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Construction of dipole type singular solutions for a biharmonic equation with critical Sobolev exponent

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Abstract

In this paper, we construct positive weak solutions of a fourth order conformally invariant equation on S^N. These solutions give rise to complete metrics on the complement of finitely many points of S^N whose Q-curvature is constant.

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1 Introduction and statement of the results

Let (S^N, g_0) be the unit N-sphere with its standard metric and let $\Lambda \subset S^N$ be a closed set. In the singular Yamabe problem, one looks for a metric $\tilde{g}$ conformal to g_0 which is complete on $S^N - \Lambda$ and whose scalar curvature $\mathrm{Scal}_{\tilde{g}}$ is constant. Since $\tilde{g}$ is assumed to be conformal to g_0, we may write $\tilde{g} := u^{\frac{4}{N-2}} g_0$ for some positive function u. The scalar curvature of the metrics g_0 and $\tilde{g}$ are then related by the equation

$$\Delta_{g_0} u - \frac{N-2}{4(N-1)} \mathrm{Scal}_{g_0} u + \frac{N-2}{4(N-1)} \mathrm{Scal}_{\tilde{g}} u^{\frac{N+2}{N-2}} = 0. \tag{1}$$

Therefore the singular Yamabe problem on $S^N - \Lambda$ reduces to the existence of positive solutions of (1) which blow up sufficiently fast at any point of Λ in order to ensure the completeness of the metric $\tilde{g}$.

The linear operator which appears in the left hand side of (1),

$$\mathcal{L}_{g_0} := \Delta_{g_0} - \frac{N-2}{4(N-1)} Scal_{g_0}, \tag{2}$$

is commonly known as the conformal Laplacian of the metric g_0. It enjoys the following conformal equivalence property : If $g_2 := u^{\frac{4}{N-2}} g_1$ for some positive function u and if $\phi \in C^\infty(S^N)$, then

$$\mathcal{L}_{g_1}(u\phi) = u^{\frac{N+2}{N-2}} \mathcal{L}_{g_2}(\phi). \tag{3}$$

Observe that (1) corresponds to the special case of (3), when $\phi \equiv 1$.

The above equation is very sensitive to the sign of $\mathrm{Scal}_{\tilde{g}}$. Indeed, it is known that when $\mathrm{Scal}_{\tilde{g}} > 0$, then for solutions to exist it is necessary that the Hausdorff dimension of Λ is less than or equal to $\frac{N-2}{2}$. Whereas, in the case where $\mathrm{Scal}_{\tilde{g}} < 0$ it is necessary that the Hausdorff dimension of Λ is greater than $\frac{N-2}{2}$.

We now concentrate our attention on the case where $\mathrm{Scal}_{\tilde{g}} > 0$. The fact that, near any isolated singularity, solutions of (1) are asymptotic to radial solutions of (1) was proved in [5]. Using this fact, examples of solutions were constructed by R. Schoen [24] when Λ is either a finite set points or a rectifiable set arising as the limit set of Kleinian groups (see also [18] for further constructions). Later on, solutions of the singular Yamabe problem when Λ is a smooth submanifold of dimension between 1 and $\frac{N-2}{2}$ were constructed by R. Mazzeo and F. Pacard in [17].

In this paper, we address the problem of the construction of solutions of the conformally invariant semilinear equation

$$\Delta^2 u = u^{\frac{N+4}{N-4}} \quad \text{in} \quad \mathbb{R}^N - \Lambda, \tag{4}$$

when Λ is a finite collection points. This equation is a special case of a fourth order conformally invariant equation which can be defined on any manifold.

Let (M, g) be a smooth 4-dimensional Riemmanian manifold. The following fourth order operator has been introduced by Paneitz in [20]

$$P_g^4 u := \Delta_g^2 u - \mathrm{div}_g\Big(\frac{2}{3}\mathrm{Scal}_g g - 2\,\mathrm{Ric}_g\Big) du,$$

where Scal_g and $\mathrm{R}ic_g$ denote the scalar curvature and the Ricci curvature respectively. Originally the introduction of this operator was motivated by physical considerations and this operator has been found in many applications in mathematical physics, see [12], [21] and [1]. The operator introduced by Paneitz is conformally invariant and can be seen as a natural extension of the Laplace-Beltrami operator on 2-manifold. Later on, this operator was studied by T. Branson [2], [3] and S.A. Chang and P. Yang [10]. We also refer to [4], [6], [8], [9], [11] and [13] for further works on this operator. The operator P_g^4 has been generalized to manifolds of dimension greater than 4 by T. Branson [3]. For $N \geq 5$, the generalization of P_g^4 is

given by

$$P_g^N u := \Delta_g^2 u - \mathrm{div}_g\Big(a_N \,\mathrm{Scal}_g g - b_N \,\mathrm{Ric}_g\Big)\, du + \frac{N-4}{2} Q_g^N u, \tag{5}$$

where the constant a_N and b_N are given by

$$a_N := \frac{(N-2)^2+4}{2(N-1)(N-2)}, \qquad b_N := \frac{4}{N-2},$$

and where

$$Q_g^N := \frac{1}{2(N-1)}\Delta_g \mathrm{Scal}_g + \frac{N^3-4N^2+16N-16}{8(N-1)^2(N-2)^2}\,\mathrm{Scal}_g^2 - \frac{2}{(N-2)^2}\,|\mathrm{Ric}_g|^2. \tag{6}$$

As mentioned before, P_g^N is also a conformally invariant operator, namely if $\tilde{g}$ and g are conformally related, $\tilde{g} := u^{\frac{8}{N-4}} g$, then for all function $v \in C^\infty(M)$ we have

$$P_g^N(vu) = u^{\frac{N+4}{N-4}}\, P_{\tilde{g}}^N(v).$$

In particular, if we consider $v \equiv 1$, we obtain

$$P_g^N(u) = \frac{N-4}{2} Q_{\tilde{g}}^N\, u^{\frac{N+4}{N-4}}. \tag{7}$$

In view of these formulae, it should be clear that the quantity Q_g^N plays, for the Paneitz operator P_g^N, the rôle played by the scalar curvature for the conformal Laplacian. It is natural to ask if one can deform conformally a given metric to produce a metric whose Q^N-curvature is constant.

The problem we would like to address in this paper is a generalization of the singular Yamabe problem for the operator $P_{g_0}^N$. This problem can be stated as follows : Assume that Λ is a closed subset of S^N. Can one find a metric $\tilde{g}$ which is conformal to g_0, complete on $S^N - \Lambda$ and for which $Q_{\tilde{g}}^N$ is constant, positive ?

