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On the Principal Eigenvalue of Disconjugate BVPs with $\mathbf{L}^1$-Coefficients

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Abstract

Establishing some general results on the principal eigenvalues for a class of multipoint boundary value problems, we prove existence and uniqueness theorems for positive solutions of nonlinear multipoint boundary value problems. Our argument is based mainly on the maximum principle [4] (highlighting new ordered Banach spaces of continuous functions) and on the continuity of the spectral radius mapping for compact linear operators.

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1 Introduction

This paper is a consequence of Degla [4]. It is devoted to the study of the principal eigenvalue problem for the conjugate multipoint BVP

$$\begin{aligned} &Ly = \lambda q(t)y \\ &y^{(j)}(a_i) = 0, \quad 1 \le i \le m, \quad 0 \le j \le k_i - 1 \end{aligned} \tag{P}$$

with $q \in \mathbf{L}^1$. The final aim is to apply the follow-up to nonlinear conjugate BVPs.

In [4], we proved the following maximum principle

$$Ly \geq 0 \text{ and } y^{(j)}(a_i) = 0\,,\ 1 \leq i \leq m\,,\ 0 \leq j \leq k_i - 1 \implies S_P(t)y(t) \geq \varphi(t)||y||_\infty$$

where S_P indicates the sign of the Levin's polynomial $P(t)$, cf. §2 bellow, and φ is a continuous function with the property

$$\frac{\varphi(t)}{|P(t)|} \geq c_0 > 0$$

for a suitable c_0. This maximum principle will be used several times in this paper. In §2 it is the starting point to introduce an ordered Banach space structure on a suitable vector subspace of $\mathcal{C}^0$ in such a way that the cone has interior points and the integral operator associated to the Green function turns out to be strongly positive. As a consequence, in §3 it is shown that $||q||_{\mathrm{L}^1} > 0$ implies that (P) has a positive eigenvalue which is the inverse of the spectral radius of the integral operator associated to the Green function, a fact that provides comparison results for the principal eigenvalues related to different q's. These comparison results are used to study nonlinear conjugate BVPs: in §4 to search solutions that are strictly positive in a suitable order connected to the boundary conditions, and in §5 to state uniqueness and existence when the nonlinearity is below the principal eigenvalue (in analogy to non-resonant elliptic problems). We emphasize that the discontinuity of the q's is crucial in the proofs of some of these applications. Previous results about the principal eigenvalue require the continuity of q, cf. [6].

2 Technical preliminaries

Following an intuition suggested by the above maximum principle, in this section we introduce a new ordered Banach space of continuous functions and we study its properties, mainly with respect to the linear operator associated to Green funtions.

Our terminology is based on Coppel [3] and Elias [5]. Let n, m and $k_1, \cdots, k_m$ be fixed positive integers such that $2 \leq m \leq \sum_{i=1}^m k_i = n$ and let $a = a_1 < \cdots < a_m = b$. Let, moreover, P denote the Levin's polynomial $P(t) = \prod_{i=1}^m (t - a_i)^{k_i}$. We shall consider disconjugate n^{th} order differential operators of the form

$$Lx := x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x$$

where the coefficients $p_1, \cdots, p_n$ are given continuous functions on $[a, b]$. Besides, given $f \in C([a, b])$, we set $||f||_\infty = \sup\limits_{a \leq t \leq b} |f(t)|$ and

$$S_f(t) = \begin{cases} \frac{f(t)}{|f(t)|} & if \quad f(t) \neq 0 \\ 0 & if \quad f(t) = 0 \end{cases}.$$

Let G be the Green's function of the boundary value problem

$$\begin{aligned} &Ly = 0, \\ &y^{(j)}(a_i) = 0, \quad 1 \leq i \leq m, \quad 0 \leq j \leq k_i - 1. \end{aligned}$$

Therefore

$$\begin{array}{l} Ly = \lambda q(t)y, \quad y \not\equiv 0, \\ y^{(j)}(a_i) = 0, \;\; 1 \le i \le m, \;\; 0 \le j \le k_i - 1 \end{array} \tag{P}$$

if and only if

$$\lambda \neq 0 \quad \text{and} \quad 0 \not\equiv \int_a^b G(t,s)q(s)y(s)\,ds \;=\; \tfrac{1}{\lambda}y(t), \quad a \le t \le b.$$

There is a relationship between the Green function and the Levin polynomial that we shall need and is proved at the bottom of page 108 of Coppel [3]: the quotient $\frac{G(t,s)}{P(t)}$ is a bounded and strictly positive continuous function on $[a,b]\times]a,b[$, hence

$$0 \;<\; \frac{G(t,s)}{P(t)} \;\le\; c_1 \;<\infty. \tag{2.1}$$

From this, we have the following property for eigenfunctions of (P):

$$\begin{aligned} |y(t)| \quad &\le \quad |\lambda| \int_a^b |G(t,s)| \cdot |q(s)| \cdot |y(s)|\,ds \\ &\le \quad |\lambda| \cdot ||y||_\infty \int_a^b c_1 |P(t)| \cdot |q(s)|\,ds \\ &\le \quad c_1 |\lambda| \cdot ||y||_\infty |P(t)| \cdot ||q||_{\mathbf{L}^1} \\ &= \quad \text{const.} ||y||_\infty |P(t)|. \end{aligned}$$

Motivated by this, we introduce the following definitions :

$$X \;=\; \Big\{x \in C([a,b]) : |x(t)| \le c|P(t)|,\; a \le t \le b\,;\; \text{for a positive constant } c = c(x)\Big\},$$

$$||x||_P \;=\; ||x||_\infty + \sup_{t \neq a_1,\cdots,a_m} \frac{|x(t)|}{|P(t)|},$$

$$\mathcal{K} \;=\; \Big\{x \in X : S_P(t)x(t) \ge 0\;,\;\; a \le t \le b\Big\}.$$

We shall denote by $\prec$ the order defined by $\mathcal{K}$ on X, i.e.

$$x \preceq y \iff y - x \in \mathcal{K}.$$

Lemma 2.1 *$(X,\; ||\;||_P)$ is an ordered Banach space with $\mathcal{K}$ as the cone of positive elements. The cone $\mathcal{K}$ has nonempty interior. In fact*

$$x \in \text{int}(\mathcal{K}) \iff \inf_{t \neq a_1,\cdots,a_m} \frac{x(t)}{P(t)} \;>\; 0$$

so that at least $P \in \text{int}(\mathcal{K})$.

