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Prescribing the Scalar Curvature Problem on Three and Four Manifolds

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Abstract

This paper is devoted to the prescribed scalar curvature problem on 3 and 4-dimensional Riemannian manifolds. We give a new class of functionals which can be realized as scalar curvature. Our proof uses topological arguments and the tools of the theory of the critical points at infinity.

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1 Introduction and the main results

Let (M^n, g_0) be a compact n-dimensional Riemannian manifold without boundary, with a nonnegative scalar curvature R_{g_0}, $n \geq 3$, and let $K : M^n \to \mathbb{R}$ be a C^3 positive function. The prescribed scalar curvature problem is to find suitable conditions on K such that K is the scalar curvature for some metric g on M^n conformally equivalent to g_0. If we set

$$g = u^{\frac{4}{n-2}} g_0,$$

where u is a positive function on M^n, then the problem is to solve the following partial differential equation

$$(P) \quad \begin{cases} -L_{g_0} u & = K(x) u^{\frac{n+2}{n-2}} \\ \quad u & > 0 \end{cases}$$

where $L_{g_0} = \Delta - ((n-2)/(4n-4))R_{g_0}$ is the conformal Laplacian of M^n.

This problem has been studied in various works previous to ours, in dimension 2, 3 and 4 (see [5],[8],[12],[13],[15],[16],[18]) as well as in high dimensions (see [1],[3],[9],[10],[14],[19],[23]).

In this paper, we study problem (P) for $n = 3$ and 4. We provide some existence results which use in a basic way the topological tools introduced by A. Bahri [2], [3]. Precisely, we follow closely the ideas developed in Aubin-Bahri [1], Bahri [3] and Ben Ayed-Chtioui-Hammami [9], where the problem of prescribing the scalar curvature on closed manifolds has been studied, using some algebraic topological tools.

For 3-dimensional manifolds which are not conformally equivalent to the 3-round sphere, an optimal result was obtained by J. Escobar and R. Schoen [17], namely in this case a function K can be realized as a scalar curvature if and only if K is positive somewhere. Such optimal result has been proved using the positive mass theorem of R. Schoen and S. T. Yau [26]. For spheres, besides the condition that the function has to be positive somewhere, there are topological obstructions known as Kazdan-Warner obstructions [19]. When it comes to the 3-dimensional case, we consider only spheres. Thus, we are reduced to looking for positive solutions u of the following problem

$$(1) \quad \begin{cases} -8\Delta u + 6u & = K(x)u^5 \\ u & > 0 \text{ on } S^3. \end{cases}$$

Assume K has only nondegenerate critical points $y_0, y_1, ..., y_l$, such that for each $i = 0, 1, ..., l$, we have $\Delta K(y_i) \neq 0$. Each y_i is assumed to be of index: $i(y_i) = 3 - k_i$. Regarding problem (1), A. Bahri and J. M. Coron [5] have proved that if

$$\sum_{i \text{ such that } -\Delta K(y_i)>0} (-1)^{k_i} \neq 1,$$

then (1) has a solution. Their method consists of studying the critical points at infinity of the associated variational problem, computing their total Morse index, and comparing this total index to the Euler-Poincaré characteristic of the space of variations.

Before stating our first result, we introduce the following assumptions
Assume that $K(y_0) \geq K(y_1) \geq ... \geq K(y_l)$ and let $I = \{y_i / -\Delta K(y_i) > 0\}$.
(H_1) Assume that $y_1 \in I$ and $i(y_1) = 3 - k$, $k \geq 1$.
Our first main result is the following

Theorem 1.1 *Under the assumption* (H_1), *if*
(G_1) $K(y)^{-1/2} > K(y_0)^{-1/2} + K(y_1)^{-1/2} \quad \forall y \in I \setminus \{y_0, y_1\}$
then (1) has a solution of index k *or* $k+1$.

Remark 1.1 Observe that if $y_1 \notin I$, taking the first critical point y_{i_1} in I below the maximum of K and assuming that $W_s(y_{i_1}) \cap W_u(y) = \emptyset$ for each critical point $y \notin I$; then, if $K(y)^{-1/2} > K(y_0)^{-1/2} + K(y_{i_1})^{-1/2}$ for each $y \in I \setminus \{y_0, y_{i_1}\}$, (1) has a solution.

Remark 1.2 The result of Theorem 1.1 is true if we change the assumption (G_1) by the following assumption:
(G_1') For each $y \in I - \{y_1\}$ such that $K(y)^{-1/2} \leq K(y_0)^{-1/2} + K(y_1)^{-1/2}$, we have $i(y) \notin \{3-k, 3-(k+1)\}$.

