must be the case that $j = k$. Hence if $(\mu, w) \in \mathcal{C}_k \backslash \{(k, 0)\}$ then $\deg(\psi(P(\mu, w)) = k$, as we wished to show. It follows immediately that $\mathcal{C}_k \cap \mathcal{C}_j = \varnothing$ if $j \neq k$. This completes the proof of the theorem. ■

## 2 Bifurcation with other boundary conditions

The method of proof used in Theorem 1 is not restricted to the periodic problem. We will apply the ideas to two other boundary value problems. Let $J$ be as before and let $g : \mathbb{R} \times \mathbb{R}^2 \times [0, \pi] \to \mathbb{R}^2$ be a continuous function. As before, we assume $g(\mu, 0, t) = 0$ and $|g(\mu, w, t)| = o(|w|)$ as $|w| \to 0$, uniformly with respect to $\mu$ and $t$ in compact sets. Let $w = (u, v)^T$ and consider

$$\frac{dw}{dt} = \mu J w + g(\mu, w, t),\ t \in [0, \pi]. \tag{9}$$

Letting $w = (u, v)^T$, we consider the following boundary value problems for (9):

$$u(0) = 0 \text{ and } v(\pi) = 0 \tag{10}$$

and

$$u(0) = 0 \text{ and } u(\pi) = 0. \tag{11}$$

It is not hard to find the eigenvalues and eigenfunctions for the linear problems $dw/dt = \mu J w$. With boundary conditions (11) the eigenvalue set is $\mathbb{Z}$. In this case each eigenspace is

one dimensional and is spanned by $(-\sin kt, \cos kt)$ for $k \in \mathbb{Z}$. With boundary conditions (10) the set of eigenvalues is $\mathbb{Z} + \frac{1}{2}$, that is, all numbers of the form $k + \frac{1}{2}$, $k \in \mathbb{Z}$. Again each eigenspace is one dimensional, and for $\mu = k + \frac{1}{2}$ is spanned by the eigenfunction $(-\sin(k + \frac{1}{2})t, \cos(k + \frac{1}{2}t))$. The arguments for bifurcation and the existence of continua $\mathcal{C}_k$ meeting the bifurcation points $\mu = k$ for (11) or $\mu = k + \frac{1}{2}$ for (10) are the same as in Theorem 1, with $\mathcal{C}_k$ being either unbounded or or meeting a different bifurcation point. The arguments that the latter cannot happen and $\mathcal{C}_j \cap \mathcal{C}_k = \varnothing$ are slightly different from those of Theorem 1.

We first consider problem (9) (10). It may be handled in a manner similar to that used for (3), 4).

We let

$$X_1 = \{w = (u, v)^T : w \in C([0, \pi], \mathbb{R}^2),\ u(0) = 0, v(\pi) = 0\}.$$

and $Y = C([0, \pi], \mathbb{R}^2)$, with the maximum norms on each space. Let $D(L) = X_1 \cap C^1([0, \pi], \mathbb{R}^2)$. With $c \in (0, \frac{1}{2})$ let $Lw = dw/dt + cJw$ for $w \in X_1$. We may again write our problem as

$$Lw = \lambda \widetilde{J} w + G(\lambda, w)$$

where $\lambda = \mu + c$. As in Theorem 1, we find continua $\mathcal{C}_k$ of solutions to (9) (10). This time $\mathcal{C}_k$ meets the bifurcation point $(k + \frac{1}{2}, 0)$. Each $\mathcal{C}_k$ is either unbounded in $\mathbb{R} \times X_1$ or else meets another point $(j + \frac{1}{2}, 0)$, $j \neq k$. We must show the latter possibility is excluded. Let $(\mu, w)$ be a solution to (9) (10), $w \neq 0$. In order to associate a topological degree or integer rotation number with $w$ we will construct a $4\pi$-periodic extension of $w$. If $(\mu, w)$ is a solution to (9), (10) and $w = (u, v) \neq (0, 0)$ then $u(0) = 0$. Extend $u$ to $[0, 2\pi]$ so that the extension $u_1(t)$is even with respect to $t = \pi$, that is $u_1(\pi - t) = u_1(\pi + t)$ for $t \in [0, \pi]$. Then $u_1(0) = 0 = u_1(2\pi)$ and $u_1$ may be extended as an odd function $\widetilde{u}$ to $[-2\pi, 2\pi]$, and finally extend $\widetilde{u}$ as a continuous $4\pi-$periodic function on $\mathbb{R}$. Note that $\widetilde{u}$ will be an odd function (with respect to 0). Next extend $v$ to $[-\pi, \pi]$ so that the extension $v_1$ is even with respect to $t = 0$, that is, $v_1(-t) = v_1(t)$ for $t \in [0, \pi]$. Now $v_1(-\pi) = 0 = v_1(\pi)$ and $v_1$ may be extended to $[-\pi, 3\pi]$ as a continuous function $\widetilde{v}$ odd with respect to $t = \pi$, so that $\widetilde{v}(\pi - t) = -\widetilde{v}(\pi + t)$ for $t \in [0, 2\pi]$. Finally extend $\widetilde{v}$ as a continuous $4\pi$-periodic function on $\mathbb{R}$. Notice $\widetilde{v}$ will be an even function. Let $\widetilde{w}(t) = (\widetilde{u}(t), \widetilde{v}(t))$. Then $\widetilde{w}(t) \neq 0$ for all $t$. Otherwise, $\widetilde{w}(t_0) = 0$ for some $t_0 \in [-2\pi, 2\pi]$. If $t_0 \in [0, \pi]$ then $w(t_0) = \widetilde{w}(t_0) = 0$, which is impossible, since $w$ is a nontrivial solution and solutions to initial value problems are unique. If $t_0 \in (\pi, 2\pi]$ then $t_0 = \pi + t_1$ for some $t_1 \in (0, \pi]$. Since $\widetilde{u}$ is even about $\pi$, $\widetilde{u}(t_0) = \widetilde{u}(\pi + t_1) = \widetilde{u}(\pi - t_1) = u(\pi - t_1) = 0$. Now $0 = \widetilde{v}(t_0) = \widetilde{v}(\pi + t_1) = -\widetilde{v}(\pi - t_1) = -v(\pi - t_1)$ since $\widetilde{v}$ is odd with respect to $t = \pi$. Thus $u(\pi - t_1) = v(\pi - t_1) = 0$ for some $0 < \pi - t_1 < \pi$, again contradicting uniqueness of solutions to initial value problems. Now suppose $t_0 \in [-2\pi, 0)$. Then $0 = \widetilde{u}(t_0) = -\widetilde{u}(-t_0)$ and $0 = \widetilde{v}(t_0) = \widetilde{v}(-t_0)$ and $-t_0 \in (0, 2\pi]$. We have just shown that this is impossible. Thus $\widetilde{w}(t) \neq 0$ for all $t \in \mathbb{R}$. Since $\widetilde{w}$ is $4\pi$-periodic if we identify $S^1$ with $[0, 4\pi]/\{0, 4\pi\}$ then for each $(\mu, w) \in \mathcal{C}_k$, $w \neq 0$, we obtain a map $\varphi(\mu, w) \in C(S^1, S^1)$ defined by $t \mapsto \widetilde{w}(t)/|\widetilde{w}(t)|$. Now it is important to note that if $w(t) = (-\sin(k + \frac{1}{2})t, \cos(k + \frac{1}{2}t))$ for $t \in [0, \pi]$ then the $4\pi$-periodic extension