Proof. It is quite clear that X is a linear space, that $\| \ \|_P$ is a norm on X and that $\mathcal{K}$ is a closed cone in X. To show that X is complete, consider a Cauchy sequence $(x_k)_k$. Since it is also a Cauchy sequence in $\mathcal{C}^0 := \mathcal{C}([a,b])$, we have $x_k \to x_0$ uniformly for a suitable $x_0 \in \mathcal{C}^0$. Now, to every $\epsilon > 0$, there corresponds k_ϵ such that

$$\sup_{t \neq a_1, \cdots, a_m} \left| \frac{x_k(t) - x_l(t)}{P(t)} \right| \leq \|x_k - x_l\|_P < \epsilon$$

for $k, l \geq k_\epsilon$. Here we fix $t \neq a_1, \cdots, a_m$ and k, and take $\lim_l$ obtaining

$$\left| \frac{x_k(t) - x_0(t)}{P(t)} \right| \leq \epsilon$$

for $k \geq k_\epsilon$. It follows that $x_0 \in X$ and $x_k \to x_0$ in the norm $\| \ \|_P$. Thus X is complete. To prove the statement about $\operatorname{int}(\mathcal{K})$, define

$$\mathcal{U} = \left\{ x \in \mathcal{K} : \inf_{t \neq a_1, \cdots, a_m} \frac{x(t)}{P(t)} > 0 \right\}.$$

We desire to show that $\mathcal{U}$ is an open subset of $(X, \| \ \|_P)$ and $\mathcal{K} \setminus \mathcal{U}$ coincides with the boundary of $\mathcal{K}$. To this end, let $u \in \mathcal{U}$. Then there exists $\epsilon > 0$ such that

$$\frac{u(t)}{P(t)} \geq 2\epsilon \quad \text{for} \quad t \neq a_1, \cdots, a_m$$

so that given any $x \in X$ satisfying $\|x - u\|_P < \epsilon$, we have

$$\begin{aligned} \frac{x(t)}{P(t)} &= \frac{x(t)-u(t)}{P(t)} + \frac{u(t)}{P(t)} \\ &\geq -\|x-u\|_P + \frac{u(t)}{P(t)} \\ &> -\epsilon + 2\epsilon \\ &= \epsilon \end{aligned}$$

for $t \neq a_1, \cdots, a_m$, which implies that

$$\inf_{t \neq a_1, \cdots, a_m} \frac{x(t)}{P(t)} \geq \epsilon > 0$$

showing that $x \in \mathcal{U}$. Thus $\mathcal{U}$ is open. It follows that $\mathcal{K} \setminus \mathcal{U}$ is closed, and then to complete the proof it is enough to prove that

$$u \in \mathcal{K} \setminus \mathcal{U} \implies u \in \partial\mathcal{K}$$

where $\partial\mathcal{K}$ is the boundary of $\mathcal{K}$. Now, $u \in \mathcal{K} \setminus \mathcal{U}$ implies $\inf_{t \neq a_1, \cdots, a_m} \frac{u(t)}{P(t)} = 0$. Hence there exists $t_l \neq a_1, \cdots, a_m$ such that $\frac{u(t_l)}{P(t_l)} \to 0$. Fix $\epsilon > 0$. We have

$$\begin{aligned} S_P(t_l)\Big(u(t_l) - \epsilon P(t_l)\Big) &= S_P(t_l) \cdot P(t_l) \cdot \left\{\tfrac{u(t_l)}{P(t_l)} - \epsilon\right\} \\ &= |P(t_l)| \left\{\tfrac{u(t_l)}{P(t_l)} - \epsilon\right\} \\ &< 0 \end{aligned}$$

for l large enough. Therefore $u - \epsilon P \notin \mathcal{K}$. Letting $\epsilon \downarrow 0$, we deduce $u \in \partial\mathcal{K}$, as desired. □

Lemma 2.2 *Let $q \in \mathbf{L}^1([a,b])$. Then the operator*

$$T_q \ : \ x \mapsto T_q x$$

defined by

$$T_q x(t) = \int_a^b G(t,s) q(s) x(s) \ ds$$

is a compact linear operator on $(X, \ || \ ||_P)$.

In the sequel T_q will always denote the above operator for a given q.

Proof. It follows from (2.1) that T_q maps X into X. To state its compactness, fix a sequence $(x_l)_l$ in X with $||x_l||_P = 1$ for all l. We need to show the existence of a convergent subsequence of $\Big(T_q x_l\Big)_l$. Standard arguments based on Ascoli theorem and the uniform continuity of G guarantee the existence of a subsequence such that

$$T_q x_{l_k} \ \longrightarrow \ y \ \text{ uniformly} \tag{2.2}$$

for a suitable $y \in \mathcal{C}^0$. To simplify notations, we set $T := T_q$ and $x_k := x_{l_k}$. So we are done if we show that

$$\left|\frac{T x_k(t) - y(t)}{P(t)}\right| \ \longrightarrow \ 0$$

uniformly for $t \neq a_1, \cdots, a_m$. We have

$$\left|\frac{T x_k(t) - y(t)}{P(t)}\right| = \left|\int_a^b \frac{G(t,s)}{P(t)} q(s) x_k(s) \ ds \ - \ \frac{y(t)}{P(t)}\right|.$$

By Coppel's result mentioned for (2.1), the bounded quotient $\frac{G(t,s)}{P(t)}$ is uniformly continuous on compact subsets of $[a,b] \times]a,b[$. Thus we can apply again Ascoli theorem (directly) and obtain a subsequence such as

$$\int_a^b \frac{G(t,s)}{P(t)} q(s) x_{k_j}(s) \ ds \ \longrightarrow \ z(t)$$

uniformly on t for a suitable $z \in C^0$. For $t \neq a_1, \cdots, a_m$, we deduce from (2.2) that

$$\int_a^b \frac{G(t,s)}{P(t)} q(s) x_{k_j}(s)\, ds \quad \longrightarrow \quad \frac{y(t)}{P(t)}.$$

Thus $z(t) = \frac{y(t)}{P(t)}$ for $t \neq a_1, \cdots, a_m$, and we are done. □

Lemma 2.3 *If* $q \in \mathbf{L}^1([a,b])$ *satisfies*

$$q \neq 0 \textit{ on a set of positive measure, and } S_P(t)q(t) \geq 0 \textit{ for a.e. } t \in [a,b],$$

then the operator T_q *is positive on the ordered Banach space* X. *In fact*

$$x \in \mathcal{K} \quad \Longrightarrow \quad \textit{either } T_q x = 0 \textit{ or } T_q x \in \operatorname{int}(\mathcal{K}),$$

hence particularly

$$x \in \operatorname{int}(\mathcal{K}) \quad \Longrightarrow \quad T_q x \in \operatorname{int}(\mathcal{K}).$$

In case q *does not vanish identically in any subinterval of* $[a,b]$, T_q *is strongly positive, i.e.*

$$x \in \mathcal{K} \setminus \{0\} \quad \Longrightarrow \quad T_q x \in \operatorname{int}(\mathcal{K}).$$

Proof. The set of continuous functions with compact support contained in $[a_i, a_{i+1}]$ is dense in $\mathbf{L}^1([a_i, a_{i+1}])$. Glueing together functions chosen on each interval $[a_i, a_{i+1}]$, $1 \leq i \leq m-1$, we deduce the existence of a sequence $(g_k)_k$ of continuous functions vanishing at $a_1, \cdots, a_m$ and converging to q in $\mathbf{L}^1([a,b])$. Set

$$q_k = |g_k| S_P \quad \text{for } k = 1, 2, \cdots .$$

Since g_k vanishes at $a_1, \cdots, a_m$, q_k is continuous. Let us show that $q_k \to q$ in $\mathbf{L}^1([a,b])$. We have

$$\begin{aligned} \textstyle\int_a^b |q_k(s) - q(s)|\, ds &= \textstyle\int_a^b \big|\, S_P(s)|g_k(s)| - S_P(s)|q(s)| \,\big|\, ds \\ &\leq \textstyle\int_a^b \big|\, |g_k(s)| - |q(s)| \,\big|\, ds \\ &\leq \|g_k - q\|_{\mathbf{L}^1} \end{aligned}$$

from which the desired conclusion follows.