To state our next result in three dimension, let Z be a pseudogradient of K of Morse-Smale type (that is the intersections of the stable and the unstable manifolds of the critical points of K are transverse). We assume that

$$W_s(y_i) \cap W_u(y_j) = \emptyset \text{ for each } y_i \in I \text{ and } y_j \notin I.$$

(H_2) Assume that there exists $k \geq 1$ such that $3 - k = \min\{i(y_i), y_i \in I\}$.
Let

$$X = \cup_{y_i \in B_k} \overline{W_s(y_i)} \text{ where } B_k = \{y \in I / \, i(y) = 3 - k\}.$$

(H_3) Assume that X is a stratified set without boundary (in the topological sense, i.e., $X \in S_k(S^3)$, the group of chains of dimension k and $\partial X = 0$).
We then have the following result:

Theorem 1.2 *Under the assumptions (H_2) and (H_3), (1) admits a solution of Morse index k or $k+1$.*

Next, we want to consider a situation where the result of [5] does not give solution to problem (1). But, by our results (Theorem 1.1 and Theorem 1.2) we derive that problem (1) admits a solution. For this, let $K : S^3 \to \mathbb{R}$ be a function such that $I = \{y_0, y_1, y_2\}$ with $K(y_0) \geq K(y_1) \geq K(y_2)$, $i(y_0) = 3$, $i(y_1) \neq i(y_2) \in \{1, 2\}$ and $K(y) < K(y_1)$ for any critical point y of K which is not in I. It is easy to see that

$$\sum_{y \in I} (-1)^{3-i(y)} = 1.$$

We distinguish two cases:
First case. $K(y_2)^{-1/2} > K(y_0)^{-1/2} + K(y_1)^{-1/2}$. Using Theorem 1.1, we deduce that K is the scalar curvature for some metric conformally equivalent to g_0.
Second case. $i(y_1) = 1$. From Theorem 1.2, we derive that (1) has a solution.

Now, our main goal in the second part of this work is to study the problem (P) for $n = 4$. Thus, we are reduced to solving the following equation:

$$(2) \quad \begin{cases} -\Delta u + \frac{1}{6} R_{g_0} u & = K(x) u^3 \\ u & > 0 \text{ on } M^4. \end{cases}$$

In order to state our next result, we need to fix some notation and state our assumptions.

Let $G(a,.)$ be a Green's function for L_{g_0} on M^4 and A_a be the value of the regular part of G evaluated at a. Assume that K has only nondegenerate critical points $y_0, y_1, ..., y_m$ such that $K(y_0) \geq K(y_1) \geq ... \geq K(y_m)$ and satisfying

$$-\frac{\Delta K(y_i)}{3K(y_i)} - 2Ay_i \neq 0 \quad i = 0, ..., m.$$

Let $H = \left\{ y_i \in \{y_0, ..., y_m\} / \ -\dfrac{\Delta K(y_i)}{3K(y_i)} - 2A_{y_i} > 0 \right\}$. We assume that y_0 is the unique absolute maximum of the function K on M^4.
We then have

Theorem 1.3 *If $y_0 \notin H$, then (2) has a solution.*

In the above result, we have assumed that $y_0 \notin H$. Next we want to state some existence results for problem (2) when $y_0 \in H$. First, we introduce the following matrix. For any N-tuple $\tau_N = (y_{i_1}, ..., y_{i_N}) \in H^N$, define a matrix $M(\tau_N) = (M_{pq})$ with:

$$M_{pp} = -\frac{\Delta K(y_{i_p})}{3K(y_{i_p})^2} - \frac{2A_{y_{i_p}}}{K(y_{i_p})}$$

$$M_{pq} = -\frac{2G(y_{i_p}, y_{i_q})}{[K(y_{i_p})K(y_{i_q})]^{1/2}} \quad 0 \leq p \neq q \leq N.$$

It was first pointed out by A. Bahri [2], see also [8], that when the interaction between the different bubbles is of the same order as the self interaction, a matrix as $M(\tau_N)$ plays a fundamental role in the theory of the critical points at infinity. For problem (2), this kind of phenomenon appears in dimension 4. Let
$H^+ = \{(y_{i_1}, ..., y_{i_s}) \ / \ y_{i_j} \in H, \ y_{i_j} \neq y_{i_k} \text{ for } j \neq k, \ M(y_{i_1}, ..., y_{i_s}) > 0 \text{ and } s \in \mathbb{N}^*.\}$
Assume that:
(A_1) $y_0, y_1 \in H$ and $i(y_1) = 4 - k$, $k \geq 1$, where $i(y_1)$ denotes the Morse index of the function K at y_1.
(A_2) $K(y)^{-1} > K(y_0)^{-1} + K(y_1)^{-1} \quad \forall \, y \in H \setminus \{y_0, y_1\}$.
Then, we have the following result:

Theorem 1.4 *Under the assumptions (A_1) and (A_2) if $(y_0, y_1) \notin H^+$ then (2) has a solution of index k or $k+1$.*

