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Removable Singularities and Existence for a Quasilinear Equation with Absorption or Source Term and Measure Data

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Abstract

We study the properties of the following equations with absorption or source term

$$-\Delta_p u = \pm |u|^{q-1} u + \mu,$$

in a domain Ω of $\mathbb{R}^N$, where $1 < p < N$, $q > p-1$, and μ is a Radon measure on Ω. We introduce a notion of local entropy solution, and give necessary conditions on μ for the existence of solutions, in terms of capacity. The question of removability of sets is also considered, as well as some stability results. Finally we give existence results in $\mathbb{R}^N$ for the case of absorption.

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1 Introduction

Let Ω be a regular domain in $\mathbb{R}^N$, which may be unbounded, and μ a Radon measure in Ω. We are interested in the elliptic equation with an absorption term:

$$-\Delta_p u + |u|^{q-1} u = \mu, \qquad \text{in } \Omega, \tag{1.1}$$

as well as a source term

$$-\Delta_p u = |u|^{q-1}u + \mu, \qquad \text{in } \Omega, \tag{1.2}$$

where $1 < p < N$, $q > p-1$, and

$$u \longmapsto \Delta_p u = div(|\nabla u|^{p-2}\nabla u)$$

is the p-Laplace operator. We study under what conditions there exist local or global solutions, or under what conditions a given subset $E \subset \Omega$ is removable. They are expressed in terms of capacity. This leads us to return to the equation without reaction term,

$$-\Delta_p u = \mu, \qquad \text{in } \Omega, \tag{1.3}$$

for which we define a notion of *local entropy solution*, and give stability properties, essential to our proofs.

We denote by $\mathcal{M}(\Omega)$ the set of all the Radon measures in Ω , $\mathcal{M}_b(\Omega)$ the space of bounded Radon measures in Ω, and by $\mathcal{M}^+(\Omega)$, $\mathcal{M}_b^+(\Omega)$) the respective subsets of nonnegative ones. The capacity $cap_{m,r}$ associated to $W_0^{m,r}(\Omega)$, for any $m \geq 1, r > 1$ is defined by

$$cap_{m,r}(K,\Omega) = \inf\left\{\|\psi\|^r_{W_0^{m,r}(\Omega)} : \psi \in \mathcal{D}(\Omega), 0 \leq \psi \leq 1, \psi = 1 \text{ on } K\right\},$$

for any compact set $K \subset \Omega$.

We first consider the problem of finding conditions on the measure μ ensuring the existence of a solution. In the case $p = 2$, a necessary and sufficient condition was found in [3] for the problem with absorption with Dirichlet data on $\partial\Omega$: for any $\mu \in \mathcal{M}_b(\Omega)$, problem

$$\begin{cases} -\Delta u + |u|^{q-1}u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

has a (weak) solution if and only if μ does not charge the sets E such that $cap_{2,q'}(E, \mathbb{R}^N) = 0$. In the case of the problem with a source term, this condition is also necessary. A necessary and sufficient condition was given in [4] for the existence of (integral) solutions of problem

$$\begin{cases} -\Delta u = (u^+)^q + \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

When μ has a compact support and $\mu \geq 0$, it is equivalent to the existence of a constant $C > 0$, such that

$$\int_K d\mu \leq C\, cap_{2,q'}(K, \mathbb{R}^N), \qquad \text{for any compact set } K \subset \Omega,$$

(see [2]). It implies in particular a limitation of the size of the measure. In the general case $p > 1$, the question becomes more difficult, because the full duality argument used in [3] and [4] is no longer available. In the sequel we set

$$q^* = q/(q-p+1), \tag{1.5}$$

so that $q^* = q'$ when $p = 2$. Concerning problem (1.1) with Dirichlet data, it was recently shown in [21] that if μ charges the sets E such that $cap_{1,R}(E, \mathbb{R}^N) = 0$, for some $R > pq^*$, then sequences of approximate solutions do not converge to a "reasonable" solution. This suggested that the problem might have no solution. Using our notion of local entropy solution, we show that the result is true. Actually, we prove a more general statement:

Theorem 1.1 *Let $\mu \in \mathcal{M}(\Omega)$, and $q > p - 1$. Suppose that μ charges some set E such that $cap_{1,R}(E, \mathbb{R}^N) = 0$ for some $R > pq^*$. Then equations (1.1) and (1.2) admit no local entropy solution. More generally, there exists no local entropy solution to equation (1.3) such that $|u|^q \in L^1_{loc}(\Omega)$.*

Notice that there is no assumption on the sign of u and μ, which is unusual in the case of source term. This result applies in the supercritical case $q \geq \overline{P}$, where

$$\overline{P} = \frac{N(p-1)}{N-p} \tag{1.6}$$

is the first critical exponent (indeed in the subcritical case $q < \overline{P}$, any non-empty set $E \subset \Omega$ satisfies $cap_{1,R}(E, \mathbb{R}^N) > 0$). As a consequence, equations (1.1) or (1.2) have no solution if μ charges points and $q > \overline{P}$. Notice that when $p = 2$, any set E, such that $cap_{1,R}(E, \mathbb{R}^N) = 0$ for some $R > 2q'$, satisfies $cap_{2,q'}(E, \mathbb{R}^N) = 0$, from [1].

Under the assumption of positivity on u and μ, we have a stronger result for equation (1.2):

Theorem 1.2 *Let $\mu \in \mathcal{M}^+(\Omega)$. If equation*

$$-\Delta_p u = u^q + \mu, \qquad \text{in } \Omega, \tag{1.7}$$

has a nonnegative local entropy solution, then for any $R > pq^*$,

$$\int_K d\mu \leq C \, (cap_{1,R}(K, \Omega))^{pq^*/R}, \qquad \text{for any compact set } K \subset \Omega, \tag{1.8}$$

where $C = C(N, p, q, R, \Omega)$.

The second problem that we consider is the characterization of removable sets. When $p = 2$, it was shown in [3] that they are precisely the sets E with $cap_{2,q'}(E, \mathbb{R}^N) = 0$. In the general case $p > 1$, a recent result of [22], for problem (1.1) with Dirichlet data, suggested that the compact sets K such that $cap_{1,R}(K, \mathbb{R}^N) = 0$, for some $R > pq^*$, might be removable. We show that it is true:

Theorem 1.3 *Let F be a relatively closed set in Ω, such that $cap_{1,R}(F, \mathbb{R}^N) = 0$ for some $R > pq^*$. Let $\mu \in \mathcal{M}(\Omega)$ such that μ does not charge the set F. Then F is removable, in the sense that any local entropy solution of equation*

$$-\Delta_p u + |u|^{q-1} u = \mu, \qquad \text{in } \Omega \backslash F,$$

is a local entropy solution of equation (1.1).

In particular any point set is removable when $q > \overline{P}$. We have a similar result for problem with a source term:

Theorem 1.4 *Let F be a relatively closed set in Ω, such that $cap_{1,R}(F, \mathbb{R}^N) = 0$ for some $R > pq^*$. Let $\mu \in \mathcal{M}^+(\Omega)$ such that does not charge the set F. Then any local nonnegative entropy solution of equation*

$$-\Delta_p u = u^q + \mu, \qquad in\ \Omega\backslash F,$$

is a local entropy solution of equation (1.7).

These results are based on local a priori estimates of the solutions (given in Theorems 4.1 and 4.2). In the case of equation (1.1), we also prove the following stability result:

Theorem 1.5 *Let F be a relatively closed set in Ω, such that $cap_{1,R}(F, \mathbb{R}^N) = 0$ for some $R > pq^*$. Let $f_\nu, f \in L^1_{loc}(\Omega)$ such that f_ν converges strongly to f in $L^1_{loc}(\Omega\backslash F)$. Let u_ν be a local entropy solution of equation*

$$-\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = f_\nu, \qquad in\ \Omega\backslash F.$$

Then up to a subsequence, u_ν converges a.e. in Ω to a local entropy solution u of equation (1.1).

We also prove stability properties for global solutions when Ω is bounded (see Theorem 7.3). Thus we improve a result of [16] given for $p = 2$, and the result of [22] quoted above. The proof is based upon a local stability result for equation (1.3) (see Theorem 3.3), following the global one of [18, Theorem 3.4]. Notice that Theorem 1.5 has no equivalent in the case of source term; this is due to the fact that equation (1.7) may have no solution in the supercritical case for some functions $\mu \in L^1(\Omega)$, independently of their size.

Another consequence of the local a priori estimates and stability properties is an existence result in whole $\mathbb{R}^N$ for the equation (1.1), without any growth condition on the data. This improves the results of [14]:

Theorem 1.6 *Assume $q > p - 1 > 0$. Then for every $f \in L^1_{loc}(\mathbb{R}^N)$, there exists at least one local entropy solution u of equation*

$$-\Delta_p u + |u|^{q-1} u = f, \qquad in\ \mathbb{R}^N. \tag{1.9}$$

Moreover this entropy solution is nonnegative when f is nonnegative.

We also give existence results in the subcritical case $q < \overline{P}$ with a measure data when Ω is bounded, or $\Omega = \mathbb{R}^N$ (see Theorems 8.2 and 8.3). For the equations (1.2) and (1.7), the existence was proved in [20] when $q < \overline{P}$, at least when $p > P_0$, and Ω, μ are bounded, and the size of μ is small enough. The problem is open in the case $q \geq \overline{P}$, even when $\mu \in L^s(\Omega)$ with s large enough.

For simplicity, all our results are given for the p-Laplace operator, with reaction term $\pm |u|^{q-1} u$, but they can be extended to elliptic operators $A(x, \nabla u)$, with power growth in $|\nabla u|$ of the order $p-1$, and a reaction term $\pm g(x, u)$ such that $g(x, u)u \geq 0$ and growth in u like $|u|^q$.

2 Global and local entropy solutions

In this section, we study the equation (1.3) without reaction term. We recall some well-known results concerning the Dirichlet problem associated to the equation when Ω is bounded and $\mu \in \mathcal{M}_b(\Omega)$. Then we define a notion of local entropy solution without boundedness assumptions, which leads to our main results.

We call $\mathcal{M}_0(\Omega)$, the set of measures $\mu_0 \in \mathcal{M}(\Omega)$ such that

$$\mu_0(B) = 0 \qquad \text{for any Borel set } B \subset \Omega \text{ such that } cap_{1,p}(B, \Omega) = 0. \tag{2.1}$$

Any measure $\mu \in \mathcal{M}(\Omega)$ admits a unique decomposition as

$$\mu = \mu_0 + \mu_s^+ - \mu_s^- \tag{2.2}$$

where $\mu_0 \in \mathcal{M}_0(\Omega)$ and $\mu_s^\pm$ are nonnegative and singular, concentrated on sets $E^\pm$ with $cap_{1,p}(E^\pm, \mathbb{R}^N) = 0$ (see [19]). Moreover μ_0 is nonnegative, and $\mu_s^- = 0$, if μ is nonnegative. Notice also that

$$\mu_s^+ \le \mu^+, \quad \mu_s^- \le \mu^- \quad \text{and} \quad |\mu_0| \le |\mu|. \tag{2.3}$$

For any $n, k > 0$ and any $r \in \mathbb{R}$, we set

$$\begin{aligned} &S_{n,k}(r) = \min((|r| - n)^+/k, 1) \text{ sign } r, \\ &h_{n,k}(r) = \min((r - n)^+/k, 1), \qquad H_{n,k}(r) = 1 - \min((|r| - n)^+/k, 1). \end{aligned} \tag{2.4}$$

2.1 Global entropy solutions

Here it is assumed that Ω is bounded and $\mu \in \mathcal{M}_b(\Omega)$. We consider the problem

$$\begin{cases} -\Delta_p u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

We set

$$P_0 = \frac{2N}{N+1}, \qquad P_1 = 2 - \frac{1}{N},$$

thus $1 < P_0 < P_1$, and $p > P_0 \iff \overline{P} > 1$, where $\overline{P}$ is given by (1.6). When $p > P_1$, problem (2.5) admits at least a solution in the sense of distributions, such that $u \in W_0^{1,m}(\Omega)$ for any $1 \le m < (p-1)N/(N-1)$. In the general case the gradient may not be defined in $L^1(\Omega)$. One needs to introduce a notion of entropy solutions, or renormalized solutions, to give a sense to the gradient. They are functions such that $\nabla T_k(u) \in L^p(\Omega)$ for any $k > 0$, where

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k \text{ sign } s & \text{if } |s| > k, \end{cases} \tag{2.6}$$

and the gradient of u, denoted by $y = \nabla u$ is defined by

$$\nabla(T_k(u)) = y \times 1_{\{|u| \le k\}} \quad \text{a.e. in } \Omega. \tag{2.7}$$

Such solutions u may not belong to $L^1(\Omega)$ when $p \leq P_0$.

More precisely, a function u is a (global) entropy solution of problem (2.5), if

$$u \text{ is measurable and finite a.e. in } \Omega, \tag{2.8}$$

$$T_k(u) \in W_0^{1,p}(\Omega) \qquad \text{for every } k > 0, \tag{2.9}$$

$$|\nabla u|^{p-1} \in L^r(\Omega), \qquad \text{for any } 1 \leq r < N/(N-1), \tag{2.10}$$

where the gradient is defined by (2.7), and u satisfies

D1: *for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and any $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, such that $h(u)\varphi \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla(h(u)\varphi)dx = \int_\Omega h(u)\varphi d\mu_0 + h(+\infty)\int_\Omega \varphi d\mu_s^+ - h(-\infty)\int_\Omega \varphi d\mu_s^-. \tag{2.11}$$

The solutions can be defined in four equivalent ways:

Theorem 2.1 ([18]) *Let u be a function such that (2.8), (2.9) and (2.10) hold. Then **D1** is equivalent to any of the following conditions:*

D2: *if $\omega \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and if there exists $k > 0$ and $\omega^+, \omega^- \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$ with $r > N$, such that $\omega = \omega^+$ a.e. on the set $\{u > k\}$ and $\omega = \omega^-$ a.e. on the set $\{u < -k\}$, then*

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\omega dx = \int_\Omega \omega d\mu_0 + \int_\Omega \omega^+ d\mu_s^+ - \int_\Omega \omega^- d\mu_s^-. \tag{2.12}$$

D3: *for any $k > 0$, there exist $\alpha_k, \beta_k \in \mathcal{M}_0(\Omega) \cap \mathcal{M}_b^+(\Omega)$, concentrated on the sets $\{u = k\}$ and $\{u = -k\}$ respectively, converging weakly to μ_s^+, μ_s^- such that*

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k.\nabla\psi dx = \int_{\{|u|\leq k\}} \psi d\mu_0 + \int_\Omega \psi d\alpha_k - \int_\Omega \psi d\beta_k, \tag{2.13}$$

for any $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

D4: *for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support, and any $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $h(u)\varphi \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla(h(u)\varphi)dx = \int_\Omega h(u)\varphi d\mu_0, \tag{2.14}$$

and for any $\varphi \in C(\Omega)$ and bounded,

$$\lim_{n\to\infty} \frac{1}{n}\int_{\{n\leq u\leq 2n\}} |\nabla u|^p \varphi dx = \int_\Omega \varphi d\mu_s^+,$$

$$\lim_{n\to\infty} \frac{1}{n}\int_{\{-2n\leq u\leq -n\}} |\nabla u|^p \varphi dx = \int_\Omega \varphi d\mu_s^-. \tag{2.15}$$

Then for any $p > 1$, there exists at least an entropy solution of (2.5), and it is unique if $\mu \in L^1(\Omega)$. Moreover any entropy solution satisfies the equation in the sense of distributions.