Now choose any $x \in \mathcal{K}$ and set

$$y_k = T_{q_k} x \quad \text{and} \quad y = T_q x.$$

From $q_k \to q$ in $\mathbf{L}^1([a,b])$, it follows

$$y_k \longrightarrow y \quad \text{uniformly}.$$

For every k, we have

$$\begin{array}{l} Ly_k = q_k(t)x = S_P(t)q_k(t) \cdot S_P(t)x \ \geq \ 0 \\ y_k^{(j)}(a_i) = 0, \ \ 1 \leq i \leq m, \ \ 0 \leq j \leq k_i - 1. \end{array}$$

Then the Maximum Principle of Degla [4] implies

$$\begin{aligned} S_P(t)y_k(t) \ &\geq \ \varphi(t)||y_k||_\infty \\ &\geq \ c_0|P(t)| \cdot ||y_k||_\infty. \end{aligned}$$

Taking limits, we deduce:

$$S_P(t)y(t) \ \geq \ c_0|P(t)| \cdot ||y||_\infty$$

showing that either $y = T_q x = 0$, or $y = T_q x \in \mathrm{int}(\mathcal{K})$ by virtue of Lemma 2.1. It also follows from the above inequality and from Lemma 2.1 that T_q maps $\mathrm{int}(\mathcal{K})$ into itself, while in case q does not vanish identically in any subinterval of $[a, b]$, we have

$$x \succ 0 \ \Longrightarrow \ T_q x \in \mathrm{int}(\mathcal{K})$$

since

$$x \succ 0 \ \Longrightarrow \ T_q x \neq 0$$

by (2.1). □

Finally, we shall need the following result which generalizes Lemma 2 of Nussbaum [8].

Lemma 2.4 *Let Y be a Banach space. Denote by $L(Y)$ its space of bounded linear operators and by $K(Y)$ the subspace of $L(Y)$ consisting of compact linear operators. Then the spectral radius $r(T)$ depends upper semi-continuously on $T \in L(Y)$ and continuously on $T \in K(Y)$ in the operator topology.*

Proof. We need only to show that $r : L(Y) \to [0, \infty[$ is upper semi-continuous in view of the proof given by Nussbaum [8]. To this end, fix arbitrarily $T \in L(Y)$ and let $(T_i)_i$, $i = 1, 2, \cdots$, be any sequence of $L(Y)$ converging to T in the operator topology. We show that

$$\limsup_{i \to \infty} r(T_i) \ \leq \ r(T).$$

For any positive integer $k \geq 2$ and any $H \in L(Y)$, we have

$$\begin{aligned} ||(T+H)^k|| \quad &\leq \ ||T^k|| + \textstyle\sum_{j=0}^{k-1} C_k^j ||T||^j ||H||^{k-j} \\ &\leq \ ||T^k|| + \tfrac{k}{k-j} ||H|| \textstyle\sum_{j=0}^{k-1} C_{k-1}^j ||T||^j ||H||^{k-1-j} \\ &\leq \ ||T^k|| + k||H||(||T|| + ||H||)^{k-1}. \end{aligned}$$

Thus

$$||(T+H)^k||^{\frac{1}{k}} \ \leq \ 2^{\frac{1}{k}} \left(||T^k||^{\frac{1}{k}} + k^{\frac{1}{k}} ||H||^{\frac{1}{k}} (||T|| + ||H||)^{\frac{k-1}{k}} \right)$$

which implies

$$||(T+H)^k||^{\frac{1}{k}} \leq 2^{\frac{1}{k}}\left(||T^k||^{\frac{1}{k}} + e||H||^{\frac{1}{k}}(||T||+||H||+1)\right). \tag{2.3}$$

Suppose now, by contradiction, that

$$r(T) < \limsup_{i\to\infty} r(T_i).$$

Then there exists a subsequence (T_{i_l}) of (T_i) such that

$$\lim_{l\to\infty} r(T_{i_l}) = \limsup_{i\to\infty} r(T_i) > r(T).$$

Now fix $\epsilon > 0$ such that

$$r(T)+\epsilon < \lim_{l\to\infty} r(T_{i_l}).$$

Thus there exists l_ϵ such that for every $l \geq l_\epsilon$,

$$r(T)+\epsilon < r(T_{i_l}) = \inf_k ||T_{i_l}^k||^{\frac{1}{k}} = \lim_k ||T_{i_l}^k||^{\frac{1}{k}}.$$

Hence, for every positive integer k, by taking an integer $l_k \geq l_\epsilon$ such that

$$||T_{i_{l_k}} - T|| < e^{-k^2}$$

and using (2.3), we get

$$r(T)+\epsilon < ||T_{i_{l_k}}^k||^{\frac{1}{k}} \leq 2^{\frac{1}{k}}\left(||T^k||^{\frac{1}{k}} + e^{1-k}(||T||+e^{-k^2}+1)\right).$$

Consequently we have

$$r(T)+\epsilon \leq \inf_k ||T_{i_{l_k}}^k||^{\frac{1}{k}} \leq r(T)$$

which is absurd. □

3 Existence and properties of the eigenvalues

Here we apply the lemmas of the previous sections in order to study the eigenvalue problem.

Theorem 3.1 *If $q \in \mathbf{L}^1([a,b])$ satisfies*

(*) *$q \neq 0$ on a set of positive measure, and $S_P(t)q(t) \geq 0$ for a.e. $t \in [a,b]$,*

then (P) has a positive eigenvalue $\lambda_1(q)$ with an eigenfunction $y \in \mathrm{int}(\mathcal{K})$ (which is unique up to normalization). Moreover, $\lambda_1(q)$ is the inverse of the spectral radius of T_q and $\lambda_1(q)$ depends continuously with respect to the $\mathbf{L}^1$-norm on the q's satisfying (*).

Proof. At first, we prove the existence part of the eigenvalues, dividing the argument in two steps.

Step 1: $S_P(t)q(t) > 0$ for a.e. t. In this case, Lemmas 2.2 and 2.3 guarantee that T_q is a strongly positive compact linear operator on the ordered Banach space X. Therefore the classical version of the Krein-Rutman theorem, cf. Amann [2], implies that the spectral radius of T_q is the inverse of the principal eigenvalue $\lambda_1(q)$ of (P) and that it admits an eigenfunction $y \in \mathrm{int}(\mathcal{K})$.