In contrast to Theorem 1.4, we have the following two results based on a topological invariant denoted by μ for some Yamabe type problems. This topological invariant was first introduced by A. Bahri [3]. To state those results, we need to state the assumptions that we are using and to fix some notation.
(B_1) Assume that $(y_0, y_1) \in H^+$.
Let

$$X = \overline{W}_s(y_1) = W_s(y_1) \cup W_s(y_0).$$

We denote by $C_{y_0}(X)$ the following set

$$C_{y_0}(X) = \{\alpha\delta_{y_0} + (1-\alpha)\delta_x \ / \ \alpha \in [0,1], \quad x \in X\}$$

where δ_x is the Dirac measure at x.
For λ large enough we introduce a map

$$\begin{aligned} f_\lambda : C_{y_0}(X) &\longrightarrow \Sigma^+ \\ \alpha\delta_{y_0} + (1-\alpha)\delta_x &\longmapsto \frac{\alpha\varphi_{(y_0,\lambda)} + (1-\alpha)\varphi_{(x,\lambda)}}{|\alpha\varphi_{(y_0,\lambda)} + (1-\alpha)\varphi_{(x,\lambda)}|}, \end{aligned}$$

where $\varphi_{(x,\lambda)}$ is defined in the next section. For λ large enough, we also define the intersection number (modulo 2) of $f_\lambda(C_{y_0}(X))$ with $W_s(y_0,y_1)_\infty$

$$\mu(y_1) = f_\lambda(C_{y_0}(X)).W_s(y_0,y_1)_\infty,$$

where $W_s(y_0,y_1)_\infty$ is the stable manifold of the critical point at infinity $(y_0,y_1)_\infty$ (notation of [3]) for a decreasing pseudogradient V for J which is transverse to $f_\lambda(C_{y_0}(X))$. This number is well defined [23].
(B_2) Assume that $i(y_1) = 4-k$ with $k \geq 1$ and $K(y_0) > 2K(y_1)$.
Then we have the following result:

Theorem 1.5 *Under the assumptions* (B_1) *and* (B_2), *if* $\mu(y_1) = 0$ *then (2) has a solution of index* k *or* $k+1$.

Now, we give a statement more general than Theorem 1.5. For $k \in \mathbb{N}^*$ we define X as

$$X = \bigcup_{y_i \in B_k} \overline{W}_s(y_i) \quad \text{with} \quad B_k \subset \{y_i \in H \ / \ i(y_i) = n-k\}.$$

We assume that X is a stratified set without boundary.
(C_1) Assume that for each z critical point of K in $X \setminus \{y_0\}$ we have $(y_0, z) \in H^+$.
For each $y_i \in B_k$, we define for λ large enough the intersection number (modulo 2)

$$\mu(y_i) = f_\lambda(C_{y_0}(X)).W_s(y_0,y_i)_\infty.$$

(C_2) Assume that $K(y_0) > 2K(y) \quad \forall y \in H \setminus \{y_0\}$.
We then have

Theorem 1.6 *Under the assumptions* (C_1) *and* (C_2), *if* $\mu(y_i) = 0$ *for each* $y_i \in B_k$, *then (2) has a solution of index* k *or* $k+1$.

Before stating another kind of existence result, we use some assumptions which were first introduced by Aubin-Bahri [1].
(D_1) Assume that

$$K(y_0) \geq K(y_1) \geq \ldots \geq K(y_\ell) > K(y_{\ell+1}) \geq \ldots \geq K(y_m)$$

with $H = \{y_0, y_1, \ldots, y_\ell\}$ (recall that $H = \left\{ y_i \in \{y_0, \ldots, y_m\} / \ -\dfrac{\Delta K(y_i)}{3K(y_i)} - 2A_{y_i} > 0 \right\}$).
Let

$$X = \bigcup_{y_i \in H} \overline{W}_s(y_i).$$

(D_2) Assume that X is not contractible and denote by k the dimension of the first nontrivial reduced homological group.
(D_3) Assume that there exists a positive constant $c < K(y_\ell)$ such that X is contractible in $K^c = \{x \in M^4 \ / \ K(x) \geq c\}$.
We then have:

Theorem 1.7 *Under the assumptions $(D_1), (D_2)$ and (D_3), there exists a constant c_0 independent of K such that if $K(y_0)/c \leq 1 + c_0$ then (2) has a solution of index $\geq k$.*

We organize the remainder of the present paper as follows. In section 2, we set up the variational structure of the problem (P), we give some preliminaries tools and we recall the characterization of the critical points at infinity of the associated variational problem in 3 and 4 dimensions. In section 3 we provide the proofs of our results

2 Variational structure and preliminaries

Problem (P) has a variational structure, the functional being

$$J(u) = \frac{\displaystyle\int_M -L_{g_0} u.u \; dv_{g_0}}{2\left(\displaystyle\int_M K(x) u^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{n-2}{n}}} \quad , u \in H^1(M^n).$$

Problem (P) is equivalent to finding the critical points of J subjected to the constraint $u \in \Sigma^+$, where:

$$\Sigma^+ = \{u \in \Sigma, u \geq 0\}, \quad \Sigma = \left\{u \in H^1(M^n), \quad |u|^2_{H^1} = \int_M -L_{g_0} u.u \; dv_{g_0} = 1\right\}.$$

The Palais-Smale condition fails to be satisfied for J on Σ^+. Its failure has been studied by various authors (see Brezis-Coron [11], Lions[22], Struwe [25]).

We introduce now a family of potential critical points at infinity. The following construction of the potential critical points at infinity was first introduced by Bahri-Brezis [4] in normal coordinates of Reimannian manifolds.

