Research Article

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Fractional Hardy-Sobolev equations with nonhomogeneous terms

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Abstract: This paper deals with existence and multiplicity of positive solutions to the following class of nonlocal equations with critical nonlinearity:

$$\begin{cases} (-\Delta)^s u - \gamma \dfrac{u}{|x|^{2s}} = K(x)\dfrac{|u|^{2_s^*(t)-2}u}{|x|^t} + f(x) \quad \text{in} \quad \mathbb{R}^N, \\ u \in \dot{H}^s(\mathbb{R}^N), \end{cases}$$

where $N > 2s$, $s \in (0,1)$, $0 \le t < 2s < N$ and $2_s^*(t) := \frac{2(N-t)}{N-2s}$. Here $0 < \gamma < \gamma_{N,s}$ and $\gamma_{N,s}$ is the best Hardy constant in the fractional Hardy inequality. The coefficient K is a positive continuous function on $\mathbb{R}^N$, with $K(0) = 1 = \lim_{|x|\to\infty} K(x)$. The perturbation f is a nonnegative nontrivial functional in the dual space $\dot{H}^s(\mathbb{R}^N)'$ of $\dot{H}^s(\mathbb{R}^N)$. We establish the profile decomposition of the Palais-Smale sequence associated with the functional. Further, if $K \ge 1$ and $\|f\|_{(\dot{H}^s)'}$ is small enough (but $f \not\equiv 0$), we establish existence of at least two positive solutions to the above equation.

Keywords: Nonlocal equations, fractional Laplacian, Hardy-Sobolev equations, profile decomposition, Palais-Smale decomposition, energy estimate, positive solutions, min-max method.

MSC: 35R11, 35A15, 35B33, 35J60

1 Introduction

The paper deals with the following fractional Hardy-Sobolev equation with nonhomogeneous term

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = K(x)\frac{|u|^{2_s^*(t)-2}u}{|x|^t} + f(x) \qquad \text{in} \quad \mathbb{R}^N, \quad u \in \dot{H}^s(\mathbb{R}^N), \tag{$E^{\gamma}_{K,t,f}$}$$

where $N > 2s$, $s \in (0,1)$, $0 \le t < 2s < N$ and $2_s^*(t) := \frac{2(N-t)}{N-2s}$. Clearly, $2 < 2_s^*(t) \le \frac{2N}{N-2s} = 2_s^*$. Here $0 < \gamma < \gamma_{N,s}$, where $\gamma_{N,s}$ is the best Hardy constant in the fractional Hardy inequality

$$\gamma_{N,s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\,dx \le \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2\,d\xi, \qquad \gamma_{N,s} = 2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

Throughout the paper $\mathcal{F}(u)$ denotes the Fourier transform of u. Moreover,

$$\lim_{s\to 1} \gamma_{N,s} = \left(\frac{N-2}{2}\right)^2,$$

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which is exactly the best Hardy constant in the classical case $s = 1$. The symbol $(-\Delta)^s$ denotes the fractional Laplace operator which can be defined for any function u of the Schwartz class functions $\mathcal{S}(\mathbb{R}^N)$ as follows:

$$(-\Delta)^s u(x) := C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad C_{N,s} = \frac{4^s\Gamma(N/2+s)}{\pi^{N/2}|\Gamma(-s)|}.$$

For the sharp Hardy inequalities in general fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$, $1 < p < \infty$, as well as for historical comments in the case $p = 2$, we refer the interested reader to [16] and the references therein. While for fractional Hardy-Sobolev-Maz'ya inequality, we mention the recent contribution [20] and for fractional Hardy inequality on the Heisenberg group we refer to [4]. Throughout the paper the homogeneous fractional Sobolev space is denoted by

$$\dot{H}^s(R^N) := \left\{ u \in L^{2_s^*}(\mathbb{R}^N) \ : \ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y < \infty \right\},$$

and it is endowed with the inner product $\langle \cdot, \cdot \rangle_{\dot{H}^s}$ and corresponding Gagliardo norm

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)} := \left(\frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2\, \mathrm{d}\xi \right)^{\frac{1}{2}}.$$

In literature there are several definitions of the fractional Laplacian in which different normalizing constants $C_{N,s}$ appear. The constant $C_{N,s}$ is chosen so that the above definition is equivalent with the one via the Fourier transform, which is called *classical*. The definition via Fourier transform recovers the standard Laplacian as $s \to 1$. Actually, the standard Laplacian can also be represented by other nonlocal formulas, such as the one in (2.46) and Appendix M of [1].

In $(E^{\gamma}_{K,t,f})$, the functions K and f satisfy the properties:

(K) $0 < K \in C(\mathbb{R}^N)$, $K(0) = 1 = \lim_{|x|\to\infty} K(x)$.

(F) $f \not\equiv 0$ *is a nonnegative functional in the dual space* $\dot{H}^s(\mathbb{R}^N)'$ *of* $\dot{H}^s(\mathbb{R}^N)$, i.e. whenever u is a nonnegative function in $\dot{H}^s(\mathbb{R}^N)$ then ${}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \geq 0$.

Using the Hardy inequality, it is easy to see that the operator $L_{\gamma,s} := (-\Delta)^s - \frac{\gamma}{|x|^{2s}}$ with $0 \leq \gamma < \gamma_{N,s}$ is a positive operator. The request $\gamma < \gamma_{N,s}$ is fairly natural since we are looking for positive solutions. In this case the Hardy-Sobolev inequality holds for $L_{\gamma,s}$, which states that if $0 \leq t < 2s < N$, then

$$S_{\gamma,t,s} = S_{\gamma,t,s}(\mathbb{R}^N) := \inf_{u\in\dot{H}^s(\mathbb{R}^N)\setminus\{0\}} \frac{\frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y - \gamma \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}}\, \mathrm{d}x}{\left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\, \mathrm{d}x \right)^{\frac{2}{2_s^*(t)}}} \tag{1.1}$$

is finite, strictly positive and achieved (see [17, 18]). Observe that thanks to [17], any minimizer for (1.1) leads (up to a constant) to a nonnegative variational solution of the equation

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(t)-2}u}{|x|^t}, \quad u \in \dot{H}^s(\mathbb{R}^N). \tag{$E^{\gamma}_{1,t,0}$}$$

If $\gamma = 0 = t$, then $S_{\gamma,t,s}$ reduces to the best Sobolev constant $S_{0,0,s} = S$ which is known to be achieved by $C_{N,s}(1+|x|^2)^{-\frac{N-2s}{2}}$ and any minimizer of S leads (up to a constant) to a nonnegative solution of equation $(E^0_{1,0,0})$ i.e., $(E^{\gamma}_{1,t,0})$ with $\gamma = 0 = t$.

Definition 1.1. We say $u \in \dot{H}^s(\mathbb{R}^N)$ is a *positive weak solution of* $(E^{\gamma}_{K,t,f})$ if $u > 0$ in $\mathbb{R}^N$ and for every $\phi \in \dot{H}^s(\mathbb{R}^N)$, we have

$$\frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\mathrm{d}x\, \mathrm{d}y - \gamma \int_{\mathbb{R}^N} \frac{u\phi}{|x|^{2s}}\mathrm{d}x$$

$$= \int_{\mathbb{R}^N} K(x) \frac{|u|^{2_s^*(t)-2} u\phi}{|x|^t} \mathrm{d}x + {}_{(\dot{H}^s)'}\langle f, \phi \rangle_{\dot{H}^s},$$

where ${}_{(\dot{H}^s)'}\langle \cdot, \cdot \rangle_{\dot{H}^s}$ denotes the duality bracket between $\dot{H}^s(\mathbb{R}^N)$ and its dual $\dot{H}^s(\mathbb{R}^N)'$.

Remark 1.1. For $0 < \gamma < \gamma_{N,s}$,

$$\|u\|_\gamma := \left(\frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y - \gamma \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}} \mathrm{d}x \right)^{\frac{1}{2}}$$

defines a norm in $\dot{H}^s(\mathbb{R}^N)$ which is equivalent to the standard norm in $\dot{H}^s(\mathbb{R}^N)$. In particular,

$$\sqrt{1 - \frac{\gamma}{\gamma_{N,s}}} \|u\|_{\dot{H}^s} \le \|u\|_\gamma \le \|u\|_{\dot{H}^s}.$$

The corresponding equivalent inner product $\langle \cdot, \cdot \rangle_\gamma$ in the fractional homogeneous Hilbert space $\dot{H}^s(\mathbb{R}^N)$ is given by

$$\langle u, v \rangle_\gamma := \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y - \gamma \int_{\mathbb{R}^N} \frac{uv}{|x|^{2s}} \mathrm{d}x.$$

Finally, for simplicity we endow in what follows the weighted Lebesgue space $L^{2_s^*(t)}(\mathbb{R}^N, |x|^{-t})$ with the norm $\|u\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})} = \left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t} \mathrm{d}x \right)^{1/2_s^*(t)}$.

We are going to prove existence and multiplicity of positive solutions of $(E^\gamma_{K,t,f})$ in the spirit of [7, 8]. Under the conditions on K and f stated above, equation $(E^\gamma_{K,t,f})$ can be regarded as a perturbation problem of the homogeneous equation $(E^\gamma_{1,t,0})$. It is known from [18] that when $0 < \gamma < \gamma_{N,s}$ or $\{\gamma = 0$ and $0 < t < 2s\}$, then any nonnegative minimizer for $S_{\gamma,t,s}$ is positive, radially symmetric, radially decreasing, and approaches zero as $|x| \to \infty$. The main question to be addressed is whether positive solution can survive after a perturbation of type $(E^\gamma_{K,t,f})$ or not.

For $\gamma = 0 = t$, this kind of question was recently studied by the first and third author of the current paper in [8]. For the Schrödinger operator (without Hardy term), the same type of questions were addressed in [7] with sub-critical nonlinearity. However for $\gamma \neq 0$ the presence of the Hardy potential requires a new argument to dealt with. One of the key steps to prove the multiplicity result is a careful analysis of the Palais-Smale level. Theorem 2.1 studies the profile decomposition of any Palais-Smale sequence possessed by the underlying functional associated to $(E^\gamma_{K,t,f})$. We show that concentration takes place along a single profile when $t > 0$, while concentration takes place along two different profiles when $t = 0$. In the local case $s = 1$, $t = 0$ and $f = 0$ Smets deals with the profile decomposition in [25]. In bounded domains and again in the local case $s = 1$, paper [9] treats the case of all $t \ge 0$. However, extension of the latter results in the nonlocal case $s \in (0, 1)$ and in the entire space $\mathbb{R}^N$ is highly nontrivial and requires several delicate estimates and techniques to deal with.

In the local case $s = 1$, we refer to [14, 25], where the authors have studied the local version of $(E^\gamma_{K,0,0})$ in $\mathbb{R}^N$. In the nonlocal case, when the domain is a bounded subset of $\mathbb{R}^N$, existence of positive solutions of $(E^\gamma_{K,t,f})$ in Ω with $\gamma = 0 = t$ (i.e., without Hardy and Hardy-Sobolev terms) and Dirichlet boundary conditions has been proved in [24]. Existence of sign changing solutions of

$$(-\Delta)^s u = |u|^{\frac{4s}{N-2s}} u + \varepsilon f \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $f \ge 0, f \in L^\infty(\Omega)$ has been studied in [5] and existence of two positive solutions have been established in [28] when f is a continuous function with compact support in Ω. In the nonlocal case, when the domain is the entire space $\mathbb{R}^N$, but $\gamma = 0$, we refer to [7, 8], where multiplicity of positive solutions have been studied in presence of a nonhomogeneous term.

There is a wide literature regarding problems involving the fractional Hardy potential. Avoiding to disclose the discussion we refer to the following (far from being complete) list of works and references therein [2, 3, 6, 10, 13, 15, 18]. In [13] Dipierro, et al. study the equation $(E^{\gamma}_{1,0,0})$ (i.e., $(E^{\gamma}_{1,t,0})$ with $t = 0$) and prove existence of a ground state solution, qualitative properties of positive solutions and asymptotic behavior of solutions at both 0 and infinity. In [6], the authors deal with the Green function for $L_{\gamma,s}$ $(0 < \gamma < \gamma_{N,s})$ and show when the integral representation of the weak solution is valid.

It is worth noting that solutions of $(E^{\gamma}_{1,t,0})$ do not belong to $L^{\infty}(\mathbb{R}^N)$ as soon as $\gamma > 0$, because of the singularity at zero. In fact solutions blow up at origin (see [13, 18]). For this reason, it seems more difficult to handle $(E^{\gamma}_{K,t,f})$ in the general case using the fine analysis of blow up technique quoted above.

To the best of our knowledge, so far there has been no papers in the literature, where existence and multiplicity of positive solutions of Hardy-Sobolev type equations (with $\gamma \neq 0$ and $t \geq 0$) in $\mathbb{R}^N$, have been established in the nonhomogeneous case $f \neq 0$. Also the profile decomposition in the nonlocal case with the Hardy term is completely new and the proof is very involved, delicate and complicated compared with the local case $s = 1$. The proofs are not at all an easy adaptation of the local case or the case $\gamma = 0$. The multiplicity results in this paper is new even in the local case $s = 1$, but we leave the obvious changes, when $s = 1$, to the interested reader.

Below we state the main result.

Theorem 1.1. *Assume that* (**F**) *and* (**K**) *are satisfied, with $K \geq 1$ in $\mathbb{R}^N$. If*

$$\|f\|_{(\dot{H}^s)'} < C_t \sqrt{1 - \frac{\gamma}{\gamma_{N,s}}}\, S_{\gamma,t,s}^{\frac{N-t}{4s-2t}}, \quad \textit{where} \quad C_t = \left(\frac{4s-2t}{N-2t+2s}\right)\left((2^{\star}_s(t)-1)\|K\|_{L^{\infty}(\mathbb{R}^N)}\right)^{-\left(\frac{N-2s}{4s-2t}\right)},$$

then

(i) For $t > 0$, equation $(E^{\gamma}_{K,t,f})$ admits two positive solutions;

(ii) For $t = 0$, equation $(E^{\gamma}_{K,t,f})$ admits a positive solution. In addition, if $\|K\|_{L^{\infty}(\mathbb{R}^N)} < \left(\frac{S}{S_{\gamma,0,s}}\right)^{\frac{N}{N-2s}}$ then $(E^{\gamma}_{K,t,f})$ admits two positive solutions.

Remark 1.2. It is worth mentioning that $S > S_{\gamma,0,s}$ for any $\gamma > 0$. To see this, we denote by W the unique positive solution of $(E^{0}_{1,0,0})$ and let $W_{\gamma,0}$ be a minimum energy positive solution (ground state solution) of $(E^{\gamma}_{1,t,0})$ with $t = 0$. Then,

$$I^{\gamma}_{1,0,0}(W_{\gamma,0}) \leq I^{\gamma}_{1,0,0}(W) < I^{0}_{1,0,0}(W).$$

A straightforward computation yields that $I^{0}_{1,0,0}(W) = \frac{s}{N} S^{\frac{N}{2s}}$ and $I^{\gamma}_{1,0,0}(W_{\gamma,0}) = \frac{s}{N} S^{\frac{N}{2s}}_{\gamma,0,s}$. Consequently, $S > S_{\gamma,0,s}$ for any $\gamma > 0$. From this observation, it immediately follows that if $K \equiv 1$, then $(E^{\gamma}_{1,t,f})$ admits two positive solutions for all $t \geq 0$ under the given assumption (**F**) on f.

Note that the Hardy-Sobolev embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2^{\star}_s(t)}(\mathbb{R}^N, |x|^{-t})$ for any $0 \leq t < 2s$ is continuous, but not compact. This noncompactness of the embedding even locally in any neighbourhood of zero leads to other additional difficulties, and more importantly, to new phenomenon concerning the possibility of blow up. Thus the variational functional associated to $(E^{\gamma}_{K,t,f})$ does not satisfy the Palais-Smale condition, briefly called (PS) condition. The lack of compactness of the functional associated to $(E^{\gamma}_{K,t,f})$ is due to a concentration phenomenon. We analyze this noncompactness in Theorem 2.1, which is one of the most important theorems of the paper. Using this theorem we prove existence and multiplicity of positive solutions to $(E^{\gamma}_{K,t,f})$ in Theorem 1.1. For that we first decompose $\dot{H}^s(\mathbb{R}^N)$ into three components which are homeomorphic to the interior, the boundary and the exterior of the unit ball in $\dot{H}^s(\mathbb{R}^N)$ respectively. Then we prove that the energy functional associated to $(E^{\gamma}_{K,t,f})$ attains its infimum on one of the components which serves as our first positive solution. The second positive solution is obtained via a careful analysis on the (PS) sequences associated to the energy functional and we construct a min–max critical level κ_t, where the (PS) condition holds.

This paper has been organised in the following way. In Section 2, we prove the Palais-Smale decomposition theorem associated with the functional corresponding to $(E^{\gamma}_{K,t,f})$ (see Theorem 2.1). In Section 3, we show existence of two positive solutions of $(E^{\gamma}_{K,t,f})$, namely Theorem 1.1. Appendix A contains some basic estimates which are used in proving the Palais-Smale characterization theorem in Section 2.

Notation: In this paper $C, C', C'', C''', \cdots$ denote a generic constant which may vary from line to line. The symbol $B_r(y)$ stands for the ball centered at $y \in \mathbb{R}^N$ and of radius r. For simplicity B_r means $B_r(0)$. Moreover, $u_+ := \max\{u, 0\}$ and $u_- := -\min\{u, 0\}$. Therefore, according to our notation $u = u_+ - u_-$.

2 Palais-Smale decomposition

In this section we study the Palais-Smale sequences (in short, (PS) sequences) of the functional $\bar{I}^{\gamma}_{K,t,f}$ associated to $(E^{\gamma}_{K,t,f})$

$$\begin{aligned}\bar{I}^{\gamma}_{K,t,f}(u) &:= \frac{C_{N,s}}{4}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{\gamma}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}\,dx \\ &\quad - \frac{1}{2^\star_s(t)}\int_{\mathbb{R}^N} K(x)\frac{|u|^{2^\star_s(t)}}{|x|^t}\,dx - {}_{(\dot H^s)'}\langle f, u\rangle_{\dot H^s} \\ &= \frac{1}{2}\|u\|_\gamma^2 - \frac{1}{2^\star_s(t)}\int_{\mathbb{R}^N} K(x)\frac{|u|^{2^\star_s(t)}}{|x|^t}\,dx - {}_{(\dot H^s)'}\langle f, u\rangle_{\dot H^s},\end{aligned} \tag{2.1}$$

where K and f satisfy (**K**) and (**F**) respectively.

We say that the sequence $(u_n)_n \subset \dot H^s(\mathbb{R}^N)$ is a (PS) sequence for $\bar{I}_{K,t,f}$ at level β if $\bar{I}_{K,t,f}(u_n) \to \beta$ and $(\bar{I}_{K,t,f})'(u_n) \to 0$ in $(\dot H^s)'$. It is easy to see that the weak limit of a (PS) sequence solves $(E^{\gamma}_{K,t,f})$ except the positivity.