Remark 2.1. In the definition of global entropy solution, (2.10) can be weakened in

$$|\nabla u|^{p-1} \in L^1(\Omega), \tag{2.16}$$

which also implies

$$|u|^{p-1} \in L^s(\Omega), \qquad \text{for any } 1 \leq s < N/(N-p), \tag{2.17}$$

from [5, Lemma 4.1]. Also condition (2.11) has to be satisfied only for any $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi \geq 0$. Indeed this implies that **D3** holds for such a φ, and by density for any nonnegative $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then for any $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Thus **D3** holds, hence also **D1.**

Remark 2.2 The different notions of entropy solutions are given in general for bounded measures, since they appear to be a natural framework for existence theorems. Notice however that when $p = 2$, the problem

$$\begin{cases} -\Delta u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

is well posed for possibly unbounded measures μ, such that $\int_\Omega \rho(x) d\mu(x) < \infty$, where $\rho(x)$ is the distance from x to $\partial\Omega$. In the following we are interessed by local solutions as well as global ones with Dirichlet data. When it is possible, it will not be required that the measures in consideration are bounded, contrarily to most of the litterature on the subject.

2.2 Local entropy solutions

Here Ω is any domain of $\mathbb{R}^N$,and $\mu \in \mathcal{M}(\Omega)$, possibly unbounded. We will say that u is a local entropy solution of equation (1.3), if u satisfies (2.8),

$$T_k(u) \in W^{1,p}_{loc}(\Omega), \qquad \text{for any } k > 0, \tag{2.18}$$

$$|u|^{p-1} \in L^s_{loc}(\Omega), \qquad \text{for any } 1 < s < N/(N-p), \tag{2.19}$$

$$|\nabla u|^{p-1} \in L^r_{loc}(\Omega), \qquad \text{for any } 1 \leq r < N/(N-1), \tag{2.20}$$

and

D1-loc: *for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and any $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, with compact support, such that $h(u)\varphi \in W^{1,p}(\Omega)$,*

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla(h(u)\varphi) dx = \int_\Omega h(u)\varphi d\mu_0 + h(+\infty) \int_\Omega \varphi d\mu_s^+ - h(-\infty) \int_\Omega \varphi d\mu_s^-. \tag{2.21}$$

Here also we can give equivalent definitions.

Theorem 2.2 *Let u be a function such that (2.8), (2.19), (2.18) and (2.20) hold. Then* ***D1-loc*** *is equivalent to any of the following conditions:*

D2-loc: *if $\omega \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω, and if there exists $k > 0$ and $\omega^+, \omega^- \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$ with $r > N$, such that $\omega = \omega^+$ a.e. on the set $\{u > k\}$ and $\omega = \omega^-$ a.e. on the set $\{u < -k\}$, then*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \omega dx = \int_\Omega \omega d\mu_0 + \int_\Omega \omega^+ d\mu_s^+ - \int_\Omega \omega^- d\mu_s^-. \tag{2.22}$$

D3-loc: *there exist $\alpha_k, \beta_k \in \mathcal{M}_0(\Omega) \cap \mathcal{M}^+(\Omega)$, concentrated on the sets $\{u = k\}$ and $\{u = -k\}$ respectively, converging weakly to μ_s^+, μ_s^- such that*

$$\int_\Omega |\nabla u_k|^{p-2} \nabla u_k.\nabla \psi dx = \int_{\{|u| \le k\}} \psi d\mu_0 + \int_\Omega \psi d\alpha_k - \int_\Omega \psi d\beta_k, \tag{2.23}$$

for any $\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω.

D4-loc: *for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support, and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω, such that $h(u)\varphi \in W^{1,p}(\Omega)$,*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla (h(u)\varphi) dx = \int_\Omega h(u)\varphi d\mu_0, \tag{2.24}$$

and for any $\varphi \in C(\Omega)$ with compact support in Ω,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \int_{\{n \le u \le 2n\}} |\nabla u|^p \varphi dx &= \int_\Omega \varphi d\mu_s^+, \\ \lim_{n\to\infty} \frac{1}{n} \int_{\{-2n \le u \le -n\}} |\nabla u|^p \varphi dx &= \int_\Omega \varphi d\mu_s^-. \end{aligned} \tag{2.25}$$

The proof follows the one of [18], with some modifications due to the fact that Lemma 4.3 of [17] does not apply. It is given in Appendix A for a better comprehension.

2.3 Solutions with a reaction term

We end this section by defining solutions of equations with a reaction term. Let f be a Caratheodory function on $\Omega \times \mathbb{R}$. For a bounded domain Ω, and $\mu \in \mathcal{M}_b(\Omega)$, an entropy solution u of problem

$$\begin{cases} -\Delta_p u = f(x,u) + \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

is a function u such that $f(x,u) \in L^1(\Omega)$ and u is an entropy solution of problem in the sense **D1**. For a general domain Ω and $\mu \in \mathcal{M}(\Omega)$, a local entropy solution of equation

$$-\Delta_p u = f(x,u) + \mu, \qquad \text{in } \Omega,$$

will be a function u such that $f(x,u) \in L^1_{loc}(\Omega)$ and u is a local entropy solution of equation in the sense **D1-loc.**

3 Local solutions without absorption

In this section we show that the assumptions on the local entropy solutions of equation (1.3) can be weakened. *This is the key point for all the sequel.* Recall that Ω may be unbounded.

Theorem 3.1 *Let u be a function satisfying (2.8), (2.18), and*

$$|u|^q \in L^1_{loc}(\Omega), \text{ for some } q > p-1, \tag{3.1}$$

$$|\nabla u|^{p-1} \in L^1_{loc}(\Omega), \tag{3.2}$$

*and **D1-loc**. Then*

$$(|u|+1)^{\alpha-1}|\nabla u|^p \in L^1_{loc}(\Omega), \qquad \text{for any } \alpha < 0, \tag{3.3}$$

and u satisfies (2.19) and (2.20). Moreover

$$|\nabla u|^{p-1} \in L^\sigma_{loc}(\Omega), \qquad \text{for any } 1 \le \sigma < p'q/(q+1). \tag{3.4}$$

Proof **Step 1: Estimate (3.3).** Let $\alpha < 0$. We set $u_k = T_k(u)$, for any $k > 0$. We take

$$h_k(u) = (1 - (|u_k|+1)^\alpha) \operatorname{sign} u \tag{3.5}$$

in (2.21), and get for any $\varphi \in \mathcal{D}(\Omega), \varphi \ge 0$,

$$\begin{aligned}
&|\alpha| \int_\Omega (|u_k|+1)^{\alpha-1} |\nabla u_k|^p \varphi dx \\
&= -\int_\Omega h_k(u) \, |\nabla u|^{p-2} \nabla u . \nabla \varphi dx + \int_\Omega h_k(u) \varphi d\mu_0 + (1-(k+1)^\alpha) \int_\Omega \varphi d\,|\mu_s| \\
&\le \int_\Omega |\nabla u_k|^{p-1} |\nabla \varphi| \, dx + \int_{\{|u|>k\}} |\nabla u|^{p-1} |\nabla \varphi| \, dx + \int_\Omega \varphi d\,|\mu| \,.
\end{aligned} \tag{3.6}$$

Now from Hölder inequality,

$$\begin{aligned}
&\int_\Omega |\nabla u_k|^{p-1} |\nabla \varphi| \, dx \\
&= \int_\Omega (|u_k|+1)^{(\alpha-1)/p'} |\nabla u_k|^{p-1} (|u_k|+1)^{(1-\alpha)/p'} |\nabla \varphi| \, dx \\
&\le \left(\int_\Omega (|u_k|+1)^{\alpha-1} |\nabla u_k|^p \ \varphi dx \right)^{1/p'} \left(\int_\Omega (|u_k|+1)^{(1-\alpha)(p-1)} \varphi^{1-p} |\nabla \varphi|^p \, dx \right)^{1/p} .
\end{aligned} \tag{3.7}$$

Since u satisfies (3.1), we can fix α such that

$$\tau = q/(p-1)(1-\alpha) > 1. \tag{3.8}$$

Then we derive

$$\int_\Omega |\nabla u_k|^{p-1} |\nabla \varphi| \, dx$$
$$\leq \left(\int_\Omega (|u_k|+1)^{\alpha-1} |\nabla u_k|^p \ \varphi dx \right)^{1/p'} \left(\int_\Omega (|u_k|+1)^q \varphi dx \right)^{1/\tau p} \times$$
$$\left(\int_\Omega \varphi^{1-\tau' p} |\nabla \varphi|^{\tau' p} \, dx \right)^{1/\tau' p}; \tag{3.9}$$

therefore

$$\int_\Omega |\nabla u_k|^{p-1} |\nabla \varphi| \, dx \leq \frac{|\alpha|}{2} \int_\Omega (|u_k|+1)^{\alpha-1} |\nabla u_k|^p \varphi dx$$
$$+ C \left(\int_\Omega (|u|+1)^q \varphi dx \right)^{1/\tau} \times \left(\int_\Omega \varphi^{1-\tau' p} |\nabla \varphi|^{\tau' p} \, dx \right)^{1/\tau'}, \tag{3.10}$$

where $C = C(\alpha)$. Reporting (3.10) into (3.6), we get

$$\frac{|\alpha|}{2} \int_\Omega (|u_k|+1)^{\alpha-1} |\nabla u_k|^p \varphi dx \leq \int_{\{|u|>k\}} |\nabla u|^{p-1} |\nabla \varphi| \, dx + \int_\Omega \varphi d|\mu|$$
$$+ C \left(\int_\Omega (|u|+1)^q \varphi dx \right)^{1/\tau} \times \left(\int_\Omega \varphi^{1-\tau' p} |\nabla \varphi|^{\tau' p} \, dx \right)^{1/\tau'}. \tag{3.11}$$

Letting k tend to ∞, we obtain, from (3.2),

$$\frac{|\alpha|}{2} \int_\Omega (|u|+1)^{\alpha-1} |\nabla u|^p \varphi dx \leq C \left(\int_\Omega (|u|+1)^q \varphi dx \right)^{1/\tau} \left(\int_\Omega \varphi^{1-\tau' p} |\nabla \varphi|^{\tau' p} \, dx \right)^{1/\tau'}$$
$$+ \int_\Omega \varphi d|\mu|. \tag{3.12}$$

Moreover, from (3.9),

$$\int_\Omega |\nabla u|^{p-1} |\nabla \varphi| \, dx$$
$$\leq \left(\int_\Omega (|u|+1)^{\alpha-1} |\nabla u|^p \ \varphi dx \right)^{1/p'} \left(\int_\Omega (|u|+1)^q \varphi dx \right)^{1/\tau p} \times$$
$$\left(\int_\Omega \varphi^{1-\tau' p} |\nabla \varphi|^{\tau' p} \, dx \right)^{1/\tau' p}, \tag{3.13}$$

where the gradient is defined in (2.7). Taking

$$\varphi = \zeta^{p\tau'}, \quad \text{with } \zeta \in \mathcal{D}(\Omega), \zeta \geq 0, \tag{3.14}$$

so that $\varphi^{1-\tau' p}\,|\nabla\varphi|^{\tau' p} \in L^1(\Omega)$, we deduce that $(|u|+1)^{\alpha-1}|\nabla u|^p \in L^1_{loc}(\Omega)$ for $|\alpha|$ small enough. This implies (3.3).

Step 2: other estimates. Let U be any regular domain such that $U \subset\subset \Omega$. For any $v \in L^1_{loc}(\Omega)$ we set

$$\overline{v} = |U|^{-1} \int_U v dx.$$

Recall the Galliardo-Niremberg estimate: for any $\lambda \geq 1$, for any $v \in W^{1,p}(U) \cap L^\lambda(U)$,

$$\|v - \overline{v}\|_{L^\gamma(U)} \leq C\, \||\nabla v|\|^\theta_{L^p(U)}\, \|v - \overline{v}\|^{1-\theta}_{L^\lambda(U)}, \tag{3.15}$$

for any $\gamma \in [1, +\infty)$ and $\theta \in [0,1]$ such that

$$\frac{1}{\gamma} = \theta\left(\frac{1}{p} - \frac{1}{N}\right) + \frac{1-\theta}{\lambda}, \tag{3.16}$$

where $C = C(N, p, \lambda, \theta, U)$. Let us take $\alpha \in (1-p, 0)$ and

$$v = (1+|u|)^\beta, \qquad \beta = (\alpha + p - 1)/p, \qquad \lambda = (p-1)/\beta. \tag{3.17}$$

Then $v \in L^p(U)$, since $\beta p < p - 1 < q$ and $u \in L^q(U)$. Moreover $u_k = T_k(u) \in W^{1,p}(U)$, then from the chain rule, $v_{k+1} = T_{k+1}(v) = (1+|u_k|)^\beta \in W^{1,p}(U)$, and

$$\nabla v_{k+1} = \beta(|u_k|+1)^{\beta-1} \nabla u_k \text{ sign } u_k.$$

By definition of the gradient, we have $\nabla v_{k+1} = \nabla v \times 1_{\{|u| \leq k\}}$, hence

$$\int_{U \cap \{|u| \leq k\}} |\nabla v|^p\, dx = \int_U |\nabla v_{k+1}|^p\, dx \leq \beta^p \int_U (|u_k|+1)^{\alpha-1} |\nabla u_k|^p dx.$$

Since $(u+1)^{\alpha-1}|\nabla u|^p \in L^1(U)$ from (3.3), we can let k go to ∞, and derive $|\nabla v| \in L^p(U)$. This implies $v \in W^{1,p}(U)$, and the gradient above coincides with the distributional gradient of v (see [18, Remark 2.10]). Moreover $u \in L^{\beta\gamma}(U)$, since

$$\begin{aligned} &\int_U (|u|+1)^\beta - \overline{v}^\gamma\, dx \\ &\leq c^\gamma \left(\int_U (|u|+1)^{\alpha-1} |\nabla u|^p\, dx\right)^{\theta\gamma/p} \left(\int_U (|u|+1)^\beta - \overline{v}^{(p-1)/\beta}\, dx\right)^{(1-\theta)\gamma\beta/(p-1)}, \end{aligned} \tag{3.18}$$

and $|u|^{p-1} \in L^1(U)$. Chosing θ close to 1, and α close to 0, we deduce (2.19). Also, for any $0 < \eta < p$, we find

$$\int_\Omega |\nabla u|^\eta\, dx \leq \left(\int_U (|u|+1)^{\alpha-1} |\nabla u|^p\, dx\right)^{\eta/p} \left(\int_\Omega (|u|+1)^{(1-\alpha)\eta/(p-\eta)} dx\right)^{(p-\eta)/p}, \tag{3.19}$$

from Hölder inequality. Choosing again α close to 0, the left-hand side is finite for any η such that $\eta/(p-\eta) < (p-1)N/(N-p)$, that means $0 < \eta < N(p-1)/(N-1)$. Hence (2.20) holds. Finally, for any $1 \leq \sigma < p'$,

$$\int_U |\nabla u|^{(p-1)\sigma}\varphi dx \leq \left(\int_U (|u|+1)^{\alpha-1}|\nabla u|^p\varphi dx\right)^{\sigma/p'} \times \left(\int_U (|u|+1)^{(1-\alpha)/(p'/\sigma-1)}\varphi dx\right)^{1-\sigma/p'}. \tag{3.20}$$

The left-hand side is finite if $1/(p'/\sigma-1) < q$, or equivalently $\sigma < p'q/(q+1)$, which proves (3.4). ∎

Remark 3.1 Estimate (3.4) was first observed in [14] for equation (1.1) in $\mathbb{R}^N$. It is stronger than (2.20) whenever $q > \overline{P}$.