$\widetilde{w}(t)$ obtained by the above construction is just $(-\sin(k+\frac{1}{2})t, \cos(k+\frac{1}{2}t))$ for $t \in \mathbb{R}$. A simple calculation shows that if $\psi : [0,4\pi]/\{0,4\pi\} = S^1 \to S^1$ is given by $t \mapsto (-\sin(k+\frac{1}{2})t, \cos(k+\frac{1}{2}t))$, $t \in [0,4\pi]$ then $\deg(\psi) = 2k+1$ for $k \in \mathbb{Z}$.

Now suppose $(\mu_n, w_n) \in \mathcal{C}_k$, $w_n \neq 0$, and $(\mu_n, w_n) \to (k+\frac{1}{2}, 0)$. Without loss of generality we may assume that $y_n = w_n / \|w_n\|$ converges to some $y^*$, $\|y^*\| = 1$ and $Ly^* = (k+\frac{1}{2}+c)Jy^*$. Thus $y^* = \pm(-\sin(k+\frac{1}{2})t, \cos(k+\frac{1}{2}t))$, $t \in [0,\pi]$. With each $w_n$ we associate the $4\pi$-periodic extension $\widetilde{w}_n$ constructed above. It is clear that if $\widetilde{y}_n = \widetilde{w}_n / \|\widetilde{w}_n\|$ then $\{\widetilde{y}_n\}$ converges on $\mathbb{R}$ to $\widetilde{y}^*(t) = \pm(\sin(k+\frac{1}{2})t, \cos(k+\frac{1}{2}t))$, $t \in \mathbb{R}$. The topological degree of the map $t \mapsto \widetilde{y}^*(t)$ from $[0,4\pi]/\{0,4\pi\} \to S^1$ is $2k+1$. Thus it follows that there is a number $\delta > 0$ such that if $|\mu_n| + \|w_n\| < \delta$ then $\mathbf{rot}(\varphi(\mu_n, w_n)) = \deg \varphi(\mu_n, w_n) = 2k+1$. It now follows, using that $\mathcal{C}_k$ is a continuum, and with arguments analogous to those of Theorem 1, that if $(\mu, w) \in \mathcal{C}_k$ and $w \neq 0$, then $\mathbf{rot}(\varphi(\mu, w)) = \deg \varphi(\mu, w) = 2k+1$. Thus if $k \neq j$ then $\mathcal{C}_k \cap \mathcal{C}_j = \varnothing$ and each $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times X_1$.

We now consider the problem (9), (11). We work in the Banach spaces $X_2$ and $Y$ where $Y = C([0,\pi], \mathbb{R}^2)$, and $w = (u,v)^T \in X_2$ if and only if $u, v \in C([0,\pi])$ and $u(0) = 0 = u(\pi)$. The norm $\|w\|$ for $w \in X_2$ or $Y$ is the maximum of $|w(t)|$, $t \in [0,\pi]$. Let $D(L) = X_2 \cap C^1([0,\pi], \mathbb{R}^2)$. Let c$\in ]0,1[$ and $L : D(L) \to Y$ be defined by

$$Lw := \frac{dw}{dt} + cw$$

for $w \in D(L)$. Let $G(\lambda, w)(t) := g(\lambda - c, w(t), t)$ for $w \in X_2$ and $t \in \mathbb{R}$. The operator $G$ maps $\mathbb{R} \times X_2$ into $Y$. It is easy to see that it is continuous and maps bounded sets in $\mathbb{R} \times X_2$ into bounded sets in $Y$. Let $\widetilde{J}$ denote the bounded linear operator mapping $X_2$ into $Y$ defined by the action of the matrix $J$, so $(\widetilde{J}w)(t) := J(w(t))$ for $w \in X_2$ and $t \in \mathbb{R}$. Letting $\lambda = \mu + c$, we may rewrite our problem (9), (11) as

$$Lw = \lambda \widetilde{J}w + G(\lambda, w),$$

or, letting $K = L^{-1}$,

$$w = \lambda K \widetilde{J} w + KG(\lambda, w). \tag{12}$$

As in Theorem 1, there are continua of solutions to (12) bifurcating from each point of the form $(\lambda, 0) = (k+c, 0)$, and these of course correspond to continua $\mathcal{C}_k$ of solutions to (9), (11) bifurcating from each point $(k,0)$, $k \in \mathbb{Z}\backslash\{0\}$. We must show that $\mathcal{C}_k \cap \mathcal{C}_j = \varnothing$ if $j \neq k$.