We give a partial answer to this problem when Λ is a discrete set of points. Our main result reads :

Theorem 1 *For all $k \geq 1$, there exists a finite set of points $\Lambda \subset S^N$ of cardinal $2\,k$ and there exists a complete metric $\tilde{g}$ which is conformal to g_0 on $S^N - \Lambda$ for which $Q_{\tilde{g}}^N \equiv \frac{2}{N-4}$.*

The metrics we construct arise as connected sum of cylindrical metrics on the cylinder $S^{N-1} \times \mathbb{R}$.

Before we proceed with the proof of the result, let us mention that, on (S^N, g_0), $N \geq 5$, the unit sphere endowed with the standard metric, the expression of $P_{g_0}^N$ is given by

$$P_{g_0}^N u = \Delta_{g_0}^2 u + c_N\, \Delta_{g_0} u + d_N\, u, \tag{8}$$

where the constants c_N and d_N are given by

$$c_N := \frac{1}{2}(N^2 - 2N - 4) \qquad d_N := \frac{N-4}{16} N(N^2-4).$$

2 Fowler-type solutions

We would like to find solutions to the Q^N-curvature problem on the complement of 2 points p_1 and p_2 on the sphere S^N. Using the invariance of our problem with respect to conformal transformations, we may assume that the points p_i are antipodal. Next, we can use the stereographic projection of S^N on $\mathbb{R}^N$ to reduce the problem to find a positive solution u of

$$\Delta^2 u = u^{\frac{N+4}{N-4}} \qquad \text{on} \qquad \mathbb{R}^N - \{0\},$$

with a non removable singularity at the origin (and a hidden singularity at infinity). A recent result of C.S. Lin [14] implies that necessarily, u is rotationally symmetric, and thus the above partial differential equation reduces to an ordinary differential equation. It is standard to write

$$u(x) := |x|^{\frac{4-N}{2}} v(-\log|x|). \tag{9}$$

In that case, the function v satisfies an autonomous 4-th order ordinary differential equation

$$\partial_t^4 v - \frac{N^2-4N+8}{2}\partial_t^2 v + \frac{N^2(N-4)^2}{16} v - v^{\frac{N+4}{N-4}} = 0. \tag{10}$$

There is one quantity which is preserved along trajectories of the solutions of (10). This quantity reads

$$H(v) := (\partial_t^3 v)(\partial_t v) - \frac{1}{2}(\partial_t^2 v)^2 - \frac{N^2-4N+8}{4}(\partial_t v)^2 + \frac{N^2(N-4)^2}{32} v^2 - \frac{N-4}{2N} v^{\frac{2N}{N-4}}. \tag{11}$$

Observe that we have an explicit constant solution of (10) which is given by

$$v_0 \equiv \Big(\frac{N(N-4)}{4}\Big)^{\frac{N-4}{4}}. \tag{12}$$

This corresponds to a standard metric on a cylinder $S^{N-1} \times \mathbb{R}$.

Let us recall that, the above strategy to find solutions of the Q^N-curvature problem on the sphere minus 2 points, is modeled after the strategy which is used in the case of the scalar curvature and the conformal Laplacian. Indeed, in this later case, radial solutions are known as Fowler's solutions. The study of these solutions is greatly simplified by the fact that the ordinary differential equation is a second order Hamiltonian ordinary differential equation, for which it is easy to show that there are periodic solutions.

In contrast, in our case, it does not seem possible to integrate completely the above ODE. Namely, beside the Hamiltonian energy, it does not seem possible to obtain other quantities which are preserved along the trajectories of (10). This is the reason why we will rely on the following result where we prove that close to the constant solution v_0 there exists a one parameter family of periodic solution of (10). This is the content of the following :

Proposition 1 *There exists $\varepsilon_0 > 0$, and for all $\varepsilon \in (0, \varepsilon_0)$, there exists v_ε, a periodic solution of (10), such that*

$$v_\varepsilon(t) = v_0 + \varepsilon \cos(\sqrt{\mu_-}\, t) + \mathcal{O}(\varepsilon^2), \tag{13}$$

where $\mu_- > 0$ only depends on N. Moreover, one can ask that v_ε depend smoothly on ε and that

$$\partial_\varepsilon v_\varepsilon(t)_{\varepsilon=0} = \cos(\sqrt{\mu_-}\, t).$$

Proof If we define $Z := (v, \partial_t v, \partial_t^2 v, \partial_t^3 v)$ and $Z_0 = (v_0, 0, 0, 0)$, we are left to find periodic solutions of

$$\dot{Z} = F(Z) \tag{14}$$

in $\mathbb{R}^4$, which are close to the constant solution Z_0. To this aim , we will use the fact that H is preserved along the trajectories of this dynamical system and we will prove that there exists a 2-dimensional center manifold which passes through Z_0 and which is transverse to the hypersurfaces $H \equiv c$.

Existence of the center manifold : The linearized equation about Z_0 is given by

$$\dot{X} = DF_{Z_0} X,$$

where

$$DF_{Z_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{N^2(N-4)}{2} & 0 & \frac{N^2-4N+8}{2} & 0 \end{pmatrix}.$$

The eigenvalues of DF_{Z_0} are given by

$$\lambda^\pm = \pm i\sqrt{\mu_-} \qquad \text{and} \qquad \tilde{\lambda}^\pm = \pm\sqrt{\mu_+} \tag{15}$$

where

$$\mu_\pm = \frac{\sqrt{N^4 - 64N + 64} \pm (N^2 - 4N + 8)}{4}, \tag{16}$$

with $N \geq 5$. Note that $\mu_\pm > 0$.

Near Z_0, the dynamical system (14) has a 1-dimensional stable manifold corresponding to the eigenvalue $\tilde{\lambda}^-$, a 1-dimensional unstable manifold corresponding to the eigenvalue $\tilde{\lambda}^+$, and a 2-dimensional center manifold corresponding to the eigenvalues $\lambda^\pm$.

