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Existence of a Closed Geodesic on Non-compact Riemannian Manifolds with Boundary*

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Abstract

A new result about the existence of a closed geodesic on a Riemannian manifold with boundary is given. A detailed comparison with previous results is carried out.

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1 Introduction

One of the most venerable topics in Differential Geometry and Global Analysis is to ensure the existence of a closed geodesic in a Riemannian manifold. Some techniques has proven to be fruitful, [2], [6], [8], [10], [17], even though it is not always clear the relations among their different hypotheses. The compact case for manifolds without boundary was solved in the classical article [10], and, in principle, all the results in the other references are applicable to unbounded manifolds. Aim of this article is to give a result on the existence of a closed geodesic in a (non–compact) Riemannian manifold M with a boundary ∂M, and to compare this result with previous ones. Our basic assumptions for the boundary are differentiability, compactness and *convexity*, i.e.: the second fundamental form with respect to the inner normal is positive semi–definite, see also Section 2. We also assume non–trivial topology and a condition on the sectional curvatures at infinity.

More precisely, the following result will be proved in Section 3.

Theorem 1.1 *Let $\overline{M} = M \cup \partial M$ be a complete Riemannian manifold with compact convex boundary ∂M, and associated distance $d(\cdot,\cdot)$. Assume that:*

(i) for some $x_0 \in M$

$$\limsup_{d(x,x_0)\to\infty} K(x) \leq 0$$

where $K(x) = \sup\{K(\pi) \mid \pi \subset T_xM\}$ for any $x \in M$, and $K(\pi)$ is the sectional curvature of the 2–dimensional plane π;

(ii) there exist a field $\mathcal{F}$ and $q \in \mathbf{N}$, $q > 2\dim M$, such that

$$H_q(\Lambda(M), \mathcal{F}) \neq \{0\} \tag{1.1}$$

where $H_q(\cdot, \mathcal{F})$ is the q–th singular homology group with coefficients in a field $\mathcal{F}$ and $\Lambda(M)$ is the free loop space with the compact–open topology on M.

Then there exists a non–trivial closed geodesic on $\overline{M}$.

Moreover, in this case if the closed geodesic touches the boundary ∂M then it is completely contained in ∂M (Remark 3.1).

In order to prevent this possibility the convexity of the boundary is not sufficient (see Example 4.1) and other assumpions on the boundary are needed. We refer to Subsection 4.1 for a discussion about the hypothesis which guarantees that the closed geodesic is entirely contained in M (also when ∂M is not differentiable).

Notice that when M is bounded (i.e., with finite diameter), no assumption on the curvature is imposed; moreover, condition (ii) can also be replaced by other topological assumptions, see Theorem 4.3.

We will compare these results with the ones in [2], [6], [8], [17]. This comparison is carried out in Section 4, and can be summarized as follows:

- We follow the variational approach introduced by Benci and Giannoni [6], which use Morse theory. In fact, these authors proved Theorem 1.1 when

M is complete and without boundary. Later on, the authors of the present article extended this result to non–complete Riemannian manifolds, by introducing the concept of CCB manifold [2]. This concept is applicable when the boundary of the manifold is non–compact. What is more, the boundary of a CCB manifold may be quite general and even non–differentiable. But, because of this reason, the closed geodesic obtained under CCB assumption cannot touch the boundary. This is a clear difference with Theorem 1.1; a detailed discussion will be given in Subsection 4.1.

- Candela and Salvatore [8] also use variational methods, but with a different technique based on Ljusternik–Schnirelman category. They impose global conditions on non–compact Riemannian manifolds with compact (Cauchy) boundary, by assuming the existence of an additional convex function U (Theorem 4.2 below). We discuss this result in Subsection 4.2, and we will see that it can be reduced to the case of a bounded Riemannian manifold (this also holds for more general problems, Remark 4.2). Therefore, it can be regarded as a consequence of Theorem 4.3; moreover, an extension is given, Corollary 4.1.

- Recall that the existence of a closed geodesic on a simply connected compact Riemannian manifold (without boundary) is a classical result proved first by Lyusternik and Fet [10] (in the non–simply connected case, the result was known in Cartan and Hadamard's era). Their technique relies on a shortening procedure and a min–max reasoning. This technique was extended by Thorbergsson [17]. Even though Thorbergsson's assumptions can be satisfied by non–compact manifolds, he uses explicitly a compact subset where the closed geodesic appears (see Theorem 4.4 below). This result will be discussed in Subsection 4.3, showing that it is essentially equivalent to our Theorem 4.3.

Summing up, as far as we know, the only technique properly applicable to general unbounded manifolds relies essentially on variational methods and Morse theory. Our results in the present article and in [2] try to solve with reasonable generality the problem of the existence of a closed geodesic, taking into account: (a) for bounded manifolds, they extend or complement previous results in [8] and [17], and (b) they are appliable when the manifold is unbounded and, in this case, they cannot be reduced to a result in the bounded case, as those in previous references.

2 Preliminaries

First, we will define some concepts related to the notion of convexity. We refer to [16] (see also [3]) for a complete discussion of these concepts.

Let $(M, \langle\cdot,\cdot\rangle_R)$ be a Riemannian manifold with boundary ∂M; we will assume differentiability C^3 for $\overline{M} = M \cup \partial M$. It is not difficult to show that there exists a

function $\phi \in C^2(\overline{M}, \mathbf{R}_+)$ such that

$$\begin{cases} \phi^{-1}(0) = \partial M & \\ \phi > 0 & \text{on } M \\ \nabla\phi(x) \neq 0 & \forall x \in \partial M \end{cases} \tag{2.1}$$

where $\nabla\phi(x)$ denotes the gradient of ϕ at x with respect to $\langle \cdot, \cdot \rangle_R$ (see for example [1, Proposition 3.1.3]). We say that such a manifold M has *convex* boundary ∂M if and only if for one, and then for all, function ϕ as above one has

$$H_R^\phi(x)[v, v] \leq 0 \quad \forall x \in \partial M, v \in T_x \partial M \tag{2.2}$$

where $H_R^\phi(x)[v, v]$ denotes the Hessian of ϕ at x in the direction of v. This condition is clearly equivalent to the fact that the second fundamental form with respect to the inner normal is positive semi–definite and, at least when $\overline{M}$ is complete and C^3, to *geometric convexity*, i.e. for any $p, q \in M$ the range of any geodesic joining them on $\overline{M}$ is contained in M. If inequality (2.2) strictly holds, the boundary is called *variationally strictly convex*. It is easy to prove that if ∂M is variationally strictly convex, then it is *geometrically strictly convex* (i.e. for any $p, q \in \overline{M}$ the range of any geodesic joining them on $\overline{M}$ is contained in M, up to p, q), but the converse is not true. Of course, if the boundary is geometrically strictly convex then no closed geodesic in $\overline{M}$ can touch the boundary ∂M.