Step 2: The general case $S_P(t)q(t) \geq 0$ a.e. with $||q||_{\mathbf{L}^1} > 0$. Referring to the proof of Lemma 2.3, we know the existence of a sequence $(g_k)_k$ of continuous functions vanishing at $a_1, \cdots,\ a_m$ and converging to q in $\mathbf{L}^1([a,b])$. Set

$$q_k \;=\; |g_k|S_P \;+\; \frac{P}{k} \quad \text{for } \ k = 1, 2, \ \cdots$$

so that q_k is continuous (since g_k vanishes at $a_1, \cdots,\ a_m$) and $q_k \to q$ in $\mathbf{L}^1([a,b])$ with

$$S_P(t)q_k(t) \;>\; 0 \quad \text{for a.e. } \ t.$$

Therefore, Step 1 implies that the spectral radius of T_{q_k} is the inverse of the principal eigenvalue $\lambda_1(q_k)$ of (P), with $q = q_k$, and that there is an eigenfunction $y_k \in \mathrm{int}(\mathcal{K})$ such that

$$y_k(t) \;=\; \lambda_1(q_k) \int_a^b G(t,s)q_k(s)y_k(s)\,ds, \quad a \leq t \leq b. \tag{3.1}$$

We assume $||y_k||_\infty = 1$. From the maximum principle of Degla [4] we have

$$S_P(t)y_k(t) \;\geq\; \varphi(t) \;\;, \quad a \leq t \leq b\ . \tag{3.2}$$

Multiplying (3.1) by S_P, we get

$$\begin{aligned} |y_k(t)| = S_P(t)y_k(t) \;&= \lambda_1(q_k) \textstyle\int_a^b S_P(t)G(t,s) \cdot S_P(s)q_k(s) \cdot S_P(s)y_k(s)\,ds \\ &\leq\; \lambda_1(q_k) \textstyle\int_a^b |G(t,s)| \cdot |q_k(s)|\,ds \end{aligned}$$

and then from (3.2)

$$\varphi(t) \;\leq\; \lambda_1(q_k) \int_a^b |G(t,s)| \cdot |q_k(s)|\,ds, \quad a \leq t \leq b. \tag{3.3}$$

Clearly

$$\int_a^b |G(t,s)| \cdot |q_k(s)|\,ds \;\longrightarrow\; \int_a^b |G(t,s)| \cdot |q(s)|\,ds \quad \text{uniformly in } \ t,$$

so that

$$\max_{a \leq t \leq b} \int_a^b |G(t,s)| \cdot |q_k(s)|\,ds \;\longrightarrow\; \max_{a \leq t \leq b} \int_a^b |G(t,s)| \cdot |q(s)|\,ds.$$

Since the last quantity is positive, taking $\sup_t$ in (3.3) we get

$$\lambda_1(q_k) \;\geq\; \text{const.} \;>\; 0. \tag{3.4}$$

Using again (3.1) and (3.2) we have

$$1 \;=\; ||y_k||_\infty \;\geq\; \lambda_1(q_k) \int_a^b |G(t,s)| \cdot |q_k(s)| \varphi(s)\, ds, \quad a \leq t \leq b,$$

and from here we can repeat the argument leading to (3.4) in order to deduce

$$\lambda_1(q_k) \;\leq\; \text{const.}. \tag{3.5}$$

On the other hand, $q_k \to q$ in $\mathbf{L}^1$ implies $T_{q_k} \longrightarrow T_q$ in the operator norm [since its topology corresponds to uniform convergence on the unit ball], hence from Lemma 2.4 it follows that

$$r(T_{q_k}) = \frac{1}{\lambda_1(q_k)} \quad\longrightarrow\quad r(T_q) =: \frac{1}{\lambda_0}. \tag{3.6}$$

From this, (3.4) and (3.5), we deduce that λ_0 is a positive real number. Besides, set for every k,

$$z_k(t) \;=\; \int_a^b G(t,s) q_k(s) y_k(s)\, ds, \quad a \leq t \leq b.$$

It is easily seen from $||q_k||_{\mathbf{L}^1} \leq$ const. and $||y_k||_\infty = 1$ that $(z_k)_k$ is equicontinuous and pointwise bounded. Therefore by Ascoli theorem, there exists a subsequence and a suitable $z_0 \in \mathcal{C}^0$ such that

$$z_{k_l} \;\longrightarrow\; z_0 \quad \text{uniformly}.$$

Combining with (3.6), we take limits in (3.1) and we find that $y_{k_l} \to y$ uniformly, y being a suitable continuous function such that $||y||_\infty = 1$, and

$$y(t) \;=\; \lambda_0 \int_a^b G(t,s) q(s) y(s)\, ds, \quad a \leq t \leq b.$$

Since $y_k \in \mathcal{K}$, also $y \in \mathcal{K}$. Moreover, (3.2) implies that $S_P(t)y(t) \geq \varphi(t)$ for $a \leq t \leq b$. Thus $y \in \text{int}(\mathcal{K})$ by Lemma 2.1. We conclude that y is an eigenfunction of (P) corresponding to the eigenvalue $\lambda_0 = \frac{1}{r(T_q)} =: \lambda_1(q)$.

It remains to state the continuous dependence. To this end, assume that $q_k \to q$ in $\mathbf{L}^1([a,b])$ with $q_k(t) \neq 0$ on a set of positive measure, and $S_P(t) q_k(t) \geq 0$ for a.e. $t \in [a,b]$. By the above, $\lambda_1(q_k) = \frac{1}{r(T_{q_k})}$ and $r(T_{q_k}) = \frac{1}{\lambda_1(q_k)}$. Therefore, since $q_k \to q$ implies $T_{q_k} \longrightarrow T_q$ in the operator norm, we need only to apply Lemma 2.4. □

The following result implies a type of variational characterization of the first eigenvalue:

Corollary 3.1 *If* $S_P(t)q(t) > 0$ *for a.e.* t , *then:*
(i) $\lambda_1(q) = \min\left\{\lambda > 0 : \ x \preceq \lambda T_q x \ \text{for some} \ x \succ 0\right\}$
(ii) $0 \prec x \prec \lambda T_q x \implies \lambda_1(q) < \lambda$.

Proof. (i) From $0 \prec x \preceq \lambda T_q x$, it follows that $\frac{1}{\lambda}x \preceq T_q x$. This implies $r(T_q) \geq \frac{1}{\lambda}$ by the normality of the ordered Banach space $(X, \| \|_P)$, and the conclusion follows.