If $(\mu, w)$ is a nontrivial solution of (9), (11) then $w = (u,v)$ and $u(0) = u(\pi) = 0$. Thus $u$ may be extended as a continuous odd function on $[-\pi,\pi]$, and $v$ may be extended as a continuous even function on $[-\pi,\pi]$. Let $\widetilde{u}, \widetilde{v}$ denote these extensions. We will refer to $\widetilde{w} = (\widetilde{u}, \widetilde{v})$ as the odd/even $2\pi$ periodic extension of $w$. Then $\widetilde{w}(t) = (\widetilde{u}(t), \widetilde{v}(t)) \neq (0,0)$ for any $t \in [-\pi,\pi]$, for both $u$ and $v$ cannot be simultaneously zero at any point in $[0,\pi]$, for then uniqueness of solutions would imply $w = 0$. Thus the extensions also cannot be simultaneously zero at any point in $[-\pi,\pi]$. Now $\widetilde{w}(-\pi) = \widetilde{w}(\pi)$, and $\widetilde{w}(t)/|\widetilde{w}(t)| \in S^1$ for all $t \in [-\pi,\pi]$. Identifying $S^1$ with $[-\pi,\pi]/\{-\pi,\pi\}$ we obtain a mapping $t \mapsto \widetilde{w}(t)/|\widetilde{w}(t)|$ of $S^1$ into itself, which we will denote by $\varphi(\mu, w)$. The rotation of the

map is defined. Now the argument is similar to that of Theorem 1 for periodic boundary conditions. Let $(\mu_n, w_n) \in \mathcal{C}_k$ with $w_n \neq 0$ and $(\mu_n, w_n) \to (k, 0)$ as $n \to \infty$. Let $y_n = w_n(t)/|w_n(t)|$ and $\widetilde{y}_n(t) = \widetilde{w}_n(t)/|\widetilde{w}_n(t)|$ then a subsequence of $(\mu_n, y_n)$ converges to some $(k, y^*)$ such that $\|y^*\| = 1$ and $y^*$ solves $dy/dt = kJy$ with boundary conditions (11). The corresponding subsequence of $(\mu_n, \widetilde{y}_n)$ converges to $(k, \widetilde{y}^*)$. We must have $y^*(t) = \pm(-\sin kt, \cos kt)$, $t \in [0, \pi]$, and $\widetilde{y}^*(t) = \pm(-\sin kt, \cos kt)$, $t \in [-\pi, \pi]$. The rotation of the map $t \mapsto \widetilde{y}^*(t)$ as a map $S^1 \to S^1$ is $k$. Repeating arguments from Theorem 1, or almost repeating them (there are small differences), we see that $\mathbf{rot}(\varphi(\mu, w)) = \deg(\varphi(\mu, w)$ associated with any $w \neq 0$ with $(\mu, w) \in \mathcal{C}_k$ must be $k$. Thus $\mathcal{C}_k \cap \mathcal{C}_j = \varnothing$ if $j \neq k$, and each $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times X_2$.

We have proved the following two theorems.

Let $\mathcal{S}_0$ denote the set of all nontrivial solutions $(\mu, w)$ of (9) with boundary conditions (11) or (10), depending on context, and let $\mathcal{S}$ denote the closure of $\mathcal{S}_0$.

**Theorem 2** *Let $g = (g_1, g_2)^T \in C(\mathbb{R} \times \mathbb{R}^2 \times [0, \pi], \mathbb{R}^2)$, $g(\mu, 0, t) = 0$, and $g(\mu, w, t) = o(|w|)$ as $|w| \to 0$. Then*

*(C1) For each $k \in \mathbb{Z}$, $(k + \frac{1}{2}, 0)$ is a bifurcation point of (9), (10). That is, in every neighborhood of $(k + \frac{1}{2}, 0) \in \mathbb{R} \times X_1$ there is a solution $(\mu, w) \in \mathbb{R} \times D(L)$, $w \neq 0$, of (3).*

*(C2) For each $k \in \mathbb{Z}$ let $\mathcal{C}_k \subset \mathbb{R} \times X_1$ denote the component of $\mathcal{S}$ which meets $(k + \frac{1}{2}, 0)$. Each $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times X_1$ and if $(\mu, w) \in \mathcal{C}_k$, $w \neq 0$, then $w$ may be extended to $\mathbb{R}$ as a $4\pi$ periodic function in such a way that if $t \in [0, 4\pi]/\{0, 4\pi\}$ then $t \mapsto w(t)/|w(t)|$ defines a mapping $\varphi(\mu, w)$ of the circle $S^1$ into itself, and $\mathbf{rot}(\varphi(\mu, w)) = 2k + 1$. It follows that if $j \neq k$ then $\mathcal{C}_j \cap \mathcal{C}_k = \varnothing$.*

**Theorem 3** *Let $g = (g_1, g_2)^T \in C(\mathbb{R} \times \mathbb{R}^2 \times [0, \pi], \mathbb{R}^2)$, $g(\mu, 0, t) = 0$, and $g(\mu, w, t) = o(|w|)$ as $|w| \to 0$. Then*

*(C1) For each $k \in \mathbb{Z}$, $(k, 0)$ is a bifurcation point of (9), (11). That is, in every neighborhood of $(k, 0) \in \mathbb{R} \times X_2$ there is a solution $(\mu, w) \in \mathbb{R} \times D(L)$, $w \neq 0$, of (3).*

*(C2) For each $k \in \mathbb{Z}$ let $\mathcal{C}_k \subset \mathbb{R} \times X_2$ denote the component of $\mathcal{S}$ which meets $(k, 0)$. Each $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times X_2$ and if $(\mu, w) \in \mathcal{C}_k$, $w \neq 0$, then $w$ may be extended to $\mathbb{R}$ as a $2\pi$ periodic function in such a way that if $t \in [0, 2\pi]/\{0, 2\pi\}$ then $t \mapsto w(t)/|w(t)|$ defines a mapping $\varphi(\mu, w)$ of the circle $S^1$ into itself, and $\mathbf{rot}(\varphi(\mu, w)) = k$. It follows that if $j \neq k$ then $\mathcal{C}_j \cap \mathcal{C}_k = \varnothing$.*

We give one simple example. Consider the system, for $t \in [0, \pi]$:

$$\begin{aligned} \frac{du}{dt} &= \mu v + \cos(3t)v^3 - \mu^3 \sin(t)u^3 \\ \frac{dv}{dt} &= -\mu u + t^2 \cos(t)u^3 + \mu^2 \sin(3t)v^4 \\ u(0) &= 0,\ v(\pi) = 0. \end{aligned}$$

It follows from Theorem 2 that $(\mu, 0) = (2k + 1, 0)$ is a bifurcation point, for each $k \in \mathbb{Z}$, and there is a continuum $\mathcal{C}_k$ of solutions $(\mu, (u, v))$ such that $\mathcal{C}_k$ meets the line of trivial

solutions only at $(k+\frac{1}{2},0)$, and $\mathcal{C}_k$ is unbounded in $\mathbb{R}\times X_1$. If $(\mu,w)\in\mathcal{C}_k$, $w=(u,v)\neq(0,0)$ then $\deg\varphi(\mu,w)=2k+1$.