Now, consider the following elements in the functional framework for the problem of closed geodesics. By the Nash embedding theorem [13], $\overline{M}$ can be considered a submanifold of $\mathbf{R}^N$ equipped with the Euclidean metric $\langle \cdot, \cdot \rangle$, for N sufficiently large. It is well known that closed geodesics are the critical points of the action functional

$$f(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds$$

on the infinite dimensional manifold

$$\Lambda^1(M) = \left\{ x \in H^1([0,1], M) \mid x(0) = x(1) \right\}$$

where

$$H^1([0,1], M) = \left\{ x \in H^1([0,1], \mathbf{R}^N) \mid x([0,1]) \subset M \right\}$$

and

$$H^1([0,1], \mathbf{R}^N) \equiv H^{1,2}([0,1], \mathbf{R}^N) =$$
$$\left\{ x \in L^2([0,1], \mathbf{R}^N) \mid x \text{ is absolutely continuous}, \dot{x} \in L^2([0,1], \mathbf{R}^N) \right\}.$$

Let us point out that, due to the periodic nature of the problem, when M is not compact, the functional f does not satisfy the well–known *Palais–Smale condition* (unbounded Palais–Smale sequences exist). We recall that if $\mathcal{H}$ is a C^2–Hilbert manifold endowed with a Riemannian metric and $F \in C^1(\mathcal{H}, \mathbf{R})$, then F satisfies the Palais–Smale condition if every sequence $\{x_m\}$ such that

$$\{F(x_m)\} \text{ is bounded}$$

and

$$|\nabla F(x_m)| \to 0$$

contains a converging subsequence (where $\nabla F(x)$ denotes the gradient of F at the point x with respect to the metric on $\mathcal{H}$ and $|\cdot|$ is the norm on the tangent bundle induced by the metric).

3 Proof of Theorem 1.1

In order to use a penalization technique, fixed $\epsilon \in]0,1]$, consider a non–negative increasing function $\psi_\epsilon \in C^2(\mathbf{R}_+, \mathbf{R}_+)$ such that

$$\begin{cases} \psi_\epsilon(s) = 0 & \text{if} \quad s \leq 1/\epsilon \\ \lim\limits_{s\to\infty} \psi_\epsilon(s) = \infty \end{cases} \tag{3.1}$$

and

$$\psi_\epsilon(s) \geq a_\epsilon s - b_\epsilon \tag{3.2}$$

for some $a_\epsilon > 0, b_\epsilon \geq 0$. Then we set

$$U_\epsilon(x) = \psi_\epsilon\left(\frac{1}{\phi^2(x)}\right), \tag{3.3}$$

where ϕ is as in (2.1). Now let $W \in C^2(M, \mathbf{R})$ be a non–negative function such that

$$\lim_{d(x,x_0)\to\infty} W(x) = \infty \tag{3.4}$$

and set for any $\epsilon \in]0,1]$

$$W_\epsilon(x) = \psi_\epsilon(W(x)). \tag{3.5}$$

Let us equip $\Lambda^1(M)$ with the Riemannian structure

$$\langle v, v\rangle_x = \int_0^1 \langle D_s v, D_s v\rangle ds + \langle v(0), v(0)\rangle \tag{3.6}$$

for any $x \in \Lambda^1(M)$, $v \in T_x\Lambda^1(M)$, where $D_s v$ is the covariant derivative of v. Let us consider on $\Lambda^1(M)$ the penalized functionals

$$f_\epsilon(x) = f(x) + \int_0^1 U_\epsilon(x) ds + W_\epsilon(x(0)). \tag{3.7}$$

In order to find critical points of functionals (3.7) some lemmas are needed.

Lemma 3.1 *Let ϕ be as in (2.1) and assume that ∂M is bounded. Let $\{x_m\}$ be a sequence in $\Lambda^1(M)$ such that*

$$\sup_{m\in\mathbf{N}} \int_0^1 \langle \dot{x}_m, \dot{x}_m\rangle < \infty \tag{3.8}$$

and assume that $\{s_m\}$ *is a sequence in* $[0,1]$ *such that*

$$\lim_{m\to\infty} \phi(x_m(s_m)) = 0. \tag{3.9}$$

Then for any $\epsilon \in]0,1]$

$$\lim_{m\to\infty} \int_0^1 U_\epsilon(x_m)\, ds = \infty.$$

For the convenience of the reader, we discuss this result in Appendix B.

The previous lemma allows one to prove the following proposition (see e.g. [5, Lemma 3.3]).

Proposition 3.1 *Let* f_ϵ *be as in (3.7). Then,*

(i) for any $\epsilon \in]0,1]$ *and for any* $c \in \mathbf{R}$ *the sublevels* $f_\epsilon^c = \{x \in \Lambda^1(M) \mid f_\epsilon(x) \le c\}$ *are complete metric subspaces of* $\Lambda^1(M)$*;*
(ii) for any $\epsilon \in]0,1]$ f_ϵ *satisfies the Palais–Smale condition.*

It can be proved (see [2, Lemma 2.5]) that for any $\epsilon \in]0,1]$, if x_ϵ is a critical point of f_ϵ, the Hessian $H^{f_\epsilon}(x_\epsilon)$ defines a Fredholm operator of index 0, thus, by Proposition 3.1 we can find critical points of the penalized functionals thanks to the following abstract critical point theorem (see e.g. [6, Theorem 3.4]).