(ii) Suppose $0 \prec x \prec \lambda T_q x$ and set $y = T_q x$. Since $x \succ 0$, we have $y \succ 0$. Since $\lambda y - x = \lambda T_q x - x \succ 0$, we have $T_q(\lambda y - x) \in \operatorname{int}(\mathcal{K})$ by Lemma 2.3. By the continuity of $\phi(t) = tT_q y - y$, there exists $0 < \lambda_0 < \lambda$ such that $\lambda_0 T_q y - y \in \mathcal{K}$; hence $y \preceq \lambda_0 T_q y$. Then (i) implies $\lambda_1(q) \leq \lambda_0 < \lambda$. □

Theorem 3.2 *The following comparison results hold.*
(a) If L is disconjugate and q_1, $q_2 \in \mathbf{L}^1([a,b])$ *do not vanish on a set of positive measure, are different on a set of positive measure and satisfy*

$$0 \ \leq \ S_P(t)q_1(t) \ \leq \ S_P(t)q_2(t) \quad for \ \ a.e.\ t \in [a,b],$$

then

$$0 \ < \ \lambda_1(q_2) \ < \ \lambda_1(q_1).$$

(b) If L is disconjugate and $q \in \mathbf{L}^1([a,b])$ *does not vanish on a set of positive measure and satisfies* $S_P(t)q(t) \geq 0$ *for a.e.* t, *with* $L - q$ *as a disconjugate operator, then*

$$\lambda_1(q) \ > \ 1.$$

Proof. (a) For simplicity of notations, set $T_i = T_{q_i}$ for $i = 1, 2$. On the basis of the assumptions on q_1 and q_2, we see from Lemma 2.3 that the compact linear operators T_1, T_2 and $T_2 - T_1$ are positive with respect to $\mathcal{K}$ and

$$(T_2 - T_1)(\operatorname{int}(\mathcal{K})) \ \subset \ \operatorname{int}(\mathcal{K}).$$

Therefore, letting $y_1 \in \operatorname{int}(\mathcal{K})$ be an eigenvector of T_1 corresponding to $r(T_1) > 0$ (cf. Theorem 3.1), we have

$$T_2 y_1 - T_1 y_1 = T_2 y_1 - r(T_1) y_1 \ \in \ \operatorname{int}(\mathcal{K}),$$

which implies that

$$T_2 y_1 - (r(T_1) + \epsilon) y_1 \ \in \ \mathcal{K}$$

for sufficiently small $\epsilon > 0$, and hence $r(T_2) > r(T_1)$ [[7], Theorem 9.4], that is

$$0 < \lambda_1(q_2) < \lambda_1(q_1).$$

(b) To show that $\lambda_1(q) > 1$, it suffices to prove that $\lambda_1(q)$ is not less than 1 due to the disconjugacy of $L - q$. For each $\alpha \in]a_l, a_{l+1}]$, $1 \leq l \leq m-1$, define the operator $T_\alpha : C([a,b]) \to C([a,b])$ by

$$T_\alpha x(t) = \begin{cases} \int_a^\alpha G_\alpha(t,s)q(s)x(s)\,ds & if \ \ a \leq t \leq \alpha \\ 0 & if \ \ \alpha < t \leq b \end{cases},$$

where $G_\alpha(t,s)$ is the Green's function of the boundary value problem

$$\begin{aligned}&Ly=0,\\&y^{(j)}(a_i)=0,\quad 1\le i\le l,\quad 0\le j\le k_i-1,\\&y^{(j)}(\alpha)=0,\quad 0\le j\le k_{l+1}+\cdots+k_m-1.\end{aligned}$$

Note that the mapping $\alpha\mapsto T_\alpha$ is continuous, and then by continuity of the spectral-radius mapping on compact linear operators (cf. Lemma 2.4), the mapping $\alpha\mapsto r(T_\alpha)$ is also continuous. Since $T_\alpha\to 0$ as $\alpha\to a$, if $r(T_b)=\frac{1}{\lambda_1(q)}$ were not less than 1, there would exist $\alpha\in]a,b]$ and $l\in\{1,\cdots,m-1\}$ such that $r(T_\alpha)=1$ which would imply that the linear problem

$$\begin{aligned}&Ly=q(t)y,\\&y^{(j)}(a_i)=0,\quad 1\le i\le l,\quad 0\le j\le k_i-1,\\&y^{(j)}(\alpha)=0,\quad 0\le j\le k_{l+1}+\cdots+k_m-1\end{aligned}$$

has a nontrivial solution by Theorem 3.1, contradicting the disconjugacy of $L-q$. □

As a consequence we have

Corollary 3.2 *Let $q\in \mathbf{L}^1([a,b])$ be such that $||q||_{\mathbf{L}^1}>0$ and $S_P(t)q(t)\ge 0$ for a.e. $t\in[a,b]$, and assume that L and the perturbed differential operator $L-q$ are both disconjugate on $[a,b]$. Moreover let $f\in\mathbf{L}^1([a,b])$ satisfy $||f||_{\mathbf{L}^1}>0$ with $f\ge 0$ for a.e. $t\in[a,b]$. Then there exists a unique function y such that*

$$\begin{aligned}&Ly-q(t)y=f(t),\\&y^{(j)}(a_i)=0,\quad 1\le i\le m,\quad 0\le j\le k_i-1,\end{aligned}$$

and the function $t\mapsto y(t)/P(t)$, $t\neq a_1,\cdots,a_m$, has a continuous extension to $[a,b]$ with positive infimum. Furthermore, given any $y_0\in C([a,b])$ with $S_P(t)y_0(t)\ge 0$, the sequence $(y_l)_l$; $l=1,2,\cdots$, defined recursively by

$$\begin{aligned}&Ly_{l+1}=q(t)y_l+f(t),\quad \textit{for}\ \ a.e.\ t\in[a,b],\\&y_{l+1}^{(j)}(a_i)=0,\quad 1\le i\le m,\quad 0\le j\le k_i-1,\end{aligned}$$

converges uniformly (and thus with respect to the C^{n-1}-norm) to y.

Proof. The nonhomogeneous problem

$$\begin{aligned}&Ly-q(t)y=f(t),\quad \text{for a.e. } t\in[a,b],\\&y^{(j)}(a_i)=0,\quad 1\le i\le m,\quad 0\le j\le k_i-1,\end{aligned}$$

is equivalent to

$$(I-T_q)y=\tilde f,\quad \tilde f\in\mathcal{K},\quad \text{where}\ \ \tilde f(t)=\int_a^b G(t,s)f(s)\,ds.$$

And since T_q is positive and $r(T_q)<1$ by Theorem 3.1, the inverse operator $(I-T_q)^{-1}$ is well-defined and can be represented as the series $I+T_q+T_q^2+\cdots$ and so is positive with respect to $\mathcal{K}$. Therefore the conclusions of the Proposition follow from [[7], Theorem 15.1]. □

4 Application to the existence of strictly positive solutions to nonlinear BVPs

In this section we apply the previous results to the search of strictly positive solutions, in the order defined by $\mathcal{K}$, of nonlinear BVPs.