The tangent space of the center manifold at Z_0 is spanned by the vectors

$$X_1 = (1, 0, -\mu_-, 0) \qquad \text{and} \qquad X_2 = (0, 1, 0, -\mu_-)$$

Conservation law: The quantity

$$H(Z) = z_4\, z_2 - \frac{1}{2} z_3^2 - \frac{N^2 - 4N + 8}{4} z_2^2 + \frac{N^2(N-4)^2}{32} z_1^2 - \frac{N-4}{2N} z_1^{\frac{2N}{N-4}}$$

is preserved along trajectories of (14). The expansion of H at $Z = Z_0$ is given by

$$H(Z_0 + X) = \frac{2}{N}\Big(\frac{N(N-4)}{4}\Big)^{\frac{N}{2}} + q(X) + \mathcal{O}(|X|^3), \tag{17}$$

where the quadratic form q is given by

$$q(X) = x_4\, x_2 - \frac{x_3^2}{2} - \frac{N^2 - 4N + 8}{2} x_2^2 - \frac{N^2(N-4)}{4} x_1^2. \tag{18}$$

One can check that this quadratic form q restricted to the tangent space of the center manifold at Z_0 is negative definite. Indeed, if $X = \alpha(1, 0, -\mu_-, 0) + \beta(0, 1, 0, -\mu_-)$, then

$$q(X) = -\left(\frac{\mu_-^2}{2} + \frac{N^2(N-4)}{4}\right)\alpha^2 - \left(\mu_- + \frac{N^2 - 4N + 8}{2}\right)\beta^2 = -(a\alpha^2 + b\beta^2). \tag{19}$$

Existence of periodic solutions: It follows from the above analysis that, for all $\varepsilon > 0$ small enough, the hypersurface $H(Z) = H(Z_0) - \varepsilon^2$ is transverse to the center manifold and the intersection produces closed curves which correspond to periodic solutions of (14). Indeed, close to Z_0 the center manifold can be parameterized by $(\alpha, \beta) \longrightarrow Y(\alpha, \beta)$ such that $Y(0) = Z_0$, $\partial_\alpha Y(0) = X_1$ and $\partial_\beta Y(0) = X_2$. Then the equation

$$H(Y(\alpha, \beta)) = H(Z_0) - \varepsilon^2$$

reduces to

$$a\,\alpha^2 + b\,\beta^2 + \mathcal{O}(\alpha^2 + \beta^2)^{3/2} = \varepsilon^2,$$

and the claim follows at once. ◇

3 Linear analysis

We consider the nonlinear equation (4) which we linearize about the solution $u_0 := |x|^{\frac{4-N}{2}} v_0$. This yields the linear operator

$$L := \Delta^2 - \frac{N+4}{N-4} u_0^{\frac{8}{N-4}}$$

3.1 Geometric Jacobi fields

We describe some explicit Jacobi fields for the linearized operator L. Thanks to the result of Proposition 1, we know that u_0 belongs to a 2-dimensional family of solutions of (4) which are given by

$$u_{\varepsilon,t_0}(x) := |x|^{\frac{4-N}{2}} v_\varepsilon(-\log|x| + t_0).$$

Differentiation with respect to t_0 or ε give two Jacobi fields which are denoted by $\Phi^{0,\pm}$. More precisely, we have

$$\Phi^{0,+} := \left(\frac{1}{\varepsilon}\,\partial_{t_0} u_{\varepsilon,t_0}\right)|_{\varepsilon=t_0=0} \qquad \text{and} \qquad \Phi^{0,-} := \partial_\varepsilon u_{\varepsilon,t_0}|_{\varepsilon=t_0=0}. \tag{20}$$

3.2 Indicial roots

Given any function ω, we define

$$\bar{\omega}(x) := |x|^{\frac{4-N}{2}}\, \omega(-\log|x|, \theta)$$

so that

$$L\,\bar{\omega}(x) = |x|^{-\frac{(N+4)}{2}}\, (\mathcal{L}\,\omega)(-\log|x|, \theta). \tag{21}$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L} := \partial_t^4 + \Delta_{S^{N-1}}^2 + 2(\partial_t^2 - \frac{N(N-4)}{4})\,\Delta_{S^{N-1}} - \frac{N^2-4N+8}{2}\partial_t^2 - \frac{N^2(N-4)}{2}. \tag{22}$$

In particular, the study of L reduces to the study of $\mathcal{L}$.

We introduce (φ_j, λ_j), the eigendata of the Laplacian on S^{N-1}. That is, $\Delta_{S^{n-1}}\varphi_j = -\lambda_j\varphi_j$. We will assume that the eigenvalues are counted with multiplicity and that the eigenfunctions are normalized so that their L^2 norm is equal to 1. It is well known that (φ_j, λ_j^2) are the eigendata of the Bilaplacian on S^{N-1}. Projection of $\mathcal{L}$ over the eigenspace generated by φ_j yields the operator

$$\mathcal{L}_j := \partial_t^4 - \left(2\,\lambda_j + \frac{N^2-4N+8}{2}\right)\partial_t^2 - \frac{N^2\,(N-4)}{2} + 2\,\lambda_j\,\frac{N\,(N-4)}{4} + \lambda_j^2. \tag{23}$$

The homogeneous equation $\mathcal{L}_j w = 0$ has 4 independent solutions which are given by $t \to e^{\gamma t}$ where γ is one of the 4 indicial roots of $\mathcal{L}_j$. We will denote these indicial roots by $\pm\gamma_j$ and $\pm\tilde{\gamma}_j$. These indicial roots can be computed explicitly and we find the :

Lemma 1 *For all $j \geq 1$, we have*

$$\gamma_j \geq 1, \qquad \textit{and} \qquad \tilde{\gamma}_j \geq 1.$$

In the case where $j = 0$, we have

$$\gamma_0 = i\sqrt{\mu_-}, \qquad \textit{and} \qquad \tilde{\gamma}_0 = \sqrt{\mu_+}.$$

Observe that $\pm\gamma_0$ are associated to the Jacobi fields $\Phi^{0,\pm}$ which have been defined in (20).

3.3 Mapping properties of $\mathcal{L}$

We consider the action of the linearized operator on weighted Hölder spaces, which are defined in the :

Definition 1 Given $k \in \mathbb{R}$, $0 < \alpha < 1$ and $\delta \in \mathbb{R}$, we define

$$\mathcal{C}^{k,\alpha}_\delta(\,\mathbb{R}\times S^{N-1}) = (\cosh t)^\delta\, \mathcal{C}^{k,\alpha}(\mathbb{R}\times S^{N-1}) \tag{24}$$

endowed with the norm

$$\|\omega\|_{\mathcal{C}^{k,\alpha}_\delta} = \|(\cosh t)^{-\delta}\,\omega\|_{\mathcal{C}^{k,\alpha}}.$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a cut-off function such that $\eta \equiv 1$ if $t \leq -1$ and $\eta \equiv 0$ if $t \geq 1$. We set

$$D_0 := \mathrm{Vect}\{\eta\, \Phi^{0,\pm}\}.$$

Proposition 2 *Assume that $\delta \in (-1,0)$ is fixed. Then, the operator*

$$\mathcal{L} : \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R} \times S^{n-1}) \oplus D_0 \longrightarrow \mathcal{C}^{0,\alpha}_{\delta}(\mathbb{R} \times S^{N-1})$$

is an isomorphism.