In the same way we deduce convergence properties.

Theorem 3.2 *Let $\{\mu_\nu\}$ be a sequence of Radon measures in Ω, uniformly locally bounded. Let (u_ν) be a sequence of local entropy solutions of*

$$-\Delta_p u_\nu = \mu_\nu, \qquad in\ \Omega,$$

such that $(|u_\nu|^q)$ is bounded in $L^1_{loc}(\Omega)$, for some $q > p-1$. Then

$$(T_k(u_\nu)) \quad is\ bounded\ in\ W^{1,p}_{loc}(\Omega), \quad for\ any\ k > 0, \tag{3.21}$$

$$(|u_\nu|^{p-1}) \quad is\ bounded\ in\ L^s_{loc}(\Omega), \quad for\ any\ 1 \leq s < \frac{N}{N-p}, \tag{3.22}$$

$$(|\nabla u_\nu|^{p-1}) \quad is\ bounded\ in\ L^r_{loc}(\Omega), \quad for\ any\ 1 \leq r < \max(\frac{N}{N-1}, \frac{p'q}{q+1}). \tag{3.23}$$

And, up to a subsequence, μ_ν converges weakly to a measure μ, and u_ν converges locally in measure and a.e. in Ω to some function u, which satisfies (2.8), (2.19), (2.18) and (2.20), (3.4). And ∇u_ν converges to ∇u locally in measure in Ω.

Proof **Step 1: a priori estimates.** Taking the same notations as above, and the test function φ defined in (3.14), we derive from (3.12) and (3.13),

$$\frac{|\alpha|}{2}\int_\Omega (|u_\nu|+1)^{\alpha-1}|\nabla u|^p\varphi dx \leq \int_\Omega \varphi d|\mu_\nu| + C\left(\int_\Omega (|u_\nu|+1)^q\varphi dx\right)^{1/\tau},$$

with a constant $C = C(\alpha,\varphi) > 0$. Then for any regular domain $U \subset\subset \Omega$,

$$\left(\int_U (|u_\nu|+1)^{\alpha-1}|\nabla u|^p dx\right) \quad \text{is bounded.} \tag{3.24}$$

In particular $(u_{\nu,k})$ is bounded in $W^{1,p}(U)$, and (3.21) holds. Then (3.22) and (3.23) follow from (3.18), (3.19) and (3.20) applied to (u_ν), after noticing that the sequence (v_ν) defined by

$$v_\nu = (|u_\nu| + 1)^\beta, \qquad \beta = (\alpha + p - 1)/p,$$

is bounded in $L^1_{loc}(\Omega)$. Thus the sequence $(\overline{v_\nu})$ of its mean values on U is bounded. Moreover considering the two functions

$$v'_\nu = (u^+_\nu + 1)^\beta, \quad v''_\nu = (u^-_\nu + 1)^\beta,$$

one has

$$|\nabla v'_\nu| \le \beta(|u_\nu| + 1)^{(\alpha-1)/p} |\nabla u_\nu|, \qquad |\nabla v''_\nu| \le \beta(|u_\nu| + 1)^{(\alpha-1)/p} |\nabla u_\nu|,$$

from the chain rule applied to the trucatures $u_{\nu,k}$, and by definition of the gradient. Hence $(|\nabla v'_\nu|)$ and $(|\nabla v''_\nu|)$ are bounded in $L^p_{loc}(\Omega)$, so that

$$(v'_\nu) \text{ and } (v''_\nu) \quad \text{are bounded in } W^{1,p}_{loc}(\Omega). \tag{3.25}$$

Up to a subsequence, μ_ν converges weakly to a measure μ, and v'_ν and v''_ν converge weakly in $W^{1,p}_{loc}(\Omega)$, strongly in $L^p_{loc}(\Omega)$, and v'_νconvergesv', $v''_\nu \to v''$ *a.e.* in Ω. Then u_ν converges locally in measure and *a.e.* in Ω to a function u, such that $|u|^q \in L^1_{loc}(\Omega)$, and (2.19) holds from the Fatou lemma. For fixed $k > 0$, $(u_{\nu,k})$ is bounded in $W^{1,p}_{loc}(\Omega)$, and $u_{\nu,k}$ converges *a.e.* to u_k. We can extract a subsequence (depending on k) converging weakly in $W^{1,p}_{loc}(\Omega)$ and *a.e.* in Ω, and necessarily to u_k. Then $u_k \in W^{1,p}_{loc}(\Omega)$, and the whole sequence $(u_{\nu,k})$ converges.

Step 2: convergence of the gradients. We set

$$\mu_\nu = \mu_{\nu,0} + \mu^+_{\nu,s} - \mu^-_{\nu,s},$$

where $\mu_{\nu,0} \in \mathcal{M}_0(\Omega)$ and $\mu^\pm_{\nu,s}$ are nonnegative and singular. Here we use the equation satisfied by the truncated functions $u_{\nu,k}$ as in [18]. From Theorem 2.2, there exist $\alpha_{\nu,k}$ and $\beta_{\nu,k} \in \mathcal{M}_0(\Omega) \cap \mathcal{M}^+(\Omega)$, respectively concentrated on the sets $\{u_\nu = k\}$ and $\{u_\nu = -k\}$, such that

$$\int_\Omega |\nabla u_{\nu,k}|^{p-2} \nabla u_{\nu,k}.\nabla\psi dx = \int_{\{|u_\nu| \le k\}} \psi d\mu_{\nu,0} + \int_\Omega \psi d\alpha_{\nu,k} - \int_\Omega \psi d\beta_{\nu,k},$$

for any $\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω. Taking $\psi = u_{\nu,k}\varphi$, with $\varphi \in \mathcal{D}(\Omega), \varphi \ge 0$, we obtain

$$\int_\Omega |\nabla u_{\nu,k}|^p \varphi dx + \int_\Omega u_{\nu,k} |\nabla u_{\nu,k}|^{p-2} \nabla u_{\nu,k} \nabla\varphi dx = \int_{\{|u_\nu| \le k\}} u_{\nu,k}\varphi d\mu_{\nu,0} + k\left(\int_\Omega \varphi d\alpha_{\nu,k} + \int_\Omega \varphi d\beta_{\nu,k}\right).$$

Moreover $|u_{\nu,k}| \leq k$, *a.e.* in Ω, and $\mu_{\nu,0}$- a.e. in Ω, and therefore

$$\int_\Omega \varphi d\alpha_{\nu,k} + \int_\Omega \varphi d\beta_{\nu,k} \leq \int_\Omega \varphi d\,|\mu_{\nu,0}| + \frac{1}{k}\int_\Omega |\nabla u_{\nu,k}|^p \varphi dx + \int_\Omega |\nabla u_{\nu,k}|^{p-1} |\nabla \varphi|\, dx.$$

Taking φ such that $\varphi = 1$ on $U \subset\subset \Omega$, we deduce, from (3.24),

$$\int_U d\alpha_{\nu,k} + \int_U d\beta_{\nu,k} \leq C\left(1 + k^{|\alpha|}\right)$$

for any $k > 1$, where $C > 0$ does not depend on ν, since $|\mu_{\nu,0}| \leq |\mu_\nu|$ from (2.3). Then $\alpha_{\nu,k}$ and $\beta_{\nu,k}$ are locally bounded independently on ν, for fixed k, thus

$$\mu_{\nu,k} = \chi_{\{|u_\nu| \leq k\}}\mu_{\nu,0} + \alpha_{\nu,k} - \beta_{\nu,k}.$$

Now $u_{\nu,k}$ is a solution (in the sense of $\mathcal{D}'(\Omega)$) of equation

$$-\Delta_p u_{\nu,k} = \mu_{\nu,k},$$

and $(u_{\nu,k})$ is uniformly bounded in $L^\infty(\Omega)$ and in $W^{1,p}_{loc}(\Omega)$. Following the proof of [15, Theorem 2.1], we deduce that, after extracting a diagonal subsequence (which may depend on k), $\nabla u_{\nu,k}$ converges *a.e.* in Ω and strongly in $(L^\lambda_{loc}(\Omega))^N$ to ∇u_k, for any $1 \leq \lambda < p$. Then the whole sequence converges. For any $\varepsilon > 0$, and any $\nu, \rho \in \mathbb{N}$,

$$\{|\nabla u_\nu - \nabla u_\rho| > \varepsilon\} \subset \{|u_\nu| > k\} \cup \{|u_\rho| > k\} \cup \{|\nabla u_{\nu,k} - \nabla u_{\rho,k}| > \varepsilon\},$$

and $|u_\nu|^q$ is bounded in $L^1_{loc}(\Omega)$. Hence (∇u_ν) is a locally a Cauchy sequence in measure. Up to a subsequence, ∇u_ν converges *a.e.* in Ω, and the limit is ∇u, by definition of the gradient. Then ∇u satisfies (2.20) and (3.4). Thus $|\nabla u_\nu|^{p-2}\nabla u_\nu$ converges strongly in $L^r_{loc}(\Omega)$ to $|\nabla u|^{p-2}\nabla u$, for any $1 \leq r < N/(N-1)$, from the Vitali theorem. ∎

Now we give a local stability result, following the global result of [18, Theorem 3.4]. We have not searched to extend it completely to the local problem, because it was not needed in our situations, where the reaction term requires stronger convergences properties. We give the proof in Appendix B.

Theorem 3.3 *Let $\lambda \in \mathcal{M}(\Omega)$. Let $f_\nu, f \in L^1_{loc}(\Omega)$, such that f_ν converges weakly to f in $L^1_{loc}(\Omega)$. Let u_ν be a local entropy solution of equation*

$$-\Delta_p u_\nu = f_\nu + \lambda, \qquad \text{in } \Omega, \tag{3.26}$$

such that $(|u_\nu|^q)$ is bounded in $L^1_{loc}(\Omega)$, for some $q > p-1$. Then, up to a subsequence, u_ν converges a.e. in Ω to a local entropy solution u of

$$-\Delta_p u = f + \lambda, \qquad \text{in } \Omega. \tag{3.27}$$

4 Estimates for absorption or source term

Here we give universal a priori estimates for equations (1.1) and (1.7). First consider the case of absorption :

Theorem 4.1 *Let u be any local entropy solution of equation (1.1). Then for any $R > pq^*$, there exists $C = C(N,p,q,R,\Omega)$, such that for any $\zeta \in \mathcal{D}(\Omega), \zeta \geq 0$,*

$$\int_\Omega (|u|+1)^q \zeta^R dx \leq C \left(\int_\Omega \zeta^R dx + \int_\Omega \zeta^R d|\mu| + \int_\Omega |\nabla \zeta|^R dx \right). \tag{4.1}$$

And for any $\alpha < 0$, there exists $C = C(\alpha,N,p,q,R,\Omega)$, such that

$$\int_\Omega (|u|+1)^{\alpha-1} |\nabla u|^p \zeta^R dx \leq C \left(\int_\Omega \zeta^R dx + \int_\Omega \zeta^R d|\mu| + \int_\Omega |\nabla \zeta|^R dx \right). \tag{4.2}$$

Proof Let $R > pq^*$. By hypothesis $|u|^q \in L^1_{loc}(\Omega)$, and $q > p-1$, thus Theorem 3.1 applies. We take the test functions h_k defined in (3.5), and obtain, for any $\varphi \in \mathcal{D}(\Omega), \varphi \geq 0$,

$$\begin{aligned}
&|\alpha| \int_\Omega (|u_k|+1)^{\alpha-1} |\nabla u_k|^p \varphi dx + \int_\Omega |u|^{q-1} u h_k(u) \varphi dx \\
&= - \int_\Omega h_k(u) |\nabla u|^{p-2} \nabla u . \nabla \varphi dx + \int_\Omega h_k(u) \varphi d\mu_0 + (1-(k+1)^\alpha) \int_\Omega \varphi d|\mu_s| \\
&\leq \int_\Omega |\nabla u_k|^{p-1} |\nabla \varphi| \, dx + \int_{\{|u|>k\}} |\nabla u|^{p-1} |\nabla \varphi| \, dx + \int_\Omega \varphi d|\mu| .
\end{aligned}$$

This estimate is similar to (3.6), with an additional nonnegative term $\int_\Omega |u|^{q-1} u h_k(u)$ φdx in the left-hand side, since $|u|^{q-1} u$ is an absortion term. By Fatou lemma, we find, as in (3.12),

$$\begin{aligned}
&\frac{|\alpha|}{2} \int_\Omega (|u|+1)^{\alpha-1} |\nabla u|^p \varphi dx + \int_\Omega |u|^q (1-(|u|+1)^\alpha) \varphi dx \\
&\leq \int_\Omega \varphi d|\mu| + C \left(\int_\Omega (|u|+1)^q \varphi dx \right)^{1/\tau} \times \left(\int_\Omega \varphi^{1-\tau' p} |\nabla \varphi|^{\tau' p} dx \right)^{1/\tau'},
\end{aligned}$$

where τ is defined in (3.8). We can choose $\alpha < 0$ small enough such that

$$\tau' p = R,$$

and take $\varphi = \zeta^R$, where $\zeta \in \mathcal{D}(\Omega), \zeta \geq 0$. Defining

$$X = \int_\Omega (|u|+1)^q \zeta^R dx, \qquad Y = \int_\Omega (|u|+1)^{\alpha-1} |\nabla u|^p \zeta^R dx,$$

and observing that $1-(|u|+1)^\alpha \geq 1-2^\alpha > 0$ on the set $\{|u| \geq 1\}$, we find

$$X + Y \leq C \left(\int_\Omega \zeta^R dx + \int_\Omega \zeta^R d|\mu| + C(\zeta) X^{1/\tau} \right) \tag{4.3}$$

with $C(\zeta) = \left(\int_\Omega |\nabla\zeta|^R\, dx\right)^{p/R}$ and $C = C(\alpha, q)$. Then, from Young inequality, with a new $C = C(\alpha, p, q)$,

$$X + Y \le C\left(\int_\Omega \zeta^R dx + \int_\Omega \zeta^R d|\mu| + \int_\Omega |\nabla\zeta|^R\, dx\right). \tag{4.4}$$

Then (4.1) follows for such α, from (4.3), and then (4.2) follows for any $\alpha < 0$. ∎

Now we consider the problem with a source term. In case of nonnegative μ and u, we have more precise results.