Theorem 4.1 *Assume that* L *is disconjugate and that* $F : [a,b] \times \mathbf{R} \to \mathbf{R}$ *is a continuous function such that*

1. $F(t, S_P(t)y) \geq 0, \ \ a \leq t \leq b, \ \ y \geq 0.$

2. $F(t,0) = 0, \ \ a \leq t \leq b.$

3. *The function* $y \mapsto F(t,y)$ *is differentiable at* 0 *with*

$$\lim_{y \to 0} \frac{F(t,y)}{y} = F_y(t,0) =: q(t) \quad \textit{uniformly for} \quad t \in [a,b]$$

and is such that $L - q$ *is disconjugate.*

4.

$$\lim_{y \to +\infty} \frac{F(t, S_P(t)y)}{y} \equiv +\infty \ \textit{uniformly for } t \textit{ in a subset of } [a,b] \textit{ with positive measure.}$$

Then the conjugate BVP

$$\begin{aligned} &Ly = F(t,y), \\ &y^{(j)}(a_i) = 0, \ \ 1 \leq i \leq m, \ \ 0 \leq j \leq k_i - 1, \end{aligned}$$

has at least one nontrivial solution such that the quotient $y(t)/P(t)$ *has a continuous extension to* $[a,b]$ *with positive infimum.*

Proof. The nonlinear problem

$$\begin{aligned} &Ly = F(t,y), \\ &y^{(j)}(a_i) = 0, \ \ 1 \leq i \leq m, \ \ 0 \leq j \leq k_i - 1, \end{aligned}$$

is equivalent to

$$y(t) = \int_a^b G(t,s) F(s, y(s)) \, ds, \ \ a \leq t \leq b,$$

and we plan to seek a solution y such that $S_P(t)y(t) \geq 0$ which will yield the result. Recall, for later use, that by virtue of the maximum principle for multipoint boundary value problems [[4], Theorem 1] there exists $\varphi \in C([a,b])$ positive on $\bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$

and such that for every $y \in C^n([a,b])$ satisfying the differential inequality $Ly \geq 0$ and the homogeneous Hermite boundary condition

$$y^{(j)}(a_i) = 0, \quad 1 \leq i \leq m, \quad 0 \leq j \leq k_i - 1,$$

we have

$$S_P(t)y(t) \geq \varphi(t)||y||_\infty, \quad a \leq t \leq b.$$

Now let

$$Y = \{y \in C[a,b] : \; y(a_1) = \cdots = y(a_m) = 0\}$$
$$\mathcal{K}_\varphi = \{y \in Y : \; S_P(t)y(t) \geq \varphi(t)||y||_\infty, \quad a \leq t \leq b\}$$
$$\mathcal{K}_0 = \{y \in Y : \; S_P(t)y(t) \geq 0, \quad a \leq t \leq b\}$$

and note that the sup-norm $|| \; ||_\infty$ is monotone (increasing) in $\mathcal{K}_\varphi$ and $\mathcal{K}_0$ with respect to their corresponding orders. Consider moreover the integral operator A defined for every $y \in \mathcal{K}_0$ by

$$Ay : \; t \mapsto \int_a^b G(t,s)F(s,y(s)) \, ds.$$

Then A maps $\mathcal{K}_0$ into $\mathcal{K}_\varphi$ since given any $y \in \mathcal{K}_0$ and putting $z = Ay$, we have via the property of the Green's function G that

$$\begin{aligned} &Lz = F(t,y(t)) = F(t, S_P(t)|y(t)|) \; \geq 0, \\ &z^{(j)}(a_i) = 0, \quad 1 \leq i \leq m, \quad 0 \leq j \leq k_i - 1, \end{aligned}$$

which implies that

$$S_P(t)z(t) \geq \varphi(t)||z||_\infty, \quad a \leq t \leq b.$$

In particular A maps $\mathcal{K}_\varphi$ into itself. Since furthermore A is completely continuous, to conclude that $A : \mathcal{K}_\varphi \to \mathcal{K}_\varphi$ has a fixed point with a positive norm, it suffices to show the existence of two positive real numbers $\epsilon < R$ such that:
(i) if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = \epsilon$, then $Ay - y \notin \mathcal{K}_\varphi$, while
(ii) if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = R$, then $y - Ay \notin \mathcal{K}_\varphi$
as shown by the Krasnosel'skii theorem on the expansion and compression of cones in the version of [[1], Theorem 11.2]. Indeed, on the one hand, by assumption (3) above, A is Frechet differentiable at 0 with $A'(0) = T_q =: T$ (for simplicity). If $q = 0$, then $r(T) = 0$. If otherwise $q \neq 0$, then using the above assumptions (1) and (3), we have $r(T) = \frac{1}{\lambda_1(q)} < 1$ by Theorem 3.1 and Theorem 3.2-b. Thus $r(T) < 1$ in any case, hence $I - T$ is invertible. Now suppose by the way of contradiction that (i) is not satisfied for any $\epsilon > 0$. Then there would exist sequences $\epsilon_l \downarrow 0$ and $y_l \in \mathcal{K}_\varphi$ such that $||y_l||_\infty = \epsilon_l$ whereas $Ay_l - y_l \in \mathcal{K}_\varphi$. Putting $z_l = y_l/\epsilon_l$ (so that $||z_l|| = 1$ for all l) and $My_l = (Ay_l - Ty_l)/\epsilon_l$, we would then have

$$My_l - (I-T)z_l \; \in \; \mathcal{K}_\varphi \subset \mathcal{K}_0,$$

which, by the positivity of $(I-T)^{-1} = I + T + T^2 + \cdots$, would imply

$$(I-T)^{-1}My_l - z_l \in \mathcal{K}_0, \quad \text{already with} \quad z_l \in \mathcal{K}_0,$$

and so for all l we would get, by the monotonicity of the sup norm,

$$||(I-T)^{-1}|| \cdot ||My_l||_\infty \ \geq \ ||(I-T)^{-1}My_l||_\infty \ \geq \ ||z_l||_\infty \ = \ 1$$

contradicting the fact that $\lim_l ||My_l||_\infty = 0$ since $T = A'(0)$. Hence (i) holds for some $\epsilon > 0$ that we fix for the sequel. On the other hand, by assumption (4) above, there exists a compact set $\Omega \subset [a,b]$ with positive measure such that

$$\lim_{y\to+\infty} \frac{F(t, S_P(t)y)}{y} \equiv +\infty \ \text{ uniformly for } \ t \in \Omega$$

and moreover, $a_j \notin \Omega$ for $j = 1, \cdots, m$. Now choose $\delta \notin \{a_1, \cdots, a_m\}$ and $\alpha > 0$ such that

$$\alpha \int_\Omega |G(\delta, s)|\varphi(s)\, ds \ > \ 1$$

(this is possible since $|G(\delta, s)|\varphi(s) > 0$ for $s \in \Omega$). Therefore, there exists $R_0 > 0$ such that

$$F(t, S_P(t)y) \ \geq \ \alpha y, \quad t \in \Omega, \quad y \geq R_0.$$

Now set

$$R = \epsilon + \frac{R_0}{\inf_\Omega \varphi}.$$

Hence, if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = R$, then

$$|y(t)| \ \geq \ \varphi(t)||y||_\infty \ \geq \ R_0, \ \ t \in \Omega$$

which implies

$$F(t, y(t)) \ \geq \ \alpha\varphi(t)||y||_\infty, \quad t \in \Omega.$$

Therefore

$$\begin{aligned} |Ay(\delta)| \ = \ S_P(\delta)Ay(\delta) \ &= \ \textstyle\int_a^b |G(\delta, s)|F(s, y(s))\, ds \\ &\geq \ \textstyle\int_\Omega |G(\delta, s)|F(s, y(s))\, ds \\ &\geq \ \left(\alpha \textstyle\int_\Omega |G(\delta, s)|\varphi(s)\, ds \right) ||y||_\infty \end{aligned}$$

and so

$$||Ay||_\infty \ > \ ||y||_\infty$$

implying in its turn that $y - Ay \notin \mathcal{K}_\varphi$ and showing (ii). □

As a consequence we have:

Corollary 4.1 *Assume that L is disconjugate and let $q : [a,b] \to \mathbf{R}$ and $h : \mathbf{R} \to [0, \infty[$ be continuous functions such that*

1. $q \neq 0$, $S_P(t)q(t) \geq 0$ *and* $L - q$ *is disconjugate.*
2. $h(0) = 1$ *and* $\lim_{|y|\to+\infty} h(y) = +\infty$.