However the main difficulty is that the (PS) sequence may not converge strongly and hence the weak limit can be zero even if $\beta > 0$. The main purpose of this section is to classify (PS) sequences of the functional $\bar{I}^{\gamma}_{K,t,f}$. Classification of (PS) sequences has been done for various problems having lack of compactness, to quote a few, we cite [8, 21, 22] in the nonlocal case with $\gamma = 0 = t$, while in the local case [9, 25] with Hardy potentials and in [26] without Hardy potentials. We also refer to [27] for a more abstract approach of the profile decomposition in general Hilbert spaces. We establish a classification theorem for the (PS) sequences of (2.1) in the spirit of the above results. In [8, 21], the noncompactness is completely described by the single blow up profile W, which is a solution of

$$(-\Delta)^s W = |W|^{2^\star_s - 2} W \quad \text{in} \quad \mathbb{R}^N, \quad W \in \dot H^s(\mathbb{R}^N). \tag{$E^0_{1,0,0}$}$$

In [9, 25] (the local case $s = 1$), the noncompactness are due to concentration occurring through two different profiles. Possibility of two different type of profiles are still present for $(E^{\gamma}_{K,t,f})$ in the case $t = 0$.

Let $t = 0$ and let W be any solution of $(E^0_{1,0,0})$. Then, it can be easily verified that any sequence of the form

$$W^{r_n, y_n}(x) := K(y)^{-\frac{N-2s}{4s}} r_n^{-\frac{N-2s}{4s}} W\Big(\frac{x-y_n}{r_n}\Big), \tag{2.2}$$

is a (PS) sequence for $\bar{I}^{\gamma}_{K,0,0}$ if $y_n \to y \neq 0$ and $r_n \to 0$. If $y = 0$, then W^{r_n, y_n} remains a (PS) sequence for $\bar{I}^{\gamma}_{K,0,0}$ provided that $\frac{|y_n|}{r_n} \to \infty$. Also $W^{r_n, y_n} \rightharpoonup 0$ in $\dot H^s(\mathbb{R}^N)$ by [21, Lemma 3].

Further, let $W_{\gamma,t}$ be any solution of $(E^{\gamma}_{1,t,0})$ (where $t \geq 0$). Define a sequence $(W^{R_n,0}_{\gamma,t})_n$ of the form

$$W^{R_n,0}_{\gamma,t}(x) := R_n^{-\frac{N-2s}{4s}} W_{\gamma,t}\Big(\frac{x}{R_n}\Big), \tag{2.3}$$

where $R_n \to 0$. Then $W^{R_n,0}_{\gamma,t} \rightharpoonup 0$ in $\dot H^s(\mathbb{R}^N)$ and $(W^{R_n,0}_{\gamma,t})_n$ is a (PS) sequence for $\bar{I}^{\gamma}_{K,t,0}$ for $t \geq 0$.

Theorem 2.1. *Let $(u_n)_n$ be a (PS) sequence for $\bar{I}^{\gamma}_{K,t,f}$ at the level β. Then up to a subsequence, still denoted by $(u_n)_n$, the next properties hold.*

If $t = 0$, then there exist $n_1, n_2 \in \mathbb{N}$, n_2 sequences $(R^k_n)_n \subset \mathbb{R}^+$ $(1 \le k \le n_2)$, n_1 sequences $(r^j_n)_n \subset \mathbb{R}^+$ and $(y^j_n)_n \subset \mathbb{R}^N \setminus \{0\}$ $(1 \le j \le n_1)$ and $0 \le \bar{u} \in \dot{H}^s(\mathbb{R}^N)$ such that

$$
\begin{aligned}
&(i) && u_n = \bar{u} + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} (W^j)^{r^j_n,\, y^j_n} + \sum_{k=1}^{n_2} (W^k_{\gamma,t})^{R^k_n,0} + o(1)\\
&(ii) && (\bar{I}^{\gamma}_{K,t,f})'(\bar{u}) = 0\\
&(iii) && R^k_n \to 0\ (1 \le k \le n_2) \text{ and } r^j_n \to 0\,(1 \le j \le n_1)\\
&(iv) && \text{either } y^j_n \to y^j \in \mathbb{R}^N \text{or } |y^j| \to \infty \text{ and } \frac{r^j_n}{|y^j_n|} \to 0\ (1 \le j \le n_1)\\
&(v) && \beta = \bar{I}^{\gamma}_{K,t,f}(\bar{u}) + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{2s}} \bar{I}^0_{1,0,0}(W^j) + \sum_{k=1}^{n_2} \bar{I}^{\gamma}_{1,t,0}(W^k_{\gamma,t}) + o(1)\\
&(vi) && \left|\log\Big(\frac{r^i_n}{r^j_n}\Big)\right| + \left|\frac{y^i_n - y^j_n}{r^j_n}\right| \underset{n\to\infty}{\longrightarrow} \infty \quad \text{for } i \ne j\\
&(vii) && \left|\log\Big(\frac{R^k_n}{R^l_n}\Big)\right| \underset{n\to\infty}{\longrightarrow} \infty \quad \text{for } k \ne l,
\end{aligned}
$$

where $o(1) \to 0$ in $\dot{H}^s(\mathbb{R}^N)$ as $n \to \infty$, $(W^j)^{r^j_n,\, y^j_n}$ and $(W^k_{\gamma,t})^{R^k_n,0}$ are (PS) sequences of the form (2.2) and (2.3) respectively, with $W = W^j$ and $W_{\gamma,t} = W^k_{\gamma,t}$.

When $t > 0$, the same conclusions hold, with $W^j = 0$ for all j.

In the case $n_1 = 0$, $n_2 = 0$ the above properties (i)–(vii) are valid without W, W_γ, R^k_n, r^j_n. In addition, if $u_n \ge 0$, then $\bar{u} \ge 0$ and $W^j \ge 0$ for all $1 \le j \le n_1$, $W^k_{\gamma,t} \ge 0$ for all $1 \le k \le n_2$. Therefore, $W^j = W_0$ for all $1 \le j \le n_1$ due to the uniqueness up to the translation and dilation for the positive solutions of $(E^0_{1,0,0})$.

Proof. We prove the theorem in several steps.

Step 1: Using standard arguments it follows that there exists $M > 0$ such that

$$\|u_n\|_\gamma < M \quad \text{for all } n \in \mathbb{N}.$$

More precisely, as $n \to \infty$

$$
\begin{aligned}
\beta + o(1) + o(1)\|u_n\|_\gamma &\ge \bar{I}^{\gamma}_{K,t,f}(u_n) - \frac{1}{2^\star_s(t)}\, {}_{(\dot{H}^s)'}\big\langle (\bar{I}^{\gamma}_{K,t,f})'(u_n), u_n \big\rangle_{\dot{H}^s}\\
&= \left(\frac{1}{2} - \frac{1}{2^\star_s(t)}\right)\|u_n\|^2_\gamma - \left(1 - \frac{1}{2^\star_s(t)}\right) {}_{(\dot{H}^s)'}\langle f, u_n\rangle_{\dot{H}^s}\\
&\ge \left(\frac{1}{2} - \frac{1}{2^\star_s(t)}\right)\|u_n\|^2_\gamma - \left(1 - \frac{1}{2^\star_s(t)}\right) \|f\|_{(\dot{H}^s)'}\|u_n\|_{\dot{H}^s}.
\end{aligned}
$$

As $2^\star_s(t) > 2$, from the above estimate it follows that $(u_n)_n$ is bounded in $\dot{H}^s(\mathbb{R}^N)$. Consequently, there exists $\bar{u}$ in $\dot{H}^s(\mathbb{R}^N)$ such that, up to a subsequence, still denoted by $(u_n)_n$, $u_n \rightharpoonup \bar{u}$ in $\dot{H}^s(\mathbb{R}^N)$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^N$. Moreover, as ${}_{(\dot{H}^s)'}\big\langle (\bar{I}^{\gamma}_{K,t,f})'(u_n), v\big\rangle_{\dot{H}^s} \to 0$ as $k \to \infty$ for all $v \in \dot{H}^s(\mathbb{R}^N)$, then

$$(-\Delta)^s u_n - \gamma\frac{u_n}{|x|^{2s}} - K(x)|u_n|^{2^\star_s(t)-2}u_n - f \longrightarrow 0 \quad \text{in} \quad \dot{H}^s(\mathbb{R}^N)'. \tag{2.4}$$

Step 2: From (2.4), letting $n \to \infty$, we get

$$\langle u_n, v\rangle_\gamma - \int_{\mathbb{R}^N} K(x)\frac{|u_n|^{2^\star_s(t)-2}u_n v}{|x|^t}\,dx - {}_{(\dot{H}^s)'}\langle f, v\rangle_{\dot{H}^s} \to 0. \tag{2.5}$$

As $u_n \rightharpoonup \bar{u}$ in $\dot{H}^s(\mathbb{R}^N)$, it is easy to see that $\langle u_n, v\rangle_\gamma \to \langle \bar{u}, v\rangle_\gamma$ for all $v \in \dot{H}^s(\mathbb{R}^N)$.

Claim 1: $\int_{\mathbb{R}^N} K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,dx \longrightarrow \int_{\mathbb{R}^N} K(x)\frac{|\bar{u}|^{2^*_s(t)-2}\bar{u}v}{|x|^t}dx$ for all $v \in \dot{H}^s(\mathbb{R}^N)$.

Indeed, $u_n \to \bar{u}$ a.e. in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,dx = \int_{B_R} K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,dx + \int_{\mathbb{R}^N\setminus B_R} K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,dx. \tag{2.6}$$

On B_R we will show the convergence using Vitali's convergence theorem. For that, given any $\varepsilon > 0$, we choose $\Omega \subset B_R$ such that $\left(\int_\Omega \frac{|v|^{2^*_s(t)}}{|x|^t}dx\right)^{\frac{1}{2^*_s(t)}} < \frac{\varepsilon}{\|K\|_{L^\infty}(MS_{\gamma,t,s}^{-\frac{1}{2}})^{2^*_s(t)-1}}$. Since $\frac{|v|^{2^*_s(t)}}{|x|^t}$ is in $L^1(\mathbb{R}^N)$, the above choice makes sense. Therefore,

$$\begin{aligned}
\left|\int_\Omega K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,dx\right| &\le \|K\|_{L^\infty(\mathbb{R}^N)}\int_\Omega \frac{|u_n|^{2^*_s(t)-1}|v|}{|x|^t}\,dx \\
&\le \|K\|_{L^\infty(\mathbb{R}^N)}\left(\int_\Omega \frac{|u_n|^{2^*_s(t)}}{|x|^t}dx\right)^{\frac{2^*_s(t)-1}{2^*_s(t)}}\left(\int_\Omega \frac{|v|^{2^*_s(t)}}{|x|^t}dx\right)^{\frac{1}{2^*_s(t)}} \\
&\le \|K\|_{L^\infty(\mathbb{R}^N)}S_{\gamma,t,s}^{-\frac{2^*(t)-1}{2}}\|u_n\|_\gamma^{2^*_s(t)-1}\left(\int_\Omega \frac{|v|^{2^*_s(t)}}{|x|^t}dx\right)^{\frac{1}{2^*_s(t)}} < \varepsilon
\end{aligned}$$

Thus $K\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}$ is uniformly integrable in B_R. Therefore, using Vitali's convergence theorem, we can pass the limit in the 1st integral on RHS of (2.6).

To estimate the integral now on B_R^c, we first set $v_n = u_n - \bar{u}$. Then $v_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$. It is not difficult to see that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left||v_n + \bar{u}|^{2^*_s(t)-2}(v_n + \bar{u}) - |\bar{u}|^{2^*_s(t)-2}\bar{u}\right| < \varepsilon|v_n|^{2^*_s(t)-1} + C_\varepsilon|\bar{u}|^{2^*_s(t)-1}.$$

Therefore,

$$\begin{aligned}
&\left|\int_{B_R^c} K(x)\left\{\frac{|u_n|^{2^*_s(t)-2}u_n}{|x|^t} - \frac{|\bar{u}|^{2^*_s(t)-2}\bar{u}}{|x|^t}\right\}v\,dx\right| \\
&\le \|K\|_{L^\infty(\mathbb{R}^N)}\left[\varepsilon\int_{B_R^c}\frac{|v_n|^{2^*_s(t)-1}|v|}{|x|^t}\,dx + C_\varepsilon\int_{B_R^c}\frac{|\bar{u}|^{2^*_s(t)-1}|v|}{|x|^t}\right] \\
&\le \|K\|_{L^\infty(\mathbb{R}^N)}\left[\varepsilon\left(\int_{B_R^c}\frac{|v_n|^{2^*_s(t)}}{|x|^t}\,dx\right)^{\frac{2^*_s(t)-1}{2^*_s(t)}}\left(\int_{B_R^c}\frac{|v|^{2^*_s(t)}}{|x|^t}\right)^{\frac{1}{2^*_s(t)}}\right. \\
&\qquad \left. + C_\varepsilon\left(\int_{B_R^c}\frac{|\bar{u}|^{2^*_s(t)}}{|x|^t}\right)^{\frac{2^*_s(t)-1}{2^*_s(t)}}\left(\int_{B_R^c}\frac{|v|^{2^*_s(t)}}{|x|^t}\right)^{\frac{1}{2^*_s(t)}}\right] \\
&\le C\|K\|_{L^\infty(\mathbb{R}^N)}\left[\varepsilon\|v_n\|_\gamma^{2^*_s(t)-1}\left(\int_{B_R^c}\frac{|v|^{2^*_s(t)}}{|x|^t}\right)^{\frac{1}{2^*_s(t)}} + C_\varepsilon\|\bar{u}\|_\gamma^{2^*_s(t)-1}\left(\int_{B_R^c}\frac{|v|^{2^*_s(t)}}{|x|^t}\right)^{\frac{1}{2^*_s(t)}}\right].
\end{aligned}$$

Since $(\|v_n\|_\gamma)_n$ is uniformly bounded and $\frac{|v|^{2^*_s(t)}}{|x|^t} \in L^1(\mathbb{R}^N)$, given $\varepsilon > 0$, we can choose $R > 0$ so large that

$$\left|\int_{B_R^c} K(x)\left\{\frac{|u_n|^{2^*_s(t)-2}u_n}{|x|^t} - \frac{|\bar{u}|^{2^*_s(t)-2}\bar{u}}{|x|^t}\right\}v\,dx\right| < \varepsilon.$$

This completes the proof of claim 1.

Hence (2.5) yields that $\bar{u}$ is a solution of $(E^\gamma_{K,t,f})$.

Step 3: Here we show that $(u_n - \bar{u})_n$ is a (PS) sequence for $\bar{I}^\gamma_{K,t,0}$ at the level $\beta - \bar{I}^\gamma_{K,t,f}(\bar{u})$. To see this, first we observe that as $n \to \infty$

$$\|u_n - \bar{u}\|_\gamma^2 = \|u_n\|_\gamma^2 - \|\bar{u}\|_\gamma^2 + o(1),$$

and by the Brézis-Lieb lemma as $n \to \infty$

$$\int_{\mathbb{R}^N} K(x)\frac{|u_n - \bar{u}|^{2_s^\star(t)}}{|x|^t}\mathrm{d}x = \int_{\mathbb{R}^N} K(x)\frac{|u_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x - \int_{\mathbb{R}^N} K(x)\frac{|\bar{u}|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x + o(1).$$

Further as $u_n \rightharpoonup u$ and $f \in \dot{H}^s(\mathbb{R}^N)'$, we also have

$$_{(\dot{H}^s)'}\langle f, u_n\rangle_{\dot{H}^s} \longrightarrow {}_{(\dot{H}^s)'}\langle f, \bar{u}\rangle_{\dot{H}^s}.$$

Therefore, as $n \to \infty$

$$\begin{aligned}
\bar{I}^\gamma_{K,t,0}(u_n - \bar{u}) &= \frac{1}{2}\|u_n - \bar{u}\|^2_\gamma - \frac{1}{2_s^\star(t)}\int_{\mathbb{R}^N} K(x)\frac{|u_n - \bar{u}|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x \\
&= \frac{1}{2}\|u_n\|^2_\gamma - \frac{1}{2_s^\star(t)}\int_{\mathbb{R}^N} K(x)\frac{|u_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u_n\rangle_{\dot{H}^s} \\
&\quad - \left\{\frac{1}{2}\|\bar{u}\|^2_\gamma - \frac{1}{2_s^\star(t)}\int_{\mathbb{R}^N} K(x)\frac{|\bar{u}|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, \bar{u}\rangle_{\dot{H}^s}\right\} + o(1) \\
&= \bar{I}^\gamma_{K,t,f}(u_n) - \bar{I}^\gamma_{K,t,f}(\bar{u}) + o(1) \\
&\longrightarrow \beta - \bar{I}^\gamma_{K,t,f}(\bar{u}).
\end{aligned}$$

Further, as $_{(\dot{H}^s)'}\langle(\bar{I}^\gamma_{K,t,f})'(\bar{u}), v\rangle_{\dot{H}^s} = 0$ for all $v \in \dot{H}^s(\mathbb{R}^N)$, we obtain

$$\begin{aligned}
{}_{(\dot{H}^s)'}\langle(\bar{I}^\gamma_{K,t,0})'(u_n - \bar{u}), v\rangle_{\dot{H}^s} &= \langle u_n - \bar{u}, v\rangle_\gamma - \int_{\mathbb{R}^N} K(x)\frac{|u_n - \bar{u}|^{2_s^\star(t)-2}(u_n - \bar{u})v}{|x|^t}\,\mathrm{d}x \\
&= \langle u_n, v\rangle_\gamma - \int_{\mathbb{R}^N} K(x)\frac{|u_n|^{2_s^\star(t)-2}u_n v}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, v\rangle_{\dot{H}^s} \\
&\quad - \left(\langle \bar{u}, v\rangle_\gamma - \int_{\mathbb{R}^N} K(x)\frac{|\bar{u}|^{2_s^\star(t)-2}\bar{u}v}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, v\rangle_{\dot{H}^s}\right) \\
&\quad + \int_{\mathbb{R}^N} K(x)\left\{\frac{|u_n|^{2_s^\star(t)-2}u_n}{|x|^t} - \frac{|\bar{u}|^{2_s^\star(t)-2}\bar{u}}{|x|^t}\right. \\
&\qquad\qquad \left. - \frac{|u_n - \bar{u}|^{2_s^\star(t)-2}(u_n - \bar{u})}{|x|^t}\right\}v\mathrm{d}x \\
&= o(1) + \int_{\mathbb{R}^N} K(x)\left\{\frac{|u_n|^{2_s^\star(t)-2}u_n}{|x|^t} - \frac{|\bar{u}|^{2_s^\star(t)-2}\bar{u}}{|x|^t}\right. \\
&\qquad\qquad \left. - \frac{|u_n - \bar{u}|^{2_s^\star(t)-2}(u_n - \bar{u})}{|x|^t}\right\}v\mathrm{d}x.
\end{aligned} \tag{2.7}$$

We observe that

$$\begin{aligned}
&\left|K\left\{|u_n|^{2_s^\star(t)-2}u_k - |\bar{u}|^{2_s^\star(t)-2}\bar{u} - |u_n - \bar{u}|^{2_s^\star-2}(u_n - \bar{u})\right\}\right| \\
&\qquad\qquad \le C\left(|u_n - \bar{u}|^{2_s^\star(t)-2}|\bar{u}| + |u|^{2_s^\star(t)-2}|u_n - \bar{u}|\right).
\end{aligned}$$

Therefore, following the same method as in the proof of Claim 1 in Step 2, we show that as $n \to \infty$

$$\int_{\mathbb{R}^N} K(x)\left\{\frac{|u_n|^{2_s^\star(t)-2}u_n}{|x|^t} - \frac{|\bar{u}|^{2_s^\star(t)-2}\bar{u}}{|x|^t} - \frac{|u_n - \bar{u}|^{2_s^\star(t)-2}(u_n - \bar{u})}{|x|^t}\right\}v\mathrm{d}x = o(1) \tag{2.8}$$

for all $v \in \dot{H}^s(\mathbb{R}^N)$. Plugging this back into (2.7), we complete the proof of Step 3.