Theorem 4.2 *Let $\mu \in \mathcal{M}^+(\Omega)$ and u be any nonnegative local entropy solution of equation (1.7). Then for any $R > pq^*$, there exists $C = C(N, p, q, R, \Omega) > 0$ such that for any $\zeta \in \mathcal{D}(\Omega)$, with $0 \le \zeta \le 1$ in Ω,*

$$\int_\Omega \zeta^R d\mu + \int_\Omega u^q \zeta^R dx \le C\left(\int_\Omega |\nabla\zeta|^R\, dx\right)^{pq^*/R}. \tag{4.5}$$

Moreover for any $\alpha < 0$, there exists $C = C(\alpha, N, p, q, R, \Omega) > 0$ such that

$$\int_\Omega (u+1)^{\alpha-1} |\nabla u|^p \zeta^R dx \le C\left(1 + \int_\Omega u^q \zeta^R dx\right)\left(\int_\Omega |\nabla\zeta|^R\, dx\right)^{p/R}. \tag{4.6}$$

Proof It has been given in [3] in case of global entropy solutions of the problem with Dirichlet data, but it is still valid for local solutions. It is based on the use of test functions

$$\tilde{h}_k(r) = (T_k(r^+) + \varepsilon)^\alpha, \qquad k > 0,\ \varepsilon > 0, \tag{4.7}$$

which are nondecreasing, contrarily to the ones defined in (3.5), and then of the test function $h(r) = 1$. ∎

Remark 4.1 These ideas were already used in [10] to obtain upper estimates for more general problems with possible singularities, and also in [9] to study the initial trace problem for a parabolic equation with absorption.

5 Necessary conditions of existence

In this section we prove our general necessary conditions for existence:

Proof [Proof of Theorem 1.1]. It is enough to consider a local entropy solution u of (1.3), such that $|u|^q \in L^1_{loc}(\Omega)$. Then Theorem 3.1 applies. Let $R > pq^*$, we still can choose $\alpha < 0$ such that τ, given by (3.8), satisfies

$$p\tau' = R.$$

Let E be a Borel set such that $cap_{1,R}(E, \mathbb{R}^N) = 0$. There exist two measurable disjoint sets A, B such that $\Omega = A \cup B$ and $\mu^+(B) = \mu^-(A) = 0$. We claim that

$$\mu^+(A \cap E) = \mu^-(B \cap E) = 0.$$

Let K be any fixed compact set in $A \cap E$. Since $\mu^-(K) = 0$, there exists an open set $\omega \subset\subset \Omega$ containing K, such that $\mu^-(\omega) < \varepsilon$. From [3, Lemma 2.1], there exists $\zeta_n \in \mathcal{D}(\omega)$ such that $0 \leq \zeta_n \leq 1$, and $\zeta_n = 1$ in a neighborhood of K contained in ω, and $\zeta_n \to 0$ in $W^{1,R}(\mathbb{R}^N)$. We choose $\varphi = \varphi_n = \zeta_n^R$ in (2.21). From (3.12) we have

$$\frac{|\alpha|}{2}\int_\Omega (|u|+1)^{\alpha-1}|\nabla u|^p \varphi_n dx \leq \int_\Omega \varphi_n d|\mu| + C\left(\int_\Omega (|u|+1)^q \varphi_n dx\right)^{1/\tau}\left(\int_\Omega |\nabla \zeta_n|^R dx\right)^{p/R},$$

where $C = C(\alpha, R)$. But

$$\int_\Omega \varphi_n d|\mu| \leq \int_{supp\Psi_K} d|\mu|,$$

$$\lim_{n\to\infty}\int_\Omega (|u|+1)^q \varphi_n dx = 0, \tag{5.1}$$

since $u \in L^q_{loc}(\Omega)$. Then

$$\int_\Omega (|u|+1)^{\alpha-1}|\nabla u|^p \varphi_n dx \text{ is bounded.} \tag{5.2}$$

From (3.13),

$$\int_\Omega |\nabla u|^{p-1}|\nabla \varphi_n| dx \leq C\left(\int_\Omega (|u|+1)^{\alpha-1}|\nabla u|^p \varphi_n dx\right)^{1/p'}\left(\int_\Omega (|u|+1)^q \varphi_n dx\right)^{1/\tau p} \times \left(\int_\Omega |\nabla \zeta_n|^R dx\right)^{1/R},$$

therefore

$$\lim_{n\to\infty}\int_\Omega |\nabla u|^{p-1}|\nabla \varphi_n| dx = 0.$$

Let us apply (2.21) with now $h(u) = 1$ and the same φ_n. We find

$$\int_\Omega \varphi_n d\mu = \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla \varphi_n dx,$$

then $\lim_{n\to\infty}\int_\Omega \varphi_n d\mu = 0$. Now

$$\mu^+(K) \leq \int_\omega \varphi_n d\mu^+ = \int_\omega \varphi_n d\mu + \int_\omega \varphi_n d\mu^-,$$

hence $\mu^+(K) \leq \varepsilon$, for any $\varepsilon > 0$. This implies $\mu^+(A \cap E) = 0$, and the claim is proved. ∎

Proof [Proof of Theorem 1.2]. We apply the estimates of Theorem 4.2. Let $R > pq^*$ and K be a compact set contained in Ω. Let $\psi_n \in \mathcal{D}(\Omega)$ such that

$$0 \leq \psi_n \leq 1, \quad \psi_n \geq \chi_K, \quad \text{and } \|\psi_n\|^R_{W^{1,R}(\Omega)} \to cap_{1,R}(K,\Omega).$$

Choosing $\varphi = \psi_n^R$ in (4.5), we deduce that

$$\int_K d\mu \leq C\left(\int_\Omega |\nabla\psi_n|^R dx\right)^{pq^*/R} \leq C\,\|\psi_n\|^{pq^*}_{W^{1,R}(\Omega)},$$

with new constants $C = C(N,p,q,R,\Omega)$, and (1.8) follows. ∎

6 Removable singularities

In this section we study under what condition a subset of Ω is removable. The estimates of Section 4 play a key role for solving this question.

Proof [Proof of Theorem 1.3]. By assumption, u is measurable and finite, *a.e.* in $\Omega\backslash F$, thus in whole Ω, since F has a Lebesgue measure zero. Moreover $|u|^q \in L^1_{loc}(\Omega\backslash F)$, $T_k(u) \in W^{1,p}_{loc}(\Omega\backslash F)$ for every $k > 0$, $|\nabla u|^{p-1} \in L^1_{loc}(\Omega\backslash F)$, and u satisfies

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla(h(u)\varphi)dx + \int_\Omega |u|^{q-1}\,uh(u)\varphi dx\\ &= \int_\Omega h(u)\varphi d\mu_0 + h(+\infty)\int_\Omega \varphi d\mu_s^+ - h(-\infty)\int_\Omega \varphi d\mu_s^- \qquad (6.1)\end{aligned}$$

for any $\varphi \in \mathcal{D}(\Omega\backslash F), \varphi \geq 0$, and any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support.

Step 1: $|u|^q \in L^1_{loc}(\Omega)$. From Theorem 4.1 in $\Omega\backslash F$, for any $\phi \in \mathcal{D}(\Omega\backslash F), \phi \geq 0$,

$$\int_\Omega (|u|+1)^q \phi^R dx \leq C\,\left(\int_\Omega \phi^R dx + \int_\Omega \phi^R d\,|\mu| + \int_\Omega |\nabla\phi|^R\,dx\right), \qquad (6.2)$$

where $C = C(N,p,q,R,\Omega,F)$, and for any $\alpha < 0$,

$$\int_\Omega (|u|+1)^{\alpha-1}|\nabla u|^p\phi^R dx \leq C\left(\int_\Omega \phi^R dx + \int_\Omega \phi^R d\,|\mu| + \int_\Omega |\nabla\phi|^R\,dx\right), \qquad (6.3)$$

with $C = C(\alpha,N,p,q,R,\Omega,F)$. Let $\zeta \in \mathcal{D}(\Omega), \zeta \geq 0$, and $K = F \cap \text{supp}\zeta$. Then K is compact, and $cap_{1,R}(K,\mathbb{R}^N) = 0$. Let $\xi_n \in \mathcal{D}(\mathbb{R}^N)$ such that

$$0 \leq \xi_n \leq 1 \quad \text{in } \mathbb{R}^N, \quad \xi_n = 1 \text{ in a neighborhood of } K,$$

and ξ_n converges to 0 in $W^{1,R}(\mathbb{R}^N)$ and everywhere on $\mathbb{R}^N\backslash N$, where $cap_{1,R}(N,\mathbb{R}^N) = 0$ (see [3]). We take

$$\phi = \zeta_n = \zeta(1-\xi_n), \qquad (6.4)$$

and obtain

$$\int_\Omega (|u|+1)^q \zeta_n^R dx \le C \left(\int_\Omega \zeta_n^R dx + \int_\Omega \zeta_n^R d|\mu| + \int_\Omega |\nabla \zeta_n|^R dx \right)$$
$$\le C \left(\int_\Omega \zeta^R dx + \int_\Omega \zeta^R d|\mu| + \int_\Omega |\nabla \zeta_n|^R dx \right).$$

Notice that $(|\nabla \zeta_n|)$ is bounded in $L^R(\mathbb{R}^N)$, and φ_n converges to ζ^R *a.e.* in Ω. Then, by the Fatou lemma,

$$\int_\Omega (|u|+1)^q \zeta^R dx < \infty,$$

therefore $|u|^q \in L^1_{loc}(\Omega)$.

Step 2: $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$. As above, from (6.3),

$$\left(\int_\Omega (|u|+1)^{\alpha-1} |\nabla u|^p \zeta_n^R dx \right) \quad \text{is bounded.} \tag{6.5}$$

Now

$$\int_\Omega |\nabla u|^{p-1} \zeta_n^R dx \le \left(\int_\Omega (|u|+1)^{\alpha-1} |\nabla u|^p \zeta_n^R dx \right)^{1/p'} \left(\int_\Omega (|u|+1)^{(1-\alpha)(p-1)} \zeta_n^R dx \right)^{1/p},$$

from Hölder inequality. Since $(1-\alpha)(p-1) < q$, we deduce that

$$\left(\int_\Omega |\nabla u|^{p-1} \zeta_n^R dx \right) \quad \text{is bounded.}$$

The conclusion follows.

Step 3: $u_k = T_k(u) \in W^{1,p}_{loc}(\Omega)$ **and** $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$. Let $k > 0$ be fixed. From (6.5),

$$\left(\int_\Omega |\nabla u_k|^p \zeta_n^R dx \right) \quad \text{is bounded.} \tag{6.6}$$

Let us set, for fixed $k > 0$,

$$w_n = u_k \zeta_n^{R/p} \in W^{1,p}(\Omega).$$

Since $|w_n| \le k \zeta^{R/p}$, (w_n) is bounded in $L^\infty(\Omega)$. And w_n converges to $u_k \zeta^{R/p}$ *a.e.* in Ω. Now

$$\nabla w_n = \zeta_n^{R/p} \nabla u_k + \frac{R}{p} u_k \zeta_n^{R/p-1} \nabla \zeta_n;$$

and $(\nabla \zeta_n)$ is bounded in $L^p_{loc}(\mathbb{R}^N)$, then (∇w_n) is bounded in $L^p(\Omega)$, from (6.6). It follows that $u_k \zeta^{R/p} \in W^{1,p}(\Omega)$, and $u_k \in W^{1,p}_{loc}(\Omega)$. The gradient of u has a sense in Ω : it is defined by (2.7), and it coincides with the gradient, still defined *a.e.* in $\Omega \backslash F$.

Step 4: $|\nabla u|^{p-1} \in L^{R'}_{loc}(\Omega)$. For any $1 \le \sigma < p'$, from Hölder inequality,

$$\int_U |\nabla u|^{(p-1)\sigma} \zeta_n^R dx \le \left(\int_U (|u|+1)^{\alpha-1} |\nabla u|^p \zeta_n^R dx \right)^{\sigma/p'} \times \left(\int_U (|u|+1)^{(1-\alpha)/(p'/\sigma-1)} \zeta_n^R dx \right)^{1-\sigma/p'}.$$

Then, as in (3.20),

$$\left(\int_\Omega |\nabla u|^{(p-1)\sigma} \zeta_n^R dx \right) \qquad \text{is bounded,}$$

for any $\sigma < p'q/(q+1)$, and $|\nabla u|^{p-1} \in L^\sigma_{loc}(\Omega)$. This holds in particular for $\sigma = R'$.

Step 5: u is a local entropy solution in Ω. Let $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support. Let $\psi \in \mathcal{D}(\Omega), \psi \ge 0$, be fixed. Taking now as test function

$$\psi_n = \psi(1-\xi_n)^R,$$

we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla(h(u)\psi_n) dx + \int_\Omega |u|^{q-1} u h(u) \psi_n dx = \int_\Omega h(u)\psi_n d\mu_0 + h(+\infty) \int_\Omega \psi_n d\mu_s^+ - h(-\infty) \int_\Omega \psi_n d\mu_s^-. \tag{6.7}$$

Since ξ_n converges to 0 everywhere on $\mathbb{R}^N \backslash N$, and μ does not charge N, we have

$$\lim_{n\to\infty} \int_\Omega \psi_n d|\mu_0| = \int_\Omega \psi d|\mu_0|, \qquad \lim_{n\to\infty} \int_\Omega \psi_n d\mu_s^\pm = \int_\Omega \psi d\mu_s^\pm.$$

Then

$$\left| \int_\Omega h(u)(\psi - \psi_n) d\mu_0 \right| \le \|h\|_{L^\infty(\mathbb{R})} \int_\Omega (\psi - \psi_n) d|\mu_0|,$$

which yields

$$\lim_{n\to\infty} \int_\Omega h(u)\psi_n d\mu_0 = \int_\Omega h(u)\psi d\mu_0.$$

Moreover $|u|^q \in L^1_{loc}(\Omega)$, then by Lebesgue theorem,

$$\lim_{n\to\infty} \int_\Omega |u|^{q-1} u h(u) \psi_n dx = \int_\Omega |u|^{q-1} u h(u) \psi dx.$$

Also $|\nabla u|^{p-1} \in L^{R'}_{loc}(\Omega)$, and

$$\left| \int_\Omega h(u) |\nabla u|^{p-2} \nabla u.\nabla(\psi - \psi_n) dx \right| \le C \left\| |\nabla u|^{p-1} \right\|_{L^{R'}(\Omega')} \| |\nabla(\psi - \psi_n)| \|_{L^R(\Omega')},$$

where Ω' contains supp ψ. But

$$\nabla(\psi - \psi_n) = (1-(1-\xi_n)^R)\nabla\psi - R\psi(1-\xi_n)^{R-1}\nabla\xi_n,$$

then $\nabla(\psi - \psi_n)$converges to 0 *a.e.* in Ω, and

$$|\nabla(\psi - \psi_n)|^R \leq C(|\nabla\psi|^R + |\nabla\xi_n|^R)$$

where $C > 0$ does not depend on n. Again by Lebesgue theorem, $\nabla(\psi - \psi_n)$ converges to 0 strongly in $L^R(\Omega)$, and

$$\lim_{n\to\infty} \int_\Omega h(u)\,|\nabla u|^{p-2}\,\nabla u.\nabla\psi_n dx = \int_\Omega h(u)\,|\nabla u|^{p-2}\,\nabla u.\nabla\psi dx.$$

At last

$$\lim_{n\to\infty} \left| \int_\Omega h'(u)\,|\nabla u|^p\,(\psi - \psi_n)dx \right| = 0,$$

since $|\nabla u_k| \in L^p_{loc}(\Omega)$ for any k, and h' has a compact support. Therefore u satisfies

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla(h(u)\psi)dx + \int_\Omega |u|^{q-1}\,uh(u)\psi dx\\ &= \int_\Omega h(u)\psi d\mu_0 + h(+\infty)\int_\Omega \psi d\mu_s^+ - h(-\infty)\int_\Omega \psi d\mu_s^-,\end{aligned}$$

for any $\psi \in \mathcal{D}(\Omega), \psi \geq 0$. Consequently u is a local entropy solution of the problem in Ω. ∎

Proof [Proof of Theorem 1.4]. From (4.5), for any $\phi \in \mathcal{D}(\Omega\backslash F)$, with $0 < \phi \leq 1$ in Ω,

$$\int_\Omega \phi^R d\mu + \int_\Omega u^q\phi^R dx \leq C\left(\int_\Omega |\nabla\phi|^R\,dx\right)^{pq^*/R},$$

with $C = C(N,p,q,R,\Omega,F)$, and for any $\alpha < 0$, there exists > 0 such that

$$\int_\Omega (u+1)^{\alpha-1}|\nabla u|^p\phi^R dx \leq C\left(1 + \int_\Omega u^q\phi^R dx\right)\left(\int_\Omega |\nabla\phi|^R\,dx\right)^{p/R},$$

with $C = C(\alpha,N,p,q,R,\Omega,F)$. Taking $\phi^R = \zeta_n$ defined in (6.4), we deduce that

$$\int_\Omega \zeta_n^R d\mu + \int_\Omega u^q\zeta_n^R dx + \int_\Omega (u+1)^{\alpha-1}|\nabla u|^p\zeta_n^R dx \quad \text{is bounded,}$$

and the proof follows as above, after minor change due to the signs. ∎

7 Stability properties

We recall below a well-known stability property of global solutions:

Theorem 7.1 ([5]) *Let $f_\nu, f \in L^1(\Omega)$, with Ω bounded, such that f_ν converges strongly to f in $L^1(\Omega)$. Let u_ν be the unique entropy solution of problem*

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = f_\nu, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega. \end{cases}$$

Then u_ν converges a.e. in Ω to the unique entropy solution u of problem

$$\begin{cases} -\Delta_p u + |u|^{q-1} u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Next we prove analogous stability properties for the local entropy solutions, which we need for proving Theorem 1.5. Here Ω may be unbounded.