# 3 Global behavior of continua

In this section we give conditions which imply stronger conclusions regarding the global behavior of the continua $\mathcal{C}_k$ bifurcating from $(\lambda_k,0)$ found in the preceding sections. We show that under some conditions on the signs of the nonlinearities the $\mathcal{C}_k$ bend to the left or to the right of $\lambda_k$. We first consider linear systems of the form

$$\begin{aligned}\frac{du}{dt}&=-p(t)v\\ \frac{dv}{dt}&=q(t)u.\end{aligned}\tag{13}$$

where $p,q\in L^\infty(\mathbb{R})$ are $2\pi$-periodic functions on $\mathbb{R}$. If $w(t)=(u(t),v(t))$ is a nontrivial $2\pi$-periodic solution of (13) then $u(t)$ and $v(t)$ are absolutely continuous and $\widetilde{w}(t)=w(t)/|w(t)|$ is a continuous $2\pi$-periodic function of norm 1 and may be viewed as a map $\varphi(w)$from $S^1$ into itself. The angle $\theta$ of $\widetilde{w}(t)$ with respect to the origin is $\tan^{-1}(v(t)/u(t))$. We have

$$\frac{d\theta}{dt}=\frac{uv'-u'v}{u^2+v^2}=\frac{q(t)u^2+p(t)v^2}{u^2+v^2}\ \text{a.e.}$$

It follows that

$$\deg(\varphi)=\mathbf{rot}(\varphi)=\frac{1}{2\pi}\int_0^{2\pi}\frac{q(t)u^2+p(t)v^2}{u^2+v^2}\,dt.$$

We wish to compare the rotations associated with solutions of two different systems. Let $p_j,q_j\in L^\infty(\mathbb{R})$ for $j=1,2$ be $2\pi-$periodic functions and let $w_j=(u_j,v_j)$ $(j=1,2)$ be a non-trivial $2\pi$-periodic solution of the system

$$\begin{aligned}\frac{du_j}{dt}&=-p_j(t)v_j\\ \frac{dv_j}{dt}&=q_j(t)u_j.\end{aligned}\tag{14}$$

Let $\varphi(w_j)$ $(j=1,2)$ be the associated map of $S^1$ into itself for $j=1,2$ respectively. We have the lemma

**Lemma 4** *Let $p_j,q_j\in L^\infty(\mathbb{R})$, $j=1,2$, be $2\pi-$periodic functions and $w_j=(u_j,v_j)$ be a non trivial $2\pi$-periodic solution of (14) for $j=1,2$ respectively. Suppose that for a.a. $t\in\mathbb{R}$ we have*

$$r_1(t):=\max(p_1(t),q_1(t))\leq\min(p_2(t),q_2(t))=:r_2(t)\tag{15}$$

*for all $t\in\mathbb{R}$. Then $\mathbf{rot}(\varphi(w_1))\leq\mathbf{rot}(\varphi(w_2))$. If there is a set $E\subset[0,2\pi]$ of positive Lebesgue measure such that strict inequality holds in (15) for $t\in E$, then $\mathbf{rot}(\varphi(w_1))<\mathbf{rot}(\varphi(w_2))$.*

*Proof.* We have

$$\begin{aligned}\mathbf{rot}(\varphi(w_1)) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{q_1(t)u_1^2 + p_1(t)v_1^2}{u_1^2+v_1^2}\,dt \le \frac{1}{2\pi}\int_0^{2\pi} \frac{r_1(t)(u_1^2+v_1^2)}{u_1^2+v_1^2}\,dt \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{r_2(t)(u_2^2+v_2^2)}{u_2^2+v_2^2}\,dt \le \frac{1}{2\pi}\int_0^{2\pi} \frac{p_2(t)u_2^2 + q_2(t)v_2^2}{u_2^2+v_2^2}dt = \mathbf{rot}(\varphi(w_2))\end{aligned}$$

which proves the first part of the claim. If $r_1(t) < r_2(t)$ for $t \in E$, then the second integral inequality above is also strict, and this implies that $\mathbf{rot}(\varphi(w_1)) < \mathbf{rot}(\varphi(w_2))$. ∎

**Remark 5** The same conclusion holds if we impose the boundary conditions (10) or (11). In those cases we consider the appropriate periodic extensions of the nontrivial solutions $w$ to define $\varphi(w)$. For example, the $2\pi$-periodic extension $\widetilde{w}$ we constructed for boundary conditions (11) has $\widetilde{u}$ odd and $\widetilde{v}$ even. If we extend $p(t)$ and $q(t)$ to be even $2\pi$-periodic functions $\widetilde{p}(t)$ and $\widetilde{q}(t)$, respectively, then $\widetilde{w} = (\widetilde{u}, \widetilde{v})$ satisfies

$$\begin{aligned}\frac{d\widetilde{u}}{dt} &= -\widetilde{p}(t)\widetilde{v} \\ \frac{d\widetilde{v}}{dt} &= \widetilde{q}(t)\widetilde{u}.\end{aligned}$$

Using this and similar arguments one may establish the conclusion of the lemma if the boundary conditions (10) or (11) are imposed on (14).