Step 4: Define $v_n := u_n - \bar{u}$. Then $v_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ and by Step 3, $(v_n)_n$ is a (PS) sequence for $\bar{I}^\gamma_{K,t,0}$ at the level $\beta - \bar{I}^\gamma_{K,t,f}(\bar{u})$. Thus,

$$\sup_{n\in\mathbb{N}}\|v_n\|_\gamma \le C \quad \text{and} \quad \langle v_n, \varphi\rangle_\gamma = \int_{\mathbb{R}^N} K(x)\frac{|v_n|^{2_s^\star(t)-2}v_n\varphi}{|x|^t}\,\mathrm{d}x + o(1) \tag{2.9}$$

as $n \to \infty$ for all $\varphi \in \dot{H}^s(\mathbb{R}^N)$. Therefore, $\|v_n\|_\gamma^2 = \int_{\mathbb{R}^N} K(x)\frac{|v_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x + o(1)$. Thus, if $\int_{\mathbb{R}^N} K(x)\frac{|v_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x \longrightarrow 0$, then we are done when $k = l = 0$ and the (PS) sequence $(u_n)_n$ admits a strongly convergent subsequence.

If not, let $0 < \delta < S_{\gamma,t,s}^{\frac{N-t}{2s-t}}\|K\|_{L^\infty(\mathbb{R}^N)}^{-\frac{N-2s}{2s-t}}$ be such that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N} K(x)\frac{|v_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x > \delta.$$

Up to a subsequence, let $R_n > 0$ be such that

$$\int_{B_{R_n}} K(x)\frac{|v_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x = \delta$$

and R_n being minimal with this property. Define

$$w_n(x) := R_n^{\frac{N-2s}{2}} v_n(R_n x). \tag{2.10}$$

Therefore, $\|w_n\|_\gamma = \|v_n\|_\gamma$ and

$$\delta = \int_{B_{R_n}} K(x)\frac{|v_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x = \int_{B_1} K(R_n x)\frac{|w_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x. \tag{2.11}$$

Therefore, up to a subsequence

$$w_n \rightharpoonup w \text{ in } \dot{H}^s(\mathbb{R}^N) \quad \text{and} \quad w_n \to w \text{ a.e. in } \mathbb{R}^N.$$

Let us now distinguish two cases $w \neq 0$ and $w = 0$.

Step 5: Assume that $w \neq 0$.

Since, $w_n \rightharpoonup w \neq 0$ and $v_n \rightharpoonup 0$, it follows that $R_n \to 0$ as $n \to \infty$. Next, we show that w is a solution of $(E^\gamma_{1,t,0})$. Indeed, thanks to (2.9), for any $\phi \in C_c^\infty(\mathbb{R}^N)$

$$\begin{aligned}
\langle w,\phi\rangle_\gamma &= \lim_{n\to\infty}\langle w_n,\phi\rangle_\gamma\\
&= \lim_{n\to\infty}\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{(w_n(x)-w_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{w_n\phi}{|x|^{2s}}\,\mathrm{d}x\\
&= \lim_{n\to\infty}\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{R_n^{\frac{N-2s}{2}}(v_n(R_nx)-v_n(R_ny))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&\qquad - \gamma\int_{\mathbb{R}^N}\frac{R_n^{\frac{N-2s}{2}}v_n(R_nx)\phi(x)}{|x|^{2s}}\,\mathrm{d}x\\
&= \lim_{n\to\infty}\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{R_n^{-\frac{N-2s}{2}}(v_n(x)-v_n(y))(\phi(\frac{x}{R_n})-\phi(\frac{y}{R_n}))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&\qquad - \gamma\int_{\mathbb{R}^N}\frac{R_n^{-\frac{N-2s}{2}}v_n(x)\phi(\frac{x}{R_n})}{|x|^{2s}}\,\mathrm{d}x\\
&= \lim_{n\to\infty}\int_{\mathbb{R}^N}K(x)\frac{|v_n|^{2_s^\star(t)-2}v_n}{|x|^t}R_n^{-\frac{N-2s}{2}}\phi\Big(\frac{x}{R_n}\Big)\,\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^N}K(R_nx)\frac{|w_n|^{2_s^\star(t)-2}w_n}{|x|^t}\phi(x)\,\mathrm{d}x.
\end{aligned} \tag{2.12}$$

Clearly $K(R_nx)\frac{|w_n|^{2_s^\star(t)-2}w_n}{|x|^t}\phi \to \frac{|w|^{2_s^\star(t)-2}w}{|x|^t}\phi$ a.e. in $\mathbb{R}^N$, since $K \in C(\mathbb{R}^N)$, with $K(0) = 1$, and $w_n \to w$ a.e. in $\mathbb{R}^N$. Further, arguing as in the proof of Claim 1 in Step 2, we have $K(R_nx)\frac{|w_n|^{2_s^\star(t)-2}w_n}{|x|^t}\phi$ is uniformly integrable. Therefore, as ϕ has compact support, using Vitali's convergence theorem we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}K(R_nx)\frac{|w_n|^{2_s^\star(t)-2}w_n}{|x|^t}\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^N}\frac{|w|^{2_s^\star(t)-2}w\phi}{|x|^t}\mathrm{d}x. \tag{2.13}$$

Combining (2.13) along with (2.12), we conclude that w is a solution of $(E^\gamma_{1,t,0})$.

Define

$$z_n(x) := v_n(x) - R_n^{-\frac{N-2s}{2}} w(\tfrac{x}{R_n}).$$

Claim 2: $(z_n)_n$ is a (PS) sequence for $\bar{I}^{\gamma}_{K,t,0}$ at the level $\beta - \bar{I}^{\gamma}_{K,t,f}(\bar{u}) - \bar{I}^{\gamma}_{1,0,0}(w)$.

To prove the claim, set

$$\tilde{z}_n(x) := R_n^{\frac{N-2s}{2}} z_n(R_n x).$$

Then

$$\tilde{z}_n(x) = w_n(x) - w(x) \quad \text{and} \quad \|\tilde{z}_n\|_\gamma = \|w_n - w\|_\gamma = \|z_n\|_\gamma.$$

As $K(0) = 1$ and K is a continuous function, the Brézis-Lieb lemma and a straight forward computation yield as $n \to \infty$

$$\begin{aligned}\int_{\mathbb{R}^N} K(R_n x)\frac{|w_n(x)|^{2^*_s(t)}}{|x|^t}\,dx - \int_{\mathbb{R}^N}\frac{|w|^{2^*_s(t)}}{|x|^t}\,dx &= \int_{\mathbb{R}^N}\frac{\big|K^{\frac{1}{2^*_s(t)}}(R_n x)w_n - w\big|^{2^*_s(t)}}{|x|^t}\,dx + o(1)\\ &= \int_{\mathbb{R}^N} K(R_n x)\frac{|w_n - w|^{2^*_s(t)}}{|x|^t}\,dx + o(1).\end{aligned}$$

Therefore, using the above relations, as $n \to \infty$

$$\begin{aligned}\bar{I}^{\gamma}_{K,t,0}(z_n) &= \frac{1}{2}\|z_n\|^2_\gamma - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{|z_n|^{2^*_s(t)}}{|x|^t}\,dx\\ &= \frac{1}{2}\|w_n - w\|^2_\gamma - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(R_n x)\frac{|w_n - w|^{2^*_s(t)}}{|x|^t}\,dx\\ &= \frac{1}{2}\left(\|w_n\|^2_\gamma - \|w\|^2_\gamma\right) - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(R_n x)\frac{|w_n(x)|^{2^*_s(t)}}{|x|^t}\,dx + \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N}\frac{|w|^{2^*_s(t)}}{|x|^t}\,dx + o(1)\\ &= \frac{1}{2}\|v_n\|^2_\gamma - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{|v(x)|^{2^*_s(t)}}{|x|^t}\,dx - \left(\frac{1}{2}\|w\|^2_\gamma - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N}\frac{|w|^{2^*_s(t)}}{|x|^t}\,dx\right) + o(1)\\ &= \bar{I}^{\gamma}_{K,t,0}(v_n) - \bar{I}^{\gamma}_{1,0,0}(w) + o(1)\\ &= \beta - \bar{I}^{\gamma}_{K,t,f}(\bar{u}) - \bar{I}^{\gamma}_{1,0,0}(w) + o(1).\end{aligned}$$

Next, let $\phi \in C_0^\infty(\mathbb{R}^N)$ be arbitrary and set $\phi_n(x) := R_n^{\frac{N-2s}{2}}\phi(R_n x)$. This in turn implies that $\|\phi_n\|_\gamma = \|\phi\|_\gamma$ and $\phi_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$. Therefore,

$$\begin{aligned}{}_{(\dot{H}^s)'}\big\langle(\bar{I}^{\gamma}_{K,t,0})'(z_n), \phi\big\rangle_{\dot{H}^s} &= \langle z_n, \phi\rangle_\gamma - \int_{\mathbb{R}^N} K(x)\frac{|z_n|^{2^*_s(t)-2}z_n\phi}{|x|^t}\,dx\\ &= \langle \tilde{z}_n, \phi_n\rangle_\gamma - \int_{\mathbb{R}^N} K(R_n x)\frac{|\tilde{z}_n|^{2^*_s(t)-2}\tilde{z}_n\phi_n}{|x|^t}\,dx\\ &= \langle w_n - w, \phi_n\rangle_\gamma - \int_{\mathbb{R}^N} K(R_n x)\frac{|w_n - w|^{2^*_s(t)-2}(w_n - w)\phi_n}{|x|^t}\,dx\\ &= \langle w_n, \phi_n\rangle_\gamma - \int_{\mathbb{R}^N} K(R_n x)\frac{|w_n|^{2^*_s(t)-2}w_n\phi_n}{|x|^t}\,dx\\ &\quad - \left(\langle w, \phi_n\rangle_\gamma - \int_{\mathbb{R}^N}\frac{|w|^{2^*_s(t)-2}w\phi_n}{|x|^t}\,dx\right)\\ &\quad + \int_{\mathbb{R}^N}(K(R_n x) - 1)\frac{|w|^{2^*_s(t)-2}w\phi_n}{|x|^t}dx\\ &\quad + \int_{\mathbb{R}^N} K(R_n x)\left(\frac{|w_n|^{2^*_s(t)-2}w_n - |w|^{2^*_s(t)-2}w - |w_n - w|^{2^*_s(t)-2}(w_n - w)}{|x|^t}\right)\phi_n dx\\ &= \langle v_n, \phi\rangle_\gamma - \int_{\mathbb{R}^N} K(x)\frac{|v_n|^{2^*_s(t)-2}v_n\phi}{|x|^t}\,dx - {}_{(\dot{H}^s)'}\big\langle\bar{I}^{\gamma}_{1,0,0}{}'(w), \phi_n\big\rangle_{\dot{H}^s} + I^1_n + I^2_n\\ &= {}_{(\dot{H}^s)'}\big\langle(\bar{I}^{\gamma}_{K,t,0})'(v_n), \phi\big\rangle_{\dot{H}^s} - 0 + I^1_n + I^2_n = o(1) + I^1_n + I^2_n.\end{aligned} \tag{2.14}$$

Now

$$I_n^1 := \int_{B_R} \big(K(R_n x) - 1\big) \frac{|w|^{2_s^\star(t)-2} w \phi_n}{|x|^t} \,\mathrm{d}x + \int_{B_R^c} \big(K(R_n x) - 1\big) \frac{|w|^{2_s^\star(t)-2} w \phi_n}{|x|^t} \,\mathrm{d}x.$$

Note that as $\frac{|w|^{2_s^\star(t)}}{|x|^t} \in L^1(\mathbb{R}^N)$, for $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$\begin{aligned}\left| \int_{B_R^c} \big(K(R_n x) - 1\big) \frac{|w|^{2_s^\star(t)-2} w \phi_n}{|x|^t} \,\mathrm{d}x \right| &\le C \bigg(\int_{B_R^c} \frac{|w|^{2_s^\star(t)}}{|x|^t} \mathrm{d}x \bigg)^{\frac{2_s^\star(t)-1}{2_s^\star(t)}} \bigg(\int_{\mathbb{R}^N} \frac{|\phi_n|^{2_s^\star(t)}}{|x|^t} \,\mathrm{d}x \bigg)^{\frac{1}{2_s^\star(t)}} \\ &\le C \bigg(\int_{B_R^c} \frac{|w|^{2_s^\star(t)}}{|x|^t} \mathrm{d}x \bigg)^{\frac{2_s^\star(t)-1}{2_s^\star(t)}} \|\phi\|_\gamma < \varepsilon.\end{aligned}$$

On the other hand, as $K \in C(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} K(x) = 1$ implies that $K \in L^\infty(\mathbb{R}^N)$, applying the Hölder inequality followed by the Hardy-Sobolev inequality, it is easy to see that

$$\big(K(R_n x) - 1\big) \frac{|w|^{2_s^\star(t)-2} w \phi_n}{|x|^t}$$

is uniformly integrable. Therefore, using Vitali's convergence theorem, we get

$$\int_{B_R} \big(K(R_n x) - 1\big) \frac{|w|^{2_s^\star(t)-2} w \phi_n}{|x|^t} \,\mathrm{d}x = o(1).$$

Hence, $I_n^1 = o(1)$ as $n \to \infty$.

Next, we aim to show that

$$I_n^2 := \int_{\mathbb{R}^N} K(R_n x) \bigg\{ \frac{|w_n|^{2_s^\star(t)-2} w_n - |w|^{2_s^\star(t)-2} w - |w_n - w|^{2_s^\star(t)-2}(w_n - w)}{|x|^t} \bigg\} \phi_n \mathrm{d}x = o(1).$$

Indeed, this follows as in the proof of (2.8), since $\int_{\mathbb{R}^N} \frac{|\phi_n|^{2_s^\star(t)}}{|x|^t} \,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{|\phi|^{2_s^\star(t)}}{|x|^t} \,\mathrm{d}x < \infty$. Hence, from (2.14) we conclude the proof of Claim 2.

Step 6: Assume that $w = 0$.

Let $\varphi \in C_0^\infty(B_1)$, with $0 \le \varphi \le 1$. Set $\psi_n(x) := [\varphi(\frac{x}{R_n})]^2 v_n(x)$. Clearly $(\psi_n)_n$ is a bounded sequence in $\dot{H}^s(\mathbb{R}^N)$. Thus,

$$\begin{aligned} o(1) &= {}_{(\dot{H}^s)'}\big\langle (\bar{I}^\gamma_{K,t,0})'(v_n), \psi_n \big\rangle_{\dot{H}^s} \\ &= \langle v_n, \psi_n \rangle_\gamma - \int_{\mathbb{R}^N} K(x) \frac{|v_n|^{2_s^\star(t)-2} v_n \psi_n}{|x|^t} \mathrm{d}x \\ &= \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y)\big(\varphi^2(\frac{x}{R_n}) v_n(x) - \varphi^2(\frac{y}{R_n}) v_n(y)\big)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y - \gamma \int_{\mathbb{R}^N} \frac{v_n^2(x) \varphi^2(\frac{x}{R_n})}{|x|^{2s}} \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} K(x) \frac{\varphi^2(\frac{x}{R_n}) |v_n|^{2_s^\star(t)}}{|x|^t} \mathrm{d}x \\ &= \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{\big(v_n(R_n x) - v_n(R_n y)\big)\big(\varphi^2(x) v_n(R_n x) - \varphi^2(y) v_n(R_n y)\big) R_n^{N-2s}}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\quad - \gamma \int_{\mathbb{R}^N} \frac{v_n^2(R_n x) \varphi^2(x) R_n^{N-2s}}{|x|^{2s}} \mathrm{d}x - \int_{\mathbb{R}^N} K(x) \frac{|v_n|^{2_s^\star(t)-2} \big(\varphi(\frac{x}{R_n}) v_n\big)^2}{|x|^t} \mathrm{d}x. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{\big(v_n(R_n x) - v_n(R_n y)\big)\big(\varphi^2(x) v_n(R_n x) - \varphi^2(y) v_n(R_n y)\big) R_n^{N-2s}}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\quad - \gamma \int_{\mathbb{R}^N} \frac{v_n^2(R_n x) \varphi^2(x) R_n^{N-2s}}{|x|^{2s}} \mathrm{d}x = \int_{\mathbb{R}^N} K(x) \frac{|v_n|^{2_s^\star(t)-2} \big(\varphi(\frac{x}{R_n}) v_n\big)^2}{|x|^t} \mathrm{d}x + o(1). \end{aligned} \tag{2.15}$$

Now,

$$\begin{aligned}
\text{RHS of (2.15)} &= \int_{B_1} K(R_n x)\frac{|v_n(R_n x)|^{2_s^\star(t)-2}\big(\varphi(x)v_n(R_n x)\big)^2 R_n^{N-t}}{|x|^t}\,\mathrm{d}x + o(1)\\
&= \int_{B_1} \frac{\Big|K(R_n x)^{\frac{1}{2_s^\star(t)-2}} w_n(x)\Big|^{2_s^\star(t)-2}\big(\varphi(x)w_n(x)\big)^2}{|x|^t}\,\mathrm{d}x + o(1)\\
&\le \left(\int_{B_1} K(R_n x)^{\frac{2_s^\star(t)}{2_s^\star(t)-2}}\frac{|w_n(x)|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{2_s^\star(t)-2}{2_s^\star(t)}}\cdot\\
&\quad\times\left(\int_{\mathbb{R}^N}\frac{|\varphi w_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{2}{2_s^\star(t)}} + o(1) \qquad (2.16)\\
&\le \frac{\|K\|_{L^\infty}^{\frac{2}{2_s^\star(t)}}}{S_{\gamma,t,s}}\left(\int_{B_1}K(R_n x)\frac{|w_n|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{2_s^\star(t)-2}{2_s^\star(t)}}\|\varphi w_n\|_\gamma^2 + o(1)\\
&\le \frac{\|K\|_{L^\infty}^{\frac{2}{2_s^\star(t)}}\,\delta^{\frac{2s-t}{N-t}}}{S_{\gamma,t,s}}\|\varphi w_n\|_\gamma^2 + o(1)\\
&< \|\varphi w_n\|_\gamma^2 + o(1)\ \ \text{(By the choice of } \delta \text{ fixed in Step 4).}
\end{aligned}$$