Theorem 7.2 *Let $f_\nu, f \in L^1_{loc}(\Omega)$, such that f_ν converges strongly to f in $L^1_{loc}(\Omega)$, and u_ν be any local entropy solution of equation*

$$-\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = f_\nu, \qquad \text{in } \Omega. \tag{7.1}$$

Then, up to a subsequence, u_ν converges a.e. in Ω to a local entropy solution u of equation

$$-\Delta_p u + |u|^{q-1} u = f, \qquad \text{in } \Omega. \tag{7.2}$$

Proof **Step 1: A priori estimates.** Let u_ν be a local entropy solution of (7.1). From Theorem 4.1, for any $R > pq^*$, there exists $C = C(N, p, q, R, \Omega)$ such that, for any $\zeta \in \mathcal{D}(\Omega), \zeta \geq 0$,

$$\int_\Omega (|u_\nu| + 1)^q \zeta^R dx \leq C \left(\int_\Omega \zeta^R dx + \int_\Omega \zeta^R |f_\nu| \, dx + \int_\Omega |\nabla \zeta|^R \, dx \right). \tag{7.3}$$

Hence $(|u_\nu|^q)$ is bounded in $L^1_{loc}(\Omega)$. Writing the equation under the form

$$-\Delta_p u_\nu = \mu_\nu, \quad \text{where } \mu_\nu = f_\nu - |u_\nu|^{q-1} u_\nu, \tag{7.4}$$

then, up to a subsequence, u_ν converges to some u satisfying the conclusions of Theorem 3.2.

Step 2: Convergence of the nonlinear term. Following [5] and [14], we prove the local uniform integrability of $(|u_\nu|^q)$: for any domain $U \subset\subset \Omega$ and any $\varepsilon > 0$, any subset $A \subset U$ such that $meas A \leq \varepsilon (k+1)^{-q}$, there holds

$$\int_A |u_\nu|^q \, dx \leq \varepsilon + \int_{A \cap \{|u_\nu| \geq k+1\}} |u_\nu|^q \, dx,$$

for any $k \geq 0$. From **D1-loc**,

$$\int_\Omega |\nabla u_\nu|^{p-2} \nabla u_\nu . \nabla (h(u_\nu)\varphi) dx + \int_\Omega |u_\nu|^{q-1} u_\nu \varphi dx = \int_\Omega h(u_\nu) f_\nu \varphi dx,$$

for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and any $\varphi \in \mathcal{D}(\Omega)$, such that $h(u_\nu)\varphi \in W^{1,p}(\Omega)$. Taking $\varphi \in \mathcal{D}(\Omega), \varphi \geq 0$, such that $\varphi = 1$ on U and $h = S_{k,k}$ defined in (2.4), we derive

$$\int_{U\cap\{k\leq|u_\nu|\leq k+1\}} |\nabla u_\nu|^p \varphi dx + \int_{U\cap\{|u_\nu|\geq k+1\}} |u_\nu|^q \varphi dx \leq \int_{\{|u_\nu|\geq k\}} |f_\nu| \varphi dx + \int_{\{|u_\nu|\geq k\}} |\nabla u_\nu|^{p-1} |\nabla\varphi| dx.$$

Since f_ν and $|\nabla u_\nu|^{p-1}$ converge strongly in $L^1_{loc}(\Omega)$, there exists $g \in L^1(U)$ such that

$$\int_{U\cap\{|u_\nu|\geq k+1\}} |u_\nu|^q dx \leq \int_{U\cap\{|u_\nu|\geq k\}} g dx.$$

Now $(|u_\nu|^q)$ is bounded in $L^1_{loc}(\Omega)$, hence $meas\{|u_\nu| \geq k\} \leq Ck^{-q}$, where $C > 0$ does not depend on ν. Hence

$$\int_{A\cap\{|u_\nu|\geq k+1\}} |u_\nu|^q dx \leq \varepsilon,$$

for k large enough. Then $|u_\nu|^{q-1} u_\nu$ converges strongly in $L^1_{loc}(\Omega)$ to $|u|^{q-1} u$.

Step 3. Conclusion. Let us apply Theorem 3.3 with f_ν replaced by μ_ν, and $\mu = 0$. Since μ_ν converges (strongly) in $L^1_{loc}(\Omega)$ to $f - |u|^{q-1} u$, we deduce that u is a local entropy solution of

$$-\Delta_p u = f - |u|^{q-1} u, \qquad \text{in } \Omega,$$

thus a solution of (7.2). ∎

Theorem 1.5 follows as a direct consequence of Theorems 7.2 and 1.3:

Proof [of Theorem 1.5]. From Theorem 7.2, and up to a subsequence, u_ν converges to a local entropy solution u of

$$-\Delta_p u + |u|^{q-1} u = f, \qquad \text{in } \Omega\backslash F.$$

By Theorem 1.3, it is a solution in Ω, since f does not charge F, as $measF = 0$. ∎

This implies also a global result:

Theorem 7.3 *Assume that Ω is bounded. Let K be a compact subset of Ω, such that $cap_{1,R}(K, \mathbb{R}^N) = 0$ for some $R > pq^*$. Let $f_\nu, f \in L^1(\Omega)$ such that f_ν converges strongly to f in $L^1_{loc}(\overline{\Omega}\backslash K)$. Let u_ν be the entropy solution of problem*

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = f_\nu, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

Then u_ν converges to the entropy solution u of

$$\begin{cases} -\Delta_p u + |u|^{q-1} u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.6}$$

Proof By Theorem 1.5, and up to a subsequence, u_ν converges *a.e.* in Ω to a local entropy solution u of equation

$$-\Delta_p u + |u|^{q-1}\, u = f, \qquad \text{in } \Omega.$$

Let U, U', U'' be any regular bounded open sets such that $K \subset U \subset\subset U' \subset\subset U'' \subset\subset \Omega$. By hypothesis $u_{\nu,k} = T_k(u_\nu) \in W_0^{1,p}(\Omega)$ for any $k > 0$. From **D1**,

$$\int_\Omega |\nabla u_\nu|^{p-2}\, \nabla u_\nu . \nabla (h(u_\nu)\varphi) dx + \int_\Omega |u_\nu|^{q-1}\, u_\nu h(u_\nu)\varphi dx = \int_\Omega h(u_\nu) f_\nu \varphi dx, \quad (7.7)$$

for any $h \in W^{1,\infty}(\mathbb{R})$ and h' has a compact support, and $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, such that $h(u)\varphi \in W_0^{1,p}(\Omega)$. We can take $h = T_k$, and $\varphi \in C^1(\overline{\Omega})$ with values in $[0,1]$, such that

$$\varphi = 1 \text{ on } \Omega\backslash U', \quad \text{and supp}\varphi \subset \Omega\backslash U.$$

Denoting $\Omega' = \Omega\backslash\overline{U'}$, we obtain

$$\int_\Omega |\nabla u_{\nu,k}|^p\, \varphi dx + \int_\Omega |u_\nu|^{q-1}\, u_\nu u_{\nu,k}\varphi dx = \int_\Omega u_{\nu,k}\, |\nabla u_\nu|^{p-2}\, \nabla u_\nu . \nabla \varphi dx + \int_\Omega u_{\nu,k} f_\nu \varphi dx.$$

Therefore

$$\frac{1}{k}\int_{\Omega'} |\nabla u_{\nu,k}|^p\, dx \le \int_{\Omega\backslash U} \varphi\, |f_\nu|\, dx + \int_{U'\backslash U} |\nabla u_\nu|^{p-1}\, |\nabla \varphi|\, dx \le C, \quad (7.8)$$

where $C > 0$ does not depend on ν and k. Indeed $\overline{U}'\backslash U$ is compact, contained in $\Omega\backslash K$, and $(|\nabla u_\nu|^{p-1})$ is bounded in $L^1_{loc}(\Omega\backslash K)$ from Theorem 3.2 applied in $\Omega\backslash K$; and f_ν converges to f in $L^1(\Omega\backslash U)$. Thus for fixed $k > 0$, $(u_{\nu,k})$ is bounded in $W_0^{1,p}(\Omega')$, and $u_{\nu,k}$ converges *a.e.* to u_k.Then $u_k \in W_0^{1,p}(\Omega') \cap W^{1,p}_{loc}(\Omega)$, that means $u_k \in W_0^{1,p}(\Omega)$, and u satisfies (2.9). Letting ν go to ∞ in (7.8), we derive

$$\int_{\Omega'} |\nabla u_k|^p\, dx \le Ck.$$

As u is a local entropy solution in Ω, it satisfies $(|u|+1)^{\alpha-1}|\nabla u|^p \in L^1_{loc}(\Omega)$ for any $\alpha < 0$ from Theorem 3.1. As a consequence,

$$\int_{U''} |\nabla u_k|^p\, dx \le C(1+k)^{1+|\alpha|},$$

with another $C > 0$ depending on U'', α, but not on k. Thus

$$\int_\Omega |\nabla u_k|^p\, dx \le C(1+k)^{1+|\alpha|}.$$

This estimate implies (2.10) and (2.17) (see for example [6, Lemma 6.2]). Since u is a local solution of the problem, it is the unique entropy solution of (7.6), from Remark 2.1, and the whole sequence converges to u. ∎

Remark 7.1 Let us recall the result of [22]: assume that Ω is bounded and K_1, K_2 are two disjoint compact sets with $cap_{1,R}(K_i, \mathbb{R}^N) = 0$ for some $R > pq^*$, and $f \in L^1(\Omega)$ and $f_{1,\nu}, f_{2,\nu} \in L^\infty(\Omega)$ are nonnegative, such that $f_{1,\nu}$ converges strongly to f^+ in $L^1_{loc}(\overline{\Omega}\backslash K_1)$, and $f_{2,\nu}$ converge strongly to f^- in $L^1_{loc}(\overline{\Omega}\backslash K_2)$; then the entropy solution u_ν of problem

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = f_{1\nu} - f_{2,\nu}, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega, \end{cases}$$

converges to a solution, in the sense of distributions, of problem (7.6). Corollary 7.3 improves this result since it does not require sign conditions on the approximating sequence, and proves that the limit u is an entropy solution.

As a consequence we deduce the following, which extends to the case $p \neq 2$ the result of Brézis [16] , except for the critical case $q = \overline{P}$:

Corollary 7.4 *Assume that Ω is bounded, and $q > \overline{P}$. Let $a \in \Omega$, and $f_\nu, f \in L^1(\Omega)$ such that f_ν converges strongly to f in $L^1_{loc}(\overline{\Omega}\backslash\{a\})$. Let u_ν be the entropy solution of problem (7.5). Then u_ν converges to the entropy solution of problem (7.6).*

Remark 7.2 Concerning the equation with source term (1.7), there is no global (or local) stability result, even if $p = 2$. Indeed when $q \geq N/(N-2)$, there exists a nonnegative function $f \in L^1(\Omega)$, with compact support, such that problem

$$\begin{cases} -\Delta u = u^q + \lambda f, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega, \end{cases}$$

has no solution, for any $\lambda > 0$ (see [4, Corollary 3.3]). Furthermore, there exists $f_\nu \in L^1(\Omega)$, such that f_ν converges strongly to f in $L^1(\Omega)$, such that for $\lambda > 0$ small enough, problem

$$\begin{cases} -\Delta u_\nu = u_\nu^q + \lambda f_\nu, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega, \end{cases} \tag{7.9}$$

has a solution for any $\nu \in \mathbb{N}$. Indeed if (ρ_ν) is a sequence of mollifiers, we can take $f_\nu = \rho_\nu * f$. By [4, Corollary 3.2], there exists $\lambda_\nu > 0$ such that problem (7.9) admits a solution for $\lambda = \lambda_\nu$. Then from [11, Remark 3.4] it has a solution for any $\lambda > 0$ such that $\lambda \|f_\nu\|_{L^1(\Omega)} \leq C$, where $C = C(N, q, \Omega)$, independent on ν; hence for any $\lambda > 0$ such that $\lambda < C/\|f\|_{L^1(\Omega)}$.

8 Existence results

In the case of absorption the a priori estimates allow to give existence results in $\mathbb{R}^N$ for data in $L^1_{loc}(\mathbb{R}^N)$, without any growth conditions. For the first time this was observed in [14], where the case $p > P_1$ was solved:

Theorem 8.1 ([14]) *Assume $p > P_1$ and $q > p-1$. Then, for any $f \in L^1_{loc}(\mathbb{R}^N)$, there exists a function $u \in W^{1,r}_{loc}(\mathbb{R}^N)$ for any $r \in [1, \overline{P})$ solution of*

$$-\Delta_p u + |u|^{q-1} u = f, \qquad in\ \mathcal{D}'(\mathbb{R}^N),$$

Moreover this u is nonnegative if f is nonnegative.

An existence result was also proved when $p \leq P_1$, but under the restrictive condition $q > 1/(p-1)$, which ensures that the gradient is well defined in $L^1_{loc}(\Omega)$, from (3.4). With our stability result, we now prove Theorem 1.6, which does not assume this condition, and moreover show the existence of a local entropy solution.

Proof [Proof of Theorem 1.6]. Let $B_r = \{|x| < r\}$ for any $r > 0$. For any $\nu \in \mathbb{N}$, there exists a unique global entropy solution of problem

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = f, & \text{in } B_\nu, \\ u_\nu = 0, & \text{on } \partial B_\nu, \end{cases}$$

since $f \in L^1(B_r)$; this follows from [5].