Proposition 2 will be a consequence of the following results :

Lemma 2 *Assume that $\delta < 1$ is fixed. Let $\omega \in \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R} \times S^{N-1})$, assume that, for all $t \in \mathbb{R}$, the function $\omega(t,\cdot)$ is orthogonal in the L^2 sense to 1 on S^{N-1} and that $\mathcal{L}\omega = 0$. Then $\omega \equiv 0$.*

Proof We decompose w over the eigenfunction φ_j

$$\omega = \sum_{j=1}^{\infty} \omega_j \varphi_j.$$

For any $j \geq 1$, the function ω_j satisfies $\mathcal{L}_j \omega_j = 0$. Hence, it is a linear combination of the functions $e^{\pm\gamma_j t}$ and $e^{\pm\tilde{\gamma}_j t}$. Since for $j \geq 1$, the absolute values of the indicial roots are greater than 1 and since we have assumed that ω is bounded by a constant times $(\cosh s)^{\delta}$, we conclude easily that $\omega_j = 0$. Hence $\omega \equiv 0$. ◇

The previous Lemma then implies some surjectivity result for $\mathcal{L}$. More precisely, we have the :

Corollary 1 *Assume that $\delta \in (-1,1)$ is fixed. Let $f \in \mathcal{C}^{0,\alpha}_{\delta}(\mathbb{R} \times S^{N-1})$, assume that, for all $t \in \mathbb{R}$, the function $\omega(t,\cdot)$ is orthogonal in the L^2 sense to 1 on S^{N-1}. Then, there exists a unique $\omega \in \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R} \times S^{N-1})$ satisfying $\mathcal{L}\omega = f$. Moreover, we have*

$$||\omega||_{\mathcal{C}^{4,\alpha}_{\delta}} \leq c\, ||f||_{\mathcal{C}^{0,\alpha}_{\delta}}. \tag{25}$$

Finally, we treat the case where the functions do not depend on the θ variable. This is the content of the following :

Lemma 3 *Assume that $\delta \in (-\sqrt{\mu_+}, 0)$ is fixed. Let $f_0 \in C^{0,\alpha}_{\delta}(\mathbb{R})$. Then there exists a unique ω_0 solution of*

$$\mathcal{L}_0 \omega_0 = f_0 \ \ in\, \mathbb{R} \tag{26}$$

such that $\omega_0 \in C^{4,\alpha}_{\delta}(\mathbb{R}) \oplus D_0$. Moreover, if we decompose $\omega_0 = \bar{\omega}_0 + \eta\,(a\,\Phi^{0,+} + b\,\Phi^{0,-})$, we have

$$||\bar{\omega}_0||_{C^{4,\alpha}_{\delta}} + |a| + |b| \leq c\, ||f_0||_{C^{0,\alpha}_{\delta}}. \tag{27}$$

Proof To begin with, we define the functions v_+ and v_- respectively on $(0,\infty)$ and $(-\infty,0)$ by

$$v_+(t) := \frac{1}{(\mu_+ + \mu_-)}\left(\frac{1}{\sqrt{\mu_-}}\int_t^{+\infty}\sin(\sqrt{\mu_-}(t-s))f_0(s)ds - \frac{1}{2\sqrt{\mu_+}}\int_t^{+\infty}e^{\sqrt{\mu_+}(t-s)}f_0(s)ds - \frac{1}{2\sqrt{\mu_+}}\int_{-\infty}^{t}e^{\sqrt{\mu_+}(s-t)}f_0(s)ds\right)$$

and

$$v_-(t) := \frac{1}{(\mu_+ + \mu_-)}\left(\frac{1}{\sqrt{\mu_-}}\int_{-\infty}^{t}\sin(\sqrt{\mu_-}(s-t))f_0(s)ds + \frac{1}{2\sqrt{\mu_+}}\int_{-\infty}^{t}e^{\sqrt{\mu_+}(t-s)}f_0(s)ds + \frac{1}{2\sqrt{\mu_+}}\int_t^{+\infty}e^{\sqrt{\mu_+}(s-t)}f_0(s)ds\right).$$

Observe that v^+ (resp. v^-) satisfy $\mathcal{L}_0 v^{\pm} = f_0$ in $(0,\infty)$ (resp. $(-\infty,0)$). Moreover, we have the estimates

$$|v_{\pm}| \leq C||f_0||_{C^{0,\alpha}_{\delta}}(\cosh t)^{\delta}$$

in $(0,\infty)$ (resp. in $(-\infty,0)$). Once this is done, we define the functions $\omega_0^{\pm}$ by

$$\omega_0^+(t) := v_+(t) + a_0\, e^{-\sqrt{\mu_+}t} \tag{28}$$

on $(0,\infty)$, and

$$\omega_0^-(t) := v_-(t) + a_1\,\Phi^{0,+}(t) + a_2\,\Phi^{0,-}(t) + a_3\, e^{\sqrt{\mu_+}t} \tag{29}$$

on $(-\infty,0)$, where the constants a_j, $j = 0,\cdots,3$ are chosen in such a way that

$$\partial_t^j\omega_0^+(0) = \partial_t^j\omega_0^-(0)$$

for $j = 0,\cdots,3$. Observe that, since the indicial roots γ_0 and $\tilde{\gamma}_0$ are distinct, the constants a_j are uniquely determined as solution of a linear system for some invertible matrix. It should be clear that the function

$$\bar{\omega}_0 := \omega_0^{\pm} - \eta\,(a_1\,\Phi^{0,+} + a_2\,\Phi^{0,-})$$

is a solution of our problem which belongs to the space $C^{4,\alpha}_{\delta}(\mathbb{R})$. ◇

All the above results can be translated in the case where we consider the operator L acting on functions defined in $\mathbb{R}^N - \{0\}$ instead of working with the operator $\mathcal{L}$ acting on functions defined in $\mathbb{R}\times S^{N-1}$. Again we define weighted Hölder spaces on $\mathbb{R}^N - \{0\}$.