Given a point $a \in M^n$, choose a conformal metric

$$g_a = u_a^{\frac{4}{n-2}} g_0$$

such that u_a depends on a smoothly and x is a conformal normal coordinate at a. There is a uniform $\rho > 0$, independent of a, such that x is well defined on $B_{2\rho}(a)$. Let

$$\delta_{(a,\lambda)}(x) = c_n \left(\frac{\lambda}{1 + \lambda^2 |x - a|^2}\right)^{\frac{n-2}{2}} \quad , x \in B_\rho(a), \; \lambda > 0$$

where c_n is a constant such that $\delta_{(a,\lambda)}$ satisfies the equation

$$\Delta \delta_{(a,\lambda)} + \delta_{(a,\lambda)}^{\frac{n+2}{n-2}} = 0.$$

Define

$$\hat{\delta}_{(a,\lambda)}(x) = u_a(x) \omega_a(x) \delta_{(a,\lambda)}(x),$$

where $\omega_a(x)$ is a cutoff function such that

$$\omega_a(x) = 1 \text{ on } B_\rho(a), \quad \omega_a(x) = 0 \text{ on } M^n \setminus B_{2\rho}(a).$$

Define $\varphi_{(a,\lambda)}$ on M^n to be the solution of

$$-L_{g_0}\varphi_{(a,\lambda)} = \hat{\delta}_{(a,\lambda)}^{\frac{n+2}{n-2}}, \quad \varphi_{(a,\lambda)} \in C^\infty(M^n).$$

Set $H_{(a,\lambda)} = \lambda^{\frac{n-2}{2}}\left(\varphi_{(a,\lambda)} - \hat{\delta}_{(a,\lambda)}\right)$.

In order to characterize the sequences failing the Palais-Smale condition, we need to introduce the following set of potential critical points at infinity. For any $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let us define

$$V(p,\varepsilon) = \Big\{u \in \Sigma^+ / \exists a_1, ..., a_p \in M,\ \exists \lambda_1, ..., \lambda_p > \varepsilon^{-1} \text{ such that}$$

$$\left| u - \frac{1}{\lambda(u)} \sum_{i=1}^p K(a_i)^{\frac{2-n}{4}} \varphi_{(a_i,\lambda_i)} \right|_{H^1} < \varepsilon \quad \text{and } \varepsilon_{ij} < \varepsilon \ \text{ for } i \neq j \Big\}$$

where $\lambda(u) = 2J(u)$ and $\varepsilon_{ij} = \left(\lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i\lambda_j d^2(a_i,a_j)\right)^{\frac{2-n}{2}}$.

The failure of Palais-Smale condition can be described as follows

Proposition 2.1 *([6],[22],[25]) Assume that J has no critical point in Σ^+, and let $\{u_k\} \subset \Sigma^+$ be a sequence such that $\partial J \longrightarrow 0$ and $\{J(u_k)\}$ is bounded. Then there exists an integer $p \in \mathbb{N}^*$, a sequence $\varepsilon_k \longrightarrow 0$ $(\varepsilon_k > 0)$ and an extracted subsequence of $\{u_k\}$ again denoted $\{u_k\}$ such that $u_k \in V(p,\varepsilon_k)$.*

Now, we introduce a parametrization of the set $V(p,\varepsilon)$. Let $B^p_{\varepsilon,\gamma}$ be the set of $(\alpha, a, \lambda) \in \mathbb{R}^p \times (M^n)^p \times (0,\infty)^p$ such that

$\lambda_i > \varepsilon^{-1}$, $\varepsilon_{ij} < \varepsilon, \alpha_i > \gamma$ and $\alpha_i^{\frac{4}{n-2}} K(a_i)/\alpha_j^{\frac{4}{n-2}} K(a_j) > 1 - \varepsilon, \quad i \neq j \quad i,j = 1...p.$

Following [5], [22], and [25], we consider the following minimization problem for a function $u \in V(p,\varepsilon)$ with ε small:

$$\min_{(\alpha,a,\lambda)\in B^p_{\varepsilon,\gamma}} \left| u - \sum_{i=1}^p \alpha_i \varphi_{(a_i,\lambda_i)} \right|_{H^1}. \qquad (*)$$

Proposition 2.2 *For any $p \in \mathbb{N}^*$ there exists $\varepsilon_p > 0$ such that, for any $0 < \varepsilon < \varepsilon_p$, $u \in V(p,\varepsilon)$ the minimization problem $(*)$ has a unique solution $(\alpha, a, \lambda) \in B^p_{(\varepsilon,\gamma)}$ (modulo permutation). Thus, we write u as follows*

$u = \sum_{i=1}^p \alpha_i \varphi_{(a_i,\lambda_i)} + v$, where v belongs to $H^1(M^4)$ and satisfies:

$$(V_0) \left(\left(v, \varphi_{(a_i,\lambda_i)}\right)_{-L} = 0, \left(v, \frac{\partial \varphi_{(a_i,\lambda_i)}}{\partial a_i}\right)_{-L} = 0, \left(v, \frac{\partial \varphi_{(a_i,\lambda_i)}}{\partial \lambda_i}\right)_{-L} = 0\right).$$

Proof (See [2]).