Let $r > 0$ be fixed. Then for any $\nu > 2r$, u_ν is a local entropy solution in B_{2r} . Hence from Theorem 4.1, as in (7.3), $(|u_\nu|^q)_{\nu > 2r}$ is bounded in $L^1(B_r)$. Then from Theorem 3.2, $(u_\nu)_{\nu > 2r}$ satisfies (3.21), (3.22), and (3.23) in B_r, and one can extract a subsequence converging locally in measure and *a.e.* in B_r.

Therefore, we can extract a diagonal subsequence (u_ρ), such that u_ρ converges locally in measure and *a.e.* in $\mathbb{R}^N$ to a function u. It satisfies (2.8), (2.18), (2.19) and (2.20), (3.4) in $\mathbb{R}^N$, and ∇u_ρ converges locally in measure to ∇u. Clearly u_ρ is a local entropy solution in B_r for $\rho > 2r$. Then from Theorem 7.2 applied with $f_\nu = f$, there exists a subsequence, depending on r, converging *a.e.* to a local entropy solution of equation

$$-\Delta_p u + |u|^{q-1} u = f, \qquad \text{in } B_r.$$

Necessarily it coincides with u, and the whole sequence converges to u. Thus u is a local entropy solution in any ball B_r, hence it is a local entropy solution of equation (1.9). In particular it satisfies the equation

$$-\Delta_p u + |u|^{q-1} u = f, \qquad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

If $f \geq 0$, then $u_n \geq 0$ from [5, Theorem 7.1], hence $u \geq 0$ *a.e.* in $\mathbb{R}^N$. ∎

Now we assume that q is subcritical. We first give a global existence result for measures when Ω is bounded, by using the global stability properties shown in [18, Theorem 3.4]. Recall that the first results were given in [12] when $P > p_1$. We get the following:

Theorem 8.2 *Assume that Ω is bounded, and $p-1 < q < \overline{P}$. Then for any $\mu \in \mathcal{M}_b(\Omega)$, there exists an entropy solution of problem*

$$\begin{cases} -\Delta_p u + |u|^{q-1} u = \mu, & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases} \tag{8.1}$$

Proof From [13], μ can be decomposed as

$$\mu = f - divg + \mu_s^+ - \mu_s^-,$$

with $f \in L^1(\Omega)$, $g \in \left(L^{p'}(\Omega)\right)^N$, and $\mu_s^+, \mu_s^- \in \mathcal{M}_b^+(\Omega)$ are singular. There exists a sequence (μ_ν) approximating μ such that

$$\mu_\nu = f_\nu - divg + \lambda_\nu - \eta_\nu, \qquad f_\nu, \lambda_\nu, \eta_\nu \in L^{p'}(\Omega).$$

Hence $\mu_\nu \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega) \cap \mathcal{M}_0(\Omega))$, and f_ν converges weakly in $L^1(\Omega)$ to f, and λ_ν (resp. η_ν) converges weakly to μ_s^+ (resp.μ_s^-). Then there exists a (unique) weak solution u_ν of problem

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = \mu_\nu, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega, \end{cases}$$

such that $u_\nu \in W_0^{1,p}(\Omega) \cap L^{q+1}(\Omega)$. It is also an entropy solution, since it satisfies **D2**. From Theorem 4.1, $(|u_\nu|^q)$ is bounded in $L^1_{loc}(\Omega)$. From Theorem 3.2, up to a subsequence, (u_ν) converges locally in measure in Ω and *a.e.* in Ω to some function u, and $\left(|u_\nu|^{p-1}\right)$ is bounded in $L^s_{loc}(\Omega)$, for any $1 \le s < N/(N-p)$. In fact the estimate is global. Indeed, by assumption,

$$\int_\Omega |\nabla u_\nu|^{p-2} \nabla u_\nu . \nabla (h(u_\nu)\varphi) dx + \int_\Omega |u_\nu|^{q-1} u_\nu h(u_\nu)\varphi dx = \int_\Omega h(u_\nu)\varphi d\mu_\nu$$

for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, such that $h(u)\varphi \in W_0^{1,p}(\Omega)$. Taking $h = T_k$ for any $k > 0$, and $\varphi = 1$, one gets

$$\frac{1}{k} \int_{\{|u_\nu| < k\}} |\nabla u_\nu|^p \, dx \le \int_\Omega u_{\nu,k} d\mu_\nu \le \int_\Omega d\,|\mu_\nu| \le C,$$

where $C > 0$ does not depend on ν and k. Then $\left(|u_\nu|^{p-1}\right)$ is bounded in $L^s(\Omega)$, for $1 < s < N/(N-p)$, from [5, Lemma 4.1]. Therefore $|u_\nu|^{q-1} u_\nu$ converges strongly in $L^1(\Omega)$ to $|u|^{q-1} u$, since $q < \overline{P}$. We can apply [18, Theorem 3.4] with right-hand side $f_\nu - |u_\nu|^{q-1} u_\nu$ instead of f_ν. Then, up to a subsequence, u_ν converges *a.e.* in Ω to an entropy solution w of

$$\begin{cases} -\Delta_p w = \mu - |u|^{q-1} u, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Necessarily $w = u$, and u is an entropy solution of equation (1.1). ∎

In turn using this result and the local stability result of Theorem 3.3, we can show an existence result in $\mathbb{R}^N$:

Theorem 8.3 *Assume $p - 1 < q < \overline{P}$. Then for any $\mu \in \mathcal{M}(\mathbb{R}^N)$, there exists a local entropy solution of equation (1.1).*

Proof For any $\nu \in \mathbb{N}$, let $\mu \llcorner B_\nu$ be the restriction of μ to B_ν. Then there exists an entropy solution u_ν of problem

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = \mu \llcorner B_\nu, & \text{in } B_\nu, \\ u_\nu = 0, & \text{on } \partial B_\nu, \end{cases}$$

since $\mu \llcorner B_\nu \in \mathcal{M}_b(B_\nu)$, from Theorem 8.2. From Theorem 4.1, (u_ν) satisfies the same estimates as in Theorem 1.6. Then, up to a subsequence, u_ν converges *a.e.* in $\mathbb{R}^N$ to some function u. For $\nu > 2r$, u_ν is a local solution in B_r of

$$-\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = \mu \llcorner B_r.$$

Thus we can extract a diagonal subsequence, (u_ρ), such that u_ρ converges, locally in measure and *a.e.* in $\mathbb{R}^N$, to a function u, satisfying the conclusions of Theorem 3.2 in $\mathbb{R}^N$. As above, $|u_\rho|^{q-1} u_\rho$ converges strongly in $L^1_{loc}(\mathbb{R}^N)$ to $|u|^{q-1} u$, since q is subcritical. Now we apply Theorem 3.3 in B_r, with right-hand side $-|u_\rho|^{q-1} u_\rho + \mu \llcorner B_r$. We can extract a subequence which depends on r, converging *a.e.* in B_r, to a local entropy solution w of

$$-\Delta_p w + |w|^{q-1} w = \mu \llcorner B_r.$$

Necessarily $w = u$, and the whole sequence converges, and u is a local entropy solution in $\mathbb{R}^N$. ∎

9 Appendix A: Proof of Theorem 2.2

Step 1: D1-loc⟹D3-loc. Let u be a local entropy solution of equation (1.3). Consider any domain $U \subset\subset \Omega$. We first apply (2.21) with $h = h_{k,\varepsilon}$ defined in (2.4) with $0 < \varepsilon < k$, and $\varphi_U \in \mathcal{D}(\Omega), \varphi_U \geq 0$, such that $\varphi_U = 1$ on U. We obatin

$$\begin{aligned} \frac{1}{\varepsilon} \int_{U \cap \{k<u<k+\varepsilon\}} |\nabla u|^p \, dx \leq & \int_\Omega h_{k,\varepsilon}(u) \varphi_U d\mu_0 + \int_\Omega \varphi_U d\mu_s^+ - \\ & \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi_U h_{k,\varepsilon}(u) dx \\ & \leq C(U) \end{aligned}$$

where $C(U)$ does not depend on ε, k, from (2.20); and similarly

$$\frac{1}{\varepsilon} \int_{U \cap \{k<|u|<k+\varepsilon\}} |\nabla u|^p \, dx \leq C(U). \tag{9.1}$$

Then we apply (2.21) with $h = H_{k,\varepsilon}$ defined in (2.4) and $\varphi \in \mathcal{D}(U)$. It follows that

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi H_{k,\varepsilon}(u) dx = & \frac{1}{\varepsilon} \int_{\{k<u<k+\varepsilon\}} |\nabla u|^p \varphi dx - \frac{1}{\varepsilon} \int_{\{-k-\varepsilon<u<-k\}} |\nabla u|^p \varphi dx \\ & + \int_\Omega H_k(u) \varphi d\mu_0 \leq C(U) \|\varphi\|_{L^\infty(U)} . \end{aligned}$$

And $H_{k,\varepsilon}(u)$ converges to $\chi_{\{|u|\leq k\}}$ *a.e.* in Ω. Hence from (2.20), for any $\varphi \in \mathcal{D}(U)$,

$$\left| \int_\Omega |\nabla u_k|^{p-2} \nabla u_k . \nabla \varphi dx \right| \leq C(U) \|\varphi\|_{L^\infty(U)} .$$

As a consequence, there exists a measure $\mu_k \in \mathcal{M}(\Omega)$, such that

$$\int_\Omega |\nabla u_k|^{p-2} \nabla u . \nabla \psi dx = \int_\Omega \psi d\mu_k, \tag{9.2}$$

for any $\psi \in \mathcal{D}(\Omega)$, and in particular $\mu_k \in W_0^{-1,p'}(U)$. By density (9.2) holds for any $\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support. Moreover the sequence (μ_k) is uniformly bounded in U . Hence there exists a sequence (k_n) tending to ∞ such that μ_{k_n} converges weakly to a measure $\lambda \in \mathcal{M}(\Omega)$. Applying (2.21) with $h(r) = 1$, we find, for any $\psi \in \mathcal{D}(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega \psi d\mu_{k_n} = \lim \int_\Omega |\nabla u_{k_n}|^{p-2} \nabla u_{k_n} . \nabla \psi dx = \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \psi dx = \int_\Omega \psi d\mu,$$

from hence $\lambda = \mu$. Following the proof of [17, Lemma 5.1] for the set U, and using a partition of unity, there exists a measure $\nu \in \mathcal{M}_0(\Omega)$ such that the restrictions $\nu \llcorner \{|u| = k\}$ and $\mu_\ell \llcorner \{|u| = k\}$ coincide for any $\ell \geq k > 0$. Then $\nu \llcorner \{|u| = k\} = \mu \llcorner \{|u| = k\}$. Considering $\mu_k \llcorner U$ and following [17, Lemma 5.3], we find

$$\int_{\{|u|>k\}} d|\mu_k| = 0, \qquad \text{for any } k > 0.$$

Defining

$$\alpha_k = \mu_k \llcorner \{u = k\}, \qquad \beta_k = -\mu_k \llcorner \{u = -k\},$$

and taking $\psi = h_{k-\varepsilon,\varepsilon}(u)\phi$ in (9.2), where $\phi \in \mathcal{D}(\Omega)$, we derive

$$\frac{1}{\varepsilon} \int_{\{k-\varepsilon<u<k\}} |\nabla u|^p \phi dx = \int_\Omega h_{k-\varepsilon,\varepsilon}(u) \phi d\mu_k - \int_{\{k-\varepsilon<u<k\}} h_{k-\varepsilon,\varepsilon}(u) |\nabla u|^{p-2} \nabla u \nabla \phi dx.$$

Then

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{k-\varepsilon<u<k\}} |\nabla u|^p \phi dx = \int_{\{u\geq k\}} \phi d\mu_k = \int_\Omega \phi d\alpha_k,$$

so that $\alpha_k \geq 0$, and in the same way $\beta_k \geq 0$. Now we apply in (2.21) with $h = H_{k-\varepsilon,\varepsilon}$ defined in (2.4) for $0 < \varepsilon < k$, and $\varphi \in \mathcal{D}(U)$. We obtain, from (9.1),

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi H_{k-\varepsilon,\varepsilon}(u) dx = \frac{1}{\varepsilon} \int_{\{k-\varepsilon<u<k\}} |\nabla u|^p \varphi dx$$
$$- \frac{1}{\varepsilon} \int_{\{-k<u<-k+\varepsilon\}} |\nabla u|^p \varphi dx + \int_\Omega H_{k-\varepsilon,\varepsilon}(u) \varphi d\mu_0.$$

Then, letting ε tend to 0, we deduce

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k.\nabla\varphi dx = \int_\Omega \varphi d\alpha_k - \int_\Omega \varphi d\beta_k + \int_{\{|u|\le k\}} \varphi d\mu_0.$$

By density it holds for any $\varphi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ with compact support in Ω, so that (2.23) holds. Also $\alpha_k \le |\mu_k|$ and $\beta_k \le |\mu_k|$; by using a partition of unity, there exists a sequence $(k_n)\to\infty$ such that α_{k_n} and β_{k_n} converge weakly respectively to some measures α and $\beta \in \mathcal{M}^+(\Omega)$. Changing k into $2k$ in (2.23) and taking $\psi = h_{k,k}(u)\phi$, with $\phi\in\mathcal{D}(\Omega)$, we find

$$\begin{aligned}&\int_{\{0<u<2k\}} h_{k,k}(u)\,|\nabla u|^{p-2}\nabla u.\nabla\phi dx + \frac{1}{k}\int_{\{k<u<2k\}} |\nabla u|^p\,\phi dx\\ &= \int_\Omega h_{k,k}(u)\phi d\alpha_k - \int_\Omega h_{k,k}(u)\phi d\beta_k + \int_{\{|u|\le 2k\}} h_{k,k}(u)\phi d\mu_0\\ &= \int_\Omega \phi d\alpha_k + \int_{\{|u|\le 2k\}} h_{k,k}(u)\phi d\mu_0.\end{aligned}$$

Moreover

$$\lim_{k\to\infty}\int_{\{|u|\le 2k\}} h_{k,k}(u)\varphi d\mu_0 = 0,$$

since $h_{k,k}(u)$ converges to 0, μ_0-*a.e.* in Ω. From (2.21), we derive

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\phi h_{k,k}(u)dx + \frac{1}{k}\int_{\{k<u<2k\}} |\nabla u|^p\,\phi dx = \int_\Omega h_{k,k}(u)\phi d\mu_0 + \int_\Omega \phi d\mu_s^+.$$

Then

$$\lim_{k\to\infty}\int_\Omega \phi d\alpha_k = \int_\Omega \phi d\mu_s^+,$$

which means $\alpha = \mu_s^+$ and $\beta = \mu_s^-$.

Step 2: D3-loc $\Longrightarrow$ D2-loc $\Longrightarrow$ D4-loc $\Longrightarrow$ D1-loc The proofs are similar to the ones in [18], after choosing test functions with compact support in Ω.