The space $C^{k,\alpha}_{\nu}(\mathbb{R}^N - \{0\})$ is defined to be the space of functions $w \in C^{k,\alpha}_{loc}(\mathbb{R}^N - \{0\})$ such that

$$||u||_{C^{k,\alpha}_{\nu}} := \sup_{r\in(0,1/2)} r^{-\nu}||u(r\,\cdot)||_{C^{k,\alpha}(B_2(0)-B_1(0))} + \sup_{r\in(0,1/2)} r^{N-4+\nu}||u(\cdot/r)||_{C^{k,\alpha}(B_2(0)-B_1(0))}).$$

Observe that, given any function u defined in $\mathbb{R}^N - \{0\}$, if we write $u(x) = |x|^{\frac{4-N}{2}} v(-\log|x|, \theta)$, then

$$||u||_{C^{k,\alpha}_{\nu}(\mathbb{R}^N - B_1(0))} = ||v||_{C^{k,\alpha}_{\delta}(\mathbb{R}\times S^{N-1})} \tag{30}$$

where $\nu = \frac{4-N}{2} - \delta$.

Finally, we define

$$E_0 = \mathrm{Vect}\{\eta(-\log|x|)|x|^{\frac{4-N}{2}}\Phi^{0,\pm}(-\log|x|)\}.$$

Granted the above definitions, the result of Proposition 2 is equivalent to :

Proposition 3 *The operator*

$$L : \mathcal{C}^{4,\alpha}_{\nu}(\mathbb{R}^N - \{0\}) \oplus E_0 \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-4}(\mathbb{R}^N - \{0\})$$

is an isomorphism if $\frac{4-N}{2} < \nu < \frac{6-N}{2}$.

Given any smooth bounded domain $\Omega \subset \mathbb{R}^N - \{0\}$, we define the space $C^{k,\alpha}_{\nu}(\Omega)$ as the space of all restrictions of functions in $C^{k,\alpha}_{\nu}(\mathbb{R}^N - \{0\})$ to Ω. This space is naturally endowed with the induced norm. The subspace $[\mathcal{C}^{k,\alpha}_{\mu}(\Omega)]_{\mathcal{D}}$ of $\mathcal{C}^{k,\alpha}_{\mu}(\Omega)$ consists of functions ω in $\mathcal{C}^{k,\alpha}_{\mu}(\Omega)$ which satisfy $\Delta\omega = \omega = 0$ on $\partial\Omega$.

Lemma 4 *Assume that* $4 - N < \mu < 0$ *is fixed. If* $0 < R < 1$, *then*

$$\Delta^2 : [\mathcal{C}^{4,\alpha}_{\mu}(B_R - \{0\})]_{\mathcal{D}} \longrightarrow \mathcal{C}^{0,\alpha}_{\mu-4}(B_R - \{0\})$$

is an isomorphism whose norm is bounded independently of $R \in (0,1)$.

Proof In order to prove this Lemma, it is convenient to work on $(0,+\infty) \times S^{N-1}$ instead of $B_1(0) - \{0\}$. This can be achieved by changing

$$u(x) = |x|^{\frac{4-N}{2}} v(-\log|x|, \theta).$$

We consider

$$\Delta_0 := \partial_t^2 + \Delta_{S^{N-1}} - 2\,\partial_t - \frac{N(N-4)}{4} \tag{31}$$

and

$$\tilde{\Delta}_0 := \partial_t^2 + \Delta_{S^{N-1}} + 2\,\partial_t - \frac{N(N-4)}{4}. \tag{32}$$

Finally, we define δ by

$$\frac{4-N}{2} < \delta := \frac{4-N}{2} - \mu < \frac{N-4}{2}$$

and $T := -\log R > 0$. The result we want to show reduces to proving that

$$\tilde{\Delta}_0\,\Delta_0 : [\mathcal{C}^{4,\alpha}_{\delta}([T,\infty) \times S^{N-1})]_{\mathcal{D}} \longrightarrow \mathcal{C}^{0,\alpha}_{\delta}([T,\infty) \times S^{N-1})$$

is an isomorphism whose norm is bounded independently of T.

Consider the following problem

$$\begin{cases} \tilde{\Delta}_0 \, \Delta_0 \omega & = f \quad \text{in} \quad [T, \infty) \times S^{N-1} \\ \Delta_0 \omega = \omega & = 0 \quad \text{on} \quad \{T\} \times S^{N-1}. \end{cases} \tag{33}$$

In particular, $w := \Delta_0 \omega$ is a solution of

$$\begin{cases} \tilde{\Delta}_0 w & = f \quad \text{in} \quad [T, \infty) \times S^{N-1} \\ w & = 0 \quad \text{on} \quad \{T\} \times S^{N-1}. \end{cases} \tag{34}$$

One can easily check that, since we have chosen $\frac{4-N}{2} < \delta \leq \frac{N}{2}$, the function $e^{\delta t}$ can be used as a barrier to show, using the maximum principle, the existence of a solution w which satisfies

$$|w| \leq c \, ||f||_{\mathcal{C}^{0,\alpha}_{\delta}} \, e^{\delta t}.$$

Now that we know this estimate, we use the fact that ω solves

$$\begin{cases} \Delta_0 \omega & = w \quad \text{in} \quad [T, \infty) \times S^{N-1} \\ \omega & = 0 \quad \text{on} \quad \{T\} \times S^{N-1} \end{cases} \tag{35}$$

and, since we have chosen $-\frac{N}{2} \leq \delta < \frac{N-4}{2}$, the function $e^{\delta t}$ can be again used as a barrier to show, using the maximum principle, the existence of ω which satisfies

$$|\omega| \leq c \sup_{\mathbb{R}} |e^{-\delta t} \, w| \, e^{\delta t} \leq c \, ||f||_{\mathcal{C}^{0,\alpha}_{\delta}} \, e^{\delta t}.$$

Hence

$$|\omega| \leq C \, ||f||_{\mathcal{C}^{0,\alpha}_{\delta}} \, e^{\delta t}$$

for some constant C which does not depend on T. Once this estimate has been obtained the estimates for the derivatives of ω follow from standard elliptic estimates. ◇

Given any point $q_0 \in \partial B_1$, we define the space $\mathcal{D}^{k,\alpha}_{\nu,\mu}(\mathbb{R}^N - \{0, q_0\})$ as the set of functions $w \in \mathcal{C}^{k,\alpha}_{loc}(\mathbb{R}^N - \{0, q_0\})$ for which the following norm is finite

$$||w||_{\mathcal{D}^{k,\alpha}_{\nu,\mu}} := ||w||_{\mathcal{C}^{k,\alpha}_{\nu}(\mathbb{R}^N - \{0\}) - B_{1/4}(q_0)} + ||w(\cdot + q_0)||_{\mathcal{C}^{k,\alpha}_{\mu}(B_{3/4} - \{0\})}. \tag{36}$$

We will slightly abuse notations, and define the space $\mathcal{C}^{k,\alpha}_{\mu}(B_\rho(q_0) - \{q_0\})$ as the space of functions w defined in $B_\rho(q_0)$ such that $w(q_0 + \cdot) \in \mathcal{C}^{k,\alpha}_{\mu}(B_\rho - \{0\})$.

