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A Priori Estimates for Solutions of "Sub-critical" Equations on CR Sphere

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Abstract

Here we study the precise blow-up behaviour and obtain a priori estimates for the *finite energy* C^2-solutions of the equation

$$\begin{aligned} \Delta_b u - n(2n+1)u + \frac{2(2n+1)}{n+1} K u^p &= 0 \\ u &> 0 \end{aligned}$$

on the odd dimensional spheres S^{2n+1} with standard CR structure, as the exponent $p \nearrow \frac{Q+2}{Q-2}$ for $p \in (1, \frac{Q+2}{Q-2}]$, $Q = 2n+2$ is the homogeneous dimension.
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1 Introduction

We consider the odd dimensional spheres $S^{2n+1} = \{\zeta \in C^{n+1} : |\zeta| = 1\}$ with the standard CR structure defined by contact form

$$\theta_1 = i(\overline{\partial} - \partial)|\zeta|^2 = i\sum_{j=1}^{n+1}(\zeta^j d\overline{\zeta^j} - \overline{\zeta^j} d\zeta^j)$$

corresponding to the *horizontal space* $T_{1,0} = \text{span}\{\frac{\partial}{\partial\zeta^1}, \dots, \frac{\partial}{\partial\zeta^{n+1}}\}$. The Levi form given by $L_{\theta_1}(V, \overline{W}) = -2id\theta_1(V \wedge \overline{W})$ is a positive definite hermitian form on $T_{1,0}$. The *sublaplacian* operator Δ_b is defined on real functions $u \in C^\infty(S^{2n+1})$ by

$$\int_{S^{2n+1}} \Delta_b u\, v\theta_1 \wedge d\theta_1^n = \int_{S^{2n+1}} L^*_{\theta_1}(du, dv)\, \theta_1 \wedge d\theta_1^n \text{ for all } v \in C_0^\infty(S^{2n+1}). \tag{1.1}$$

In this paper, we shall study the precise blow-up behaviour and obtain a priori estimates for the *finite energy* C^2-solutions of the equation

$$\left.\begin{array}{rcl} \Delta_b u - n(2n+1)u + \frac{2(2n+1)}{n+1}Ku^p & = & 0 \text{ in } S^{2n+1} \\ u & > & 0 \end{array}\right\} \tag{1.2}$$

as $p \nearrow \frac{Q+2}{Q-2}$ for $p \in (1, \frac{Q+2}{Q-2}]$, $Q = 2n+2$ is the homogeneous dimension. Here, the energy of a solution u of (1.2)is defined by

$$\mathcal{E}(u) = \int_{S^{2n+1}} \{L^*_{\theta_1}(du, du) + n(2n+1)u^2\}\, \theta_1 \wedge d\theta_1^n. \tag{1.3}$$

Observe that for $p = \frac{Q+2}{Q-2}$, the transformation laws (equations (3.1)-(3.2) in [9]) imply that a solution u of the equation (1.2) gives a contact form $\theta = u^{2/n}\theta_1$ which is "conformal" to the contact form θ_1 and which has Webster scalar curvature equal to the function $K(\xi)$. The form θ_1 and its images under CR automorphisms of sphere(which are induced by biholomorphisms of the unit ball in C^{n+1}) have constant pseudohermition scalar curvature $n(n+1)/2$.

As in the Riemannian case, one can consider the problem of finding suitable function K such that it defines a contact form conformal to the given θ_1 on the sphere. Obtaining a priori estimates for solutions of (1.2) is a first step towards understanding this problem on CR spheres.

For S^n $(n \geq 3)$ with the standard Riemannian metric g, the corresponding equation is

$$\left.\begin{array}{rcl} \Delta u - \frac{n(n-2)}{4}u + \frac{(n-2)}{4(n-1)}Ku^p & = & 0 \text{ in } S^n \\ u & > & 0 \end{array}\right\} \tag{1.4}$$

where $p \in (1, \frac{n+2}{n-2}]$. Here for $p = \frac{n+2}{n-2}$, $u^{4/(n-2)}g$ gives a conformal metric with scalar curvature equal to K. The equation (1.4) has been studied and a priori estimates have been obtained by Schoen and Zhang [12], Li [13](also see references therein), recently by Chen and Lin [2].....