10 Appendix B: Proof of Theorem 3.3

We essentially follow the proof of [18, Theorem 3.2] and adapt it to the local case. Let

$$\lambda = \lambda_0 + \lambda_s^+ - \lambda_s^-$$

be the decomposition of λ given by (2.2), and E^+, E^- be the disjoint sets where λ_s^+, λ_s^- are concentrated. Let u_ν be any local entropy solution of (3.26). By **D2-loc**, equivalent to **D1-loc** from Theorem 2.2, if $\omega\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ with compact support in Ω, and if there exists $k>0$ and $\omega^+,\omega^-\in W^{1,r}(\Omega)\cap L^\infty(\Omega)$ with $r>N$,

such that $\omega = \omega^+$ *a.e.* on the set $\{u_\nu > k\}$ and $\omega = \omega^-$ *a.e.* on the set $\{u_\nu < -k\}$, then

$$\int_\Omega |\nabla u_\nu|^{p-2} \nabla u_\nu . \nabla \omega dx = \int_\Omega \omega f_\nu dx + \int_\Omega \omega d\lambda_0 + \int_\Omega \omega^+ d\lambda_s^+ - \int_\Omega \omega^- d\lambda_s^-. \quad (10.1)$$

By assumption, $|u_\nu|^q$ is bounded in $L^1_{loc}(\Omega)$, then (3.21), (3.22), (3.23) hold from Theorem 3.2, and up to a subsequence u_ν converges *a.e.* to some function u. Let Ω', Ω'' be two fixed regular domains such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, and

$$E''^+ = E^+ \cap \Omega'', \qquad E''^- = E^- \cap \Omega''.$$

Let $\varphi, \Psi \in \mathcal{D}(\Omega)$ be a fixed functions with values in $[0,1]$, such that

$$\varphi \equiv 1 \quad \text{in } \Omega', \quad \text{supp}\varphi \subset \Omega'', \quad \Psi \equiv 1 \quad \text{in } \Omega''.$$

For any $\delta, \eta > 0$ and $n, \nu \in \mathbb{N}$, we denote by $\varpi(\eta, \delta, n, \nu)$ any quantity such that

$$\lim_{\eta \to 0} \limsup_{\delta \to 0} \lim_{n \to \infty} \sup_{\nu \to \infty} \lim |\varpi(\eta, \delta, n, \nu)| = 0.$$

Let $\delta, \eta > 0$. From [18, Lemma 5.1], we can define two compacts sets K_δ^+, K_δ^- such that

$$\lambda_s^+(E''^+ \backslash K_\delta^+) \leq \delta, \qquad \lambda_s^-(E''^- \backslash K_\delta^-) \leq \delta,$$

and two functions $\psi_\delta^+, \psi_\delta^- \in \mathcal{D}(\Omega'')$ with values in $[0,1]$ and disjoint supports, such that $\psi_\delta^+ = 1$ on K_δ^+, $\psi_\delta^- = 1$ on K_δ^-, and

$$\int_{\Omega''} |\nabla \psi_\delta^+|^p dx + \int_{\Omega''} |\nabla \psi_\delta^-|^p dx + \int_{\Omega''} \psi_\delta^- d\lambda_s^+ + \int_{\Omega''} \psi_\delta^+ d\lambda_s^- \leq \delta, \quad (10.2)$$

$$\int_{\Omega''} (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_s^+ + \int_{\Omega''} (1 - \psi_\delta^- \psi_\eta^-) d\lambda_s^- \leq \delta + \eta. \quad (10.3)$$

Step 1. Behaviour near E''. Let $\phi_\eta^+, \phi_\eta^- \in W^{1,\infty}(\Omega)$, with values in $[0,1]$, and compact support in Ω'', such that

$$\int_{\Omega''} \phi_\eta^- d\lambda_s^+ + \int_{\Omega''} \phi_\eta^+ d\lambda_s^- \leq \eta. \quad (10.4)$$

• Claim 1: as in [18, Lemma 6.1], there holds

$$\frac{1}{n} \left(\int_{\{n \leq u_\nu \leq 2n\}} |\nabla u_\nu|^p \phi_\eta^- dx + \int_{\{-2n \leq u_\nu \leq -n\}} |\nabla u_\nu|^p \phi_\eta^+ dx \right) \leq \varpi_\eta(n, \nu) + \eta. \quad (10.5)$$

Indeed we first take $\omega = h_{n,n}(u_\nu)\Psi$ in (10.1), where $h_{n,n}$ is defined in (2.4), and obtain

$$\frac{1}{n} \int_{\{n < u_\nu < 2n\}} |\nabla u_\nu|^p \Psi dx = \int_\Omega f_\nu h_{n,n}(u_\nu) \Psi dx + \int_\Omega h_{n,n}(u_\nu) \Psi d\lambda_0$$
$$- \int_\Omega h_{n,n}(u_\nu) |\nabla u_\nu|^{p-2} \nabla u_\nu . \nabla \Psi dx + \int_\Omega \Psi d\lambda_s^+.$$

Then, from (3.23),

$$\frac{1}{n}\int_{\{n<u_\nu<2n\}} |\nabla u_\nu|^p \Psi dx \le \|f_\nu \Psi\|_{L^1(\Omega)} + \int_\Omega \Psi d|\lambda| + \int_{\text{supp }\Psi} |\nabla u_\nu|^{p-1} dx \le C$$

where C depends on λ, Ψ, f, but not on n and ν. As a consequence,

$$\frac{1}{n}\int_{\Omega''\cap\{n<u_\nu<2n\}} |\nabla u_\nu|^p dx = \int_{\Omega''} |\nabla h_{n,n}(u_\nu)|^p dx \le Cn^{1-p}. \tag{10.6}$$

For fixed n, the sequence $(h_{n,n}(u_\nu))$ is bounded in $L^\infty(\Omega)$, and $h_{n,n}(u_\nu)$ converges to $h_{n,n}(u)$, *a.e.* in Ω, as $\nu \to \infty$, and strongly in $W^{1,p}(\Omega'')$ from (10.6), so that

$$\int_{\Omega''} |\nabla h_{n,n}(u)|^p dx \le Cn^{1-p}. \tag{10.7}$$

Now the sequence $(h_{n,n}(u))$ is bounded in $L^\infty(\Omega)$, and $h_{n,n}(u)$ converges to 0, *a.e.* in Ω, as $n \to \infty$, and strongly in $W^{1,p}(\Omega'')$ from (10.7).

Then we take in (10.1) $\omega = h_{n,n}(u_\nu)\phi_\eta^-$, and obtain

$$\begin{aligned}\frac{1}{n}\int_{\{n<u_\nu<2n\}} |\nabla u_\nu|^p \phi_\eta^- dx = &\int_\Omega f_\nu h_{n,n}(u_\nu)\phi_\eta^- dx + \int_\Omega h_{n,n}(u_\nu)\phi_\eta^- d\lambda_0 \\ &- \int_\Omega h_{n,n}(u_\nu)\,|\nabla u_\nu|^{p-2}\nabla u_\nu.\nabla\phi_\eta^- dx + \int_\Omega \phi_\eta^- d\lambda_s^+.\end{aligned}$$

Since $|\nabla u_\nu|^{p-2}\nabla u_\nu$ converges strongly in $(L^r(\Omega''))^N$ for any $1 \le r < N/(N-1)$, we find

$$\begin{aligned}\int_\Omega h_{n,n}(u_\nu)\,|\nabla u_\nu|^{p-2}\nabla u_\nu.\nabla\phi_\eta^- dx &= \int_\Omega h_{n,n}(u)\,|\nabla u|^{p-2}\nabla u.\nabla\phi_\eta^- dx + \varpi_{\eta,n}(\nu) \\ &= \varpi_\eta(n,\nu),\end{aligned}$$

$$\int_\Omega h_{n,n}(u_\nu)\phi_\eta^- d\lambda_0 = \int_\Omega h_{n,n}(u)\phi_\eta^- d\lambda_0 + \varpi_{\eta,n}(\nu) = \varpi_\eta(n,\nu).$$

Also, by [18, Proposition 2.8],

$$\int_\Omega f_\nu h_{n,n}(u_\nu)\phi_\eta^- dx = \int_\Omega f h_{n,n}(u)\phi_\eta^- dx + \varpi_{\eta,n}(\nu) = \varpi_\eta(n,\nu).$$

Therefore, from (10.2),

$$\frac{1}{n}\int_{\{n\le u_\nu\le 2n\}} |\nabla u_\nu|^p \phi_\eta^- dx \le \varpi_\eta(n,\nu) + \eta,$$

and (10.5) follows, after exchanging ϕ_η^- into ϕ_η^+ and $h_{n,n}(r)$ into $h_{n,n}(-r)$.

• Claim 2: as in [18, Lemma 6.3], for fixed $k > 0$, and for any $\eta, \delta > 0$,

$$\int_\Omega |\nabla u_{\nu,k}|^p \psi_\delta^+ \psi_\eta^+ dx + \int_\Omega |\nabla u_{\nu,k}|^p \psi_\delta^- \psi_\eta^- dx = \varpi(\eta, \delta, \nu). \tag{10.8}$$

Indeed we take $\omega = H_{n,n}(u_\nu)(k - u_{\nu,k})\psi_\delta^+\psi_\eta^+$ in (10.1), where $H_{n,n}$ is defined in (2.4), and obtain

$$\begin{aligned}
&- \int_\Omega |\nabla u_{\nu,k}|^p H_{n,n}(u_\nu)\psi_\delta^+\psi_\eta^+ dx + \int_\Omega |\nabla u_\nu|^{p-2} \nabla u_\nu . \nabla \psi_\delta^+ H_{n,n}(u_\nu)(k - u_{\nu,k})\psi_\eta^+ dx \\
&+ \int_\Omega |\nabla u_\nu|^{p-2} \nabla u_\nu . \nabla \psi_\eta^+ H_{n,n}(u_\nu)(k - u_{\nu,k})\psi_\delta^+ dx \\
&+ \int_\Omega |\nabla u_\nu|^p H'_{n,n}(u_\nu)(k - u_{\nu,k})\psi_\delta^+\psi_\eta^+ dx \\
&= \int_\Omega f_\nu (k - u_{\nu,k}) H_{n,n}(u_\nu)\psi_\delta^+\psi_\eta^+ dx + \int_\Omega (k - u_{\nu,k}) H_{n,n}(u_\nu)\psi_\delta^+\psi_\eta^+ d\lambda_0.
\end{aligned} \tag{10.9}$$

For fixed n, the sequence $(H_{n,n}(u_\nu))$ is bounded. As $\nu \to \infty$, $H_{n,n}(u_\nu)$ converges to $H_{n,n}(u)$ and *a.e.* in Ω, and strongly in $W^{1,p}(\Omega'')$, from (10.6) and the same estimate on $\Omega'' \cap \{-2n < u_\nu < -n\}$. From (10.5) applied with $\phi_\eta^+ = \psi_\eta^+$,

$$\begin{aligned}
\int_\Omega |\nabla u_\nu|^p H'_{n,n}(u_\nu)(k - u_{\nu,k})\psi_\delta^+\psi_\eta^+ dx &= \frac{2k}{n} \int_{\{-2n<u\le -n\}} |\nabla u_\nu|^p \psi_\delta^+\psi_\eta^+ dx \\
&\le \frac{2k}{n} \int_{\{-2n<u\le -n\}} |\nabla u_\nu|^p \psi_\eta^+ dx \\
&\le \varpi_\eta(n, \nu) + \eta = \varpi(\eta, n, \nu).
\end{aligned}$$

As in the proof of [18, Lemma 6.3], since ψ_η^+ has a compact support, we obtain successively

$$\begin{aligned}
\int_\Omega |\nabla u_\nu|^{p-2} \nabla u_\nu . \nabla \psi_\delta^+ H_{n,n}(u_\nu)(k - u_{\nu,k})\psi_\eta^+ dx &= \varpi_{\eta,n}(\delta, \nu), \\
\int_\Omega |\nabla u_\nu|^{p-2} \nabla u_\nu . \nabla \psi_\eta^+ H_{n,n}(u_\nu)(k - u_{\nu,k})\psi_\delta^+ dx &= \varpi_{\eta,n}(\delta, \nu), \\
\int_\Omega f_\nu (k - u_{\nu,k}) H_{n,n}(u_\nu)\psi_\delta^+\psi_\eta^+ dx &= \varpi_{\eta,n}(\delta, \nu),
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega (k - u_{\nu,k}) H_{n,n}(u_\nu)\psi_\delta^+\psi_\eta^+ d\lambda_0 &= \int_\Omega (k - u_k) H_{n,n}(u)\psi_\delta^+\psi_\eta^+ d\lambda_0 + \varpi_{\eta,n,\delta}(\nu) \\
&= \varpi_{\eta,n}(\delta, \nu).
\end{aligned}$$

Indeed $\lambda_0 \in \mathcal{M}_0(\Omega)$ and $(k - u_{\nu,k})H_{n,n}(u_\nu)\psi_\delta^+\psi_\eta^+$ converges weakly in $W_0^{1,p}(\Omega'')$ to $(k - u_{\nu,k})H_{n,n}(u_\nu)\psi_\delta^+\psi_\eta^+$ as $\nu \to \infty$; this holds because $u_{\nu,k}$ converges weakly to u_k

in $W^{1,p}(\Omega'')$ from (3.21). Moreover $(k-u_k)H_{n,n}(u)\psi_\delta^+\psi_\eta^+$ converges strongly to 0 in $W_0^{1,p}(\Omega'')$, from (10.2), as $\delta \to 0$. Hence from (10.9),

$$\int_\Omega |\nabla u_{\nu,k}|^p \, H_{n,n}(u_\nu)\psi_\delta^+\psi_\eta^+ dx = \varpi(\eta, n, \delta, \nu),$$

and more precisely

$$\int_\Omega |\nabla u_{\nu,k}|^p \, h_{n,n}(u_\nu)\psi_\delta^+\psi_\eta^+ dx = \int_\Omega |\nabla u_{\nu,k}|^p \, \psi_\delta^+\psi_\eta^+ dx = \varpi(\eta, \delta, \nu)$$

since $n > k$. We deduce (10.8) after replacing the test function by $\omega = H_{n,n}(u_\nu)(k+u_{\nu,k})\psi_\delta^-\psi_\eta^-$.