Granted this definition, we now collect the results of Proposition 3 and Lemma 4 to obtain the :

Proposition 4 *Assume that* $\frac{4-N}{2} < \nu < \frac{6-N}{2}$ *and* $4 - N < \mu < 0$ *are fixed. Then, the operator*

$$L : \mathcal{D}^{4,\alpha}_{\nu,\mu}(\mathbb{R}^N - \{0, q_0\}) \oplus E_0 \longrightarrow \mathcal{D}^{0,\alpha}_{\nu-4,\mu-4}(\mathbb{R}^N - \{0, q_0\})$$

is an isomorphism.

Proof Since we will not need the fact that this operator is injective, we shall leave this fact to the reader and only prove that one can build a right inverse.

To begin with, observe that, since the function u_0 is bounded in $B_{1/2}(q_0)$ we have for all $\omega \in \mathcal{C}^{4,\alpha}_{\mu}(B_\rho(q_0) - \{q_0\})$ we have

$$||(L - \Delta^2)\omega||_{C^{0,\alpha}_{\mu-4}(B_\rho(q_0)-\{q_0\})} \leq c_0\, \rho^4\, ||\omega||_{C^{4,\alpha}_{\mu}(B_\rho(q_0)-\{q_0\})}.$$

Using this and a simple perturbation argument shows that the result of Lemma 4 remains true if the operator Δ^2 is replaced by the operator L, provided ρ is chosen small enough. Hence, we can fix ρ small enough and solve

$$\begin{cases} L\omega_0 & = f \quad \text{in } B_\rho(q_0) - \{q_0\} \\ \omega_0 = \Delta\omega_0 & = 0 \quad \text{in } \partial B_\rho(q_0). \end{cases}$$

Moreover, we have the estimate

$$||\omega_0||_{C^{4,\alpha}_{\mu}(B_\rho-\{q_0\})} \leq C\, ||f||_{C^{0,\alpha}_{\mu-4}(B_\rho-\{q_0\})} \tag{37}$$

where the constant C does not depend on ρ.

We define the cut-off function χ_ρ which is equal to 0 outside $B_\rho(q_0)$ and which is identically equal to 1 in $B_{\rho/2}(q_0)$. Now, using Proposition 3, we can define $\omega_1 \in \mathcal{C}^{4,\alpha}_{\nu}(\mathbb{R}^N - \{0\}) \oplus E_0$ to be the solution of

$$L\omega_1 = f - L(\chi_\rho \omega_0)$$

in $\mathbb{R}^N - \{0\}$. We know that

$$||\omega_1||_{C^{4,\alpha}_{\nu}(\mathbb{R}^N-\{0\})\oplus E_0} \leq C\, ||f - L(\chi_\rho\omega_0)||_{C^{0,\alpha}_{\nu-4}(\mathbb{R}^N-\{0\})}.$$

The solution of $Lw = f$ we are looking for is then given by $w = \omega_0 + \omega_1$. This is the desired result. ◇

4 Connected sum result

For $i = 1, 2$, let u_i be the solution of (4) which is defined by

$$u_i(x) = |x|^{\frac{4-N}{2}}\, v_0$$

on $\mathbb{R}^N - \{0\}$. In other words, we have two solutions to the Q^N-curvature problem on the sphere $R^N - \{0\}$. Since $\mathbb{R}^N$ is locally conformally equivalent to S^N or to a cylinder $\mathbb{R} \times S^{N-1}$, we can think that we have two solutions of the Q^N-curvature problem on S^N (with two points removed) or on $\mathbb{R} \times S^{N-1}$ (with two points removed).

For each solution, we choose a point $q_i \in \partial B_1$. The idea is that $B_{1/2}(q_1)$ is conformally equivalent to a half infinite cylinder $C_1 := (-T, \infty) \times S^{N-1}$ and $B_{1/2}(q_2)$ is conformally equivalent to a half infinite cylinder $C_2 := (-\infty, T) \times S^{N-1}$. Next, we identify the two pieces of cylinders $(-T, T) \times S^{N-1}$ to produce a manifold which is conformally equivalent to S^N with four points removed and for which we have almost solved the Q^N-curvature problem. Finally, we show that we can adjust the conformal factor in such a way that the Q^N-curvature problem is solved.

We choose $q_0 \in \partial B_1$ and $T > 0$. We can write

$$u_1(x) = r^{\frac{4-N}{2}} v_1(-\log(2r) + T, \theta) \tag{38}$$

where (r, θ) are polar coordinates about q_0. Since u_1 is regular at q_0, we have

$$||v_1(\cdot - T, \cdot)||_{C^{2,\alpha}_{\frac{4-N}{2}}((0,\infty)\times S^{N-1})} \leq C. \tag{39}$$

Similarly, we can write

$$u_2(x) = r^{\frac{4-N}{2}} v_2(\log(2r) - T, \theta) \tag{40}$$

where (r, θ) are polar coordinates about q_0. Since u_2 is regular at q_0, we have

$$||v_2(\cdot + T, \cdot)||_{C^{2,\alpha}_{\frac{4-N}{2}}((\infty,0)\times S^{N-1})} \leq C. \tag{41}$$

We now define a cutoff function χ as being identically equal to 0 for $t \geq 1$ and identically equal to 1 for $t \leq -1$. We set

$$\chi_{1,T}(t) = \chi(t - T + 1)$$

and

$$\chi_{2,T}(t) = 1 - \chi(T - 1 + t).$$

Denote by v the function defined on $\mathbb{R} \times S^{N-1}$ by

$$v_T := \chi_{1,T}\, v_1 + \chi_{2,T}\, v_2. \tag{42}$$

We recall that the functions v_i, for $i = 1, 2$, are solutions of $\mathcal{N}(v) = 0$ where

$$\begin{aligned} \mathcal{N}(v) \;=\; & \partial_t^4 v + \Delta^2_{S^{N-1}} v + 2(\partial_t^2 - \frac{N(N-4)}{4})\Delta_{S^{N-1}} v - \frac{N^2 - 4N + 8}{2}\partial_t^2 v + \\ & \frac{N^2(N-4)^2}{16} v - v^{\frac{N+4}{N-4}}. \end{aligned} \tag{43}$$

Obviously, we have $\mathcal{N}(v_T) = 0$ in $(-\infty, -T) \times S^{N-1}$ and also in $(T, \infty) \times S^{N-1}$.