Theorem 3.1 *Let* $F \in C^2(\mathcal{H}, \mathbf{R})$*. Assume that*

(i) for any critical point x *of* F*, the Hessian* $H^F(x)$ *defines a Fredholm operator of index* 0*;*
(ii) F *is bounded from below and satisfies the Palais–Smale condition;*
(iii) $\mathcal{H}$ *is complete or the sublevels of* F *are complete metric subspaces of* $\mathcal{H}$*;*
(iv) there exist $q \in \mathbf{N}$ *and a field* $\mathcal{F}$ *such that (1.1) holds.*

Then there exists a critical point $\bar{x}$ *of* F *corresponding to the critical value*

$$c = \inf_{A\in\Gamma_q} \sup_{x\in A} F(x)$$

(where $\Gamma_q = \{A \subset \mathcal{H} \mid i_*(H_q(A,\mathcal{F})) \neq \{0\}\}$*, and* i_* *is the induced map in homology by the inclusion* $i : A \to \mathcal{H}$*) and satisfying*

$$m(\bar{x}, F) \le q \le m(\bar{x}, F) + \dim\ker H^F(x)$$

where $m(\bar{x}, F)$ *denotes the Morse index of* F *at* $\bar{x}$*.*

Summing up, for any $\epsilon \in]0,1]$, there exists a critical point x_ϵ of f_ϵ corresponding to the critical value

$$c_\epsilon = \inf_{A\in\Gamma_q} \sup_{x\in A} f_\epsilon(x) \tag{3.10}$$

and such that

$$m(x_\epsilon, f_\epsilon) \le q \le m(x_\epsilon, f_\epsilon) + \dim\ker H^{f_\epsilon}(x_\epsilon) \tag{3.11}$$

where q is as in *(ii)* of Theorem 1.1.

Proof of Theorem 1.1:

Let $\{x_\epsilon\}$ be a family in $\Lambda^1(M)$ such that for any $\epsilon \in]0,1]$ x_ϵ is a critical point of f_ϵ satisfying (3.10) and (3.11).

Step 1: Let us prove that there exists $\epsilon_0 \in]0,1]$ such that for any $\epsilon \in]0,\epsilon_0]$

$$\sup_{s\in[0,1]} d(x_\epsilon(s), x_0) \leq M \tag{3.12}$$

for a suitable $M > 0$, where x_0 is as in *(i)* of Theorem 1.1. To prove (3.12) assume by contradiction that there exists a decreasing sequence $\{\epsilon_m\} \to 0$ and a sequence of critical points $\{x_m\}$ of $f_m \equiv f_{\epsilon_m}$ satisfying (3.10), (3.11) and such that

$$\lim_{m\to\infty} \sup_{s\in[0,1]} d(x_m(s), x_0) = \infty. \tag{3.13}$$

Since the singular homology has compact support, Γ_q must contain a compact element $C \in \Gamma_q$. Thus by (3.10) we have

$$f_\epsilon(x_\epsilon) \leq \sup f_\epsilon(C) \leq \sup f_1(C) \qquad \forall \epsilon \in]0,1]. \tag{3.14}$$

From (3.14) and the assumptions made, the constant $B = \sup f_1(C)$ satisfies

$$\int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle \, ds \leq B \qquad \forall m \in \mathbf{N}. \tag{3.15}$$

Hence from (3.13) and (3.15)

$$\lim_{m\to\infty} \inf_{s\in[0,1]} d(x_m(s), x_0) = \infty. \tag{3.16}$$

From (3.16), as ∂M is bounded, there exists $\mu > 0$ such that for m large

$$\phi(x_m(s)) \geq \mu > 0 \quad \forall s \in [0,1].$$

Therefore, for m large,

$$\psi_{\epsilon_m}\left(\frac{1}{\phi^2(x_m(s))}\right) = 0 \quad \forall s \in [0,1] \tag{3.17}$$

(see (3.1)) and the proof can be carried out as in the complete case (see [6, Theorem 1.1]). Indeed, let us set for any $m \in \mathbf{N}$

$$Z_m = \left\{ v \in T_{x_m}\Lambda^1(M) \mid v(0) = 0 \right\}. \tag{3.18}$$

From (3.6) we get

$$Z_m^\perp = \left\{ v \in T_{x_m}\Lambda^1(M) \mid D_s D_s v = 0 \text{ for any } s \in [0,1] \right\}.$$

Observe that the space of the solutions of the system of differential equations

$$D_s D_s v = 0$$

has dimension $2n$, where $\dim M = n$, hence

$$\dim Z_m^{\perp} \leq 2n. \tag{3.19}$$

By *(i)* of Theorem 1.1 there exists a sequence $\{\delta_m\} \to 0$ of positive real numbers such that for any $v \in Z_m$

$$\frac{\langle R_{(\dot{x}_m,v)}\dot{x}_m, v\rangle}{\langle \dot{x}_m, \dot{x}_m\rangle\langle v, v\rangle - \langle \dot{x}_m, v\rangle^2} \leq \delta_m \tag{3.20}$$

on $[0,1]$, see also [14, p.77]. Then, from (3.20) and (3.17), for any $v \in Z_m$, with m sufficiently large

$$H^{f_m}(x_m)[v,v] \geq$$
$$\int_0^1 \langle D_s v, D_s v\rangle ds - \delta_m \int_0^1 \left(\langle \dot{x}_m, \dot{x}_m\rangle\langle v, v\rangle - \langle \dot{x}_m, v\rangle^2\right) ds \geq$$
$$\int_0^1 \langle D_s v, D_s v\rangle ds - \delta_m \int_0^1 \langle \dot{x}_m, \dot{x}_m\rangle\langle v, v\rangle ds. \tag{3.21}$$

By the following Poincaré inequality

$$\int_0^1 \langle D_s v, D_s v\rangle ds \geq \frac{1}{4}\int_0^1 \langle\, v, v\rangle\, ds,$$

(3.21) and (3.20), we get

$$H^{f_m}(x_m)[v,v] \geq \left(\frac{1}{4} - B\delta_m\right)\int_0^1 \langle v, v\rangle ds. \tag{3.22}$$

Finally, from (3.22), for m sufficiently large

$$H^{f_m}(x_m)[v,v] > 0 \qquad \forall v \in Z_m \setminus \{0\},$$

and thus

$$m^*(x_m, f_m) \leq \dim Z_m^{\perp} \leq 2n,$$

in contradiction with (3.11), because $q > 2n$.