We consider systems of the form

$$\begin{aligned}\frac{du}{dt} &= -\mu v - vf(\mu,u,v,t) \\ \frac{dv}{dt} &= \mu u + ug(\mu,u,v,t).\end{aligned} \tag{16}$$

Let $f, g \in C(\mathbb{R}\times\mathbb{R}^2\times[0,\pi],\mathbb{R})$. We consider the system (16) with boundary conditions (11). Recall

$$X_2 = \{w = (u,v) : w \in C([0,\pi],\mathbb{R}^2), u(0) = 0, u(\pi) = 0\}.$$

**Theorem 6** *Suppose $f$ and $g$ satisfy the conditions of Theorem 3, and also $f, g \le 0$ ($\ge 0$) for all $(\mu,u,v,t) \in \mathbb{R}\times\mathbb{R}^2\times[0,\pi]$. Let $\mathcal{C}_k \subset \mathbb{R}\times X_2$, $k \in \mathbb{Z}$, denote the continuum bifurcating from the point $(k,0)$. Then $\mathcal{C}_k \subset [k,\infty)\times X_2$ ($(-\infty,k]\times X_2$).*

*Proof.* Suppose $f(\mu,u,v,t), g(\mu,u,v,t) \le 0$ for all $(\mu,u,v,t) \in \mathbb{R}\times\mathbb{R}^2\times[0,\pi]$. Let $(\mu,w) = (\mu,(u,v)) \in \mathcal{C}_k$, with $(u,v) \ne (0,0)$, and let $p(t) = \mu + f(\mu,u(t),v(t),t)$ and $q(t) = \mu + g(\mu,u(t),v(t),t)$. Then $u, v$ satisfy

$$\begin{aligned}\frac{du}{dt} &= -p(t)v \\ \frac{dv}{dt} &= q(t)u.\end{aligned}$$

Suppose $\mu < k$. Then $\max(p(t), q(t)) \leq \mu < k = \min(k, k)$. Thus if $\varphi(w)$ is the associated map of $S^1$ to itself, by the Lemma we must have $\mathbf{rot}(\varphi(w)) < d_1$, where $d_1$ is the rotation associated with solutions to

$$\frac{du}{dt} = -kv$$
$$\frac{dv}{dt} = ku.$$

satisfying (11). But it is easy to verify that $d_1 = k$. Hence $\mathbf{rot}(\varphi(w)) < k$. But we saw in the proof of Theorem 3 that $\mathbf{rot}(\varphi(w)) = k$. This is a contradiction and implies that $\mu \geq k$, and thus $\mathcal{C}_k \subset [k, \infty) \times X_2$.

If $f(\mu, u, v, t), g(\mu, u, v, t) \geq 0$ for all $(\mu, u, v, t) \in \mathbb{R} \times \mathbb{R}^2 \times [0, \pi]$ then very similar arguments show that $\mathcal{C}_k \subset (-\infty, k] \times X_2$. ■

**Remark 7** If we impose boundary conditions (10) or periodic boundary conditions, then the same conclusions may be drawn as in Theorem 6.

# 4 Infinitely many solutions

In this section we prove the existence of infinitely many structurally distinct solutions to boundary value problems of the form

$$\frac{du}{dt} = -f(t, u, v)v \tag{17}$$
$$\frac{dv}{dt} = g(t, u, v)u$$

$$Bw = (0, 0) \tag{18}$$

where $w = (u, v)$ and $B$ is one of the boundary operators $Bw = w(2\pi) - w(0)$, $B(u, v) = (u(0), u(\pi))$, or $B(u, v) = (u(0), v(\pi))$. To obtain the existence of infinitely many solutions we apply our earlier bifurcation results to the family of problems

$$\frac{du}{dt} = -\mu v - f(t, u, v)v \tag{19}$$
$$\frac{dv}{dt} = \mu u + g(t, u, v)u$$

for $\mu \in \mathbb{R}$ with boundary conditions (18). As always, we assume that solutions to initial value problems for (19) are unique and that $f$ and $g$ are continuous functions of their arguments. In order to be specific, we will assume that $B$ is the boundary operator

$$B(u, v) = (u(0), u(\pi))$$

but our arguments work just as well for the other boundary conditions. Recall that we call $\widetilde{w}$ the odd/even extension of $w$ provided $w = (u, v) \in C([0, \pi], \mathbb{R}^2)$, $u(0) = 0 = v(0)$

and $\widetilde{w} = (\widetilde{u}, \widetilde{v})$ where $\widetilde{u}$, $\widetilde{v}$ are, respectively, the odd extension of $u$ and the even extension of $v$ to $[-\pi, \pi]$. There is a related result in [7] using bifurcation and the number of nodal points to prove the existence of infinitely many solutions in second order Sturm-Liouville problems.

**Theorem 8** *Let $f, g \in C([0,\pi] \times \mathbb{R}^2, \mathbb{R})$ and suppose $f(t,0,0) = 0 = g(t,0,0)$ for $t \in [0,\pi]$. In addition assume:*
*(c1) There is an $H \in C(\mathbb{R}^2, \mathbb{R})$ such that $f(t,u,v)$, $g(t,u,v) \geq H(u,v) \geq 0$ for all $(t,u,v) \in [0,\pi] \times \mathbb{R}^2$ and*

$$H(u,v) \to \infty \text{ as } |u| + |v| \to \infty.$$

*(c2) There are numbers $M_0 \geq m_0 > 0$ such that*

$$m_0 |g(t,u,v)| \leq |f(t,u,v)| \leq M_0 |g(t,u,v)|$$

*for all $(t,u,v) \in [0,\pi] \times \mathbb{R}^2$.*
*Then (17), (18) has infinitely many topologically distinct solutions. Indeed, for each $k \in \mathbb{N}$ there is a solution $w_k = (u_k, v_k)$ such that the odd/even $2\pi$ periodic extension $\widetilde{w}_k$ of $w_k$ has rotation number $k$.*

**Remark 9** The proof of the preceding theorem will actually show that (19), (18) has, for each $\mu \in \mathbb{R}$, infinitely many solutions, with $\mu = 0$ being simply a special case.

**Remark 10** If instead we have $f(t,u,v)$, $g(t,u,v) \leq H(u,v) \leq 0$ for all $(t,u,v) \in [0,\pi] \times \mathbb{R}^2$ and $H(u,v) \to -\infty$ as $|u| + |v| \to \infty$ then there are again infinitely many distinct solutions, but now for each $k \in \mathbb{N}$ there is a solution whose odd/even $2\pi$ periodic extension has rotation number $-k$.

We will need the following lemma.