Lemma 5 *There exists a constant $c > 0$, which does not depend on $T > 4$ such that*

$$||\mathcal{N}(v_T)||_{\mathcal{C}^{0,\alpha}_{-\frac{N-12}{2}}([-T,T]\times S^{N-1})} \leq c\, e^{-\frac{N+4}{2}T}. \tag{44}$$

Proof On $[-T+2, T-2] \times S^{N-1}$, we have $\mathcal{N}(v_1) = \mathcal{N}(v_2) = 0$ and $v_T = v_1 + v_2$. Therefore, we can write

$$\mathcal{N}(v_T) = v_1^p + v_2^p - (v_1 + v_2)^p$$

where $p := \frac{N+4}{N-4}$, in this set. Since, in $[-T+2, 0] \times S^{n-1}$, the function $w := v_2/v_1$ is bounded from above by a constant which does not depend on T, we can write

$$v_1^p + v_2^p - (v_1 + v_2)^p = v_1^p \left(1 + w^p - |1 + w|^p\right).$$

Using the Taylor expansion together with (39) and (41), it is now easy to check that

$$||\mathcal{N}(v_T)||_{\mathcal{C}^{0,\alpha}_{-\frac{N-12}{2}}([2-T,0]\times S^{N-1})} \leq c\, e^{-\frac{N+4}{2}T}.$$

The set $[0, T-2] \times S^{N-1}$ can be dealt with similarly. Finally, one checks that the cutoff functions do not alter the estimate. We leave the details to the reader. ◇

We now explain what function space we will use to evaluate functions on the connected sum of the two $\mathbb{R}^N$. So assume that you have a function w which is of class $\mathcal{C}^{k,\alpha}$ on $\mathbb{R} \times S^{N-1}$ away from the singularities. We decompose $\mathbb{R}^N \times S^{N-1}$ into three overlapping sets: $(-\infty, -T) \times S^{N-1}$, $[-1-T, 1+T] \times S^{N-1}$ and $[T, \infty) \times S^{N-1}$. We can conformally transform $[-\infty, -T] \times S^{N-1}$ so that we are left to evaluate the norm of a function W which is defined in $(\mathbb{R}^N - \{0\}) - B_{1/2}(q_0))$. We evaluate the function W by taking its $\mathcal{C}^{k,\alpha}_\nu$ norm in $(\mathbb{R}^N - \{0\}) - B_{1/2}(q_0)$, where $\frac{4-N}{2} < \nu < \frac{6-N}{2}$ is fixed. A similar treatment can be given to $[T, \infty) \times S^{N-1}$. Finally we evaluate the norm of the function w in $(-1-T, 1+T) \times S^{N-1}$, we simply use $e^{\delta T}$ times the norm of w in $\mathcal{C}^{k,\alpha}_\delta([-1-T, 1+T] \times S^{N-1})$, where the constant $\frac{4-N}{2} < \delta < \frac{N-4}{2}$ is fixed.

In other words, what we have simply done is a patching of two copies of $\mathcal{C}^{k,\alpha}_\nu((\mathbb{R}^N - \{0\}) - B_{1/2}(q_0))$ with the space $\mathcal{C}^{k,\alpha}_\delta([-1-T, 1+T] \times S^{N-1})$, using conformal diffeomorphisms. When $k = 0$, this space will be denoted by $\mathcal{F}^{0,\alpha}_{\nu,\delta,T}$. And, when $k = 2$ we shall instead consider the space obtained by patching of two copies of $\mathcal{C}^{k,\alpha}_\nu((\mathbb{R}^N - \{0\}) - B_{1/2}(q_0)) \oplus E_0$ with the space $\mathcal{C}^{k,\alpha}_\delta([-1+T, T+1] \times S^{N-1})$, using conformal diffeomorphisms. With slight abuse of notations, this space will be denoted by $\mathcal{F}^{2,\alpha}_{\nu,\delta,T}$.

Given any function $f \in \mathcal{F}^{0,\alpha}_{\nu-4,\delta,T}$, we would like to solve the linear problem

$$\mathcal{L}_{v_T} w = f \tag{45}$$

where $\mathcal{L}_{v_T}$ is the linearized Q^N-curvature operator about v_T. To this aim, we use the fact that, thanks to the result of Proposition 2, we can solve

$$\mathcal{L}_{v_1} w_1 = \chi f$$

and we can also solve

$$\mathcal{L}_{v_2} w_2 = (1-\chi)\, f$$

where $\mathcal{L}_v$ denotes the linearized Q^N-curvature operator about v. We can therefore define the function

$$w = \chi_{1,T}\, w_1 + \chi_{2,T}\, w_2, \tag{46}$$

which clearly belongs to $\mathcal{F}^{2,\alpha}_{\nu,\delta,T}$.

In the next result, we show that w is almost a solution of the linear problem (45) and give a quantitative version of this fact.

Lemma 6 *Assume that $\nu \in (\frac{4-N}{2}, \frac{6-N}{2})$ and $\delta \in (0, \frac{N-4}{2})$ are fixed. Then*

$$||\mathcal{L}_{v_T} w - f||_{\mathcal{F}^{0,\alpha}_{\nu-4,\delta,T}} \leq C\, \left((\cosh T)^{-2\delta} + (\cosh T)^{-4}\right)\, ||f||_{\mathcal{F}^{0,\alpha}_{\nu-4,\delta,T}}$$

Proof Observe that we have

$$\mathcal{L}_{v_T} w - f = 0$$

away from $[-T,T]\times S^{N-1}$. So it only remains to estimate this function in this later set. In $[2-T,-1]\times S^{N-1}$, we can compute explicitly

$$\mathcal{L}_{v_T} w - f = \frac{N+4}{N-4}\left((v_1+v_2)^{\frac{8}{N-4}}(w_1+w_2) - w_1 - v_2^{\frac{8}{N-4}}\, w_2\right).$$