Step 2: Now, exploiting the regularity of the boundary, we shall consider the limit of the approximating solutions x_ϵ. From (3.12) and (3.5), for ϵ small enough, each x_ϵ satisfies the equation

$$D_s\dot{x}_\epsilon = -\frac{2}{\phi^3(x_\epsilon)}\psi_\epsilon'\left(\frac{1}{\phi^2(x_\epsilon)}\right)\nabla\phi(x_\epsilon).$$

From (3.14), (3.12) and the properties of the functions ψ_ϵ it can be proved that, up to a subsequence, $\{x_\epsilon\}$ converges in $H^1([0,1],\mathbf{R}^N)$ to a curve $x \in H^2([0,1],\mathbf{R}^N)(= \{x \in H^1([0,1],\mathbf{R}^N) \mid \dot{x} \text{ is absolutely continuous}, \ddot{x} \in L^2([0,1],\mathbf{R}^N)\})$. Notice that this curve x satisfies:

$$D_s\dot{x} = -\lambda(s)\nabla\phi(x) \quad \lambda \in L^2([0,1],\mathbf{R}). \tag{3.23}$$

Moreover, it results $x(s) \in M \cup \partial M$ for all $s \in [0,1]$, $\lambda(s) \geq 0$ for almost all $s \in [0,1]$ and $\lambda(s) = 0$ if $x(s) \in M$. If $s_0 \in [0,1]$ is such that $x(s_0) \in \partial M$ and there exists $D_s\dot{x}(s_0)$, then, as it is a minimum point for $\phi(x(s))$ and ∂M is convex, from (3.23) and (2.2), it results

$$\lambda(s_0)\langle \nabla\phi(x(s_0)), \nabla\phi(x(s_0))\rangle \leq H^{\phi}(x(s_0))[\dot{x}(s_0), \dot{x}(s_0)] \leq 0.$$

Thus we get $\lambda(s) = 0$ for almost $s \in [0,1]$ and by (3.23)

$$D_s\dot{x} = 0 \quad \forall s \in [0,1],$$

that is x is a closed geodesic on $\overline{M}$.

Step 3: Let us show that the closed geodesic x found in Step 2 is not trivial. Remark that under our assumptions

$$\lim_{\epsilon \to 0} \int_0^1 \psi_\epsilon \left(\frac{1}{\phi^2(x_\epsilon)} \right) ds = 0$$

hence, denoted by $\overline{f}$ the action functional on $\Lambda^1(\overline{M})$, we get

$$\lim_{\epsilon \to 0} f_\epsilon(x_\epsilon) = \overline{f}(x).$$

Then from (3.11) we get

$$m(x, \overline{f}) + \dim \ker H^{\overline{f}}(x) \geq q, \tag{3.24}$$

see e.g. [4, Proposition 3.1]. Set

$$Z = \left\{ v \in T_x\Lambda^1(\overline{M}) \mid v(0) = 0 \right\}. \tag{3.25}$$

Reasoning as for (3.19) in Step 1, we get

$$\dim Z^{\perp} \leq 2n. \tag{3.26}$$

If by contradiction x is a constant curve, we have for any $v \in T_x\Lambda^1(\overline{M})$

$$H^{\overline{f}}(x)[v,v] = \int_0^1 \langle D_s v, D_s v \rangle ds,$$

hence in particular from (3.25) it results

$$H^{\overline{f}}(x)[v,v] > 0 \qquad \forall v \in Z \setminus \{0\}. \tag{3.27}$$

It follows

$$m(x, \overline{f}) + \dim \ker H^{\overline{f}}(x) \leq \dim Z^{\perp} \leq 2n,$$

and this is in contradiction with (3.24) since $q > 2n$.

Remark 3.1 Under our assumptions, the convexity of the boundary is equivalent to geometrical convexity, because of a result in [9]. This implies that either the closed non–trivial geodesic x lies completely in the boundary ∂M, or x lies completely in M. In fact, if $x(s_0) \in M$, consider the 1–periodic extension of x. As $x(s_0 + 1) \in M$, then $x([s_0, s_0 + 1]) \subset M$, because of geometric convexity.

4 Comparison with previous results

4.1 Convexity close to the (Cauchy) boundary.

In [2] the authors have studied the existence of a closed geodesic on Riemannian manifolds M which are *convex close to their boundaries*. In order to explain this concept, let $\overline{M}^c = M \cup \partial_c M$ be the canonical Cauchy completion of M as a metric space, by using Cauchy sequences. Recall that this boundary is canonically associated to any Riemannian manifold. If $\overline{M} = M \cup \partial M$ is a complete Riemannian manifold with boundary as in the previous sections, then there is a natural identification $\overline{M}^c \equiv \overline{M}$ (see [3]).

Definition 4.1 A Riemannian manifold M is *convex close to its boundary (CCB)* if there exist $\delta, b > 0$ and a function $\phi \in C^0(\overline{M}^c, \mathbf{R}_+) \cap C^2(M, \mathbf{R}_+)$ such that:

(i) $\phi^{-1}(0) = \partial_c M$;
(ii) $0 < \langle \nabla\phi(x), \nabla\phi(x) \rangle \leq b$ for all $x \in M$ with $0 < \phi(x) < \delta$;
(iii) $H^\phi(x)[v, v] \leq 0$ for any $x \in M$ with $0 < \phi(x) < \delta$ and for all $v \in T_x M$ such that $\langle \nabla\phi(x), v \rangle = 0$.