Step 2. Behaviour far from E''. Now we define

$$\Phi_{\delta,\eta} = \psi_\delta^+\psi_\eta^+ + \psi_\delta^-\psi_\eta^-$$

and following [18, Lemma 7.1], we will show that for fixed $k > 0$,

$$\int_\Omega (|\nabla u_{\nu,k}|^p - |\nabla u_k|^p)\,(1-\Phi_{\delta,\eta})\varphi dx = \varpi(\eta, \delta, \nu). \tag{10.10}$$

• Claim 3: as in [18, Lemma 7.3], there holds

$$\begin{aligned}
&\int_\Omega |\nabla u_{\nu,k}|^p \,(1-\Phi_{\delta,\eta})\varphi dx - \int_\Omega u_k\varphi \,|\nabla u_k|^{p-2}\,\nabla u_k.\nabla\Phi_{\delta,\eta} dx + \\
&\quad \int_\Omega \Phi_{\delta,\eta} u_k \,|\nabla u_k|^{p-2}\,\nabla u_k.\nabla\varphi dx \\
&= \int_\Omega f(1-\Phi_{\delta,\eta})u_k\varphi dx + \int_\Omega (1-\Phi_{\delta,\eta})u_k\varphi d\lambda_0 + \varpi(\eta,\delta,\nu).
\end{aligned} \tag{10.11}$$

Indeed choosing $\omega = (1-\Phi_{\delta,\eta})u_{\nu,k}\varphi$ as test function, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u_{\nu,k}|^p \,(1-\Phi_{\delta,\eta})\varphi dx - \int_\Omega u_{\nu,k}\varphi \,|\nabla u_{\nu,k}|^{p-2}\,\nabla u_{\nu,k}.\nabla\Phi_{\delta,\eta} dx \\
&+ \int_\Omega \Phi_{\delta,\eta} u_{\nu,k} \,|\nabla u_{\nu,k}|^{p-2}\,\nabla u_{\nu,k}.\nabla\varphi dx \\
&= \int_\Omega f_\nu(1-\Phi_{\delta,\eta})u_{\nu,k}\varphi dx + \int_\Omega (1-\Phi_{\delta,\eta})u_{\nu,k}\varphi d\lambda_0 \\
&+ k\int_\Omega (1-\Phi_{\delta,\eta})\varphi d\lambda_s^+ - k\int_\Omega (1-\Phi_{\delta,\eta})\varphi d\lambda_s^-;
\end{aligned}$$

we derive successively, as in [18, Lemma 7.3],

$$\int_\Omega u_{\nu,k}\varphi \,|\nabla u_{\nu,k}|^{p-2}\,\nabla u_{\nu,k}.\nabla\Phi_{\delta,\eta} dx = \int_\Omega u_k\varphi \,|\nabla u_k|^{p-2}\,\nabla u_k.\nabla\Phi_{\delta,\eta} dx + \omega_{\eta,\delta}(\nu);$$

$$\int_\Omega \Phi_{\delta,\eta} u_{\nu,k} \left|\nabla u_{\nu,k}\right|^{p-2} \nabla u_{\nu,k}.\nabla\varphi dx = \int_\Omega \Phi_{\delta,\eta} u_k \left|\nabla u_k\right|^{p-2} \nabla u_k.\nabla\varphi dx + \omega_{\eta,\delta}(\nu);$$

$$\int_\Omega f_\nu(1-\Phi_{\delta,\eta}) u_{\nu,k}\varphi dx = \int_\Omega f(1-\Phi_{\delta,\eta}) u_k\varphi dx + \varpi_{\eta,\delta}(\nu);$$

$$\int_\Omega (1-\Phi_{\delta,\eta}) u_{\nu,k}\varphi d\lambda_0 = \int_\Omega (1-\Phi_{\delta,\eta}) u_k\varphi d\lambda_0 + \varpi_{\eta,\delta}(\nu);$$

since φ has a compact support in Ω'', and values in $[0,1]$, we derive from (10.3)

$$\int_\Omega (1-\psi_\delta^+\psi_\eta^+)\varphi d\lambda_s^+ + \int_{\Omega''} (1-\psi_\delta^-\psi_\eta^-)\varphi d\lambda_s^- \leq \delta+\eta;$$

therefore, since k is fixed,

$$k\int_\Omega (1-\Phi_{\delta,\eta})\varphi d\lambda_s^+ - k\int_\Omega (1-\Phi_{\delta,\eta})\varphi d\lambda_s^- = \varpi(\delta,\eta),$$

which implies (10.11).

- Claim 4: as in [18, Lemma 7.4], there holds

$$\frac{1}{n}\int_{\{n<|u_\nu|<2n\}} \left|\nabla u_\nu\right|^p (1-\Phi_{\delta,\eta})\varphi dx = \varpi(\eta,\delta,n,\nu). \tag{10.12}$$

Indeed we have

$$\begin{aligned}
&\frac{1}{n}\int_{\{n<|u_\nu|<2n\}} \left|\nabla u_\nu\right|^p (1-\Phi_{\delta,\eta})\varphi dx \\
&= \frac{1}{n}\int_{\{n<u_\nu<2n\}} \left|\nabla u_\nu\right|^p (1-\psi_\delta^+\psi_\eta^+)\varphi dx - \frac{1}{n}\int_{\{n<u_\nu<2n\}} \left|\nabla u_\nu\right|^p \psi_\delta^+\psi_\eta^+\varphi dx \\
&+ \frac{1}{n}\int_{\{-2n<u_\nu<-n\}} \left|\nabla u_\nu\right|^p (1-\psi_\delta^-\psi_\eta^-)\varphi dx - \frac{1}{n}\int_{\{-2n<u_\nu<-n\}} \left|\nabla u_\nu\right|^p \psi_\delta^+\psi_\eta^+\varphi dx.
\end{aligned}$$

Using the fact that φ has values in $[0,1]$, we can apply (10.5) to

$$\phi_{\delta+\eta}^- = (1-\psi_\delta^+\psi_\eta^+)\varphi \quad \text{and} \quad \phi_{\delta+\eta}^+ = (1-\psi_\delta^-\psi_\eta^-)\varphi,$$

since (10.3) holds. The functions $\psi_\delta^+, \psi_\delta^-$ take their values in $[0,1]$, then we can apply (10.5) to

$$\phi_{\delta,\eta}^- = \psi_\delta^-\psi_\eta^-\varphi \quad \text{and} \quad \phi_{\delta,\eta}^+ = \psi_\delta^+\psi_\eta^+\varphi,$$

since (10.2) holds for ψ_η^-, ψ_η^+. Thus we conclude to (10.12).

- Claim 5: as in [18, Lemma 7.5], there holds

$$\begin{aligned}
&\int_\Omega \left|\nabla u_k\right|^p (1-\Phi_{\delta,\eta})\varphi dx - \int_\Omega u_k\varphi \left|\nabla u\right|^{p-2} \nabla u.\nabla\Phi_{\delta,\eta} dx + \\
&\int_\Omega \Phi_{\delta,\eta} u_k \left|\nabla u\right|^{p-2} \nabla u.\nabla\varphi dx \\
&= \int_\Omega f(1-\Phi_{\delta,\eta}) u_k\varphi dx + \int_\Omega (1-\Phi_{\delta,\eta}) u_k\varphi d\lambda_0 + \varpi(\eta,\delta).
\end{aligned} \tag{10.13}$$

Indeed first observe that for fixed n, $(H_{n,n}(u_\nu))$ is bounded in $L^\infty(\Omega)$, $H_{n,n}(u_\nu)$ converges to $H_{n,n}(u)$, *a.e.* in Ω, and strongly in $W^{1,p}(\Omega'')$ as $\nu \to \infty$. Also $(H_{n,n}(u))$ is bounded in $L^\infty(\Omega)$, $H_{n,n}(u)$ converges to 1, *a.e.* in Ω, and strongly in $W^{1,p}(\Omega'')$, as $n \to \infty$. Choosing $\omega = (1-\Phi_{\delta,\eta})u_k H_{n,n}(u_\nu)\varphi$ as test function in (10.1) with $n > k$, we obtain

$$\begin{aligned}
&\int_\Omega (1-\Phi_{\delta,\eta})H_{n,n}(u_\nu)\varphi\,|\nabla u_\nu|^{p-2}\,\nabla u_\nu.\nabla u_k dx\\
&-\int_\Omega u_k H_{n,n}(u_\nu)\varphi\,|\nabla u_k|^{p-2}\,\nabla u_k.\nabla\Phi_{\delta,\eta}dx\\
&+\int_\Omega (1-\Phi_{\delta,\eta})H_{n,n}(u_\nu)u_k\,|\nabla u_\nu|^{p-2}.\nabla\varphi dx\\
&+\int_\Omega |\nabla u_\nu|^p\,u_k(1-\Phi_{\delta,\eta})H'_{n,n}(u_\nu)\varphi dx\\
&=\int_\Omega f(1-\Phi_{\delta,\eta})u_k H_{n,n}(u_\nu)\varphi dx+\int_\Omega (1-\Phi_{\delta,\eta})u_k H_{n,n}(u_\nu)\varphi d\lambda_0.
\end{aligned}$$

As in [18, Lemma 7.5], we deduce

$$\begin{aligned}
&\int_\Omega (1-\Phi_{\delta,\eta})H_{n,n}(u_\nu)\varphi\,|\nabla u_\nu|^{p-2}.\nabla u_k dx\\
&=\int_\Omega (1-\Phi_{\delta,\eta})H_{n,n}(u)\varphi\,|\nabla u_{2n}|^{p-2}.\nabla u_k dx+\omega_{\eta,\delta,n}(\nu)\\
&=\int_\Omega |\nabla u_k|^p\,(1-\Phi_{\delta,\eta})\varphi dx+\omega_{\eta,\delta}(n,\nu);
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega u_k H_{n,n}(u_\nu)\varphi\,|\nabla u_k|^{p-2}\,\nabla u_k.\nabla\Phi_{\delta,\eta}dx\\
&=\int_\Omega u_k H_{n,n}(u)\varphi\,|\nabla u|^{p-2}\,\nabla u.\nabla\Phi_{\delta,\eta}dx+\varpi_{\eta,\delta,n}(\nu)\\
&=\int_\Omega u_k\varphi\,|\nabla u|^{p-2}\,\nabla u.\nabla\Phi_{\delta,\eta}dx+\varpi_{\eta,\delta}(n,\nu);
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega (1-\Phi_{\delta,\eta})H_{n,n}(u_\nu)u_k\,|\nabla u_\nu|^{p-2}.\nabla\varphi dx\\
&=\int_\Omega (1-\Phi_{\delta,\eta})u_k\,|\nabla u|^{p-2}.\nabla\varphi h_{n,n}(u)dx+\varpi_{\eta,\delta,n}(\nu)\\
&=\int_\Omega (1-\Phi_{\delta,\eta})u_k\,|\nabla u|^{p-2}\,\nabla u.\nabla\varphi dx+\varpi_{\eta,\delta}(n,\nu);
\end{aligned}$$

moreover

$$\left|\int_\Omega |\nabla u_\nu|^p\,u_k(1-\Phi_{\delta,\eta})H'_{n,n}(u_\nu)\varphi dx\right|\le\varpi(\eta,\delta,n,\nu)$$

from (10.12), since k is fixed. And

$$\int_\Omega f(1-\Phi_{\delta,\eta})u_k H_{n,n}(u_\nu)\varphi dx = \int_\Omega f(1-\Phi_{\delta,\eta})u_k H_{n,n}(u)\varphi dx + \varpi_{\eta,\delta,n}(\nu)$$
$$= \int_\Omega f(1-\Phi_{\delta,\eta})u_k\varphi dx + \omega_{\eta,\delta}(n,\nu);$$

$$\int_\Omega (1-\Phi_{\delta,\eta})u_k H_{n,n}(u_\nu)\varphi d\lambda_0 = \int_\Omega (1-\Phi_{\delta,\eta})u_k H_{n,n}(u)\varphi d\lambda_0 + \varpi_{\eta,\delta,n}(\nu)$$
$$= \int_\Omega (1-\Phi_{\delta,\eta})u_k\varphi d\lambda_0 + \varpi_{\eta,\delta}(n,\nu).$$

Hence (10.13) holds, because none of the terms depends on n or ν.

• Conclusion: at last (10.10) follows from (10.11) and (10.13).

Step 3. Strong convergence of trucates in $W^{1,p}_{loc}(\Omega)$.

We consider the difference

$$\int_\Omega \left(|\nabla u_{\nu,k}|^p - |\nabla u_k|^p\right)\varphi dx$$
$$= \int_\Omega \left(|\nabla u_{\nu,k}|^p - |\nabla u_k|^p\right)(1-\Phi_{\delta,\eta})\varphi dx + \int_\Omega |\nabla u_{\nu,k}|^p\,\Phi_{\delta,\eta}\varphi dx - \int_\Omega |\nabla u_k|^p\,\Phi_{\delta,\eta}\varphi dx.$$

From (10.8), we find

$$\int_\Omega |\nabla u_{\nu,k}|^p\,\Phi_{\delta,\eta}\varphi dx = \varpi(\eta,\delta,\nu).$$

Since $|\nabla u_k|^p \in L^1(\Omega'')$ and $\Phi_{\delta,\eta}$ converges to 0 *a.e.* in Ω and $(|\nabla u_k|^p)$ remains bounded in $L^\infty(\Omega'')$, we obtain

$$\int_\Omega |\nabla u_k|^p\,\Phi_{\delta,\eta}\varphi dx = \varpi(\eta,\delta).$$

From (10.10), we deduce

$$\int_\Omega \left(|\nabla u_{\nu,k}|^p - |\nabla u_k|^p\right)\varphi dx = \varpi(\eta,\delta,\nu) = \varpi(\nu).$$

Moreover $|\nabla u_{\nu,k}|^{p-2}\nabla u_{\nu,k}$ converges *a.e* in Ω, and strongly in $L^1(\Omega')$, hence in $L^1_{loc}(\Omega)$; and $\nabla u_{\nu,k}$ converges strongly in $L^p(\Omega')$, and thus in $L^p_{loc}(\Omega)$.

Step 4: u is a local entropy solution of (3.27). We have, from **D1-loc,**

$$\int_\Omega |\nabla u_\nu|^{p-2}\nabla u_\nu.\nabla(h(u_\nu)\psi)dx = \int_\Omega h(u_\nu)f_\nu\psi dx + \int_\Omega h(u_\nu)\psi d\lambda_0$$
$$+ h(+\infty)\int_\Omega \psi d\lambda_s^+ - h(-\infty)\int_\Omega \psi d\lambda_s^- \qquad (10.14)$$

for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a support in some interval $[-k,k]$, and any $\psi \in \mathcal{D}(\Omega), \psi \geq 0$. One the one hand, the sequence $(h(u_\nu))$ is bounded in $L^\infty(\Omega)$, and $h(u_\nu)$ converges to $h(u)$ *a.e.* in Ω and strongly in $W^{1,p}_{loc}(\Omega)$, hence λ_0-*a.e.* in Ω; moreover $|\nabla u_\nu|^{p-2}\nabla u_\nu$ converges strongly in $L^1_{loc}(\Omega)$; hence

$$\lim_{\nu\to\infty}\int_\Omega h(u_\nu)\,|\nabla u_\nu|^{p-2}\nabla u_\nu.\nabla\psi dx = \int_\Omega h(u)\,|\nabla u|^{p-2}\nabla u.\nabla\psi dx,$$

$$\lim_{\nu\to\infty}\int_\Omega h(u_\nu)f_\nu\psi dx = \int_\Omega h(u)f\psi dx,$$

$$\lim_{\nu\to\infty}\int_\Omega h(u_\nu)\psi d\lambda_0 = \int_\Omega h(u)\psi d\lambda_0.$$

On the other hand, $(h'(u_\nu))$ is bounded in $L^\infty(\Omega)$, and $h'(u_\nu)$ converges to $h'(u)$ *a.e.* in Ω; then from Step 3,

$$\lim_{\nu\to\infty}\int_\Omega h'(u_\nu)\,|\nabla u_\nu|^p\,\psi dx = \lim_{\nu\to\infty}\int_\Omega h'(u_{\nu,k})\,|\nabla u_{\nu,k}|^p\,\psi dx = \int_\Omega h'(u)\,|\nabla u|^p\,\psi dx.$$

Thus

$$\int_\Omega h'(u)\,|\nabla u|^p\,\psi dx = \int_\Omega h(u)f\psi dx - \int_\Omega h(u)\,|\nabla u|^{p-2}\nabla u.\nabla\psi dx$$
$$+\int_\Omega h(u)\psi d\lambda_0 + h(+\infty)\int_\Omega \psi d\lambda_s^+ - h(-\infty)\int_\Omega \psi d\lambda_s^-,$$

and the conclusion follows.

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