**Lemma 11** *Let $f, g \in C([0,\pi] \times \mathbb{R}^2, \mathbb{R}^+)$ and suppose (c2) in the preceding Theorem holds. Let $w = (u,v)$ be any solution to the differential equation (19) with $\mu \geq 0$. Then there are constants $C_1$ and $C_2$ (depending on $w$ but not on $t$ ) such that*

$$\begin{aligned} |u(t)| &\leq C_1 |v(t)| + C_2 \text{ and} \\ |v(t)| &\leq C_1 |u(t)| + C_2 \end{aligned} \tag{20}$$

*for all $t$ in the domain of $w$.*

*Proof.* If $u'(t) = v'(t) = 0$ for all $t$ in the domain of $w$ then (20) is trivial. If $u'(t) = 0$ on an interval $J_0$ then $v(t) = 0$ on $J_0$ and hence $u(t) = 0$ on $J_0$ and $u(t) = v(t) = 0$ for all $t \in [0,\pi]$ and the conclusion is again trivial. Suppose $v'(t) \neq 0$ on an interval $J$. Then on $J$

$$\frac{u'(t)}{v'(t)} = \frac{-[\mu + f]v(t)}{[\mu + g]u(t)}$$

and thus

$$\left|\frac{u'(t)}{v'(t)}\right| = \left|\frac{\mu + f}{\mu + g}\right| \left|\frac{v(t)}{u(t)}\right| \leq b_0 \left|\frac{v(t)}{u(t)}\right|$$

for some $b_0 > 0$. Thus $|u(t)u'(t)| \leq b_0 |v(t)v'(t)|$ for all $t \in J$. It follows by quadrature that there are constants $b_1$ and $b_2$ such that for all $t \in J$, $u^2(t) \leq b_1 v^2(t) + b_2(u^2(t_0) + v^2(t_0))$ and $v^2(t) \leq b_1 u^2(t) + b_2(u^2(t_0) + v^2(t_0))$ (notice that $u(t)u'(t)$ and $v(t)v'(t)$ must have opposite sign, but this does not affect the conclusion). There can only be finitely many points in $[0, \pi]$ at which $v'(t) = 0$, and the conclusion follows. ■

*Proof of the Theorem:* By Theorem 3 each $(k, 0) \in \mathbb{Z} \times \{0\}$ is a bifurcation point of (19), (18), $\mathcal{C}_j \cap \mathcal{C}_k = \varnothing$ for $j \neq k$, and by Theorem 6, $\mathcal{C}_k \subset (-\infty, k] \times X_2$. We claim that if $k \in \mathbb{Z}$ and $k \geq 1$ then $\mathcal{C}_k \cap (\{0\} \times X_2) \neq \varnothing$, that is, there is a $(0, w) \in \mathcal{C}_k$ with $w \neq 0$. Suppose this is not the case for some fixed $k \geq 1$. Then $\mathcal{C}_k \subset ]0, k] \times X_2$, and as $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times X_2$ it follows that there is a sequence $\{(\mu_n, w_n)\}$ in $\mathcal{C}_k$ such that $0 < \mu_n \leq k$ for all $n \in \mathbb{N}$ and $\|w_n\| \to \infty$ as $n \to \infty$. Let $\widetilde{w}_n$ be the $2\pi$-periodic extension of $w_n = (u_n, v_n)$ ($u_n$ odd, $v_n$ even) and recall that $k = \mathbf{rot}(\widetilde{y}_n)$ for all $n \in \mathbb{N}$. Extend $f$ and $g$ to $\widetilde{f}$ and $\widetilde{g}$ so that for $-\pi \leq t \leq 0$ and $(u, v) \in \mathbb{R}^2$, $\widetilde{f}(t, u, v) = f(-t, -u, v)$ and $\widetilde{g}(t, u, v) = g(-t, -u, v)$. We then have that $\widetilde{w}_n = (\widetilde{u}_n, \widetilde{v}_n)$ satisfies

$$\begin{aligned} \frac{d\widetilde{u}}{dt} &= -\mu\widetilde{v} - \widetilde{f}(t, \widetilde{u}, \widetilde{v})\widetilde{v} \\ \frac{d\widetilde{v}}{dt} &= \mu\widetilde{u} + \widetilde{g}(t, \widetilde{u}, \widetilde{v})\widetilde{u} \end{aligned} \tag{21}$$

and $\widetilde{w}_n(2\pi) - \widetilde{w}_n(0) = 0$. Moreover if $\widetilde{H}(u, v) = \min H(\pm u, \pm v)$ for $(u, v) \in \mathbb{R}^2$ we will have $\widetilde{H}(u, v) \to \infty$ as $|u| + |v| \to \infty$. We have

$$k = \mathbf{rot}(\widetilde{w}_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{[\mu_n + \widetilde{f}(t, \widetilde{u}_n, \widetilde{v}_n)]\widetilde{v}_n^2 + [\mu_n + \widetilde{g}(t, \widetilde{u}_n, \widetilde{v}_n)]\widetilde{u}_n^2}{\widetilde{u}_n^2 + \widetilde{v}_n^2}\, dt \tag{22}$$

$$\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\widetilde{f}(t, \widetilde{u}_n, \widetilde{v}_n)\widetilde{v}_n^2 + \widetilde{g}(t, \widetilde{u}_n, \widetilde{v}_n)\widetilde{u}_n^2}{\widetilde{u}_n^2 + \widetilde{v}_n^2}\, dt \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \widetilde{H}(\widetilde{u}_n(t), \widetilde{v}_n(t))\, dt.$$

If we can show that $|\widetilde{u}_n(t)| + |\widetilde{v}_n(t)| \to \infty$ on a set of positive measure in $[-\pi, \pi]$ then we will have a contradiction. We will show that $|\widetilde{u}_n(t)| + |\widetilde{v}_n(t)| \to \infty$ uniformly on $[-\pi, \pi]$. We know there is a sequence $\{t_n\}$ in $[-\pi, \pi]$ with $|\widetilde{u}_n(t_n)| + |\widetilde{v}_n(t_n)| \to \infty$. Suppose there is a sequence $\{s_n\}$ in $[-\pi, \pi]$ and a constant $C$ such that $|\widetilde{u}_n(s_n)| + |\widetilde{v}_n(s_n)| \leq C$. Without loss of generality we may assume that $s_n \to s_0 \in [-\pi, \pi]$, $(u_n(s_n), v_n(s_n)) \to (u_0, v_0) \in \mathbb{R}^2$, and $\mu_n \to \mu_0 \in [0, k]$. Let $w_0(t) = (u_0(t), v_0(t))$ be the solution to the initial value problem (21) with $\mu = \mu_0$ and $w_0(s_0) = (u_0, v_0)$. By the Lemma there are constants $b_0, b_1$ such that