Remark 4.1 (1) It is possible to restate Definition 4.1 without mentioning $\partial_c M$ and $\overline{M}^c$. Simply, assume that ϕ is a positive C^2 function on M, and replace *(i)* by:

(i') For any smooth curve of constant speed $\gamma : [0, 1[\to M$ which cannot be continuously extended to 1, we have: $\lim_{s \to 1} \phi(\gamma(s)) = 0$.

Condition *(i')* is also equivalent to the following one, which has been used in [8]:

(i") For any $\eta > 0$ the set $\{x \in M \mid \phi(x) \geq \eta\}$ is complete.

(2) The first inequality of condition *(ii)* ensures that the level subsets $\phi^{-1}(\rho)$ of ϕ are smooth hypersurfaces for small ρ, $0 < \rho < \delta$; by the second inequality, these hypersurfaces do not separate too much. By *(iii)* each of these hypersurfaces is the convex boundary of $\phi^{-1}(]\rho, \infty[)$.

The authors obtained the following result in [2][1] which also extends to non–complete manifolds the result in [6].

Theorem 4.1 [2] *Let M be a CCB Riemannian manifold. Assume that conditions (i) and (ii) of Theorem 1.1 hold. Then there exists a non–trivial closed geodesic on M.*

The notion of CCB Riemannian manifold appears naturally when a closed geodesic which does not touch the boundary is found (in fact, recall that the Cauchy boundary may be non–differentiable). It is clear that Theorem 4.1 is not contained in

[1] See Theorem 1.3 of this reference. This result is formulated in a more general setting because it can be used to yield periodic trajectories for particles under potentials by using a Jacobi metric; this yields consequences also for static (Lorentzian) spacetimes. All this can be recovered now. Nevertheless, there is a gap in the statement of [2, Theorem 1.3] because a multiplicity result is also announced there. At any case, the multiplicity result holds in the applications (Reissner–Nordström and Schwarzschild spacetimes) because of spherical symmetry (see [1, Ch. 7] for a complete discussion).

Theorem 1.1 because the boundary may be not only non–differentiable but also non–compact. Neither holds the converse, because CCB manifolds do not include all the complete manifolds with differentiable convex and compact boundary. In fact, recalling the definitions in Section 2, we have for $\overline{M}$ complete and compact ∂M: (i) if ∂M is strictly variationally convex then M is CCB, (ii) this not necessarily holds if ∂M is just strictly geometrically convex (even though, in this case, closed geodesics cannot touch the boundary, obviously), and (iii) when ∂M is convex but close ϕ-hypersurfaces are not, then the closed geodesic obtained under Theorem 1.1 may touch the boundary (in fact, it would be completely contained in ∂M).

Example 4.1 Let us consider the warped product $M =]0,1[\times_f F$ with metric $g = dx^2 + f^2(x)g_F$, where x is the natural coordinate on $\mathbf{R}$, and (F, g_F) is a compact simply connected Riemannian manifold without boundary; so, hypothesis *(ii)* in Theorem 1.1 is satisfied (see [18]). Choose $f \in C^3([0,1], \mathbf{R}_+)$ such that $f'(x) > 0$ for $0 < x < 1$ and $f'(0) = 0 = f'(1)$. Clearly, the boundary $\partial M (= \partial_c M) = (\{0\} \times F) \cup (\{1\} \times F)$ is compact and convex, and each of its connected components can be reparametrized as a closed geodesic. Taking into account the general equation of the geodesics of a warped product (see for example [14, Ch. 7 Proposition 38]) it is easy to check the following: for any geodesic $\gamma(s) = (x(s), \gamma_F(s))$, if the first component $x(s)$ is not constant then it cannot attain a maximum $x_0 = x(s_0)$ (otherwise, $x''(s_0) = f(x_0)f'(x_0)|\gamma_F'(s_0)|^2 > 0$). Thus, there is no closed geodesic on M and the manifold cannot be CCB (no choice of ϕ can satisfy Definition 4.1). Straightforward modifications provide less trivial examples (say, with a non-totally geodesic boundary). For example, let N be the inner vector normal to ∂M, and choose a conformal metric $g^* = e^{2u}g$, $u \in C^3([0,1] \times F, \mathbf{R})$ such that $g(\nabla u, N) \leq 0$, with strict inequality at some point. It is straightforward to check that the boundary is convex and not totally geodesic for g^* (see for example [1, Equation 3.71]); however, a closed geodesic will appear for g^*. On the other hand, if $]0,1[$ is replaced by $\mathbf{R}_+$, f satisfies $\liminf_{x\to\infty}(f''/f)(x) \geq 0$ and either the curvature of (F, g_F) is nonpositive or $\lim_{x\to\infty} f = \infty$, then the hypothesis *(i)* in Theorem 1.1 will hold for g,[2] which can be used to generate new families of examples.

4.2 Candela–Salvatore result.

Hereafter we shall denote by $B_r(x_0)$ the open ball $B_r(x_0) = \{x \in M \mid d(x, x_0) < r\}$. By using Definition 4.1, the result in [8] can be stated as follows.

Theorem 4.2 [8] *Let M be a CCB Riemannian manifold having compact Cauchy boundary. Assume that:*

(i) there exist $U \in C^2(M, \mathbf{R})$ and some positive constants r, ν such that

$$H^U(x)[v,v] \geq \nu\langle v, v\rangle \quad \forall x \in M \setminus B_r(x_0), v \in T_xM \tag{4.1}$$

[2]The hypotheses on f can be weakened, see the general formulas [14, Ch. 7, Proposition 42]. Recall that manifolds of negative curvature are constructed and studied in a classical article by Bishop and O'Neill [7], and we need only that the limsup of the curvature is nonpositive (see for example Theorem 7.5 in [7], or the formula for the curvature above this theorem).

(ii) M is not contractible in itself and either its fundamental group $\pi_1(M)$ is finite or it has infinitely many conjugacy classes.

Then there exists a non–trivial closed geodesic on M.