Claim 3: As $n \to \infty$

$$\text{LHS of (2.15)} = \|\varphi w_n\|_\gamma^2 + o(1). \qquad (2.17)$$

Indeed,

$$\begin{aligned}
\text{LHS of (2.15)} &= \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{\big(v_n(R_n x)-v_n(R_n y)\big)\big(\varphi^2(x)v_n(R_n x)-\varphi^2(y)v_n(R_n y)\big)R_n^{N-2s}}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&\quad-\gamma\int_{\mathbb{R}^N}\frac{|\varphi w_n|^2}{|x|^{2s}}\mathrm{d}x\\
&= \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{\big(w_n(x)-w_n(y)\big)\big(\varphi^2(x)w_n(x)-\varphi^2(y)w_n(y)\big)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&\quad-\gamma\int_{\mathbb{R}^N}\frac{|\varphi w_n|^2}{|x|^{2s}}\mathrm{d}x \qquad (2.18)\\
&= \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{|\varphi(x)w_n(x)-\varphi(y)w_n(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y-\gamma\int_{\mathbb{R}^N}\frac{|\varphi w_n|^2}{|x|^{2s}}\,\mathrm{d}x\\
&\quad-\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{(\varphi(x)-\varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&= \|\varphi w_n\|_\gamma^2-\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{(\varphi(x)-\varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Now,

$$\begin{aligned}
\iint_{\mathbb{R}^{2N}}\frac{(\varphi(x)-\varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y &= \int_{x\in B_1}\int_{y\in B_1} + \int_{x\in B_1}\int_{y\in B_1^c} + \int_{x\in B_1^c}\int_{y\in B_1}\\
&=: \mathcal{I}_n^1+\mathcal{I}_n^2+\mathcal{I}_n^3.
\end{aligned}$$

Of course, $\mathcal{I}_n^2 = \mathcal{I}_n^3$, as the integral is symmetric with respect to x and y.

$$\begin{aligned}
\mathcal{I}_n^1 &= \int_{x\in B_1}\int_{y\in B_1}\frac{(\varphi(x)-\varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&\le C\int_{x\in B_1}\int_{y\in B_1}\frac{|w_n(x)||w_n(y)|}{|x-y|^{N+2s-2}}\,\mathrm{d}x\mathrm{d}y\\
&\le C\left(\int_{x\in B_1}\int_{y\in B_1}\frac{|w_n(x)|^2}{|x-y|^{N+2s-2}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}\cdot
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_{x\in B_1}\int_{y\in B_1}\frac{|w_n(y)|^2}{|x-y|^{N+2s-2}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \qquad (2.19)\\
&\le C\int_{x\in B_1}\int_{y\in B_1}\frac{|w_n(x)|^2}{|x-y|^{N+2s-2}}\,\mathrm{d}x\mathrm{d}y\\
&\le C\int_{x\in B_1}\left(\int_{|z|<2}\frac{1}{|z|^{N+2s-2}}\,\mathrm{d}z\right)|w_n(x)|^2\mathrm{d}x\\
&\le C\|w_n\|^2_{L^2(B_1)}=o(1)\ \ \big(\text{as } w=0 \text{ implies } w_n\to 0 \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^N)\big).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\mathcal{I}_n^2 &= \int_{x\in B_1}\int_{y\in B_1^c}\frac{(\varphi(x)-\varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&\le \int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\le 1\}}+\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\ge 1\}} \qquad (2.20)\\
&=: \mathcal{I}_n^{21}+\mathcal{I}_n^{22},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_n^{21} &\le C\left(\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\le 1\}}\frac{|w_n(x)|^2}{|x-y|^{N+2s-2}}\,\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}\cdot\\
&\qquad\times\left(\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\le 1\}}\frac{|w_n(y)|^2}{|x-y|^{N+2s-2}}\,\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}\\
&=: CJ_n^1\cdot J_n^2.
\end{aligned}
$$

Now,

$$
|J_n^1|^2\le\int_{x\in B_1}\left(\int_{|z|<1}\frac{1}{|z|^{N+2s-2}}\,\mathrm{d}z\right)|w_n(x)|^2\mathrm{d}x\le C\|w_n\|^2_{L^2(B_1)}=o(1),
$$

and

$$
\begin{aligned}
|J_n^2|^2 &= \int_{x\in B_1}\int_{y\in B_1^c}\frac{\mathbf{1}_{\{|x-y|<1\}}(x,y)|w_n(y)|^2}{|x-y|^{N+2s-2}}\,\mathrm{d}y\mathrm{d}x\\
&\le \int_{y\in B_1^c}\left(\int_{x\in B_1}\frac{\mathbf{1}_{\{|x-y|<1\}}(x,y)}{|x-y|^{N+2s-2}}\,\mathrm{d}x\right)|w_n(y)|^2\,\mathrm{d}y\\
&\le \int_{y\in B_2}\left(\int_{x\in B_1}\frac{\mathbf{1}_{\{|x-y|<1\}}(x,y)}{|x-y|^{N+2s-2}}\,\mathrm{d}x\right)|w_n(y)|^2\,\mathrm{d}y\\
&\le C\|w_n\|^2_{L^2(B_2)}\le C'.
\end{aligned}
$$

Therefore, $\mathcal{I}_n^{21}=o(1)$ as $n\to\infty$. Moreover,

$$
\begin{aligned}
\mathcal{I}_n^{22} &= \int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\ge 1\}}\frac{|w_n(x)||w_n(y)||\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x\\
&\le C\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\ge 1\}}\frac{|w_n(x)||w_n(y)|}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x\\
&\le C\left(\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\ge 1\}}\frac{|w_n(x)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}\cdot\\
&\qquad\times\left(\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\ge 1\}}\frac{|w_n(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}\\
&\le C\left(\int_{x\in B_1}\left(\int_{\{|z|\ge 1\}}\frac{1}{|z|^{N+2s}}\,\mathrm{d}z\right)|w_n(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\cdot
\end{aligned}
$$

$$\times \left(\int_{x \in B_1} \int_{\{|z| \geq 1\}} \frac{|w_n(x+z)|^2}{|x+z|^{2s}} \frac{|x+z|^{2s}}{|z|^{N+2s}} \, \mathrm{d}z\mathrm{d}x \right)^{\frac{1}{2}}$$
$$\leq C' \|w_n\|_{L^2(B_1)} \left[\int_{x \in B_1} \left(\int_{\mathbb{R}^N} \frac{|w_n(x+z)|^2}{|x+z|^{2s}} \mathrm{d}z \right) \mathrm{d}x \right]^{\frac{1}{2}},$$

since $|z| \geq 1$ and $|x| < 1$ implies $\frac{|x+z|^{2s}}{|z|^{N+2s}} \leq C$. Therefore, using the Hardy inequality, we obtain from the last of the above estimate that as $n \to \infty$

$$\mathcal{I}_n^{22} \leq C'' \|w_n\|_{L^2(B_1)} \|w_n\|^2_{\dot{H}^s(\mathbb{R}^N)} = o(1).$$

Putting the above estimates together, we obtain from (2.20) that $\mathcal{I}_n^2 = o(1)$ as $n \to \infty$. This, along with (2.19), concludes the proof of Claim 3.

Combining Claim 3 with (2.16) yields

$$\|\varphi w_n\|_\gamma = o(1) \text{ as } n \to \infty. \tag{2.21}$$

Substituting this into (2.17) and comparing with (2.15) yields as $n \to \infty$

$$\int_{\mathbb{R}^N} K(R_n x) \frac{\varphi^2(x)|w_n(x)|^{2_s^\star(t)}}{|x|^t} \, \mathrm{d}x = o(1).$$

Therefore,

$$\int_{B_r} K(R_n x) \frac{|w_n|^{2_s^\star(t)}}{|x|^t} \, \mathrm{d}x = o(1), \quad \text{for any} \quad 0 < r < 1. \tag{2.22}$$

But this contradicts (2.11) when $t > 0$. Therefore, $w = 0$ cannot happen in the case $t > 0$, i.e.,

$$t > 0 \implies w \neq 0.$$

Consequently, from now onwards, we restrict ourselves to the case $t = 0$ and $w = 0$.

Step 7: Let $t = 0$ and $w = 0$. First we consider the tight case, $(v_n)_n \subset \dot{H}^s_0(B_R)$, for some fixed ball of radius $R > 0$ (where $\dot{H}^s_0(B_R)$ is the closure of $C_0^\infty(B_R)$ with respect to the $\dot{H}^s(\mathbb{R}^N)$ norm). The remaining case will be obtained by a splitting argument together with a Kelvin transform.

Therefore, in view of (2.11) and (2.22), using the concentration-compactness principle in the tight case [19], it follows that in the sense of measure,

$$K(R_n x)|w_n|^{2_s^\star} dx\Big|_{\{|x| \leq 1\}} \overset{\star}{\rightharpoonup} \sum_j C_{x_j} \delta_{x_j}, \tag{2.23}$$

where $x_j \in \mathbb{R}^N$ satisfies $|x_j| = 1$. Let $\bar{C} := \max_j C_{x_j}$ and define

$$Q_n(r) := \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} K(R_n x)|w_n|^{2_s^\star} \, \mathrm{d}x. \tag{2.24}$$

Clearly, $Q_n(r) > C/2$ for each $r > 0$ large enough. Moreover, (2.23) gives

$$\liminf_{n \to \infty} Q_n(r) \geq \frac{C}{2}.$$

Hence, there exist sequences $(s_n)_n \subset \mathbb{R}^+$ and $(q_n)_n \subset \mathbb{R}^N$ such that $s_n \to 0$ and $|q_n| > 1/2$ and

$$\frac{C}{2} = \sup_{q \in \mathbb{R}^N} \int_{B_{s_n}(q)} K(R_n x) w_n^{2_s^\star} \mathrm{d}x = \int_{B_{s_n}(q_n)} K(R_n x) w_n^{2_s^\star} \, \mathrm{d}x. \tag{2.25}$$

Define $\theta_n(x) := s_n^{\frac{N-2s}{2}} w_n(s_n x + q_n)$. Thus $\|\theta_n\|_\gamma = \|w_n\|_\gamma$ for any $n \in \mathbb{N}$. Consequently, up to a subsequence, there exists $\theta \in \dot{H}^s(\mathbb{R}^N)$ such that $\theta_n \rightharpoonup \theta$ in $\dot{H}^s(\mathbb{R}^N)$ and $\theta_n \to \theta$ a.e. in $\mathbb{R}^N$.

First note that $\theta \neq 0$. Otherwise, choosing $\varphi \in C_0^\infty(B_1(x))$, with $0 \le \varphi \le 1$, for an arbitrary but fixed $x \in \mathbb{R}^N$, and proceeding exactly as in obtaining (2.21), we are able to show that $\theta_n \to 0$ in $L^{2_s^*}_{\mathrm{loc}}(\mathbb{R}^N)$. On the other hand, from (2.25) it follows that

$$\int_{B_1} K(s_n R_n x + q_n)\theta_n^{2_s^*}\,\mathrm{d}x = \frac{C}{2} > 0.$$

which leads to a contradiction. Thus, $\theta \neq 0$. Recall that

$$\theta_n(x) = s_n^{\frac{N-2s}{2}} w_n(s_n x + q_n) = (s_n R_n)^{(\frac{N-2s}{2})} v_n(s_n R_n x + R_n q_n).$$

Define $r_n = s_n R_n = o(1)$ and $y_n = R_n q_n$. Hence, $\frac{r_n}{|y_n|} < 2s_n = o(1)$ and, up to a subsequence, $y_n \to y$ in $\mathbb{R}^N$. From Lemma A.1, we deduce that

$$\theta = K(y)^{-\frac{N-2s}{4s}} W^{\tau,a} \quad \text{for some } \tau > 0,\ a \in \mathbb{R}^N,$$

where W is a solution of $(E^0_{1,0,0})$ and that $n \mapsto \tilde{v}_n(x) := v_n(x) - K(y)^{\frac{4s}{N-2s}} W^{r_n\tau, y_n + r_n a}(x)$ is a (PS) sequence for $\bar{I}^{\gamma}_{K,t,0}$ at level $\beta - \bar{I}^{\gamma}_{K,t,f}(\bar{u}) - K(y)^{-\frac{N-2s}{2s}} \bar{I}^{(0)}_{1,0,0}(W)$, where W is a solution of $(E^0_{1,0,0})$.

In summary, in both cases $t > 0$ and $t = 0$, starting from a (PS) sequence $(v_n)_n$ of $\bar{I}^{\gamma}_{K,t,0}$ we have found another (PS) sequence $(\tilde{v}_n)_n$ of $\bar{I}^{\gamma}_{K,t,0}$ at a strictly lower level, with a fixed minimum amount of decrease. Since $\sup_n \|v_n\|_\gamma \le C < \infty$, the process should stop after finitely many steps.

Step 8: When $t = 0$ we only dealt with the case $(v_n)_n \subset \dot{H}^s_0(B_R)$ for some fixed $R > 0$. Now we are going to relax the assumption $(v_n)_n \subset \dot{H}^s_0(B_R)$.

Let us define

$$\tilde{f}(k) := \liminf_{n\to\infty} \int_{B_{k+1}\setminus B_k} K(x)|v_n|^{2_s^*}\,\mathrm{d}x.$$

We claim that $\tilde{f}(k) = 0$ for all but finitely many k's.

Indeed, if $\tilde{f}(k) > 0$ for some k, then $\liminf_{n\to\infty} \int_{B_{k+1}\setminus B_k} K(x)|v_n|^{2_s^*}\,\mathrm{d}x > 0$. Therefore,

$$\liminf_{n\to\infty} \int_{B_{k+1}\setminus B_k} |v_n|^{2_s^*}\,\mathrm{d}x > 0. \tag{2.26}$$

By Step 6, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ as $n \to \infty$

$$\begin{aligned}\|K\|^{\frac{2}{2_s^*}}_{L^\infty(\mathbb{R}^N)} \left(\int_{\operatorname{supp}(\varphi)} K(R_n x)|w_n|^{2_s^*}\mathrm{d}x\right)^{\frac{2_s^*-2}{2_s^*}} \left(\int_{\mathbb{R}^N} |\varphi w_n|^{2_s^*}\mathrm{d}x\right)^{\frac{2}{2_s^*}} &\ge \|\varphi w_n\|_\gamma^2 + o(1)\\ &\ge S_{\gamma,0,s}\left(\int_{\mathbb{R}^N}|\varphi w_n|^{2_s^*}\right)^{\frac{2}{2_s^*}} + o(1).\end{aligned} \tag{2.27}$$

Fix any $\varepsilon > 0$ and choose $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ in $B_{k+1}\setminus B_k$ and $\operatorname{supp}(\varphi) \subseteq B_{k+1+\varepsilon}\setminus B_{k-\varepsilon}$ and $0 \le \varphi \le 1$. Define, $\varphi_n(x) = \varphi(R_n x)$. Then

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\varphi_n w_n|^{2_s^*} dx = \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\varphi v_n|^{2_s^*} dx > \liminf_{n\to\infty} \int_{B_{k+1}\setminus B_k} |v_n|^{2_s^*} dx > 0.$$

Now (2.27), with $\varphi = \varphi_n$, yields as $n \to \infty$

$$\int_{B_{k+1+\varepsilon}\setminus B_{k-\varepsilon}} K(x)|v_n|^{2_s^*}\,\mathrm{d}x \ge \|K\|^{-\frac{N-2s}{2s}}_{L^\infty(\mathbb{R}^N)} S^{\frac{N}{2s}}_{\gamma,0,s} + o(1).$$

Combining the above, as $\varepsilon > 0$ is arbitrary, we obtain $\tilde{f}(k) \ge \|K\|^{-\frac{N-2s}{2s}}_{L^\infty(\mathbb{R}^N)} S^{\frac{N}{2s}}_{\gamma,0,s}$. Therefore, since $(v_n)_n$ is bounded in $L^{2_s^*}(\mathbb{R}^N)$, it follows that $\tilde{f}(k) = 0$ for all but finitely many k's and this completes the proof of the claim.

Now given such a k for which $\tilde{f}(k) = 0$, we take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\chi \equiv 1$ on B_k and $\chi \equiv 0$ on B_{k+1}^c and $0 \le \chi \le 1$. We shall show that both $(\chi v_n)_n$ and $\big((1-\chi)v_n\big)_n$ are (PS) sequences for $I^\gamma_{K,0,0}$. Indeed for $h \in C_0^\infty(\mathbb{R}^N)$ as $n \to \infty$

$$\begin{aligned}
\langle \chi v_n, h\rangle_\gamma &= \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)v_n(x)-\chi(y)v_n(y)\big)\big(h(x)-h(y)\big)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{\chi v_n h}{|x|^{2s}}\mathrm{d}x\\
&= \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{\big(v_n(x)-v_n(y)\big)\big(\chi(x)h(x)-\chi(y)h(y)\big)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{v_n(\chi h)}{|x|^{2s}}\mathrm{d}x\\
&\quad+\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)-\chi(y)\big)h(x)v_n(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y - \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)-\chi(y)\big)h(y)v_n(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&= \langle v_n, \chi h\rangle_\gamma + C_{N,s}\iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)-\chi(y)\big)h(x)v_n(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&= \int_{\mathbb{R}^N} K(x)|v_n|^{2_s^*-2}v_n(\chi h)\,\mathrm{d}x + C_{N,s}\mathbb{I}_n + o(\|h\|),
\end{aligned} \tag{2.28}$$

where $\mathbb{I}_n := \iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)-\chi(y)\big)h(x)v_n(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y$.

Claim 4: $\mathbb{I}_n = o(\|h\|_\gamma)$ as $n \to \infty$.