In this paper, we essentially follow the techniques used by Li in [13], with suitable modifications and interpretations to obtain similar results for the CR spheres too. The main difference is that unlike as in Riemmanian case, we need to assume that the solutions have finite energy. Observe that when we study the blow up analysis of the equation (1.2), the limiting function is a solution of an equation of the type

$$\left.\begin{array}{rcl} \Delta_{\mathrm{H}^{2n+1}} u + C u^{(Q+2)/(Q-2)} & = & 0 \\ u & > & 0 \end{array}\right\} \tag{1.5}$$

on the Heisenberg group H^{2n+1}, where C is a constant. The classification of all solutions to (1.5) on H^{2n+1} is still an open problem. However, if we further assume in (1.5) that $u \in L^{2Q/(Q-2)}$, then the result of Jerison and Lee ([10]) gives complete classification of u. The finite energy assumption ensures that the limiting function lies in the right space so that Jerison and Lee's result is applicable.

Furthermore, the curvature functions are assumed to be in the right nonisotropic Lipschitz space so as to be able to apply the subelliptic regularity results. These are still in C^1, but need not be in C^2.

In the next section, we give a definition of a simple blow up point which is still consistent with one given by Schoen and works in CR case too; in fact it is general enough to work in the case of nilpotent, stratified groups(which includes $\mathbb{R}^n$). The definitions and statements of main result are also given in the next section. The third section contains the necessary information about Heisenberg group. In fourth and fifth sections, various estimates needed for the proof of the theorem are derived. The sixth section deals with some local results near an isolated blow up point while the last section contains the proof of the main theorem.

2 Main results

We consider the Heisenberg group H^{2n+1} as the set $C^n \times \mathbb{R}$ with coordinates (z,t) endowed with the group action $\circ$ defined by

$$(z_0, t_0) \circ (z, t) = (z_0 + z, t_0 + t + 2Im\sum_{i=1}^{n} \overline{z_0}^i \overline{z}^i). \tag{2.1}$$

Using the complex notations, the CR structure of H^{2n+1} is given by the horizontal bundle spanned by the vector fields

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + i\overline{z}^\alpha \frac{\partial}{\partial t} \qquad \alpha = 1, \ldots, n.$$

The standard left invariant contact form on H^{2n+1} is defined as

$$\theta_0 = dt + \sum (iz^\alpha d\overline{z}^\alpha - i\overline{z}^\alpha dz^\alpha).$$

The CR equivalence between the sphere S^{2n+1} minus a point and the Heisenberg group H^{2n+1} is given by the Cayley transform $F : S^{2n+1} \to \mathrm{H}^{2n+1} = C^n \times \mathbb{R}$, defined by

$$F(\zeta_1,\ldots,\zeta_{n+1}) = \left(\frac{\zeta_1}{\zeta_{n+1}},\ldots,\frac{\zeta_n}{\zeta_{n+1}},\frac{1-|\zeta_{n+1}|^2}{|1+\zeta_{n+1}|^2}\right).$$

Also, the contact form on the sphere θ_1 is pull back of the standard contact form θ_0 i.e., $\theta_1 = F^*(\frac{4}{|i+\omega|^2}\theta_0)$. Let u be a nonnegative function on H^{2n+1}. Then for

$$v(\zeta) = |1+\zeta_{n+1}|^{-1} u \circ F(\zeta)$$

we have the relations

$$\int_{S^{2n+1}} (b_n|dv|^2_{\theta_1} + R_n v^2)\,\theta_1 \wedge d\theta_1^n = \int_{\mathrm{H}^{2n+1}} \left(b_n \sum_{\alpha=1}^{n} |Z_\alpha u|^2\right) \theta_0 \wedge d\theta_0^n, \quad (2.2)$$

$$\int_{S^{2n+1}} v^p \theta_1 \wedge d\theta_1^n = \int_{\mathrm{H}^{2n+1}} u^p \theta_0 \wedge d\theta_0^n \quad (2.3)$$

where $b_n = (n+1)/2(2n+1)$ and $R_n = n(n+1)/2$ is the scalar curvature associated to θ_1. Define

$$\Lambda_0(z,t) = |i+\omega|^{-(Q-2)/2} \quad (2.4)$$

where $\omega = t + i|z|^2$, $z \in C^n$, $t \in \mathbb{R}$. It can be easily verified that the function Λ_0 satisfies the equation

$$\Delta_{\mathrm{H}^{2n+1}}\Lambda_0 + 4n^2\Lambda_0^{Q+2/Q-2} = 0 \quad \text{on} \quad \mathrm{H}^{2n+1}. \quad (2.5)$$

Infact, Jerison and Lee showed in [10] that every $L^{2Q/(Q-2)}$-solution of (2.5) is obtained from Λ_0 by left translations and dilations $(z,t) \to (\lambda z, \lambda^2 t)$ on the Heisenberg group.