For the relation between Theorems 1.1, 4.1 and 4.2, notice first that assumption (i) of Theorem 4.2 is also fulfilled if the Cauchy boundary is bounded and all the sectional curvatures outside a compact subset are non–positive (indeed in this case the distance function plays the role of U). Nevertheless, in Theorems 1.1 or 4.1 we only require that the *limsup* of the sectional curvatures is non–negative.

Next, we will see how Theorem 4.2 can be reproved and extended from the bounded case. First, notice the following result:

Theorem 4.3 *Let M be a Riemannian manifold and assume that one of the topological assumptions (ii) in either Theorem 4.2 or Theorem 1.1 holds. Then:*

(1) If M is CCB and bounded, there exists a closed geodesic on M.
(2) If $\overline{M} = M \cup \partial M$ is a compact Riemannian manifold with (differentiable) convex boundary, there exists a (non–trivial) closed geodesic on $\overline{M}$.

For the proof of this result, remark that, as M is bounded, it is enough to consider the penalized functionals

$$f_\epsilon(x) = f(x) + \int_0^1 U_\epsilon(x)ds \quad x \in \Lambda^1(M)$$

(see Section 3). Plainly, if *(ii)* of Theorem 1.1 holds, (1) and (2) follow respectively by Theorems 4.1 and 1.1. If *(ii)* of Theorem 4.2 holds, (1) is a particular case of Theorem 4.2, and (2) can be obtained as follows. As M is bounded, Step 1 in the proof of Theorem 1.1 in unnecessary, from Step 2 we get the existence of a closed geodesic on $\overline{M}$ and the non–triviality of this geodesic can be proved by using arguments based on Ljusternik–Schnirelman category theory (see e.g. [5]).

Recall now the following definition.

Definition 4.2 Let A be a subset of a topological space X; A is called a retract of X if there exists a continuous map $\rho : X \to A$ such that $\rho(x) = x$ for any $x \in A$. A is a *deformation retract* of X if A is a retract of X and there exists a continuous map $F : [0,1] \times X \to X$, such that

$$\begin{cases} F(0,x) = x & \forall x \in X; \\ F(1,x) = \rho(x) & \forall x \in X. \end{cases}$$

In this case, the homotopy groups of the retract A are equal to the homotopy groups of the topological space X. The proof of the following lemma will be provided in the Appendix A.

Lemma 4.1 *Let M be a CCB Riemannian manifold with compact Cauchy boundary. Assume that there exist $U \in C^2(M, \mathbf{R})$ and some positive constants r, ν such that inequality (4.1) holds outside some ball $B_r(x_0)$. Then there exists a sequence $\{D_m\}$ of open subsets of M such that:*

(i) $D_m \subset D_{m+1}$ *and* $M = \bigcup_{m\in\mathbf{N}} D_m$*;*
(ii) for any $m \in \mathbf{N}$*,* D_m *is a bounded CCB Riemannian manifold having compact Cauchy boundary;*
(iii) there exists $\overline{m} \in \mathbf{N}$ *such that* $D_m \cup \partial D_m$ *is a deformation retract of* M *for any* $m \geq \overline{m}$*.*

By using Lemma 4.1, one has that Theorem 4.2 is reduced to search closed geodesics on a *bounded* CCB Riemannian manifold $D \equiv D_m$ having the same topological properties of the whole manifold. So, Theorem 4.2 turns out to be a corollary of Theorem 4.3.

Even more, Theorem 4.2 can be improved in different senses. First, because a similar version of Lemma 4.1 works for manifolds with compact (differentiable) boundary. But also because the hypotheses on U in this lemma can be sharpened. For example, under the hypotheses of Lemma 4.1, U is *regular out a ball* in the sense that, removing a ball of sufficiently big radius, U is proper, bounded from below and without critical points (see Remark A.1). But in this case, the inequality $H^U_R(x)[v,v] \geq 0$, for all $x \in M \setminus B_r(x_0), v \in T_xM$ (rather than (4.1)), is enough to prove the conclusion of Lemma 4.1. Summing up, we have:

Corollary 4.1 *Let* M *be a Riemannian manifold and assume that there exists a function* $U \in C^2(M,\mathbf{R})$ *which is* regular out a ball *in the sense above. If*

$$H^U_R(x)[v,v] \geq 0 \quad \forall x \in M \setminus B_r(x_0), v \in T_xM$$

for some $r > 0$*, and one of the topological assumptions (ii) in either Theorem 4.2 or Theorem 1.1 holds, then:*

(1) If M *is CCB, there exist a closed geodesic on* M*.*
(2) If $\overline{M} = M \cup \partial M$ *is a Riemannian manifold with compact (differentiable) convex boundary, there exists a (non-trivial) closed geodesic on* $\overline{M}$*.*

Remark 4.2 This reduction of a problem for an unbounded manifold to a problem for a bounded manifold can be carried out in the more general case of trajectories for autonomous bounded potentials, i.e., solutions of $x''(s) = -\nabla V(x(s))$, for some potential V. Assume that we look for a periodic trajectory of fixed energy $E > \sup V$. It is well-known that these trajectories are, up to a reparametrization, geodesics for a conformal metric $\langle\cdot,\cdot\rangle_E = (E-V)\langle\cdot,\cdot\rangle_R$. Thus, closed trajectories for the potential can be found just imposing conditions for the existence of closed geodesics of $\langle\cdot,\cdot\rangle_E$. In particular, [12, Theorem 1.2] can be reobtained from the results in the present subsection.

4.3 Thorbergsson result.

Next, the results in [17] will be compared. Following the notation in [15], a closed subset $K \subseteq M$ is said to be *strongly convex* if for any $p, q \in K$ there exists a unique (up to affine reparameterization) minimal geodesic γ in M joining p, q and γ is contained completely in K. K is *locally convex* if for any $p \in K$ there is an $\epsilon(p) > 0$

such that $K \cap B_{\epsilon(p)}(p)$ is locally convex. The main result in [17] [3] can be stated as follows.