The following propositions which are proved in [5] and [8] characterize the critical points at infinity of the associated variational problem. We recall that the critical points at infinity are the orbits of the gradient flow of J which remain in $V(p, \varepsilon(s))$, where $\varepsilon(s)$ is a given function such that $\varepsilon(s)$ goes to zero when s goes to $+\infty$ (see [2] and [3]).

Proposition 2.3 (5) *Let $M^n = S^3$. Assume that J has no critical point in Σ^+, then the only critical points at infinity for J are $\delta(y_i, \infty)$ such that $y_i \in I$. For each $y_i \in I$, we have the following:*

$$J(\delta_{(y_i,\infty)}) = S^{2/3} K(y_i)^{-1/3}, \text{ where } S = \int_{S^3} \delta^6 dv.$$

Moreover, the Morse index of the critical point at infinity $\delta_{(y_i,\infty)}$ is given by $i(y_i)_\infty = 3 - i(y_i)$.

Before giving the analogous proposition in dimension 4, we need to recall the following lemmas.

Lemma 2.1 *Let $n = 4$. Then, there exists a constant $c = c(\rho) > 0$ such that*

$$\left| H_{(a,\lambda)}(x) \right|_{L^\infty} \le c, \quad \left| \lambda \frac{\partial H_{(a,\lambda)}(x)}{\partial \lambda} \right|_{L^\infty} \le c \quad \left| \lambda^{-1} \frac{\partial H_{(a,\lambda)}(x)}{\partial a} \right|_{L^\infty} \le c$$

when λ is large.

Lemma 2.2 *Let $n = 4$. Then, for ρ small enough, and λ large, we have*

$$\begin{array}{lll} H_{(a,\lambda)}(a) & \longrightarrow A_a & \text{as } \lambda \longrightarrow \infty \\ H_{(a,\lambda)}(x) & \longrightarrow G(a,x) & \text{outside } B_{3\rho}(a), \text{ as } \lambda \longrightarrow \infty. \end{array}$$

Let σ_p be the permutation group of $\{1, ..., p\}$. Let

$$\begin{array}{ll} \Psi_p : V(p, \varepsilon) & \longrightarrow M^p/\sigma_p \\ u = \sum_{i=1}^p \alpha_i \varphi_{(a_i,\lambda_i)} + v & \longmapsto (a_1, ..., a_p). \end{array}$$

Proposition 2.4 is proven in [8].

Proposition 2.4 *(Deformation lemma) Let $n = 4$. Then for any $\delta > 0$ and $\varepsilon > 0$ small enough there exists a pseudogradient vector field $(-Z_p^\delta)$. Denoting by $\eta(s, .)$ the 1-parameter group generated by $(-Z_p^\delta)$, we have*

$$\Psi_p(\eta(1, V(p, \varepsilon)) \subset \delta - \text{neighborhood of } (y_{i_1}, ..., y_{i_p})/\sigma_p,$$

where $(y_{i_1}, ..., y_{i_p}) \in H^+$ ($(y_{i_1}, ..., y_{i_p})$ is called a critical point at infinity.)

Now, we are able to prove our results.

3 Proofs of theorems

Proof of Theorem 1.1. Let Z be a pseudogradient of K of Morse-Smale type. Set

$$X = \overline{W}_s(y_1) = W_s(y_1) \cup W_s(y_0)$$

where $W_s(y_i)$ is the stable manifold of y_i for Z. Then X is a compact manifold in dimension k without boundary. Arguing by contradiction, we suppose that J has no critical points. It follows from Proposition 2.3 that under the assumption of theorem 1.1, J has two critical points at infinity under the level $c_1 = S^{2/3}(K(y_0)^{-1/2} + K(y_1)^{-1/2})^{2/3} + \varepsilon$ for ε small enough, which correspond to $\delta_{(y_0,\infty)}$ and $\delta_{(y_1,\infty)}$. The unstable manifold at infinity of such critical points at infinity, $W_u(y_i)_\infty$, $i = 0, 1$, can be described, using lemma 10 of [5] as the product of $W_s(y_i)$, $i = 0, 1$, (for a pseudogradient of K) by $[A, +\infty[$ domain of the variable λ, for some positive number A large enough. Since J has no critical point in Σ^+, it follows that $J_{c_1} = \{u \in \Sigma^+ \ / \ J(u) \le c_1\}$ retracts by deformation on $X_\infty = W_u(y_1)_\infty \cup W_u(y_0)_\infty$ (see sections 7 and 8 of [7]) which can be parameterized by $X \times [A, +\infty[$.