Indeed,

$$\mathbb{I}_n \le \left(\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 h^2(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac12}\left(\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac12}.$$

Now,

$$\begin{aligned}
\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y &= \int_{y\in B_{k+1}}\int_{x\in B_{k+1}} + \int_{y\in B_{k+1}}\int_{x\in B_{k+1}^c} + \int_{y\in B_{k+1}^c}\int_{x\in B_{k+1}}\\
&=: \mathbb{I}_n^1 + \mathbb{I}_n^2 + \mathbb{I}_n^3.
\end{aligned}$$

Since $v_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ implies $v_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we see that as $n \to \infty$

$$\begin{aligned}
\mathbb{I}_n^1 &= \int_{y\in B_{k+1}}\int_{x\in B_{k+1}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&\le C\int_{y\in B_{k+1}}\int_{x\in B_{k+1}} \frac{v_n^2(y)}{|x-y|^{N+2s-2}}\,\mathrm{d}x\mathrm{d}y\\
&\le C\int_{y\in B_{k+1}}\left(\int_{x\in B_{k+1}\cap\{|x-y|<1\}}\frac{\mathrm{d}x}{|x-y|^{N+2s-2}} + \int_{x\in B_{k+1}\cap\{|x-y|\ge 1\}}\mathrm{d}x\right)v_n^2(y)\mathrm{d}y\\
&\le C'\int_{y\in B_{k+1}} v_n^2(y)\mathrm{d}y\\
&= o(1);
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
\mathbb{I}_n^2 &= \int_{y\in B_{k+1}}\int_{x\in B_{k+1}^c} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&\le C\int_{y\in B_{k+1}}\left(\int_{x\in B_{k+1}^c\cap\{|x-y|\le 1\}}\frac{\mathrm{d}x}{|x-y|^{N+2s-2}} + \int_{x\in B_{k+1}^c\cap\{|x-y|\ge 1\}}\frac{\mathrm{d}x}{|x-y|^{N+2s}}\right)v_n^2(y)\mathrm{d}y\\
&\le C''\int_{y\in B_{k+1}} v_n^2(y)\mathrm{d}y = o(1);
\end{aligned} \tag{2.30}$$

$$\mathbb{I}_n^3 = \int_{y\in B_{k+1}^c}\int_{x\in B_{k+1}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y$$

$$
\begin{aligned}
&= \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x\\
&= \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c\cap\{|x-y|\ge 1\}} + \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c\cap\{|x-y|\le 1\}}\\
&=: \mathbb{I}_n^{31} + \mathbb{I}_n^{32}.
\end{aligned}
$$

For estimating $\mathbb{I}_n^{31}$, we choose $\varepsilon > 0$ arbitrary and $R >> k+1$ so that

$$
\begin{aligned}
\mathbb{I}_n^{31} &= \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c\cap\{|x-y|\ge 1\}} \frac{\chi^2(x)v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x\\
&\le \int_{x\in B_{k+1}}\left(\int_{\{y\,:\,|x-y|\ge 1\}} \frac{v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\right)\mathrm{d}x\\
&\le \int_{x\in B_{k+1}}\left(\int_{B_R} v_n^2(y)\mathrm{d}y + \int_{B_R^c\cap\{|x-y|\ge 1\}} \frac{v_n^2(y)}{|y|^{N+2s}}\frac{|y|^{N+2s}}{|x-y|^{N+2s}}\mathrm{d}y\right)dx\\
&\le \int_{x\in B_{k+1}}\left(o(1) + C\int_{B_R^c} \frac{v_n^2(y)}{|y|^{N+2s}}\mathrm{d}y\right)dx\\
&\le C'''\left(o(1) + C\Big(\int_{B_R^c}|v_n|^{2_s^\star}\mathrm{d}y\Big)^{\frac{2}{2_s^\star}}\Big(\int_{B_R^c}\frac{\mathrm{d}y}{|y|^{(N+2s)N/2s}}\Big)^{\frac{2s}{N}}\right) < \varepsilon \quad \text{for } R >> k+1,
\end{aligned}
\tag{2.31}
$$

since $(v_n)_n$ is uniformly bounded in $L^{2_s^\star}(\mathbb{R}^N)$ and $|y|^{(N+2s)N/2s} \in L^1(\{|y|>1\})$. Moreover,

$$
\begin{aligned}
\mathbb{I}_n^{32} &= \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c\cap\{|x-y|\le 1\}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x\\
&\le C\int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c} \frac{\mathbf{1}_{\{|x-y|\le 1\}}(x,y)v_n^2(y)}{|x-y|^{N+2s-2}}\,\mathrm{d}y\mathrm{d}x\\
&= C\int_{y\in B_{k+1}^c}\left(\int_{x\in B_{k+1}} \frac{\mathbf{1}_{\{|x-y|\le 1\}}(x,y)}{|x-y|^{N+2s-2}}\mathrm{d}x\right)v_n^2(y)\mathrm{d}y\\
&= C\int_{y\in B_{k+2}}\left(\int_{x\in B_{k+1}} \frac{\mathbf{1}_{\{|x-y|\le 1\}}(x,y)}{|x-y|^{N+2s-2}}\mathrm{d}x\right)v_n^2(y)\mathrm{d}y\\
&\le C''''\int_{y\in B_{k+2}} v_n^2(y)\mathrm{d}y = o(1).
\end{aligned}
\tag{2.32}
$$

Combining (2.29)–(2.32), we obtain

$$
\left(\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} = o(1)
$$

as $n\to\infty$. Similarly, it follows that

$$
\left(\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 h^2(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \le \|h\|_{\dot{H}^s(\mathbb{R}^N)}.
$$

Hence Claim 4 is proved.

Therefore, using (2.28) and the fact that $\tilde{f}(k) = 0$, we obtain as $n\to\infty$

$$
\begin{aligned}
{}_{(\dot{H}^s)'}\langle(\bar{I}^{\gamma}_{K,t,0})'(\chi v_n), h\rangle_{\dot{H}^s} &= \langle \chi v_n, h\rangle_\gamma - \int_{\mathbb{R}^N} K(x)|\chi v_n|^{2_s^\star-2}(\chi v_n)h\,\mathrm{d}x\\
&= \int_{\mathbb{R}^N} K(x)\{\chi - \chi^{2_s^\star-1}\}|v_n|^{2_s^\star-2}v_n h\,\mathrm{d}x + o(\|h\|_{\dot{H}^s(\mathbb{R}^N)})\\
&\le C\|K\|_{L^\infty(\mathbb{R}^N)}\left(\int_{B_{k+1}\setminus B_k}|v_n|^{2_s^\star}\mathrm{d}x\right)^{\frac{2_s^\star-1}{2_s^\star}}\|h\|_\gamma + o(\|h\|_{\dot{H}^s(\mathbb{R}^N)}) = o(\|h\|_{\dot{H}^s(\mathbb{R}^N)}).
\end{aligned}
$$

This is the required inequality.

Now, as $n \to \infty$

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)|v_n|^{2_s^\star}\mathrm{d}x &= \int_{\mathbb{R}^N} K(x)|\chi v_n + (1-\chi)v_n|^{2_s^\star}\mathrm{d}x \\ &= \int_{\mathbb{R}^N} K(x)|\chi v_n|^{2_s^\star}\mathrm{d}x + \int_{\mathbb{R}^N} K(x)|(1-\chi)v_n|^{2_s^\star}\mathrm{d}x + o(1).\end{aligned} \tag{2.33}$$

The last line in (2.33) follows from the fact that $\operatorname{supp}(\chi) \subseteq B_{k+1}$ and $\operatorname{supp}(1-\chi) \subset \mathbb{R}^N \setminus B_k$ and all the remaining terms in the expansion of $|\chi v_n+(1-\chi)v_n|^{2_s^\star}$ involves product of some powers of χv_n and $(1-\chi)v_n$ whose support lies in $B_{k+1} \setminus B_k$, but in the definition of χ we have chosen the same k for which $\tilde{f}(k) = 0$.

We know that $(v_n)_n$ is a (PS) sequence of $\bar{I}^{\gamma}_{K,0,0}$ at the level $\beta - \bar{I}^{\gamma}_{K,t,f}(\bar{u})$. Hence, from (2.33) the level of the (PS) sequence $(v_n)_n$ of $\bar{I}^{\gamma}_{K,0,0}$ is integrally split between the two new (PS) sequences $(\chi v_n)_n$ and $\big((1-\chi)v_n\big)_n$.

Let $\mathcal{K}$ denote the Kelvin transform in $\dot{H}^s(\mathbb{R}^N)$ given by

$$\mathcal{K}u(x) := \frac{1}{|x|^{N-2s}} u(|x|^{-2}x).$$

Therefore, it is known that (see Appendix of [23]),

$$(-\Delta)^s \mathcal{K}u(x) = \frac{1}{|x|^{N+2s}}(-\Delta)^s u(|x|^{-2}x).$$

Claim 5: $\|\mathcal{K}(u)\|_{\dot{H}^s(\mathbb{R}^N)} = \|u\|_{\dot{H}^s(\mathbb{R}^N)}$.

To prove the claim, first assume that $u \in C_0^\infty(\mathbb{R}^N)$. Thus

$$|(-\Delta)^s \mathcal{K}u(x)| \le \frac{C}{1+|x|^{N+2s}}. \tag{2.34}$$

Therefore,

$$\begin{aligned}\|\mathcal{K}(u)\|^2_{\dot{H}^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \mathcal{K}u(x)|^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} (-\Delta)^s \mathcal{K}(u(x))\mathcal{K}u(x)\,\mathrm{d}x \quad (\text{Using (2.34) and } \mathcal{K}(u) \in \dot{H}^s(\mathbb{R}^N)) \\ &= \int_{\mathbb{R}^N} \frac{1}{|x|^{N+2s}}(-\Delta)^s u(|x|^{-2}x)\frac{1}{|x|^{N-2s}}u(|x|^{-2}x)\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \big((-\Delta)^s u(x)\big)u(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2\mathrm{d}x \quad \big(\text{as } u \in C_0^\infty(\mathbb{R}^N)\big) \\ &= \|u\|^2_{\dot{H}^s(\mathbb{R}^N)}.\end{aligned}$$

Next for any $u \in \dot{H}^s(\mathbb{R}^N)$, let $(u_n)_n \in C_0^\infty(\mathbb{R}^N)$ be such that $u_n \to u$ in $\dot{H}^s(\mathbb{R}^N)$. Then

$$\|\mathcal{K}(u_n)\|_{\dot{H}^s(\mathbb{R}^N)} = \|u_n\|_{\dot{H}^s(\mathbb{R}^N)} \to \|u\|_{\dot{H}^s(\mathbb{R}^N)}. \tag{2.35}$$

Thus,

$$\|\mathcal{K}(u_n) - \mathcal{K}(u_m)\|_{\dot{H}^s(\mathbb{R}^N)} = \|\mathcal{K}(u_n - u_m)\|_{\dot{H}^s(\mathbb{R}^N)} = \|u_n - u_m\|_{\dot{H}^s(\mathbb{R}^N)} \underset{n,m\to\infty}{\longrightarrow} 0.$$

Hence, $(\mathcal{K}(u_n))_n$ is a Cauchy sequence in $\dot{H}^s(\mathbb{R}^N)$, so there exists $v \in \dot{H}^s(\mathbb{R}^N)$ such that $\mathcal{K}(u_n) \to v$. Now, as $u_n \to u$ a.e. in $\mathbb{R}^N$ so $\mathcal{K}(u_n) \to \mathcal{K}(u)$ a.e. in $\mathbb{R}^N$. Consequently, $v = \mathcal{K}(u)$. Therefore, passing the limit in (2.35), we have $\|\mathcal{K}(u)\|_{\dot{H}^s(\mathbb{R}^N)} = \|u\|_{\dot{H}^s(\mathbb{R}^N)}$ for all $u \in \dot{H}^s(\mathbb{R}^N)$.

Using Claim 5 along with standard change of variable, it is easy to see that

$$\bar{I}^{\gamma}_{K,0,0}\big(\mathcal{K}(u)\big) = \frac{1}{2}\|u\|^2_{\dot{H}^s} - \frac{\gamma}{2}\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\mathrm{d}x - \frac{1}{2_s^\star}\int_{\mathbb{R}^N} \mathcal{K}\big(|x|^{-2}x\big)|u|^{2_s^\star}(x)\mathrm{d}x,$$

that is, $\bar{I}^{\gamma}_{K,0,0} \circ \mathcal{K}$ has the same expression as $\bar{I}^{\gamma}_{K,0,0}$ except that $K(x)$ has to be replaced by $K(|x|^{-2}x)$. Hence, Steps 5 and 7 can be applied to $(\mathcal{K}((1-\chi)v_n))_n$, since this sequence is now a (PS) sequence for $\bar{I}^{\gamma}_{K,0,0} \circ \mathcal{K}$ in $\dot{H}^s_0(B_{\frac{1}{k}})$. Next, we can define the rescaling of the sequence $(\mathcal{K}((1-\chi)v_n))_n$ as in (2.10). If the limit of this rescaled sequence is nonzero then using Step 5 (otherwise, that is, if the limit of this rescaled sequence is zero, then applying Step 7), we obtain the characterization of $(\mathcal{K}((1-\chi)v_n))_n$ and from that we deduce the characterization of $((1-\chi)v_n)_n$; the only point which needs to be taken care of $\mathcal{K}\big(W(\frac{x-y^j_n}{r^j_n})\big)$. This is the concern in Lemma A.2.

Finally (vi) and (vii) follow as in [21, Theorem 4]. Thus the proof is completed. □

3 Proof of the main Theorem 1.1

In this section we assume without further mentioning that all the assumptions of Theorem 1.1 are satisfied. We first establish existence of two positive critical points for the functional

$$I^{\gamma}_{K,t,f}(u) = \frac{1}{2}\|u\|^2_{\gamma} - \frac{1}{2^\star_s(t)}\int_{\mathbb{R}^N} K(x)\frac{u_+^{2^\star_s(t)}}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s}.$$

Clearly, if u is a critical point of $I^{\gamma}_{K,t,f}$, then u solves

$$\begin{cases} (-\Delta)^s u - \gamma\dfrac{u}{|x|^{2s}} = K(x)\dfrac{u_+^{2^\star_s(t)-1}}{|x|^t} + f(x) \quad \text{in } \mathbb{R}^N, \\ u \in \dot{H}^s(\mathbb{R}^N). \end{cases} \tag{3.1}$$

Remark 3.1. If u is a weak solution of (3.1) and f is a nonnegative functional in $\dot{H}^s(\mathbb{R}^N)'$, then taking $v = u_-$ as a test function in (3.1), we obtain

$$-\|u_-\|^2_{\gamma} - \iint_{\mathbb{R}^{2N}} \frac{|u_+(y)u_-(x) + u_+(x)u_-(y)|}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y = {}_{(\dot{H}^s)'}\langle f, u_-\rangle_{\dot{H}^s} \geq 0,$$

which in turn implies that $u_- \equiv 0$, i.e., $u \geq 0$. Therefore, the maximum principle [12, Theorem 1.2] yields that u is a positive solution to (3.1). Hence u is a solution to $(E^{\gamma}_{K,t,f})$.

To establish the existence of two critical points for $I^{\gamma}_{K,t,f}$, we first need to prove some auxiliary results. Towards that, we partition $\dot{H}^s(\mathbb{R}^N)$ into three disjoint sets. Let $\psi_t : \dot{H}^s(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\psi_t(u) := \|u\|^2_{\gamma} - \big(2^\star_s(t) - 1\big)\|K\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N} \frac{|u|^{2^\star_s(t)}}{|x|^t}\,\mathrm{d}x$$

and set

$$\Sigma^t_1 := \big\{u \in \dot{H}^s(\mathbb{R}^N) \;:\; u = 0 \text{ or } \psi_t(u) > 0\big\}, \quad \Sigma^t_2 := \big\{u \in \dot{H}^s(\mathbb{R}^N) \;:\; \psi_t(u) < 0\big\},$$
$$\Sigma^t := \big\{u \in \dot{H}^s(\mathbb{R}^N) \;:\; \psi_t(u) = 0\big\}.$$

Remark 3.2. If $u \in \Sigma^t$, then

$$\|u\|^2_{\gamma} = \big(2^\star_s(t) - 1\big)\|K\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N} \frac{|u|^{2^\star_s(t)}}{|x|^t}\,\mathrm{d}x \leq \big(2^\star_s(t) - 1\big)\|K\|_{L^\infty(\mathbb{R}^N)} S_{\gamma,t,s}^{-\frac{2^\star_s(t)}{2}}\|u\|_{\gamma}^{2^\star_s(t)}.$$

Therefore, $\|u\|_{\gamma}$ and $\|u\|_{L^{2^\star_s(t)}(\mathbb{R}^N,|x|^{-t})}$ are bounded away from 0 for all $u \in \Sigma^t$.

Set

$$c^t_0 := \inf_{\Sigma^t_1} I^{\gamma}_{K,t,f}(u), \quad c^t_1 := \inf_{\Sigma^t} I^{\gamma}_{K,t,f}(u), \quad t \geq 0. \tag{3.2}$$

Remark 3.3. For any $\lambda > 0$ and $u \in \dot{H}^s(\mathbb{R}^N)$

$$\psi_t(\lambda u) = \lambda^2 \|u\|_\gamma^2 - \lambda^{2_s^\star(t)} \big(2_s^\star(t) - 1\big) \|K\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \frac{|u|^{2_s^\star(t)}}{|x|^t}\, \mathrm{d}x.$$

Moreover, $\psi_t(0) = 0$ and $\lambda \mapsto \psi_t(\lambda u)$ is a strictly concave function. Thus for any $u \in \dot{H}^s(\mathbb{R}^N)$ with $\|u\|_\gamma = 1$, there exists a unique $\lambda = \lambda(u)$ such that $\lambda u \in \Sigma^t$. Moreover, as $\psi_t(\lambda u) = \big(\lambda^2 - \lambda^{2_s^\star(t)}\big)\|u\|_\gamma^2$ for all $u \in \Sigma^t$, then $\lambda u \in \Sigma_1^t$ for all $\lambda \in (0,1)$ and $\lambda u \in \Sigma_2^t$ for all $\lambda > 1$.

Lemma 3.1. *Assume that C_t is defined as in Theorem 1.1. Then*

$$\frac{4s - 2t}{N - 2t + 2s} \|u\|_\gamma \geq C_t S_{\gamma,t,s}^{\frac{N-t}{4s-2t}} \quad \text{for all } \ u \in \Sigma^t, \ \ t \geq 0.$$

Proof. Fix $u \in \Sigma^t$. Then

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^\star(t)}}{|x|^t}\, \mathrm{d}x \right)^{\frac{1}{2_s^\star(t)}} = \frac{\|u\|_\gamma^{\frac{2}{2_s^\star(t)}}}{\Big(\big(2_s^\star(t) - 1\big) \|K\|_{L^\infty(\mathbb{R}^N)} \Big)^{\frac{1}{2_s^\star(t)}}}.$$

Combining this with the definition of $S_{\gamma,t,s}$ yields

$$\|u\|_\gamma \geq S_{\gamma,t,s}^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^\star(t)}}{|x|^t}\, \mathrm{d}x \right)^{\frac{1}{2_s^\star(t)}} = S_{\gamma,t,s}^{\frac{1}{2}} \frac{\|u\|_\gamma^{\frac{2}{2_s^\star(t)}}}{\Big(\big(2_s^\star(t) - 1\big) \|K\|_{L^\infty(\mathbb{R}^N)} \Big)^{\frac{1}{2_s^\star(t)}}}$$

for all $u \in \Sigma^t$. From here using the definition of C_t, we conclude the proof of the lemma. □

Lemma 3.2. *Assume that $t \geq 0$, C_t is given as in Theorem 1.1 and c_0^t, c_1^t are defined as in (3.2). Further if*

$$\inf_{\substack{u \in \dot{H}^s(\mathbb{R}^N) \\ \|u\|_{L^{2_s^\star(t)}(\mathbb{R}^N, |x|^{-t})} = 1}} \left\{ C_t \|u\|_\gamma^{\frac{N-2t+2s}{2s-t}} - {}_{(\dot{H}^s)'}\langle f, u \rangle_{\dot{H}^s} \right\} > 0, \tag{3.3}$$

then $c_0^t < c_1^t$.