Theorem 4.4 [17] *Let M be a complete Riemannian manifold and let $K \subset M$ be compact and locally convex. If there exists an $i > 0$ such that the $i-th$ homotopy group $\pi_i(K)$ is not trivial, then there exists a non-trivial closed geodesic on M.*

In order to relate this result with Theorem 1.1 notice that by [15, Theorem 5.5], the following result holds for any (not necessarily complete) Riemannian manifold:

Lemma 4.2 *Let M be a Riemannian manifold and let $K \subset M$ be compact and locally convex. Then there exists a (necessarily bounded) totally geodesic submanifold N of M such that $K = \overline{N}$. Moreover, K is also a topological manifold with boundary $\partial K (\equiv \partial_c K) = \partial N$.*

For the comparison between Theorem 4.3 (or 1.1) and Theorem 4.4, recall first that the boundary ∂N in previous lemma is not necessarily differentiable (for example: consider a convex poligon in $\mathbf{R}^n$). Even in this case, Theorem 4.4 can find a closed geodesic because one assumes that K is included in a bigger complete differentiable manifold; but this possibility is excluded in Theorem 4.3 because its hypotheses are imposed on N intrinsically. In order to simplify, assume that ∂K is differentiable (and, thus, *convex*). So, Lemma 4.2[4] reduces this comparison to the comparison between the topological assumptions in Theorems 4.3 and 4.4.

For this, recall first that if $\pi_1(K) \neq 0$ then one can easily find a closed geodesic in each non-trivial conjugacy class: simply, look for the closed curves in the class with minimum length. When $\pi_1(K) = 0$, topological assumptions in Theorem 4.3 include explicitly those in Theorem 4.4.

Nevertheless, the topological assumptions in Theorem 4.3 also imply those in Theorem 4.4, because:

any CW-complex with all its homotopy groups equal to 0 is contractible.

The reason of this result relies on a classical Theorem by Whitehead (see for example [11, Theorem 7.5.4]): *if K, L are connected CW complexes and $f : K \to L$ is a weak homotopy equivalence* (i.e. the induced map between all the homotopy groups is an isomorphism) *then f is a homotopy equivalence.* Thus, apply this result with $K = \{x_0\}$ and $L = \overline{M}$. (Recall that it is easy to find topological spaces which are not contractible but with all their homotopy groups 0; nevertheless, they are not manifolds.)

[3] This result corresponds to [17, Theorem 4.2]. Theorem 4.3 of this reference also ensure the existence of a closed geodesic, but reducing the problem to [17, Theorem 4.2]. An analogous comment can be done of [17, Theorem 3.2], see [17, Lemma 3.1].

[4] and standard arguments about the possibility to modify the metric out of K in order to obtain a complete Riemannian manifold

A Appendix: Proof of Lemma 4.1

As the Cauchy boundary $\partial_c M$ is assumed to be compact, we can choose r in (4.1) such that

$$\partial_c M \subset B_r(x_0);$$

moreover, let us set

$$B = B_{r+1}(x_0). \tag{A.1}$$

The following lemma holds.

Lemma A.1 *Let M be a* convex[5] *Riemannian manifold. Assume that there exists $U \in C^2(M, \mathbf{R})$ such that (4.1) holds and let B be as in (A.1). Then for any $\{x_m\} \subset M \setminus B$ such that*

$$\lim_{m\to\infty} d(x_m, B) = \infty \tag{A.2}$$

we obtain

$$\lim_{m\to\infty} U(x_m) = \infty \qquad \lim_{m\to\infty} \langle \nabla U(x_m), \nabla U(x_m)\rangle = \infty. \tag{A.3}$$

Proof.
Let $\{x_m\} \subset M \setminus B$ be such that (A.2) holds. As ∂B is compact, for any $m \in \mathbf{N}$ there exists $b_m \in \partial B$ such that $d(x_m, B) = d(x_m, b_m)$. Remark that as M is convex, there exists a unitary minimizing geodesic $\gamma_m : [0, d_m] \to M$ such that $\gamma_m(0) = b_m, \gamma_m(d_m) = x_m$. Clearly,

$$\gamma_m([0, d_m]) \subset M \setminus B. \tag{A.4}$$

Let us set

$$U'_{\max} = \max_{x\in\partial B} |\nabla U(x)| \qquad U_{\min} = \min_{x\in\partial B} U(x)$$

$$g_m(t) = (U \circ \gamma_m)(t) \quad \forall t \in [0, d_m],$$

where $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. From (4.1) and standard calculations we have

$$g'_m(t) = g'_m(0) + \int_0^t g''_m(s)\, ds =$$

$$\langle \nabla U(b_m), \gamma'_m(b_m)\rangle + \int_0^t H^U(\gamma_m(s))[\gamma'_m(s), \gamma'_m(s)]\, ds \geq$$

$$-|\nabla U(b_m)| + \nu t \geq -U'_{\max} + \nu t.$$

Hence, for $t = d_m$, we get the second equality in (A.3). Finally, we easily get

$$g_m(t) = g_m(0) + \int_0^t g'_m(s)\, ds \geq$$

$$U(b_m) + \int_0^t (-U'_{\max} + \nu s)\, ds \geq$$

$$U_{\min} - U'_{\max} t + \nu \frac{t^2}{2},$$

[5] In the sense that any pair of its points can be joined by a (non necessarily unique) minimizing geodesic. For a complete Riemannian manifold with boundary, this is equivalent to the convexity of the boundary (see, i.e., [16] or [3].

and for $t = d_m$ the first equality in (A.3) is proved. □

Remark A.1 By Lemma A.1 it follows that, putting

$$\hat{U} = U_{|M\setminus B}, \tag{A.5}$$

then $\hat{U}$ is a proper function, $\hat{U}$ is bounded from below and its critical points lie in a compact subset, that is, U is regular out the ball B.