$$|u_0(t)| \leq b_0 |v_0(t)| + b_1 \text{ and } |v_0(t)| \leq b_0 |u_0(t)| + b_1 \tag{23}$$

for as long as $w_0(t)$ exists. Moreover since both $\mu_0 + f \geq 0$ and $\mu_0 + g \geq 0$ it follows from the differential equations that $u_0'(t)v_0(t) \leq 0$ and $u_0(t)v_0'(t) \geq 0$ as long as $w_0(t)$ exists. We wish to show that $w_0(t)$ continues as a solution to all of $[-\pi, \pi]$. If not, then there are basically two possibilities. Either (i) there is an interval $I = ]a, b[$ on which one of $u_0(t)$ or $v_0(t)$ never vanishes and tends to one of $\pm\infty$ as $t$ approaches $a$ or $b$, or (ii) one of $u_0(t)$ or $v_0(t)$ vanishes infinitely many times in an interval $I = ]a, b[$ and has a

sequence of local extremal values at points $\{y_j\}$ which tend to one of $\pm\infty$ as the $\{y_j\}$ approach one of $a$ or $b$. Suppose first that (i) takes place, and (without loss of generality) that $u_0(t) \to \infty$ as $t \to b^-$. By (23) we must have that $0 < |v_0(t)|$ on an interval $[a_1, b[$ and $|v_0(t)| \to \infty$ as $t \to b^-$. However there must be a point $t^* \in [a_1, b[$ at which $u_0'(t^*) > 0$. Thus at this point we must have $v_0(t^*) < 0$, and hence $v_0(t) < 0$ for $t^* \leq t < b$. But $u_0(t)v_0'(t) \geq 0$ and thus $v_0(t)$ must be increasing on $t^* \leq t < b$, and we have both $|v_0(t)| \to \infty$ as $t \to b^-$ and $v_0(t^*) \leq v_0(t) < 0$ on $t^* \leq t < b$. This is a contradiction, and (i) cannot hold. Now suppose that (ii) holds. Then there are sequences $x_1 < y_1 < x_2 < y_2 < x_3 < \ldots < b$ such that, for all $j$ we have, say, $u_0(x_j) = 0$ and $|u_0(y_j)| = \max|u_0(t)|$ on $[x_j, x_{j+1}]$, and $u_0'(y_j) = 0$. Moreover $|u_0(y_j)| \to \infty$ as $j \to \infty$. We have that for each $j$, $[\mu_0 + \widetilde{f}(y_j, u_0(y_j), v_0(y_j)]v_0(y_j) = 0$. Now $[\mu_0 + \widetilde{f}(y_j, u_0(y_j), v_0(y_j)] \geq \widetilde{H}(u_0(y_j), v_0(y_j)) \to \infty$ since $|u_0(y_j)| + |v_0(y_j)| \to \infty$. Thus we must have that $v_0(y_j) = 0$ for all sufficiently large $j$. But then we have $|u_0(y_j)| \leq b_0|v_0(y_j)| + b_1 = b_1$ for all large $j$, a contradiction. This proves that $w_0(t)$ extends as a solution of (21) on all of $[-\pi, \pi]$. But now we can reach a final contradiction and show that $|\widetilde{u}_n(t)| + |\widetilde{v}_n(t)| \to \infty$ uniformly on $[-\pi, \pi]$. We had that $(u_n(s_n), v_n(s_n)) \to (u_0, v_0)$, $\mu_n \to \mu_0$, and $s_n \to s_0$. By continuity of solutions to initial value problems there is a $\delta > 0$ such that $|s_n - s_0| + |\mu_n - \mu_0| + |(u_n(s_n), v_n(s_n)) - (u_0, v_0)| < \delta$ implies that $|\widetilde{w}_n(t) - w_0(t)| < 1$ on $[-\pi, \pi]$, which is impossible since $|\widetilde{u}_n(t_n)| + |\widetilde{v}_n(t_n)| \to \infty$. This proves that we must have $|\widetilde{u}_n(t)| + |\widetilde{v}_n(t)| \to \infty$ uniformly on $[-\pi, \pi]$, and hence $\frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{H}(\widetilde{u}_n(t), \widetilde{v}_n(t))\, dt \to \infty$ as $n \to \infty$. This is a contradiction to (22), and hence we must have $\mathcal{C}_k \cap (\{0\} \times X_2) \neq \varnothing$, that is, there is a $(0, w) \in \mathcal{C}_k$ with $w \neq 0$. This proves the theorem. ■

**Remark 12** It is not clear if (c2) is needed for the conclusion. However if (c1) fails but (c2) holds the conclusion may not hold. This is shown by the simple example $u' = -v$, $v' = u$ in which $f = g = 1$ and the only solutions with $u(0) = 0 = u(\pi)$ are $(u, v) = \alpha(-\sin t, \cos t)$ for $\alpha \in \mathbb{R}$.

**Example 13** The system

$$\frac{du}{dt} = -(u^2 + v^2)v \tag{24}$$

$$\frac{dv}{dt} = (5u^2 + uv + 3v^2)u$$

$$u(0) = 0 = u(\pi) \tag{25}$$

has, for each $k \in \mathbb{N}$, a solution $w_k$ with $\mathbf{rot}(w_k) = k$.

# 5 Extensions to larger systems

There are some straight forward extensions of the preceding results to systems of dimension greater than two. We will illustrate this with a more or less special case here. Consider

boundary value problems of the form

$$\begin{aligned} &\frac{dw}{dt} = A(\mu)w + F(\mu, w, t), 0 \leq t \leq \pi \\ &Bw = 0 \end{aligned} \tag{26}$$

where $A(\mu)$ is, for each $\mu \in \mathbb{R}$, a real $N \times N$ matrix, and $F \in C(\mathbb{R} \times \mathbb{R}^N \times [0, \pi], \mathbb{R}^N)$. We assume throughout that solutions to initial value problems for the differential equation in (26) exist and are unique. Let $X = C([0, \pi], \mathbb{R}^N)$ have the sup norm. The boundary conditions are specified by a bounded linear operator $B$ mapping $X$ into $\mathbb{R}^N$. We will assume $A$ has the form