Proof. Define

$$\tilde{J}_t(u) := \frac{1}{2} \|u\|_\gamma^2 - \frac{\|K\|_{L^\infty(\mathbb{R}^N)}}{2_s^\star(t)} \int_{\mathbb{R}^N} \frac{|u|^{2_s^\star(t)}}{|x|^t}\, \mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u \rangle_{\dot{H}^s}. \tag{3.4}$$

<u>**Step I:**</u> In this step we prove that there exists $\beta_t > 0$ such that

$$\frac{d}{dp} \tilde{J}_t(pu) \bigg|_{p=1} \geq \beta_t \quad \text{for all } \ u \in \Sigma^t.$$

Indeed, using the definition of Σ^t and the value of C_t, we have for $u \in \Sigma^t$

$$\begin{aligned} \frac{d}{dp} \tilde{J}_t(pu) \bigg|_{p=1} &= \|u\|_\gamma^2 - \|K\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \frac{|u|^{2_s^\star(t)}}{|x|^t}\, \mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u \rangle_{\dot{H}^s} \\ &= \left(1 - \frac{1}{2_s^\star(t) - 1} \right) \|u\|_\gamma^2 - {}_{(\dot{H}^s)'}\langle f, u \rangle_{\dot{H}^s} \\ &= \frac{4s - 2t}{N - 2t + 2s} \|u\|_\gamma^2 - {}_{(\dot{H}^s)'}\langle f, u \rangle_{\dot{H}^s} = C_t \frac{\|u\|_\gamma^{\frac{N+2s-2t}{2s-t}}}{\|u\|_{L^{2_s^\star(t)}(\mathbb{R}^N, |x|^{-t})}^{\frac{N-t}{2s-t}}} - {}_{(\dot{H}^s)'}\langle f, u \rangle_{\dot{H}^s}. \end{aligned} \tag{3.5}$$

Furthermore, (3.3) implies that there exists $d > 0$ such that

$$\inf_{\substack{u\in \dot{H}^s(\mathbb{R}^N)\\ \|u\|_{L^{2_s^\star(t)}(\mathbb{R}^N,|x|^{-t})}=1}} \left\{ C_t \|u\|_\gamma^{\frac{N-2t+2s}{2s-t}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \right\} \geq d. \tag{3.6}$$

Observe that,

$$\begin{aligned}(3.6) \quad &\iff \quad C_t \frac{\|u\|_\gamma^{\frac{N+2s-2t}{2s-t}}}{\|u\|_{L^{2_s^\star(t)}(\mathbb{R}^N,|x|^{-t})}^{\frac{N-t}{2s-t}}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \geq d, \quad \int_{\mathbb{R}^N} \frac{|u|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x = 1\\ &\iff \quad C_t \frac{\|u\|_\gamma^{\frac{N+2s-2t}{2s-t}}}{\|u\|_{L^{2_s^\star(t)}(\mathbb{R}^N,|x|^{-t})}^{\frac{N-t}{2s-t}}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \geq d\|u\|_{L^{2_s^\star(t)}(\mathbb{R}^N,|x|^{-t})}, \quad u \in \dot{H}^s(\mathbb{R}^N)\setminus\{0\}.\end{aligned}$$

Hence, plugging back the above estimate into (3.5) and using Remark 3.2, we complete the proof of Step I.

Step II: Let $(u_n^t)_n$ be a minimizing sequence for $I_{K,t,f}^\gamma$ on Σ^t, that is,

$$I_{K,t,f}^\gamma(u_n^t) \to c_1^t \quad \text{and} \quad \|u_n^t\|_\gamma^2 = \|K\|_{L^\infty(\mathbb{R}^N)}\big(2_s^\star(t)-1\big)\int_{\mathbb{R}^N}\frac{|u_n^t|^{2_s^\star(t)}}{|x|^t}\mathrm{d}x.$$

Therefore,

$$c_1^t + o(1) = I_{K,t,f}^\gamma(u_n) \geq \tilde{J}_t(u_n^t) \geq \left(\frac{1}{2} - \frac{1}{2_s^\star(t)\big(2_s^\star(t)-1\big)}\right)\|u_n^t\|_\gamma^2 - \|f\|_{(\dot{H}^s)'}\|u_n\|_\gamma.$$

This implies that $(\tilde{J}_t(u_n^t))_n$ is bounded and $(\|u_n^t\|_\gamma)_n$, $(\|u_n^t\|_{L^{2_s^\star(t)}(\mathbb{R}^N,|x|^{-t})})_n$ are bounded.

Claim: $c_0^t < 0$ for all $t \geq 0$.

To prove this claim, it is enough to show that there exists $v^t \in \Sigma_1^t$ such that $I_{K,t,f}^\gamma(v^t) < 0$. Note that, thanks to Remark 3.3, we can choose $u^t \in \Sigma^t$ such that ${}_{(\dot{H}^s)'}\langle f, u^t\rangle_{\dot{H}^s} > 0$.

Therefore,

$$I_{K,t,f}^\gamma(pu^t) = p^2\left[\frac{\big(2_s^\star(t)-1\big)\|K\|_{L^\infty(\mathbb{R}^N)}}{2} - \frac{p^{2_s^\star(t)-2}}{2_s^\star(t)}\right]\int_{\mathbb{R}^N}\frac{|u^t|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x - p\,{}_{(\dot{H}^s)'}\langle f, u^t\rangle_{\dot{H}^s} < 0$$

for $p << 1$. Moreover, $pu^t \in \Sigma_1^t$ by Remark 3.3. Hence the claim follows.

Thanks to the above claim, $I_{K,t,f}^\gamma(u_n^t) < 0$ for large n. Consequently,

$$0 > I_{K,t,f}^\gamma(u_n^t) \geq \left(\frac{1}{2} - \frac{1}{2_s^\star(t)\big(2_s^\star(t)-1\big)}\right)\|u_n^t\|_\gamma^2 - {}_{(\dot{H}^s)'}\langle f, u_n^t\rangle_{\dot{H}^s}$$

for large n. This in turn implies that ${}_{(\dot{H}^s)'}\langle f, u_n^t\rangle_{\dot{H}^s} > 0$ for n large enough. Hence, $\frac{d}{dp}\tilde{J}_t(pu_n^t) < 0$ for $p > 0$ small enough. Thus, by Step I there exists $p_n^t \in (0,1)$ such that $\frac{d}{dp}\tilde{J}_t(p_n^t u_n^t) = 0$.

Moreover, it is easy to check that for all $u^t \in \Sigma^t$, the map $p \mapsto \frac{d}{dp}\tilde{J}_t(pu^t)$ is strictly increasing in $[0,1)$ and therefore, we can conclude that p_n^t is unique.

Step III: In this step we show that

$$\liminf_{n\to\infty}\left\{\tilde{J}_t(u_n^t) - \tilde{J}_t(p_n^t u_n^t)\right\} > 0. \tag{3.7}$$

We observe that $\tilde{J}_t(u_n^t) - \tilde{J}_t(p_n^t u_n^t) = \int_{p_n^t}^1 \frac{d}{dp}\tilde{J}_t(pu_n)\mathrm{d}p$ and that for all $n \in \mathbb{N}$ there exists $\xi_n^t > 0$ such that $p_n^t \in (0, 1-2\xi_n^t)$ and $\frac{d}{dp}\tilde{J}_t(pu_n^t) \geq \frac{\beta_t}{2}$ for $p \in [1-\xi_n^t, 1]$.

To establish (3.7), it is enough to show that $\xi_n^t > 0$ can be chosen independently of $n \in \mathbb{N}$. This is possible, since $\frac{d}{dp}\tilde{J}_t(pu_n^t)\Big|_{p=1} \geq \beta_t$, and $(u_n^t)_n$ is bounded, so that for all n and $p \in [0,1]$

$$\begin{aligned}\left|\frac{d^2}{dp^2}\tilde{J}_t(pu_n^t)\right| &= \left|\|u_n^t\|_\gamma^2 - (2_s^\star(t)-1)\|K\|_{L^\infty(\mathbb{R}^N)}p^{2_s^\star(t)-2}\int_{\mathbb{R}^N}\frac{|u_n^t|^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x\right| \\ &= \left|\left(1-p^{2_s^\star(t)-2}\right)\|u_n^t\|_\gamma^2\right| \leq C.\end{aligned}$$

Step IV: From the definition of $I_{K,t,f}^\gamma$ and $\tilde{J}_t$, it immediately follows that $\frac{d}{dp}I_{K,t,f}^\gamma(pu) \geq \frac{d}{dp}\tilde{J}_t(pu)$ for all $u \in \dot{H}^s(\mathbb{R}^N)$ and for all $p > 0$. Hence,

$$I_{K,t,f}^\gamma(u_n^t) - I_{K,t,f}^\gamma(p_n^t u_n^t) = \int_{p_n^t}^1 \frac{d}{dp}I_{K,t,f}^\gamma(pu_n^t)\,\mathrm{d}p \geq \int_{p_n^t}^1 \frac{d}{dp}\tilde{J}_t(pu_n^t)\,\mathrm{d}p = \tilde{J}_t(u_n^t) - \tilde{J}_t(p_n^t u_n^t).$$

Since $(u_n^t)_n \subset \Sigma^t$ is a minimizing sequence for $I_{K,t,f}^\gamma$ on Σ^t and $p_n^t u_n^t \in \Sigma_1^t$, then by (3.7)

$$c_0^t = \inf_{\Sigma_1^t} I_{K,,t,f}^\gamma(u) < \inf_{\Sigma^t} I_{K,t,f}^\gamma(u) = c_1^t.$$

□

Proposition 3.1. *Assume that $t \geq 0$ and (3.3) holds. Then $I_{K,t,f}^\gamma$ has a critical point $u_t \in \Sigma_1^t$ with $I_{K,t,f}^\gamma(u_t) = c_0^t$. In particular, u_t is a positive solution to $(E_{K,t,f}^\gamma)$.*

Proof. We divide the proof in few steps.
Step 1: In this step we show that $c_0^t > -\infty$.

From the definition of $\tilde{J}_t$ in (3.4), we have $I_{K,t,f}^\gamma(u) > \tilde{J}_t(u)$. Therefore, in order to prove Step 1, it is enough to show that $\tilde{J}_t$ is bounded from below. From the definition of Σ_1^t,

$$\tilde{J}_t(u) \geq \left(\frac{1}{2} - \frac{1}{2_s^\star(t)(2_s^\star(t)-1)}\right)\|u\|_\gamma^2 - \|f\|_{(\dot{H}^s)'}\|u\|_\gamma \quad \text{for all } u \in \Sigma_1^t. \tag{3.8}$$

As the RHS is a quadratic function in $\|u\|_\gamma$, then $\tilde{J}_t$ is bounded from below and thus so is $I_{K,t,f}^\gamma$.

Step 2: In this step we show that there exists a bounded nonnegative (PS) sequence $(u_n^t)_n \subset \Sigma_1^t$ for $I_{K,t,f}^\gamma$ at the level c_0^t. Let $(u_n^t)_n \subset \bar{\Sigma}_1^t$ such that $I_{K,t,f}^\gamma(u_n^t) \to c_0^t$. Since Lemma 3.2 implies $c_0^t < c_1^t$, without any restriction we can assume that $(u_n)_n \subset \Sigma_1^t$. Further, using Ekeland's variational principle, $(u_n^t)_n$ admits a (PS) subsequence, still called $(u_n^t)_n$, in Σ_1^t for $I_{K,t,f}^\gamma$ at the level c_0^t. Moreover, as $I_{K,t,f}^\gamma(u) \geq \tilde{J}_t(u)$, from (3.8) it follows that $(u_n^t)_n$ is a bounded sequence in $\dot{H}^s(\mathbb{R}^N)$. Therefore, up to a subsequence, $u_n^t \rightharpoonup u_t$ in $\dot{H}^s(\mathbb{R}^N)$ and $u_n^t \to u_t$ a.e. in $\mathbb{R}^N$. In particular, $(u_n^t)_+ \to (u_t)_+$ and $(u_n^t)_- \to (u_t)_-$ a.e. in $\mathbb{R}^N$. Moreover, the fact that f is a nonnegative functional gives as $n \to \infty$

$$\begin{aligned} o(1) &= {}_{(\dot{H}^s)'}\langle (I_{K,t,f}^\gamma)'(u_n^t), (u_n^t)_-\rangle_{\dot{H}^s} \\ &= \langle u_n^t, (u_n^t)_-\rangle_\gamma - \int_{\mathbb{R}^N}\frac{K(x)(u_n^t)_+^{2_s^\star(t)-1}(u_n^t)_-}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, (u_n^t)_-\rangle_{\dot{H}^s} \\ &\leq -\|(u_n^t)_-\|_\gamma^2 - \iint_{\mathbb{R}^{2N}}\frac{(u_n^t)_-(x)(u_n^t)_+(y) + (u_n^t)_+(x)(u_n^t)_-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\ &\leq -\|(u_n^t)_-\|_\gamma^2. \end{aligned}$$

This implies that $(u_n)_- \to 0$ in $\dot{H}^s(\mathbb{R}^N)$ and so $(u_n)_- \to 0$ in a.e. in $\mathbb{R}^N$, which in turn yields that $(u_0)_- \equiv 0$, that is, $u_0 \geq 0$ a.e. in $\mathbb{R}^N$. Consequently, without loss of generality, we can assume $(u_n^t)_n$ is a nonnegative (PS) sequence. This completes the proof of Step 2.

Step 3: In this step we show that $u_n^t \to u_t$ in $\dot{H}^s(\mathbb{R}^N)$.

Applying Theorem 2.1, we get as $n \to \infty$

$$u_n^t = u_t + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,\, y_n^j} + \sum_{k=1}^{n_2} (W_{\gamma,t}^k)^{R_n^k,0} + o(1), \quad \text{if } t = 0, \tag{3.9}$$

and

$$u_n^t = u_t + \sum_{k=1}^{n_2} (W_{\gamma,t}^k)^{R_n^k,0} + o(1), \quad \text{if } t > 0, \tag{3.10}$$

where $(I_{K,t,f}^\gamma)'(u_t) = 0$, W is the unique positive solution of $(E_{1,0,0}^0)$, where $W_{\gamma,t}^k$, $k = 1, 2, \cdots n_2$ are positive ground state solutions of $(E_{1,t,0}^\gamma)$. Moreover, $(y_n^j)_n$, $(r_n^j)_n$ and $(R_n^k)_n$ are some appropriate sequences with $R_n^k \to 0$ for each $k = 1, \cdots n_2$, $r_n^j \to 0$, $\frac{r_n^j}{y_n^j} \to 0$ and either $y_n^j \to y^j$ or $|y_n^j| \to \infty$ and for all $j = 1, \cdots n_1$, are appropriate sequences. To prove Step 3, we need to show that $n_1 = 0 = n_2$. We prove this by the method of contradiction.

Suppose $t = 0$. The case $t > 0$ is comparatively easier and the proof of that case will easily follow from arguments that we present in the case of $t = 0$. Also for $t > 0$, one can argue as in [7, Proposition 3.1].

Thus, let us assume that $t = 0$ and $u_n^t \not\to u_t$ in $\dot{H}^s(\mathbb{R}^N)$. For simplicity of notations, we denote u_n^0 by u_n.

Then either $n_1 \neq 0$ or $n_2 \neq 0$ or both $n_1,\ n_2 \neq 0$ in (3.9). Here we prove the last case that is when n_1 and n_2 both are non zero. If one of them is zero, that case is again comparatively easier and argument in that case will follow from this case. First we observe that

$$\begin{aligned}\psi_0\left(K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j}\right) &= K(y^j)^{-\frac{N-2s}{2s}} \|W\|_\gamma^2 - (2_s^\star - 1)\|K\|_{L^\infty(\mathbb{R}^N)} K(y^j)^{-\frac{N}{2s}} \|W\|_{L^{2_s^\star}(\mathbb{R}^N)}^{2_s^\star} \\ &= K(y^j)^{-\frac{N}{2s}} \left(K(y^j) - (2_s^\star - 1)\|K\|_{L^\infty(\mathbb{R}^N)}\right) \|W\|_{L^{2_s^\star}(\mathbb{R}^N)}^{2_s^\star} \\ &\quad - \gamma K(y^j)^{-\frac{N-2s}{2s}} \int_{\mathbb{R}^N} \frac{|W|^2}{|x|^{2s}} dx < 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\psi_0\big((W_{\gamma,0}^k)^{R_n^k,0}\big) = \psi_0(W_{\gamma,0}^k) &= \|W_{\gamma,0}^k\|_\gamma^2 - (2_s^\star - 1)\|K\|_{L^\infty(\mathbb{R}^N)} \|W_{\gamma,0}^k\|_{L^{2_s^\star}(\mathbb{R}^N)}^{2_s^\star} \\ &= \big(1 - (2_s^\star - 1)\|K\|_{L^\infty(\mathbb{R}^N)}\big) \|W_{\gamma,0}^k\|_\gamma^2 < 0.\end{aligned}$$

Theorem 2.1 gives

$$o(1) + c_0^0 = I_{K,0,f}^\gamma(u_n) \to I_{K,0,f}^\gamma(u_0) + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{2s}} I_{1,0,0}^0(W) + \sum_{k=1}^{n_2} I_{1,0,0}^\gamma(W_{\gamma,0}^k).$$

As $K > 0$, $I_{1,0,0}^0(W) = \frac{s}{N}\|W\|_{\dot{H}^s}^2 > 0$ and $I_{1,0,0}^\gamma(W_{\gamma,0}) = \frac{s}{N}\|W_{\gamma,0}\|_\gamma^2 > 0$, from the above expression we obtain $I_{K,0,f}^\gamma(u_0) < c_0^0$. This in turn yields $u_0 \notin \Sigma_1^0$ and

$$\psi_0(u_0) \le 0. \tag{3.11}$$

Next, we evaluate $\psi_0\left(u_0 + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,\, y_n^j} + \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}\right)$. Since $u_n \in \Sigma_1^0$, we have $\psi_0(u_n) \ge 0$. Therefore, the uniform continuity of ψ_0 and (3.9) imply

$$0 \le \liminf_{n\to\infty} \psi_0(u_n) = \liminf_{n\to\infty} \psi_0\left(u_0 + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j} + \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}\right). \tag{3.12}$$