Proof of Lemma 4.1:

(i) As $\hat{U}$ in Remark A.1 is a proper function, there exists an increasing sequence $\{u_m\}$ of regular values of $\hat{U}$ such that

$$\lim_{m\to\infty} u_m = \infty. \tag{A.6}$$

Setting

$$D_m = B \cup \hat{U}^{-1}(]-\infty, u_m[) \quad \forall m \in \mathbf{N},$$

$\{D_m\}$ is clearly an increasing sequence of open subsets exhausting M.

(ii) By Lemma A.1 (see also Remark A.1) it follows that D_m is bounded and it has compact Cauchy boundary for any $m \in \mathbf{N}$. Let us show that any D_m is CCB. The Cauchy boundary of each D_m is the union of two compact and disjoint sets: the Cauchy boundary of M and $\partial D_m = \hat{U}^{-1}(u_m)$ (the differentiable boundary of $\hat{U}^{-1}(]-\infty, u_m[)$). Let $\epsilon > 0$ and let us set

$$V_m = \hat{U}^{-1}([u_m - \epsilon, u_m]) \qquad W_m = \hat{U}^{-1}\left(\left[u_m - \frac{\epsilon}{2}, u_m\right]\right),$$

such that $V_m \cap \partial B = \emptyset$ and $\nabla U \neq 0$ on W_m. Consider a function $\mu \in C^0(\overline{M}^c, \mathbf{R}_+) \cap C^2(M, \mathbf{R}_+)$ such that

$$\begin{cases} 0 \leq \mu \leq 1 & \\ \mu = 1 & \text{on } W_m \\ \text{supp } \mu \subset V_m & \end{cases}$$

and set

$$\phi_m = \mu(u_m - U) + (1-\mu)\phi, \tag{A.7}$$

where ϕ is as in Definition 4.1, and can be chosen such that $\phi \geq 1$ on V_m. It is easy to check that $\phi_m^{-1}(0) = \partial_c D_m$, and $\phi_m(x) > \epsilon/2$ for any $x \in V_m \setminus W_m$. Set $\overline{\delta} = \min\{\epsilon/2, \delta\}$ (where δ is as in Definition 4.1); then for any $x \in D_m$ such that $0 < \phi_m(x) < \overline{\delta}$ either

$$x \in W_m \quad (\text{and } \nabla\phi_m(x) = -\nabla U(x), H^{\phi_m}(x)[v,v] = -H^U(x)[v,v])$$

or

$$x \in D_m \setminus V_m \quad (\text{and } \nabla\phi_m(x) = \nabla\phi(x), H^{\phi_m}(x)[v,v] = H^{\phi}(x)[v,v]);$$

at any case *(ii), (iii)* of Definition 4.1 are satisfied, thus D_m is CCB with respect to ϕ_m.

(iii) By Remark A.1, we can consider $r_1 > r + 1$ such that the set K of the critical points of $\hat{U}$ is contained in $B_{r_1}(x_0)$. Moreover, take $U_0 > 0$ such that

$$\forall x \in M \setminus B : U(x) \geq U_0 \Rightarrow d(x, x_0) \geq r_1.$$

Let $\overline{m} \in \mathbf{N}$ be such that $u_m > U_0$ for any $m \geq \overline{m}$, thus $K \subset D_m$ for any $m \geq \overline{m}$. As there are not critical points of U in $M \setminus D_m = \{x \in M \setminus B \mid U(x) \geq u_m\}$, the following Cauchy problem makes sense:

$$\begin{cases} \dot{\eta} = -\dfrac{\nabla U(\eta)}{|\nabla U(\eta)|^2} \\ \eta(0) = x \in M \setminus D_m. \end{cases} \tag{A.8}$$

Call $\eta(t, x)$ the flow associated to the Cauchy problem (A.8). As

$$\frac{d}{dt} U(\eta(t, x)) = -1,$$

putting

$$\tau(x) = U(x) - u_m$$

it results that $\eta(\tau(x), x) \in U^{-1}(u_m)$. Let us define $\rho : M \to D_m \cup \partial D_m$ by

$$\rho(x) = \begin{cases} \eta(\tau(x), x) & \text{if } x \in M \setminus D_m \\ x & \text{if } x \in D_m. \end{cases}$$

Clearly ρ is a retraction (see Definition 4.2). Finally, let us consider $F : [0, 1] \times M \to M$ such that for any $s \in [0, 1]$:

$$F(s, x) = \begin{cases} \eta(s\tau(x), x) & \text{if } x \in M \setminus D_m \\ x & \text{if } x \in D_m. \end{cases}$$

It is easy to verify that $D_m \cup \partial D_m$ is a deformation retract of M.

B Appendix: Note on Lemma 3.1

Some variations on Lemma 3.1 are contained in previous references. In fact, in [5] a version of this result is given but replacing the loop space $\Lambda^1(M)$ by the set of curves joining two given points. For our version, it is worth to point out that the second inequality of *(ii)* of Definition 4.1 can be imposed on all the manifold because of the following lemma.

Lemma B.1 *Let M be a CCB Riemannian manifold. Then there exists a function $\overline{\phi} \in C^0(\overline{M}^c, \mathbf{R}) \cap C^2(M, \mathbf{R})$ verifying (i)—(iii) of Definition 4.1 and such that, for some $B > 0$, $|\nabla \overline{\phi}(x)| \leq B$ for any $x \in M$.*

Then one can show the following result (see [1, Lemma 4.3.3] for detailed proofs).

Lemma B.2 *Let M be a Riemannian manifold and assume that there exist $\delta, b > 0$ and a non-negative function $\phi \in C^0(\overline{M}^c, \mathbf{R}) \cap C^1(M, \mathbf{R})$ such that:*

(i) $\phi^{-1}(0) = \partial_c M$;
(ii) $0 < |\nabla \phi(x)| \leq b$ *for all* $x \in M$ *with* $0 < \phi(x) < \delta$.

Let $\{x_m\}$ be a sequence in $\Lambda^1(M)$ such that (3.8) and (3.9) hold. Then

$$\lim_{m \to \infty} \int_0^1 \frac{1}{\phi^2(x_m)}\, ds = \infty.$$

By using (3.2), Lemma 3.1 can be regarded as a particular case of Lemma B.2.

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