$$A(\mu) = \begin{pmatrix} 0 & -\mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & D(\mu) \end{pmatrix}$$

where $D(\mu)$ is an $(N-2) \times (N-2)$ real matrix. Also assume that $F(\mu, w, t) = o(|w|)$ as $|w| \to 0$, uniformly with respect to $t$ and $\mu$ in compact sets. Let $w = (w_1, w_2, ..., w_N)^T \in X$. We assume the boundary operator has the form

$$Bw = (w_1(0), w_1(\pi), B_0\widehat{w})$$

where $B_0$ is a bounded linear operator mapping $C([0, \pi], \mathbb{R}^{N-2})$ into $\mathbb{R}^{N-2}$ and $\widehat{w} = (w_3, w_4, ..., w_N)^T$. We assume that the problem

$$\frac{d\widehat{w}}{dt} = D(\mu)\widehat{w}, \quad B_0\widehat{w} = 0$$

has, for each $\mu \in \mathbb{R}$, only the trivial zero solution. Thus the problem

$$\frac{dw}{dt} = A(\mu)w, \ Bw = 0$$

has nontrivial solutions only when $\mu = k \in \mathbb{Z}$, and then the solutions have the form $w = \alpha(-\sin kt, \cos kt, 0, 0, 0, ..., 0)$ for $\alpha \in \mathbb{R}$. Thus the eigenspaces are all one dimensional. We make the following structural assumption on $F = (F_1, F_2, ..., F_N)^T$:

$(HF)$

$$\text{For each } 1 \leq i \leq n, \ F_i(\mu, w, t) = 0 \text{ whenever } w_1^2 + w_2^2 = 0.$$

We associate a rotation number with nontrivial solutions to (26) in the following way. Suppose $(\mu, w)$ is a nontrivial solution of (26). If at any point $t_0$ we have $w_1(t_0) = 0 = w_2(t_0)$ then by $(HF)$ we have $F_i(\mu, w(t_0), t_0) = 0$ and at $t = t_0$

$$\begin{pmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \\ \frac{d\widehat{w}}{dt} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ D(\mu)\widehat{w} \end{pmatrix}.$$

Now the problem $\frac{dw}{dt} = A(\mu)w + F(\mu, w, t)$, $w(t_0) = (0, 0, \widehat{w}(t_0))$ has a unique solution. By $(HF)$ a solution is $w(t) = (0, 0, e^{D(\mu)}\widehat{w}(t_0))$. But $w$ must also satisfy the boundary

conditions $Bw = 0$, so $B_0(e^{D(\mu)}\widehat{w}(t_0)) = 0$, and by hypothesis we must have $\widehat{w}(t_0) = 0$. Thus $w = 0$ and $w$ is not a non-trivial solution of (26). It follows that if $w$ is a nontrivial solution then $w_1^2(t) + w_2^2(t) \neq 0$ for all $t \in [0, \pi]$. This provides a topological obstruction so that the rotation of $w$ can be defined. If $(\mu, w)$ is a non-trivial solution of (26) then

$$\frac{w_2'(t)w_1(t) - w_2(t)w_1'(t)}{w_1^2(t) + w_2^2(t)}$$

is defined for $t \in [0, \pi]$. Let $\widetilde{w}_1$, $\widetilde{w}_2$ be extensions of $w_1$, $w_2$, respectively, to $[-\pi, \pi]$ so that $\widetilde{w}_1$ is odd and $\widetilde{w}_2$ is even. Then $\widetilde{w}_1'\widetilde{w}_2$ and $\widetilde{w}_1\widetilde{w}_2'$ are both even functions. It follows that $\mathbf{rot}(w_1, w_2)$ is defined and we let $\mathbf{rot}(w) := \mathbf{rot}(w_1, w_2)$. We have

$$\mathbf{rot}(w) := \mathbf{rot}(w_1, w_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\widetilde{w}_2'\widetilde{w}_1 - \widetilde{w}_2\widetilde{w}_1'}{\widetilde{w}_1^2 + \widetilde{w}_2^2}\, dt = \frac{2}{2\pi}\int_0^{\pi} \frac{\widetilde{w}_2'\widetilde{w}_1 - \widetilde{w}_2\widetilde{w}_1'}{\widetilde{w}_1^2 + \widetilde{w}_2^2}\, dt =$$
$$= \frac{1}{\pi}\int_0^{\pi} \frac{w_2'w_1 - w_2w_1'}{w_1^2 + w_2^2}\, dt = \frac{1}{\pi}\int_0^{\pi} \frac{[\mu + F_2]w_1^2 + [\mu + F_1]w_2^2}{w_1^2 + w_2^2}\, dt$$

for the rotation we associate with a non-trivial solution $w$.

We use the earlier notation, so $\mathcal{S}_0$ denotes the set of non-trivial solutions $(\mu, w) \in \mathbb{R} \times X$ and $\mathcal{S}$ is the closure of $\mathcal{S}_0$.

**Theorem 14** *Let $A(\mu)$ and $F$ be as described above. Then*

*(C1) For each $k \in \mathbb{Z}$, $(k, 0)$ is a bifurcation point of (26). That is, in every neighborhood of $(k, 0) \in \mathbb{R} \times X$ there is a solution $(\mu, w) \in \mathbb{R} \times D(L)$, $w \neq 0$, of (3).*

*(C2) For each $k \in \mathbb{Z}$ let $\mathcal{C}_k \subset \mathbb{R} \times X$ denote the component of $\mathcal{S}$ which meets $(k, 0)$. Each $\mathcal{C}_k$ is unbounded in $\mathbb{R} \times X$ and if $(\mu, w) \in \mathcal{C}_k$, $w \neq 0$, then the rotation $\mathbf{rot}(w)$ as defined above is equal to $k$. It follows that if $j \neq k$ then $\mathcal{C}_j \cap \mathcal{C}_k = \varnothing$.*

*Proof.* The proof of this theorem is now reduced to mimicking the proof of Theorem 3, and so we omit the proof. ■

**Remark 15** Notice that the exact form of the boundary operator $B_0$ has no significance.

**Remark 16** There is also an extension of Theorems 6 to $\mathbb{R}^N$. For the extension of Theorem 6 we make the assumptions of the preceding theorem, and also assume that $F_1, F_2 \leq 0$ (or $F_1, F_2 \geq 0$).

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