Since $u_0,\ W,\ W_{\gamma,0}^k \ge 0$ for all $k = 1, \cdots, n_1$,

$$\psi_0\left(u_0 + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j} + \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}\right)$$

$$\begin{aligned}
&\leq \|u_0\|_\gamma^2 + \sum_{k=1}^{n_2} \|W_{\gamma,0}^k\|_\gamma^2 + \sum_{j=1}^{n_1} \|K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j}\|_\gamma^2 + 2\Big\langle u_0, \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}\Big\rangle_\gamma \\
&\quad + 2\Big\langle u_0, \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j}\Big\rangle_\gamma + 2\Big\langle \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}, \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j}\Big\rangle_\gamma \\
&\quad + \sum_{i,k=1,\, i\neq k}^{n_2} \big\langle (W_{\gamma,0}^i)^{R_n^i,0}, (W_{\gamma,0}^k)^{R_n^k,0}\big\rangle_\gamma + \sum_{l,j=1,\, l\neq j}^{n_1} \big\langle K(y^l)^{-\frac{N-2s}{4s}} W^{r_n^l,y_n^l}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j}\big\rangle_\gamma \\
&\quad - (2_s^\star - 1)\|K\|_{L^\infty(\mathbb{R}^N)} \bigg(\|u_0\|_{2_s^\star}^{2_s^\star} + \sum_{j=1}^{n_1} \|K(y^j)^{-\frac{N-2s}{4s}} (W^{r_n^j,y_n^j})\|_{2_s^\star}^{2_s^\star} + \sum_{k=1}^{n_2} \|W_{\gamma,0}^k\|_{2_s^\star}^{2_s^\star} \bigg) \\
&\leq \psi_0(u_0) + \sum_{k=1}^{n_2} \psi_0(W_{\gamma,0}^k) + \sum_{j=1}^{n_1} \psi_0\big(K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j}\big) + \text{the above inner products.}
\end{aligned} \tag{3.13}$$

We now prove that all the five inner products in the RHS of (3.13) approaches 0 as $n \to \infty$. As $r_n^j \to 0$ and $\frac{|y_n^j|}{|r_n^j|} \to \infty$, it follows that $W^{r_n^j,y_n^j} \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ (see [21, Lemma 3]) and $W^{r_n^j,y_n^j} \to 0$ a.e. in $\mathbb{R}^N$. Choosing $R > 0$ large enough as $n \to \infty$

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{u_0 W^{r_n^j,y_n^j}}{|x|^{2s}} \mathrm{d}x &\leq \int_{B_R} \frac{u_0 W^{r_n^j,y_n^j}}{|x|^{2s}} \mathrm{d}x + \int_{|x|>R} \frac{u_0 W^{r_n^j,y_n^j}}{|x|^{2s}} \mathrm{d}x \\
&\leq \int_{B_R} \frac{u_0 W^{r_n^j,y_n^j}}{|x|^{2s}} \mathrm{d}x + \int_{|x|>R} \bigg(\frac{|u_0|^2}{|x|^{2s}} \mathrm{d}x\bigg)^{\frac12} \int_{|x|>R} \bigg(\frac{|W|^2}{|x+\frac{y_n^j}{r_n^j}|^{2s}} \mathrm{d}x\bigg)^{\frac12} \\
&= o(1),
\end{aligned}$$

where in the 1st integral we have passed the limit using Vitali's convergence theorem via the Hölder inequality, while in the 2nd integral simply using the Hardy inequality. Therefore, as $n \to \infty$

$$\big\langle u_0,\ K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j}\big\rangle_\gamma = K(y^j)^{-\frac{N-2s}{4s}} \bigg[\big\langle u_0,\ W^{r_n^j,y_n^j}\big\rangle_{\dot{H}^s} - \gamma \int_{\mathbb{R}^N} \frac{u_0 W^{r_n^j,y_n^j}}{|x|^{2s}} \mathrm{d}x\bigg] = o(1). \tag{3.14}$$

Since $R_n^k \to 0$ as $n \to \infty$, similarly we also have

$$\Big\langle u_0, \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}\Big\rangle_\gamma = o(1). \tag{3.15}$$

Now,

$$\begin{aligned}
&\Big\langle K(y^l)^{-\frac{N-2s}{4s}} \big(W^{r_n^l,y_n^l}\big), K(y^j)^{-\frac{N-2s}{4s}} \big(W^{r_n^j,y_n^j}\big)\Big\rangle_\gamma \\
&= K(y^l)^{-\frac{N-2s}{4s}} K(y^j)^{-\frac{N-2s}{4s}} (r_n^l)^{-\frac{N-2s}{2}} (r_n^j)^{-\frac{N-2s}{2}} \\
&\quad \times \bigg[\iint_{\mathbb{R}^{2N}} \frac{\big(W(\frac{x-y_n^l}{r_n^l}) - W(\frac{y-y_n^l}{r_n^l})\big)\big(W(\frac{x-y_n^j}{r_n^j}) - W(\frac{y-y_n^j}{r_n^j})\big)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\qquad - \gamma \int_{\mathbb{R}^N} \frac{W(\frac{x-y_n^l}{r_n^l}) W(\frac{x-y_n^j}{r_n^j})}{|x|^{2s}} \mathrm{d}x \bigg] \\
&= K(y^l)^{-\frac{N-2s}{4s}} K(y^j)^{-\frac{N-2s}{4s}} (r_n^l)^{\frac{N-2s}{2}} (r_n^j)^{-\frac{N-2s}{2}} \\
&\quad \times \bigg[\iint_{\mathbb{R}^{2N}} \frac{\big(W(x) - W(y)\big)\big(W(\frac{r_n^l x + y_n^l - y_n^j}{r_n^j}) - W(\frac{r_n^l y + y_n^l - y_n^j}{r_n^j})\big)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\qquad - \gamma \int_{\mathbb{R}^N} \frac{W(x) W(\frac{r_n^l y + y_n^l - y_n^j}{r_n^j})}{|x + \frac{y_n^l}{r_n^l}|^{2s}} \mathrm{d}x \bigg]
\end{aligned}$$

$$= K(y^l)^{-\frac{N-2s}{4s}} K(y^j)^{-\frac{N-2s}{4s}} \left[\langle W, W_n \rangle_{\dot{H}^s(\mathbb{R}^N)} - \gamma \int_{\mathbb{R}^N} \frac{W W_n}{|x + \frac{y_n^l}{r_n^l}|^{2s}} \mathrm{d}x \right],$$

where $W_n := (\frac{r_n^l}{r_n^j})^{\frac{N-2s}{2}} W(\frac{r_n^l}{r_n^j} x + \frac{y_n^l - y_n^j}{r_n^j})$. Theorem 2.1 (vi) yields

$$\left| \log \left(\frac{r_n^l}{r_n^j}\right) \right| + \left| \frac{y_n^l - y_n^j}{r_n^j} \right| \longrightarrow \infty.$$

Thus $W_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ (see [21, Lemma 3]). Hence, as $n \to \infty$

$$\left\langle K(y^l)^{-\frac{N-2s}{4s}} \left(W^{r_n^l, y_n^l}\right), K(y^j)^{-\frac{N-2s}{4s}} \left(W^{r_n^j, y_n^j}\right) \right\rangle_\gamma = o(1). \tag{3.16}$$

Similarly,

$$\langle (W_{\gamma,0}^i)^{R_n^i,0}, (W_{\gamma,0}^k)^{R_n^k,0} \rangle_\gamma = o(1) \tag{3.17}$$

as $|\log \frac{R_n^j}{R_n^k}| \to \infty$.

Finally, we estimate $\langle (W_{\gamma,0}^k)^{R_n^k,0}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j} \rangle_\gamma$. First we note that $|\log \frac{R_n^j}{R_n^k}| \to \infty$ implies that either $\frac{R_n^j}{R_n^k} \to 0$ or $\frac{R_n^j}{R_n^k} \to \infty$. Suppose $\frac{R_n^j}{R_n^k} \to 0$. Then

$$\begin{aligned}
\langle (W_{\gamma,0}^k)^{R_n^k,0}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j} \rangle_\gamma &= K(y^j)^{-\frac{N-2s}{4s}} (R_n^k)^{\frac{N-2s}{2}} (r_n^j)^{-\frac{N-2s}{2}} \cdot \\
&\quad \times \left[\iint_{\mathbb{R}^{2N}} \frac{(W_{\gamma,0}^k(x) - W_{\gamma,0}^k(y))\left(W(\frac{R_n^k x - y_n^j}{r_n^j}) - W(\frac{R_n^k y - y_n^j}{r_n^j})\right)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right. \\
&\qquad \left. - \gamma \int_{\mathbb{R}^N} \frac{W_{\gamma,0}^k(x) W(\frac{R_n^k x - y_n^j}{r_n^j})}{|x|^{2s}} \mathrm{d}x \right] \\
&= K(y^j)^{-\frac{N-2s}{4s}} \left[\langle W_{\gamma,0}^k, W^n \rangle_{\dot{H}^s(\mathbb{R}^N)} - \gamma \int_{\mathbb{R}^N} \frac{W_{\gamma,0}^k W^n}{|x|^{2s}} \mathrm{d}x \right],
\end{aligned}$$

where $W^n := (\frac{r_n^j}{R_n^k})^{-\frac{N-2s}{2}} W\left(\frac{x - \frac{y_n^j}{R_n^k}}{r_n^j / R_n^k}\right)$. The proof of Theorem 2.1 gives $\frac{r_n^j}{R_n^k} = \frac{s_n^j R_n^j}{R_n^k}$ for any j and k. As $s_n^j \to 0$ and $\frac{R_n^j}{R_n^k} \to 0$, we have $\frac{r_n^j}{R_n^k} \to 0$. Moreover, $\frac{|y_n^j|}{r_n^j} \to \infty$ implies that $\frac{|y_n^j / R_n^k|}{r_n^j / R_n^k} \to \infty$. Thus $|\log \frac{r_n^j}{R_n^k}| + |y_n^j / R_n^k| \to \infty$. Consequently by [21, Lemma 3], $W^n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$. Hence, an argument similar to (3.14) yields

$$\langle (W_{\gamma,0}^k)^{R_n^k,0}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j} \rangle_\gamma = o(1). \tag{3.18}$$

On the other hand, if $\frac{R_n^j}{R_n^k} \to \infty$ then $\frac{R_n^k}{R_n^j} \to 0$. Then similarly, we also show that

$$\langle (W_{\gamma,0}^k)^{R_n^k,0}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j} \rangle_{\dot{H}^s(\mathbb{R}^N)} = K(y^j)^{-\frac{N-2s}{4s}} \langle W_\gamma^n, W \rangle,$$

where $W_\gamma^n(x) = \left(\frac{R_n^k}{R_n^j}\right)^{-\frac{N-2s}{2s}} W_{\gamma,0}^k \left(\frac{x - \frac{y_n^j}{r_n^j}}{R_n^k / r_n^j}\right)$. Since $\frac{R_n^k}{R_n^j} \to 0$ and $\frac{|y_n^j|}{r_n^j} \to \infty$, again applying [21, Lemma 3], we get $W_\gamma^n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$. Hence, in any case (3.18) holds.

Combining (3.14)–(3.18) along with (3.13), we have

$$\psi_0 \left(u_0 + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,y_n^j} + \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0} \right) < 0.$$

This contradicts (3.12). Therefore, $n_1 = 0$ and $n_2 = 0$ in (3.9). Hence, $u_n \to u_0$ in $\dot{H}^s(\mathbb{R}^N)$. Consequently, $\psi_0(u_n) \to \psi_0(u_0)$, which in turn implies that $u_0 \in \bar{\Sigma}_1^t$. But, since $c_0^0 < c_1^0$, we conclude $u_0 \in \Sigma_1^t$. Hence Step 3 follows. □

Proposition 3.2. *Assume that $t \geq 0$ and (3.3) holds. Then $I^{\gamma}_{K,t,f}$ has a second critical point $v_t \neq u_t$. In particular, v_t solves $(E^{\gamma}_{K,t,f})$.*

Proof. Let $t \geq 0$ and let u_t be the critical point of $I^{\gamma}_{K,t,f}$ obtained in Proposition 3.1. Let $W_{\gamma,t}$ be a positive radial ground state solution of $(E^{\gamma}_{1,t,0})$. Set $w^{\tau}_{\gamma,t}(x) := W_{\gamma,t}(\frac{x}{\tau})$. Let $\bar{x}_0 \in \mathbb{R}^N$ be such that $K(\bar{x}_0) = \|K\|_{L^\infty(\mathbb{R}^N)}$.

Claim 1: $u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t} \in \Sigma^t_2$ for $\tau > 0$ large enough.

Indeed, as $\|K\|_{L^\infty(\mathbb{R}^N)} \geq 1$, $0 \leq t < 2s$ and $u_t,\ w^{\tau}_{\gamma,t} > 0$, using Cauchy's inequality, with $\varepsilon > 0$, we have

$$\begin{aligned}
\psi_t\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) &= \|u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\|^2_{\gamma} \\
&\quad - \big(2^*_s(t) - 1\big) K(\bar{x}_0) \int_{\mathbb{R}^N} \frac{|u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x \\
&\leq \|u_t\|^2_{\gamma} + K(\bar{x}_0)^{-\frac{N-2s}{2s}} \|w^{\tau}_{\gamma,t}\|^2_{\gamma} + 2K(\bar{x}_0)^{-\frac{N-2s}{4s}} \langle u_t, w^{\tau}_{\gamma,t}\rangle_{\gamma} \\
&\quad - \big(2^*_s(t) - 1\big)\bigg\{ \int_{\mathbb{R}^N} \frac{|u_t|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x + K(\bar{x}_0)^{-\frac{N-t}{2s}} \int_{\mathbb{R}^N} \frac{|w^{\tau}_{\gamma,t}|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x \bigg\} \\
&\leq \|u_t\|^2_{\gamma} + K(\bar{x}_0)^{-\frac{N-2s}{2s}} \|w^{\tau}_{\gamma,t}\|^2_{\gamma} + 2K(\bar{x}_0)^{-\frac{N-2s}{4s}} \Big(\frac{\varepsilon}{2}\|w^{\tau}_{\gamma,t}\|^2_{\gamma} + \frac{1}{2\varepsilon}\|u_t\|^2_{\gamma}\Big) \\
&\quad - \big(2^*_s(t) - 1\big)\bigg\{ \int_{\mathbb{R}^N} \frac{|u_t|^{2^*_s(t)}}{|x|^t}\mathrm{d}x + K(\bar{x}_0)^{-\frac{N-t}{2s}} \tau^{N-t} \int_{\mathbb{R}^N} \frac{|W_{\gamma,t}|^{2^*_s(t)}}{|x|^t}\mathrm{d}x \bigg\} \\
&= \Big(1 + \frac{1}{\varepsilon}\Big)\|u_t\|^2_{\dot{H}^s(\mathbb{R}^N)} - (2^*_s(t) - 1)\|u_t\|^{2^*_s(t)}_{L^{2^*_s(t)}(\mathbb{R}^N, |x|^{-t})} \\
&\quad + \|W_{\gamma,t}\|^2_{\gamma}\Big[(1+\varepsilon)\tau^{N-2s} - \big(2^*_s(t) - 1\big) K(\bar{x}_0)^{-\frac{N-t}{2s}} \tau^{N-t}\Big] \\
&< 0 \qquad \text{for } \tau > 0 \text{ large enough.}
\end{aligned}$$

Therefore, $u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t} \in \Sigma^t_2$ for $\tau > 0$ large enough. Hence, Claim 1 follows.

Claim 2: $I^{\gamma}_{K,t,f}\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) < I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big)$ for all $\tau > 0$.

Indeed, as $u_t,\ w^{\tau}_{\gamma,t} > 0$ taking $K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}$ as the test function in $(E^{\gamma}_{K,t,f})$, we get

$$\begin{aligned}
\langle u_t, K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\rangle_{\gamma} = K(\bar{x}_0)^{-\frac{N-2s}{4s}} &\int_{\mathbb{R}^N} K(x) \frac{u_t^{2^*_s(t)-1} w^{\tau}_{\gamma,t}}{|x|^t}\mathrm{d}x \\
&+ {}_{(\dot{H}^s)'}\langle f, K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\rangle_{\dot{H}^s}.
\end{aligned} \tag{3.19}$$

Therefore, using the above equality together with the fact that $K \geq 1$ yields

$$\begin{aligned}
I^{\gamma}_{K,t,f}\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) &= \frac{1}{2}\|u_t\|^2_{\gamma} + \frac{1}{2}K(\bar{x}_0)^{-\frac{N-2s}{2s}} \|w^{\tau}_{\gamma,t}\|^2_{\gamma} + K(\bar{x}_0)^{-\frac{N-2s}{4s}} \langle u_t, w^{\tau}_{\gamma,t}\rangle_{\gamma} \\
&\quad - \frac{1}{2^*_s(t)} \int_{\mathbb{R}^N} K(x) \frac{\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big)^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x \\
&\quad - {}_{(\dot{H}^s)'}\langle f, u_t\rangle_{\dot{H}^s} - K(\bar{x}_0)^{-\frac{N-2s}{4s}} {}_{(\dot{H}^s)'}\langle f, w^{\tau}_{\gamma,t}\rangle_{\dot{H}^s} \\
&= I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) + K(\bar{x}_0)^{-\frac{N-2s}{4s}} \langle u_t, w^{\tau}_{\gamma,t}\rangle_{\gamma} \\
&\quad + \frac{1}{2^*_s(t)} \int_{\mathbb{R}^N} K(x) \frac{u_t^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x + \frac{K(\bar{x}_0)^{-\frac{N-t}{2s}}}{2^*_s(t)} \int_{\mathbb{R}^N} \frac{(w^{\tau}_{\gamma,t})^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x \\
&\quad - \frac{1}{2^*_s(t)} \int_{\mathbb{R}^N} K(x) \frac{\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big)^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x \\
&\quad - K(\bar{x}_0)^{-\frac{N-2s}{4s}} {}_{(\dot{H}^s)'}\langle f, w^{\tau}_{\gamma,t}\rangle_{\dot{H}^s} \\
&\leq I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ K(\bar{x}_0)^{-\frac{N-2s}{4s}} \int_{\mathbb{R}^N} K(x) \frac{u_t^{2_s^\star(t)-1} w_{\gamma,t}^\tau}{|x|^t}\,\mathrm{d}x + \frac{1}{2_s^\star(t)} \int_{\mathbb{R}^N} K(x) \frac{u_t^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x \\
&+ \frac{K(\bar{x}_0)^{-\frac{N-t}{2s}}}{2_s^\star(t)} \int_{\mathbb{R}^N} \frac{(w_{\gamma,t}^\tau)^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x \\
&- \frac{1}{2_s^\star(t)} \int_{\mathbb{R}^N} K(x) \frac{\left(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right)^{2_s^\star(t)}}{|x|^t}\,\mathrm{d}x \\
\le\ & I_{K,t,f}^\gamma(u_t) + I_{1,t,0}^\gamma\left(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right) \\
&+ \frac{1}{2_s^\star(t)} \int_{\mathbb{R}^N} K(x) \left[2_s^\star(t) K(\bar{x}_0)^{-\frac{N-2s}{4s}} \frac{u_t^{2_s^\star(t)-1} w_{\gamma,t}^\tau}{|x|^t} \right. \\
&\left. + \frac{u_t^{2_s^\star(t)}}{|x|^t} + K(\bar{x}_0)^{-\frac{N-t}{2s}} \frac{(w_{\gamma,t}^\tau)^{2_s^\star(t)}}{|x|^t} - \frac{\left(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right)^{2_s^\star(t)}}{|x|^t} \right] \mathrm{d}x \\
<\ & I_{K,t,f}^\gamma(u_t) + I_{1,t,0}^\gamma\left(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right).
\end{aligned}
$$