On the other hand, X_∞ is contractible in J_{c_1}. Indeed, let

$$\begin{aligned} R : H^1(S^3) \setminus \{0\} &\longrightarrow \Sigma^+ \\ u &\longmapsto \frac{u}{|u|_{-L}} \end{aligned}$$

and let

$$\begin{aligned} h : [0,1] \times X_\infty &\longrightarrow \Sigma^+ \\ (t, x, \lambda) &\longmapsto R(t\delta_{(y_0,\lambda)} + (1-t)\delta_{(x,\lambda)}). \end{aligned}$$

For $t = 0$, $\quad h(0, x, \lambda) = \dfrac{1}{S}\delta_{(x,\lambda)} \in X_\infty$, h is continuous and $h(1, x, \lambda) = \dfrac{1}{S}\delta_{(y_0,\lambda)}$. Furthermore, we have

$$J(R(t\delta_{(y_0,\lambda)} + (1-t)\delta_{(x,\lambda)})) \le \left(S(\frac{1}{K(y_0)^{1/2}} + \frac{1}{K(x)^{1/2}})\right)^{2/3} (1 + o(1))$$

(see the proof of Corollary B.3 of [6], see also [3]).
Since $K(x) \ge K(y_1)$ for each $x \in X$, we then have

$$J(R(t\delta_{(y_0,\lambda)} + (1-t)\delta_{(x,\lambda)})) < c_1.$$

Thus, the contraction h is performed under the level c_1. Therefore X_∞ is contractible leading to the contractibility of X. This yields a contradiction, since X is a manifold in dimension k without boundary. Hence (1) has a solution.

It remains to compute the Morse index of the solution. Using dimension argument, since $h([0,1] \times X_\infty)$ is a manifold in dimension $(k+1)$ then the Morse index of the solution provided by theorem 1.1 is $\le k+1$. On the other hand, assume that the Morse index of the solution is $\le k - 1$. Perturbing J,if necessary, we may assume that all the critical points of J are nondegenerate and have their Morse index $\le k - 1$. Such critical points do not change the homological group in dimension k

of the level sets of J. Now, X_∞ defines a homological class in dimension k which is nontrivial in $J_{C_\infty(y_1)+\varepsilon}$ for ε small enough where $C_\infty(y_1) = S^{2/3}K(y_1)^{-1/3}$. However, X_∞ defines a homological class in dimension k, which is trivial in J_{c_1}. Hence our theorem follows.

Proof of Remark 2. If we change the assumption (G_1) by the assumption (G_1'), under the level c_1, we can find other critical points at infinity but of index $\notin \{k, k+1\}$. Using the same arguments as those used above, Remark 2 follows.

Proof of Theorem 1.2. Let $X_\infty = \cup_{y_i \in B_k} \overline{W_u(y_i)_\infty}$. Here $W_u(y_i)_\infty$ is the unstable manifold at infinity of the critical point at infinity $\delta_{(y_i,\infty)}$, for a decreasing pseudogradient V for J (see [3] and [5]). The unstable manifold at infinity $W_u(y)_\infty$, for each $y \in I$, can be described using Lemma 10 of [5] as the product of $W_s(y)$ (for a pseudogradient of K) by the interval $[A, \infty)$, domain of the variable λ for some positive number A large enough. Thus X_∞ can be parameterized by $X \times [A, \infty)$.

We argue by contradiction. We suppose that J has no critical point in Σ^+. It follows from Proposition 7.24 and Theorem 8.2 of [7] that Σ^+ retracts by deformation on $\cup_{y_i \in I} W_u(y_i)_\infty$ (recall by $I = \{y / \nabla K(y) = 0 \text{ and } -\Delta K(y) > 0\}$). More precisely, Σ^+ retracts by deformation on $X_\infty \cup D_\infty$, where $D_\infty = \cup_{y_i \in D} W_u(y_i)_\infty$, with $D = \{y \in I / i(y) > 3 - k\}$. For each $y \in D$, the critical point at infinity $\delta_{(y,\infty)}$ has a Morse index $\leq k-1$. Thus, D_∞ is a stratified set of dimension at most $k-1$. Since Σ^+ is contractible set then $H_*(X_\infty \cup D_\infty) = 0$ for each $* \in \mathbb{N}^*$.
Using the exact homology sequence of $(X_\infty \cup D_\infty, X_\infty)$, we have :

$$\ldots \longrightarrow H_{k+1}(X_\infty \cup D_\infty) \xrightarrow{\pi} H_{k+1}(X_\infty \cup D_\infty, X_\infty) \xrightarrow{\partial} H_k(X_\infty) \xrightarrow{i} H_k(X_\infty \cup D_\infty) \longrightarrow \ldots$$

Since $H_*(X_\infty \cup D_\infty) = 0$ for all $* \in \mathbb{N}^*$ then $H_k(X_\infty) \equiv H_{k+1}(X_\infty \cup D_\infty, X_\infty)$. In addition, $(X_\infty \cup D_\infty, X_\infty)$ is stratified set of dimension at most k, so $H_{k+1}(X_\infty \cup D_\infty, X_\infty) = 0$. Thus $H_k(X_\infty) = 0$ and therefore $H_k(X) = 0$ (since $X_\infty \equiv X \times [A, \infty)$), which is in contradiction to the assumption (H_3). Hence (1) has a solution.

Now using the same arguments as those used in the proof of Theorem 1.1, we deduce that the Morse index of the solution provided in Theorem 1.2 is equal to k or $k+1$. This concludes the proof of our result.