Taking into account that v_2/v_1 is bounded by a constant independent of T in $[1-T,0]\times S^{N-1}$, we can use Taylor's expansion, (39) and (41), together with the estimates

$$||w_1||_{\mathcal{C}^{2,\alpha}_{\delta}([-T,0]\times S^{N-1})} \leq C\, e^{\delta T}\, ||f||_{\mathcal{F}^{0,\alpha}_{\nu-4,\delta,T}}$$

while

$$||w_2||_{\mathcal{C}^{2,\alpha}_{-\delta}([-T,0]\times S^{N-1})} \leq C\, e^{-\delta T}\, ||f||_{\mathcal{F}^{0,\alpha}_{\nu-4,\delta,T}}$$

to find, with little work, that

$$||\mathcal{L}_v w - f||_{\mathcal{C}^{0,\alpha}_{\delta}([2-T,-1]\times S^{N-1})} \leq C\, \left((\cosh T)^{-2\delta} + (\cosh T)^{4-N} + (\cosh T)^{-4}\right).$$

Similar estimates hold on $[1,T-2]\times S^{N-1}$. Finally, one checks that the cutoff functions which are used in $[-T,2-T)\times S^{N-1}$, in $(-1,1)\times S^{N-1}$ or in $(T-2,T]\times S^{N-1}$ do not alter the above estimate. ◇

Thanks to the previous result, we can use a standard perturbation argument to show the :

Corollary 2 *Assume that $\nu \in (\frac{4-N}{2}, \frac{6-N}{2})$ and $\delta \in (0, \frac{N-4}{2})$ are fixed. There exists $T_0 > 0$ such that, for all, if T is chosen large enough, the operator*

$$\mathcal{L}_{v_T} : \mathcal{F}^{2,\alpha}_{\nu,\delta,T} \longrightarrow \mathcal{F}^{0,\alpha}_{\nu-4,\delta,T}$$

is an isomorphism the norm of whose inverse is bounded by some constant which does not depend on $T \geq T_0$.

5 The nonlinear perturbation argument

Granted the results of the previous section, we now perturb the function v_T by some small function w so that $v_T + w$ is a solution of $\mathcal{N}(v_T + w) = 0$.

Again it is easier to decompose $\mathbb{R} \times S^{N-1}$ into three overlapping pieces as we have done in the previous section. In particular, in $[-1 - T, T + 1] \times S^{N-1}$ the equation we have to solve is precisely

$$\mathcal{N}(v_T + w) = 0$$

while, by conformal transformation, the equation we have to solve on $(-\infty, -T) \times S^{N-1}$ is equivalent to

$$\mathcal{N}(v_T + w) := \Delta^2(V_T + W) + (V_T + W)^{\frac{N+4}{N-4}} = 0$$

in $(\mathbb{R}^N - \{0\}) - B_{1/2}(q_0)$.

Unfortunately, when $W \in \mathcal{C}^{2,\alpha}_{\nu}((\mathbb{R}^N - \{0\}) - B_{1/2}(q_0)) \oplus E_0$, it is not true that $\mathcal{N}(v_T + w) \in \mathcal{C}^{0,\alpha}_{\nu-4}((\mathbb{R}^N - \{0\}) - B_{1/2}(q_0))$ because the elements of E_0 which force the nonlinearity to belong to $\mathcal{C}^{0,\alpha}_{\frac{N-4}{2}-4}((\mathbb{R}^N - \{0\}) - B_{1/2}(q_0))$.

To overcome this difficulty, we use the idea developed by R. Mazzeo, D. Pollack and K. Uhlenbeck in [20] to overcome a similar difficulty in the scalar curvature problem.

Recall that there exists a one parameter family $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ of periodic solution of (10) near the constant solution v_0. Now, if $a = \xi \cos\phi$ and $b = \xi \sin\phi$,, we define

$$U(a, b)(t) := |x|^{\frac{4-N}{2}} v_\xi(-\log|x| + \phi).$$

Which is a solution of (4). Furthermore, differentiation with respect to a and b, at $a = b = 0$ just yield the Jacobi fields $|x|^{\frac{4-N}{2}} \Phi^{0,\pm}(-\log|x|)$.

The idea is to decompose any function $W \in \mathcal{C}^{2,\alpha}_{\nu}((\mathbb{R}^N - \{0\}) - B_{1/2}(q_0)) \oplus E_0$ as

$$W(x) = V(x) + a_\pm \eta(-\log|x|)|x|^{\frac{4-N}{2}} \Phi^{0,\pm}(-\log|x|)$$

where $V \in \mathcal{C}^{2,\alpha}_{\nu}((\mathbb{R}^N - \{0\}) - B_{1/2}(q_0))$, and to replace it by

$$\tilde{W}(x) = V(x) + \eta(-\log|x|)\, U(a_+, a_-)(x).$$

Abusing notations slightly, the problem we have to solve in $(\mathbb{R}^N - \{0\}) - B_{1/2}(q_0)$, now reads

$$\mathcal{N}(v_T + w) := \Delta^2(V_T + \tilde{W}) + (V_T + \tilde{W})^{\frac{N+4}{N-4}} = 0$$

This time this is a well defined nonlinear operator between $\mathcal{C}^{2,\alpha}_{\nu}((\mathbb{R}^N - \{0\}) - B_{1/2}(q_0)) \oplus E_0$ and $\mathcal{C}^{0,\alpha}_{\nu-4}((\mathbb{R}^N - \{0\}) - B_{1/2}(q_0))$. Furthermore, the linearization of this operator about $w = 0$ is nothing but the operator L.

We solve the nonlinear equation by using a fixed point algorithm. Indeed, we can write

$$\mathcal{N}(v_T + w) = \mathcal{N}(v_T) + \mathcal{L}_{v_T} w + Q_T(w)$$

where Q collects all the nonlinear terms. Assume that $\frac{4-N}{2} < \nu < \frac{6-N}{2}$ and $\delta \in (0, \frac{N-4}{2})$ are fixed. We can use the result of Proposition 4. We rephrase this as a fixed point problem

$$w = -\mathcal{L}_{v_T}^{-1} \left(\mathcal{N}(v_T) + \mathcal{L}_{v_T} w + Q_T(w)\right)$$

and now it is a simple exercise, using Lemma 5 and Corollary 2, to show that there exists $R > 0$ and $\tilde{T}_0 > 0$ such that for all $T > \tilde{T}_0$, the mapping

$$w \longrightarrow -\mathcal{L}_{v_T}^{-1} \left(\mathcal{N}(v_T) + \mathcal{L}_{v_T} w + Q_T(w)\right)$$

is a contraction from the ball of radius $R(e^{-2\delta T} + e^{-4T})$ in $\mathcal{F}^{2,\alpha}_{\nu,\delta,T}$ into itself and hence has a unique fixed point in this ball. This proves the existence of a solution of (4) and hence completes the proof of the result in the case of 4 singularities. The general case can be handled similarly with more involved notations.

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