Then the nonlinear problem

$$\begin{aligned} &Ly = q(t)h(y)y, \\ &y^{(j)}(a_i) = 0, \quad 1 \le i \le m, \quad 0 \le j \le k_i - 1, \end{aligned}$$

has at least one nontrivial solution y *such that the function* $t \mapsto y(t)/P(t),\ \ t \notin \{a_1, \cdots, a_m\}$, *has a continuous extension to* $[a, b]$ *with positive infimum.*

Theorem 4.2 *Assume that* L *is disconjugate and let* $F : [a, b] \times \mathbf{R} \to \mathbf{R}$ *be a continuous function such that*

1. $F(t, S_P(t)y) \ge 0, \quad a \le t \le b, \ \ y \ge 0.$

2. *There exists the limit*

   $$\lim_{|y| \to +\infty} \frac{F(t, y)}{y} =: q(t) \ \ uniformly\ for\ t \in [a, b]$$

   with $L - q$ *disconjugate.*

3. *We have*

   $$\lim_{y \to 0^+} \frac{F(t, S_P(t)y)}{y} \equiv +\infty \ \ uniformly\ for\ t\ in\ a\ subset\ of\ [a, b]\ with\ positive\ measure.$$

   Then the nonlinear BVP

   $$\begin{aligned} &Ly = F(t, y), \\ &y^{(j)}(a_i) = 0, \quad 1 \le i \le m, \quad 0 \le j \le k_i - 1, \end{aligned}$$

   has a least one nontrivial solution y *such that the quotient* $y(t)/P(t)$ *has a continuous extension to* $[a, b]$ *with positive infimum.*

Proof. Let $\mathcal{K}_\varphi$ be as in the proof of Theorem 4.1 and consider the operator A assigning to every $y \in \mathcal{K}_\varphi$ the function Ay defined by

$$Ay : \ t \mapsto \int_a^b G(t, s)F(s, y(s))\, ds$$

which belongs to $\mathcal{K}_\varphi$. It suffices to show that this completely continuous operator $A : \mathcal{K}_\varphi \to \mathcal{K}_\varphi$ has a nonzero fixed point y yielding that the corresponding quotient $y(t)/P(t)$ has a continuous extension to $[a, b]$ with positive infimum by the property of the Green's function G. To this end, we prove that there exist two positive real numbers $\epsilon < R$ such that:

(i) if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = \epsilon$, then $y - Ay \notin \mathcal{K}_\varphi$.

(ii) if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = R$, then $Ay - y \notin \mathcal{K}_\varphi$

and we apply the Krasnosel'skii theorem on the expansion and compression of cones in the version of [[1], Theorem 11.2].

(i) By assumption (3), there exists a compact set $\Omega \subset \cup_{i=1}^{m-1}]a_i, a_{i+1}[$ with positive measure such that

$$\lim_{y \to 0^+} \frac{F(t, S_P(t)y)}{y} \equiv +\infty \text{ uniformly for } t \in \Omega.$$

Now choose $\delta \notin \{a_1, \cdots, a_m\}$ and $\alpha > 0$ such that

$$\alpha \int_\Omega |G(\delta, s)| \varphi(s)\, ds > 1.$$

Therefore, there exists $\epsilon > 0$ such that

$$F(t, S_P(t)y) \geq \alpha y, \quad t \in \Omega, \quad 0 \leq y \leq \epsilon.$$

Thus, if $y \in \mathcal{K}_\varphi$ with $||y||_\infty = \epsilon$ we have:

$$\begin{aligned} |Ay(\delta)| = S_P(\delta) \textstyle\int_a^b G(\delta, s) F(s, y(s))\, ds &= \textstyle\int_a^b |G(\delta, s)| F(s, S_P(s)|y(s)|)\, ds \\ &\geq \alpha \textstyle\int_a^b |G(\delta, s)||y(s)|\, ds \\ &\geq \alpha \textstyle\int_a^b |G(\delta, s)| \varphi(s) ||y||_\infty\, ds \\ &> ||y||_\infty \end{aligned}$$

which implies $||Ay||_\infty > ||y||_\infty$, and then $y - Ay \notin \mathcal{K}_\varphi$.

(ii) Suppose by contradiction that there does not exist any positive real number $R > \epsilon$ such that $Ay - y \notin \mathcal{K}_\varphi$ for every $y \in \mathcal{K}_\varphi$ satisfying $||y||_\infty = R$. Then, there would exist a sequence $(y_l)_l$ of elements of $\mathcal{K}_\varphi \setminus \{0\}$ such that $||y_l|| \uparrow \infty$ and $Ay_l - y_l \in \mathcal{K}_\varphi$. Considering $A'(\infty) = T_q =: T$ that exists and is defined (according to assumption (2)) by

$$Tx(t) = \int_a^b G(t, s) q(s) x(s)\, ds, \quad a \leq t \leq b,$$

we observe that if $q = 0$, then $r(T) = 0$, while if otherwise $q \neq 0$, then using the assumptions (1) and (2), we have $r(T) = \frac{1}{\lambda_1(q)} < 1$ as a result of Theorem 3.2-b. Thus $r(T) < 1$ in any case so that the inverse of $I - T$, $(I - T)^{-1}$, exists and is positive. It follows that

$$(Ay_l - Ty_l) + (Ty_l - y_l) = Ay_l - y_l \in \mathcal{K}_\varphi$$

and so

$$My_l + (Tz_l - z_l) \in \mathcal{K}_\varphi;$$

where

$$z_l = \frac{y_l}{||y_l||_\infty} \quad \text{and} \quad My_l = \frac{(Ay_l - Ty_l)}{||y_l||_\infty}$$

which implies that

$$(I-T)^{-1}My_l - z_l \in \mathcal{K}_\varphi, \quad \text{already with } z_l \in \mathcal{K}_\varphi.$$

Consequently, by the monotonicity of the sup norm, we have

$$||(I-T)^{-1}|| \cdot ||My_l||_\infty \geq ||(I-T)^{-1}My_l||_\infty \geq ||z_l||_\infty = 1, \quad \text{for all } l,$$

in contrast to the fact that $\lim_l ||My_l||_\infty = 0$ since $T = A'(\infty)$. □

5 Application to nonlinear problem below the principal eigenvalue

From the comparison result in §3, we deduce in this section a uniqueness and an existence theorem for nonlinear BVPs.