Proof of Theorem 1.3. Arguing by contradiction, we suppose that J has no critical points. Let $c_1 = (S_4/K(y_0))^{1/2} + \varepsilon$, ε is a positive constant small enough such that for every critical point y of K, $y \neq y_0$, we have $c_1 < c_\infty(y) = (S_4/K(y))^{1/2}$. Let $u_0 \in J_{c_1} = \{u \in \Sigma^+ / J(u) \leq c_1\}$. We denote by $\eta(s, u_0)$ the one parameter group generated by the pseudogradient W of the Morse Lemma at infinity defined in [8]. It follows that, under the assumption of the Theorem λ_{max} is bounded along the flowlines (see [8]). Thus we derive $|\partial J(\eta(s, u_0)).W(\eta(s, u_0))| \geq \beta > 0 \ \forall s \geq 0$ (β depends only on u_0). Therefore $J(\eta(s, u_0))$ goes to $-\infty$ when s goes to $+\infty$, which is a contradiction. Hence our theorem follows.

Before proving Theorems 4,...,7, we state the following lemma. Its proof is very similar to the proof of Corollary B.3 of [6], see also [3].

Lemma 3.1 *Let $a_1, a_2 \in M^4, \alpha_1, \alpha_2 > 0$ and λ large enough. For $u = \alpha_1\varphi_{(a_1,\lambda)} + \alpha_2\varphi_{(a_2,\lambda)}$, we have*

$$J\left(\frac{u}{|u|_{H^1}}\right) \leq \left(S_4(\frac{1}{K(a_1)} + \frac{1}{K(a_2)})\right)^{1/2} (1 + o(1)).$$

Proof of Theorem 1.4. We argue by contradiction. We suppose that J has no critical point. Observe that, under the assumption of Theorem 1.4, (y_0, y_1) is not critical point at infinity. It follows from Proposition 2.4 that under the assumptions (A_1) and (A_2) of the theorem, the only critical points at infinity of J under the level $c_1 = S_4^{1/2}\left(K(y_0)^{-1} + K(y_1)^{-1}\right)^{1/2} + \varepsilon$, for ε small enough, are $(y_1)_\infty$ and $(y_0)_\infty$ (notation of [3]). Let

$$X_\infty = W_u(y_1)_\infty \cup W_u(y_0)_\infty,$$

where $W_u(y_i)_\infty$ is the unstable manifold at infinity of the critical points at infinity, $(y_i)_\infty$, for a decreasing pseudogradient V for J (see [8]). The unstable manifold at infinity $W(y_i)_\infty$ can be described using Lemma 4.2 of [8] as the product of $W_s(y_i)$ (for a pseudogradient of K) by $[A, \infty)$, domain of the variable λ, for some positive number A large enough. Thus X_∞ can be parameterized by $X \times [A, \infty)$ where

$$X = \overline{W}_s(y_1) = W_s(y_1) \cup W_s(y_0).$$

Since J has no critical point in Σ^+, it follows from Proposition 7.24 and Theorem 8.2 of [7] that J_{c_1} retracts by deformation on X_∞. On the other hand, we define by $C_{y_0}(X)$ the following set:

$$C_{y_0}(X) = \{\alpha\delta_{y_0} + (1-\alpha)\delta_x \;/\; \alpha \in [0,1], \quad x \in X\},$$

where δ_x is the Dirac measure at x. For λ large enough, we define

$$\begin{aligned} f_\lambda \;: C_{y_0}(X) \quad &\longrightarrow \Sigma^+ \\ \alpha\delta_{y_0} + (1-\alpha)\delta_x \quad &\longmapsto \frac{\alpha\varphi_{(y_0,\lambda)} + (1-\alpha)\varphi_{(x,\lambda)}}{\left|\alpha\varphi_{(y_0,\lambda)} + (1-\alpha)\varphi_{(x,\lambda)}\right|_{H^1}}. \end{aligned}$$

Then $C_{y_0}(X)$ and $f_\lambda(C_{y_0}(X))$ are a contractible manifolds in dimension $k+1$. Since $K(x) \geq K(y_1)$ for each $x \in X$, it follows from lemma 3.1 that $J\left(f_\lambda(C_{y_0}(X))\right) < c_1$. Thus, X_∞ is contractible in J_{c_1}, which retracts by deformation on X_∞. Therefore X_∞ is contractible, leading to the contractibility of X. This yields a contradiction since X is a manifold in dimension k without boundary. The proof of Theorem 1.4 is thereby completed.

Proof of Theorem 1.5. Assume that (2) has no solution. Let

$$u = \alpha\varphi_{(x,\lambda)} + (1-\alpha)\varphi_{(y_0,\lambda)} \in f_\lambda(C_{y_0}(X)).$$

The action of the flow of the pseudogradient (See the pseudogradiant W of [3], see also [9]) is essentially on α.