Hence the claim follows. As

$$
\|w_{\gamma,t}^\tau\|_\gamma^2 = \tau^{N-2s} \|W_{\gamma,t}\|_\gamma^2, \quad \|w_{\gamma,t}^\tau\|_{L^{2_s^\star(t)}(\mathbb{R}^N,|x|^{-t})}^{2_s^\star(t)} = \tau^N \|W_{\gamma,t}\|_\gamma^2,
$$

and $0 \le t < 2s < N$, it is easy to see using the definition of $I_{1,t,0}^\gamma\left(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right)$ that

$$
\lim_{\tau\to\infty} I_{1,t,0}^\gamma\left(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right) = -\infty \tag{3.20}
$$

Consequently, a straight forward computation yields that

$$
\sup_{\tau>0} I_{1,t,0}^\gamma\left(K(x_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right) = I_{1,t,0}^\gamma\left(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^{\tau_{\max}}\right), \quad \text{where } \tau_{\max} = K(\bar{x}_0)^{\frac{1}{2s}}.
$$

Therefore, substituting the value of $\tau_{\max}$ in the definition of $I_{1,t,0}^\gamma$, it is not difficult to check that

$$
\sup_{\tau>0} I_{1,t,0}^\gamma\left(K(x_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right) = I_{1,t,0}^\gamma(W_{\gamma,t}).
$$

Combining the above relation with Claim 2 and (3.20), we obtain

$$
I_{K,t,f}^\gamma\left(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right) < I_{K,t,f}^\gamma(u_t) + I_{1,t,0}^\gamma(W_{\gamma,t}) \quad \text{for all } \tau > 0, \tag{3.21}
$$

$$
I_{K,t,f}^\gamma\left(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau\right) < I_{K,t,f}^\gamma(u_t) \quad \text{for } \tau \text{ large enough.} \tag{3.22}
$$

Now, fix $\tau_0 > 0$ large enough such that Claim 1 and (3.22) are satisfied. Set

$$
\kappa_t := \inf_{\theta\in\Theta_t} \max_{r\in[0,1]} I_{K,t,f}^\gamma\left(\theta(r)\right),
$$

where $\Theta_t := \left\{\theta \in C\left([0,1], \dot{H}^s(\mathbb{R}^N)\right) : \theta(0) = u_t,\ \theta(1) = u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^{\tau_0}\right\}$. As $u_t \in \Sigma_1^t$, $u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^{\tau_0} \in \Sigma_2^t$ for every $\theta \in \Theta_t$, there exists $r_\theta \in (0,1)$ such that $\theta(r_\theta) \in \Sigma^t$. Thus

$$
\max_{r\in[0,1]} I_{K,t,f}^\gamma(\theta(r)) \ge I_{K,t,f}^\gamma(\theta(r_\theta)) \ge \inf_{u\in\Sigma^t} I_{K,t,f}^\gamma(u) = c_1^t.
$$

Hence,

$$
\kappa_t \ge c_1^t > c_0^t = I_{K,t,f}^\gamma(u_t).
$$

Here in the last inequality we have used Lemma 3.2.

Claim 3: $\kappa_t < I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}(W_{\gamma,t})$.

Note that $\lim_{\tau\to 0} \|w^{\tau}_{\gamma,t}\|_{\gamma} = 0$, thus if we define $\bar{\theta}(r) := u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{r\tau_0}_{\gamma,t}$, then $\bar{\theta} \in \Theta_t$ and $\lim_{r\to 0} \|\bar{\theta}(r) - u_t\|_{\gamma} = 0$. Therefore by (3.21),

$$\kappa_t \le \max_{r\in[0,1]} I^{\gamma}_{K,t,f}\left(\bar{\theta}(r)\right) = \max_{r\in[0,1]} I^{\gamma}_{K,t,f}\left(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{r\tau_0}_{\gamma,t}\right) < I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}(W_{\gamma,t}),$$

that is,

$$I^{\gamma}_{K,t,f}(u_t) < \kappa_t < I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}(W_{\gamma,t}) \quad \text{for all } t \ge 0. \tag{3.23}$$

Using Ekeland's variational principle, there exists a (PS) sequence $(v^t_n)_n$ of $I^{\gamma}_{K,t,f}$ at level κ_t for all $t \ge 0$. Since any (PS) for $I^{\gamma}_{K,t,f}$ is bounded and $\kappa_t < I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}(W_{\gamma,t})$, using Theorem 2.1, in the case of $t > 0$, there exists $v_t \in \dot{H}^s(\mathbb{R}^N)$ such that $v^t_n \to v_t$ in $\dot{H}^s(\mathbb{R}^N)$, with $I^{\gamma}_{K,t,f}(v_t) = \kappa_t$ and $(I^{\gamma}_{K,t,f})'(v_t) = 0$. Moreover, $I^{\gamma}_{K,t,f}(u_t) < \kappa_t$ implies that $u_t \ne v_t$. Hence we have proved the proposition for $t > 0$.

Next let us assume that $t = 0$ so that we are in case (ii) of Theorem 1.1 and so

$$K(\bar{x}_0) = \|K\|_{L^\infty(\mathbb{R}^N)} < \left(\frac{S}{S_{\gamma,0,s}}\right)^{\frac{N}{N-2s}} \tag{3.24}$$

holds by assumption. Let W denote the unique positive solution of $(E^0_{1,0,0})$. As $W_{\gamma,0}$ is a minimum energy positive solution (ground state solution) of $(E^{\gamma}_{1,t,0})$, with $t = 0$, it follows that

$$I^{\gamma}_{1,0,0}(W_{\gamma,0}) \le I^{\gamma}_{1,0,0}(W) < I^0_{1,0,0}(W),$$

where the last inequality is due to the fact that $W > 0$ and so $\int_{\mathbb{R}^N} \frac{|W|^2}{|x|^{2s}} dx > 0$. Since S and $S_{\gamma,0,s}$ are achieved by W and $W_{\gamma,0}$ respectively, it is easy to see that $\|W\|^2_{\dot{H}^s(\mathbb{R}^N)} = S^{\frac{N}{2s}}$ and $\|W_{\gamma,0}\|^2_{\gamma} = S^{\frac{N}{2s}}_{\gamma,0,s}$. On the other hand, as $I^0_{1,0,0}(W) = \frac{s}{N}\|W\|^2_{\dot{H}^s}$ and $I^{\gamma}_{1,0,0}(W_{\gamma,0}) = \frac{s}{N}\|W_{\gamma,0}\|^2_{\gamma}$, we obtain $\frac{I^{\gamma}_{1,0,0}(W_{\gamma,0})}{I^0_{1,0,0}(W)} = \left(\frac{S_{\gamma,0,s}}{S}\right)^{\frac{N}{2s}}$. This together with (3.24) yields

$$I^{\gamma}_{1,0,0}(W_{\gamma,0}) < K(\bar{x}_0)^{-\frac{N-2s}{2s}} I^0_{1,0,0}(W) \le K(x)^{-\frac{N-2s}{2s}} I^0_{1,0,0}(W) \quad \text{for all } x \in \mathbb{R}^N.$$

Combining the above inequality with (3.23) yields

$$I^{\gamma}_{K,0,f}(u_0) < \kappa_0 < \min\left\{I^{\gamma}_{K,0,f}(u_0) + K(x)^{-\frac{N-2s}{2s}} I^0_{1,0,0}(W), \quad I^{\gamma}_{K,0,f}(u_0) + I^{\gamma}_{1,0,0}(W_{\gamma,0})\right\}.$$

Hence, again using Theorem 2.1 (as in the case $t > 0$), we can conclude that the (PS) sequence $(v^0_n)_n$ converges strongly to some $v_0 \in \dot{H}^s(\mathbb{R}^N)$, with $I^{\gamma}_{K,0,f}(v_0) = \kappa_0$ and $(I^{\gamma}_{K,0,f})'(v_0) = 0$. As before, $I^{\gamma}_{K,0,f}(u_0) < \kappa_0$ implies that $u_0 \ne v_0$. Hence we have completed the proof for all $t \ge 0$. □

Lemma 3.3. *If* $\|f\|_{(\dot{H}^s)'} < C_t\sqrt{1 - \frac{\gamma}{\gamma_{N,s}}}\, S^{\frac{N-t}{4s-2t}}_{\gamma,t,s}$, *then* (3.3) *holds.*

Proof. By the given assumption, there exists $\varepsilon > 0$ such that

$$\|f\|_{(\dot{H}^s)'} < C_t\sqrt{1 - \frac{\gamma}{\gamma_{N,s}}}\, S^{\frac{N-t}{4s-2t}}_{\gamma,t,s} - \varepsilon.$$

Combining this with Lemma 3.1, for all $u^t \in \Sigma^t$, it holds

$$\begin{aligned}
{}_{(\dot{H}^s)'}\langle f, u^t\rangle_{\dot{H}^s} \le \|f\|_{(\dot{H}^s)'}\|u^t\|_{\dot{H}^s(\mathbb{R}^N)} &\le \left(1 - \frac{\gamma}{\gamma_{N,s}}\right)^{-\frac{1}{2}} \|f\|_{(\dot{H}^s)'}\|u^t\|_{\gamma} \\
&< C_t S^{\frac{N-t}{4s-2t}}_{\gamma,t,s}\|u^t\|_{\gamma} - \varepsilon\left(1 - \frac{\gamma}{\gamma_{N,s}}\right)^{-\frac{1}{2}}\|u^t\|_{\gamma} \\
&\le \frac{4s-2t}{N-2t+2s}\|u^t\|^2_{\gamma} - \varepsilon\left(1 - \frac{\gamma}{\gamma_{N,s}}\right)^{-\frac{1}{2}}\|u^t\|_{\gamma}.
\end{aligned}$$

Hence,

$$\inf_{u\in\Sigma^t}\left[\frac{4s-2t}{N-2t+2s}\|u\|_\gamma^2 - {}_{(\dot H^s)'}\langle f,u\rangle_{\dot H^s}\right] \geq \varepsilon\left(1-\frac{\gamma}{\gamma_{N,s}}\right)^{-\frac{1}{2}} \inf_{u\in\Sigma^t}\|u\|_\gamma.$$

Since $\|u\|_\gamma$ is bounded away from 0 on Σ^t by Remark 3.2, the above expression implies that

$$\inf_{u\in\Sigma^t}\left[\frac{4s-2t}{N-2t+2s}\|u\|_\gamma^2 - {}_{(\dot H^s)'}\langle f,u\rangle_{\dot H^s}\right] > 0. \tag{3.25}$$

On the other hand,

$$\begin{aligned}
(3.3) \iff\ & C_t \frac{\|u\|_\gamma^{\frac{N-2t+2s}{2s-t}}}{\left(\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{N-2s}{4s-2t}}} - {}_{(\dot H^s)'}\langle f,u\rangle_{\dot H^s} > 0 \quad \text{for } \|u\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})} = 1 \\
\iff\ & C_t \frac{\|u\|_\gamma^{\frac{N-2t+2s}{2s-t}}}{\left(\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{N-2s}{4s-2t}}} - {}_{(\dot H^s)'}\langle f,u\rangle_{\dot H^s} > 0 \quad \text{for } u\in\Sigma^t \qquad (3.26)\\
\iff\ & \frac{4s-2t}{N-2t+2s}\|u\|_\gamma^2 - {}_{(\dot H^s)'}\langle f,u\rangle_{\dot H^s} > 0 \quad \text{for } u\in\Sigma^t.
\end{aligned}$$

Clearly, (3.25) ensures the RHS of (3.26) holds. The lemma now follows. □

Proof of Theorem 1.1 completed. Combining Propositions 3.1 and 3.2 with Lemma 3.3, we conclude the proof of Theorem 1.1. □

A

Lemma A.1. *Let $(v_n)_n \subset \dot H^s(\mathbb{R}^N)$ be a (PS) sequence for $\bar I^\gamma_{K,0,0}$ at the level d. Assume that, there exist sequences $(y_n)_n \to y \in \mathbb{R}^N$, $r_n \to 0 \in \mathbb{R}^+\cup\{0\}$ such that $w_n(x) = r_n^{\frac{N-2s}{2}} v_n(r_nx+y_n)$ converges weakly in $\dot H^s(\mathbb{R}^N)$ and a.e. to some $w\in\dot H^s(\mathbb{R}^N)$. If $\frac{|y_n|}{r_n}\to\infty$, then $K(y)^{\frac{N-2s}{4s}}w$ solves $(E^0_{1,0,0})$. Moreover,*

$$z_n := v_n - r_n^{-\frac{N-2s}{2}} w\Big(\frac{x-y_n}{r_n}\Big)$$

is a (PS) sequence for $\bar I^\gamma_{K,0,0}$ at the level $d - K(y)^{-\frac{N-2s}{2s}}\bar I^0_{1,0,0}(K(y)^{\frac{N-2s}{4s}}w)$.

Proof. Let $(v_n)_n\subset\dot H^s(\mathbb{R}^N)$ be a (PS) sequence for $\bar I^\gamma_{K,0,0}$ at the level d and ϕ be an arbitrary $C_c^\infty(\mathbb{R}^N)$ function. Put $\phi_n(x) := r_n^{-\frac{N-2s}{2}}\phi(\frac{x-y_n}{r_n})$. Thus,

$$\begin{aligned}
\langle w,\phi\rangle_{\dot H^s} &= \lim_{n\to\infty}\langle w_n,\phi\rangle_{\dot H^s}\\
&= \lim_{n\to\infty}\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{(w_n(x)-w_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&= \lim_{n\to\infty}\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{(v_n(x)-v_n(y))(\phi_n(x)-\phi_n(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \qquad (A.1)\\
&= \lim_{n\to\infty}\gamma\int_{\mathbb{R}^N}\frac{v_n\phi_n}{|x|^{2s}}\,\mathrm{d}x + \int_{\mathbb{R}^N}K(x)|v_n|^{2_s^*-2}v_n\phi_n\,\mathrm{d}x\\
&= \lim_{n\to\infty}\left[\gamma\int_{\mathbb{R}^N}\frac{w_n\phi}{|x+r_n^{-1}y_n|^{2s}}\,\mathrm{d}x + \int_{\mathbb{R}^N}K(r_nx+y_n)|w_n|^{2_s^*-2}w_n\phi\,\mathrm{d}x\right].
\end{aligned}$$

Since $r_n^{-1}|y_n|\to\infty$, for each fixed ϕ we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{w_n\phi}{|x+r_n^{-1}y_n|^{2s}}\,\mathrm{d}x = 0.$$

Therefore, taking the limit as $n \to \infty$ in (A.1), we obtain $(-\Delta)^s w = K(y)|w|^{2_s^*-2}w$, or equivalently $K(y)^{\frac{N-2s}{4s}} w$ solves $(E^0_{1,0,0})$. Moreover,

$$\int_{\mathbb{R}^N} \frac{w_n w}{|x - r_n^{-1} y_n|^{2s}} \mathrm{d}x = \int_{\mathbb{R}^N} \frac{|w|^2}{|x - r_n^{-1} y_n|^{2s}} \mathrm{d}x = o(1).$$

Therefore, proceeding as in Claim 2 of Step 5 in the proof of Theorem 2.1, we obtain as $n \to \infty$

$$\bar{I}^{\gamma}_{K,0,0}(z_n) = \bar{I}^{\gamma}_{K,0,0}(v_n) - K(y)^{-\frac{N-2s}{2s}} \bar{I}^0_{1,0,0}(K(y)^{\frac{N-2s}{4s}} w) + o(1).$$

To prove that ${}_{(\dot{H}^s)'}\langle \bar{I}^{\gamma}_{K,0,0}(z_n), \varphi \rangle_{\dot{H}^s} = o(\|\varphi\|)$, we proceed as in the proof of Claim 2 of Step 5 in Theorem 2.1, the only additional estimate we need to check is

$$\int_{\mathbb{R}^N} \frac{w \varphi_n}{|x - r_n^{-1} y_n|^{2s}} \mathrm{d}x = o(\|\varphi_n\|_{\dot{H}^s(\mathbb{R}^N)}),$$

where $\varphi_n = r_n^{\frac{N-2s}{2s}} \varphi(r_n x + y_n)$, $\|\varphi_n\|_{\dot{H}^s(\mathbb{R}^N)} = \|\varphi\|_{\dot{H}^s(\mathbb{R}^N)}$. This estimate follows from the Cauchy-Schwartz and the Hölder inequalities. □

Lemma A.2. *Let $\mathcal{K}$ denote the Kelvin transform in $\mathbb{R}^N$. If $(r_n)_n \subset \mathbb{R}^+ \cup \{0\}$ and $(y_n)_n \subset \mathbb{R}^N$ are sequences such that $\frac{|y_n|}{r_n} \to \infty$ and W is a positive solution of $(E^0_{1,0,0})$, then in the sense of $\dot{H}^s(\mathbb{R}^N)$-norm as $n \to \infty$*

$$r_n^{-\frac{N-2s}{2}} \mathcal{K}\left(W(\tfrac{x-y_n}{r_n}) \right) = \left(r_n |y_n|^{-2} \right)^{-\frac{N-2s}{2}} W\left(\frac{x - \frac{y_n}{|y_n|^2}}{r_n |y_n|^{-2}} \right) + o(1). \tag{A.2}$$

Proof. Let W be a positive solution of $(E^0_{1,0,0})$. Then $W(x) = C_{N,s}(1+|x|^2)^{-\frac{N-2s}{2}}$ thanks to [11]. The $\dot{H}^s(\mathbb{R}^N)$ norm is invariant under the scaling so that

$$v \to \tilde{v}(x) := \left(\frac{r_n}{|y_n|^2} \right)^{\frac{N-2s}{2}} v\left(\frac{r_n}{|y_n|^2} x + \frac{y_n}{|y_n|^2} \right), \tag{A.3}$$

we can apply it to each side of (A.2) to check the convergence. The RHS of (A.2) becomes $W + o(1)$. The LHS of (A.2), after some algebraic computation, is transformed into

$$W^n(x) := C_{N,s} \left(1 + \frac{r_n}{|y_n|} \langle x, y_n \rangle + \left(1 + r_n^2 |y_n|^{-2} \right) |x|^2 \right)^{-\frac{N-2s}{2}}.$$

As $\frac{r_n}{|y_n|} \to 0$, clearly $W^n \to W$ in $\dot{H}^s(\mathbb{R}^N)$. Hence the proof is complete. □

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