Theorem 5.1 *Assume that* $f : [a,b]\times\mathbf{R} \to \mathbf{R}$ *satisfies the generalized Caratheodory conditions and that* $f_x = \frac{\partial}{\partial x} f$ *exists and satisfies the generalized Caratheodory conditions. If there exist* $p, q \in \mathbf{L}^1$ *such that*

(a) p *and* q *do not vanish in sets of positive measures,*

(b) $0 \leq S_P(t) f_x(t,x) \leq S_P(t)p(t) \leq S_P(t)q(t)\lambda_1(q)$ *for a.e.* t *and all* x, $\lambda_1(q)$ *being the principal eigenvalue of* (P),

(c) $p \neq \lambda_1(q)q$ *in a set of positive measure,*

then the conjugate BVP

$$\begin{aligned} &Lx = f(t,x), \\ &x^{(j)}(a_i) = A_{ij}, \quad 1 \leq i \leq m, \quad 0 \leq j \leq k_i - 1 \end{aligned}$$

has at most one solution whenever $A_{ij} \in \mathbf{R}$.

Proof. Assume the existence of two different solutions x and y of the same conjugate BVP, and argue for a contradiction. Setting

$$Q(t) = \int_0^1 f_x(t, y(t) + \xi(x(t) - y(t)))\, d\xi,$$

we have

$$f(t,x(t)) - f(t,y(t)) = Q(t) \cdot (x(t) - y(t)).$$

It follows that the function $u = x - y$ is an eigenfunction corresponding to $\lambda = 1$ of the eigenvalue problem

$$\begin{aligned} &Lu = \lambda Q(t) u, \\ &u^{(j)}(a_i) = 0 \quad \text{for } 1 \leq i \leq m, \quad 0 \leq j \leq k_i - 1. \end{aligned} \tag{5.1}$$

Clearly we have $S_P(t)Q(t) \leq S_P(t)q(t)\lambda_1(q)$ a.e. and $Q(t) \neq q(t)\lambda_1(q)$ in a set of positive measure. Then an application of Theorem 3.2-a to the eigenvalue problems (5.1) and to

$$\begin{aligned}&Ly = \lambda[\lambda_1(q)q(t)]y,\\ &y^{(j)}(a_i) = 0 \quad \text{for} \quad 1 \leq i \leq m, \quad 0 \leq j \leq k_i - 1\end{aligned}$$

provides

$$\lambda_1(Q) \; > \; \lambda_1(\lambda_1(q)q) \; = \; 1. \tag{5.2}$$

But then we have a contradiction: since the principal eigenvalue is less than or equal to the absolute value of any other eigenvalue of the same BVP, and since $\lambda = 1$ is an eigenvalue of (5.1), we should have $\lambda_1(Q) \leq 1$, contradicting (5.2). □

Theorem 5.2 *Assume that* $f : [a,b] \times \mathbf{R} \to \mathbf{R}$ *satisfies the generalized Caratheodory conditions and that* $f_x = \frac{\partial}{\partial x} f$ *exists and satisfies the generalized Caratheodory conditions. If there exist* $p, q \in \mathbf{L}^1$ *and* $R > 0$ *such that*

(a) p *and* q *do not vanish in sets of positive measures,*

(b) $0 \leq S_P(t)f_x(t,x)$ *for a.e.* t *and all* x,

(c) $S_P(t)f_x(t,x) \leq S_P(t)p(t) \leq S_P(t)q(t)\lambda_1(q)$ *for a.e.* t *and* $|x| \geq R$, $\lambda_1(q)$ *being the principal eigenvalue of* (P),

(d) $p \neq \lambda_1(q)q$ *in a set of positive measure,*

then for every $g : [a,b] \times \mathbf{R} \to \mathbf{R}$ *satisfying the generalized Caratheodory conditions and*

$$\lim_{|x| \to \infty} \frac{g(t,x)}{|x|} \; = \; 0,$$

and the conjugate BVP

$$\begin{aligned}&Lx = f(t,x) + g(t,x),\\ &x^{(j)}(a_i) = 0, \quad 1 \leq i \leq m, \quad 0 \leq j \leq k_i - 1\end{aligned}$$

has at least one solution.

Proof. Applying Theorem 3.2-a to the eigenvalue problems

$$\left\{\begin{array}{l} Ly = \lambda p(t)y \\ y^{(j)}(a_i) = 0 \quad \text{for} \quad 1 \leq i \leq m, \quad 0 \leq j \leq k_i - 1 \end{array}\right. ,$$

$$\left\{\begin{array}{l} Ly = \lambda[\lambda_1(q)q(t)]y \\ y^{(j)}(a_i) = 0 \quad \text{for} \quad 1 \leq i \leq m, \quad 0 \leq j \leq k_i - 1 \end{array}\right.$$

we see that

$$\lambda_1(p) \; > \; \lambda_1(\lambda_1(q)q) \; = \; 1. \tag{5.3}$$

Let $p_k = p + \frac{S_P}{k}$. We claim the existence of $k_0 \geq 1$ such that

$$\lambda_1(p_{k_0}) > 1. \tag{5.4}$$

This follows from (5.3) and the continuous dependence of the principal eigenvalue since p_k satisfies conditon $(*)$ in Theorem 3.1. Now we define

$$q(t,x) = \int_0^1 f_x(t,\xi x)\, d\xi$$

and the given equation can be rewritten as

$$Lx = q(t,x)x + f(t,0) + g(t,x).$$

Clearly $S_P(t)q(t,x) \geq 0$ for a.e. t and all x. Let $|x| \geq R$. From the generalized Caratheodory conditions we get the existence of $r \geq R$ such that

$$\int_0^{R/r} |f_x(t,\xi x)|\, d\xi \leq \frac{1}{k_0}$$

for a.e. t and $|x| \leq R$. From this, from (b), and from

$$q(t,x) = \int_0^{R/|x|} f_x(t,\xi x)\, d\xi + \int_{R/|x|}^1 f_x(t,\xi x)\, d\xi$$

we deduce

$$0 \leq S_P(t)q(t,x) \leq \frac{1}{k_0} + |p(t)| = S_P(t)(\frac{S_P(t)}{k_0} + p(t)) = S_P(t)p_{k_0}(t)$$

for a.e. t and $|x| \geq r$. Now fix any $h \in \mathbf{L}^1$ such that $|h(t)| \geq r$ a.e. By the above, we have

$$0 \leq S_P(t)q(t,h(t)) \leq S_P(t)p_{k_0}(t) \quad \text{a.e.},$$

hence Theorem 3.2-a implies

$$\lambda_1(q(\,\cdot\,,h(\,\cdot\,))) \geq \lambda_1(p_{k_0}) > 1.$$

Then

$$\begin{aligned} &Lx = q(t,h(t))x \\ &x^{(j)}(a_i) = 0 \quad \text{for } 1 \leq i \leq m, \ \ 0 \leq j \leq k_i - 1 \end{aligned}$$

has only the trivial solution. It follows that we can apply Theorem 1 and Example 2 of Vidossich [9] by considering the first order system in $\mathbf{R}^n$ equivalent to the given equation $Lx = f(t,x) + g(t,x)$, letting X be the set corresponding in this equivalence to

$$Y = \Big\{ h \in \mathbf{L}^1 : \ 0 \leq S_P(t)h(t) \leq S_P(t)p_{k_0}(t) \ \text{ a.e.} \Big\},$$

and obtain the existence of at least one solution to the given BVP. □

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