If $\alpha < 1/2$, the flow brings α to zero and thus u goes in this case to $\overline{W}_u(y_0)_\infty \equiv \{y_0\}$. If $\alpha > 1/2$, the flow brings α to 1 and thus u goes in this case to $\overline{W}_u(y_1)_\infty \equiv X_\infty$. If $\alpha = (1-\alpha) = 1/2$, observe that we then have $u = \frac{1}{2}\varphi_{(y_0,\lambda)} + \frac{1}{2}\varphi_{(x,\lambda)}$. Since only x can move, then y_0 remains one of the points of concentration of u, and x goes to $W_s(y_i)$, where $y_i = y_1$ or $y_i = y_0$, and two cases may occur:

- In the first case $y_i = y_1$, then u goes to $W_u(y_0, y_1)_\infty$.
- In the second case $y_i = y_0$, in this case there is $s_0 \geq 0$ such that $\forall s \geq s_0$,

$u(s) = \frac{1}{2}\varphi_{(y_0,\lambda)} + \frac{1}{2}\varphi_{(x(s),\lambda)}$ with $x(s) \in N_\rho(y_0)$ for ρ small enough. Thus $J(u(s)) \leq C_\infty(y_0, y_0) + \delta = (2S_4/K(y_0))^{1/2} + \delta$, for δ small enough (see lemma 3.1). Using the vector field $(-J')$ under the level $C_\infty(y_0, y_0) + \delta$, it follows from Proposition 2.4 that under the assumption (B_2) of Theorem 1.5, $J_{C_\infty(y_0,y_0)+\delta}$ retracts by deformation on $W_u(y_0)_\infty = \{y_0\}$, and thus u goes to $W_u(y_0)_\infty$. Therefore $f_\lambda(C_{y_0}(X))$ retracts by deformation on $X_\infty \cup W_u(y_0, y_1)_\infty$. Since $\mu(y_1) = 0$, we can be more precise. This strong retract does not intersect $W_u(y_0, y_1)_\infty$ and thus it is contained in X_∞. Therefore X_∞ is contractible, and it follows that X is contractible. This yields a contradiction since X is a manifold in dimension k without boundary. Then our theorem follows.

Proof of Theorem 1.6. We argue by contradiction. Assume that (2) has no solution. Using the same arguments as those used in the proof of Theorem 1.5 it follows that, $f_\lambda(C_{y_0}(X))$ retracts by deformation on $X \bigcup \left(\bigcup_{y_i \in B_k} W_u(y_0, y_i) \right) \bigcup D$, where $D \subset \sigma$ is a stratified set and $\sigma = \bigcup_{y_i \in X \setminus (B_k \cup \{y_0\})} W_u(y_0, y_i)$, the dimension of σ is at most k.

Since $\mu(y_i) = 0$ for each $y_i \in B_k$, $f_\lambda(C_{y_0}(X))$ retracts by deformation on $X \cup D$. Therefore $H_*(X \cup D) = 0$ for all $* \in \mathbb{N}^*$, since $f_\lambda(C_{y_0}(X))$ is a contractible set. Using now the same arguments as those used in the proof of Theorem 1.2, Theorem 1.6 follows.

Proof of Theorem 1.7. Our proof follow the algebraic topological arguments introduced in [1]. Arguing by contradiction, we suppose that J has no critical points. It follows from Proposition 2.4 that under the assumptions of Theorem 1.7, the critical points at infinity of J under the level $c_1 = (S_4/k(y_\ell))^{1/2} + \varepsilon$, for ε small enough,are in one to one correspondence with the critical points of K y_i such that $y_i \in H$. Since J has no critical point, it follows that $J_{c_1} = \{u \in \Sigma^+ / J(u) \leq c_1\}$ retracts by deformation on $X_\infty = \bigcup_{y_i \in H} W_u(y_i)_\infty$ (see section 7 and 8 of [7]) which can be parametrized by $X \times [A, +\infty[$ where $X = \bigcup_{y_i \in H} W_s(y_i)$, (for a pseudogradient of K).

From another part, we have X_∞ is contractible in $J_{c_2+\varepsilon}$ where $c_2 = (S_4/c)^{1/2}$.

Indeed from (D_3), it follows that there exists a contraction

$$h : [0,1] \times X \longrightarrow K^c$$

h continuous, such that for any $a \in X$, $h(0,a) = a$ and $h(1,a) = a_0$ a point of X. Such contraction gives rise to the following contraction:

$$\begin{array}{lll}\tilde{h} : & [0,1] \times X & \longrightarrow \Sigma^+ \\ (t,a,\lambda) & & \longmapsto \varphi_{(h(t,a),\lambda)} + \bar{v}.\end{array}$$

For $t = 0$, $\tilde{h}(0,a,\lambda) + \bar{v} = \varphi(a,\lambda) + \bar{v} \in X_\infty$, $\tilde{h}$ is continuous and $\tilde{h}(1,a,\lambda) = \varphi(a_0,\lambda) + \bar{v}$. Hence our claim follows.
Now using proposition 2.4 of [8], we deduce that

$$J(\varphi_{(h(t,a),\lambda)} + \bar{v}) \sim \left[\frac{S_4}{K(h(t,a))}\right]^{1/2} (1 + O(\frac{1}{A^2}))$$

where $K(h(t,a)) \geq c$ by construction. Therefore such a contraction is performed under $c_2 + \varepsilon$ for A large enough, so X_∞ is contractible in $J_{c_2+\varepsilon}$. In addition, choosing c_0 small enough, $J_{c_2+\varepsilon}$ retracts by deformation on J_{c_1}, which retracts by deformation on X_∞. Therefore X_∞ is contractible leading to the contractibility of X, which contradicts the assumption (D_2). This concludes the proof of our result.

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