As in [9], the problem (1.2) on S^{2n+1} can be reduced to one on Heisenberg group H^{2n+1} as follows: if u satisfies the equation

$$\frac{n+1}{2(2n+1)}\Delta_b u - \frac{n(n+1)}{2}u + Ku^{Q+2/Q-2} = 0 \quad \text{on} \quad \mathrm{S}^{2n+1}$$

then $v(\xi) = \Lambda_0(\xi)u(F^{-1}\xi)$ satisfies the equation

$$\frac{n+1}{2(2n+1)}\Delta_{\mathrm{H}^{2n+1}}v + K(F^{-1}\xi)v^{Q+2/Q-2} = 0 \quad \text{on} \quad \mathrm{H}^{2n+1}.$$

Thus we need to study similar equations on the Heisenberg group H^{2n+1}. Therefore, let $\Omega \subset \mathrm{H}^{2n+1}$, $(n \geq 1)$ be a bounded domain, $\tau_i \geq 0$ satisfy $\lim_{i\to\infty}\tau_i = 0$, $p_i = \frac{Q+2}{Q-2} - \tau_i$, and $\{K_i\} \in \Gamma_{2+\alpha}(\Omega)$, $0 < \alpha < 1$ satisfy

$$1/A_1 \leq K_i(\xi) \leq A_1 \quad \text{for all} \quad \xi \in \Omega \quad (2.6)$$

for some constant $A_1 > 0$. Here $\Gamma_{2+\alpha}(\Omega)$ denotes the set of functions having horizontal derivatives up to order 2 in the nonisotropic Lipschitz space $\Gamma_\alpha(\Omega)$; see preliminaries for the precise definition. Note that $K_i \in \Gamma_{2+\alpha}(\Omega)$ implies that $K_i \in C^1(\Omega)$.

We consider solutions of equations

$$\left.\begin{array}{rcl} -\Delta_{\mathbb{H}^{2n+1}} u_i & = & c(n)K_i(\xi)u_i^{p_i} \quad \text{in} \quad \Omega \\ u_i & > & 0. \end{array}\right\} \tag{2.7}$$

If $\{\max u_i\}$ remains bounded as $i \to \infty$, then it follows from the subelliptic estimates that a subsequence of $\{u_i\}$ converges to some function u in $C^1_{loc}(\Omega)$(see Claim 5.3 in Section 5). Otherwise, $\{u_i\}$ is said to *blow up*. The following notion of isolated blow up point, introduced in [12] and [13] in the context of Riemannian manifold can also be used in the CR manifolds:

Definition 2.1 Suppose that $\{K_i\}$ satisfies (2.6) and $\{u_i\}$ satisfies (2.7). A point $\tilde{\xi} \in \Omega$ is called *an isolated blow up point* of $\{u_i\}$ if there exists $0 < \sigma < dist(\xi, \partial\Omega)$, $\overline{C} > 0$ and a sequence $\xi_i \to \tilde{\xi}$ such that ξ_i is a local maximum of u_i, $u_i(\xi_i) \to \infty$ and

$$u_i(\xi) \leq \overline{C} d(\xi, \xi_i)^{-2/(p_i-1)} \quad for\ all \quad \xi \in B(\xi_i, \sigma).$$

It will be clear from Remark 5.4 in section 5 that the points ξ_i are uniquely determined for large i.

Intuitively, according to Schoen [12], a simple blow up point on a sphere S^n is a point where the solution of (1.4) approximates the "standard solution" upto a conformal transformation, in a neighbourhood. This definition was further reformulated by Li in [13] using spherical averages. However, his definition does not seem to work for the Heisenberg group. We observe that one of the reasons is that the "standard solutions" in the case of CR sphere S^{2n+1} are "*not radial*". These standard solutions are pull back of Λ_0 defined in (2.4)(upto Heisenberg translation and dilations), via the Cayley transform. Hence we proceed as follows:

For any positive solution u_i of (2.7) and θ in H^{2n+1} with $dist(\theta, 0) = 1$, we define the function $f_{u_i,\theta}(s) : [0, R] \to \mathbb{R}$(for a fixed $R > 0$) as

$$f_{u_i,\theta}(s) = s^{2/(p_i-1)} u_i(\xi_i \circ s\theta). \tag{2.8}$$

Here $\circ$ is the group action whereas $s\theta = s(x, y, t) = (sx, sy, s^2 t)$ is the dilation in H^{2n+1}. We will use the notation $f_{i,\theta}$ to denote this function whenever the corresponding function involved is clear.

Definition 2.2 We say that $\tilde{\xi} \in \Omega$ is an isolated *simple blow up point,* if $\tilde{\xi}$ is an isolated blow up point and there exists some $\sigma > 0$(independent of i and $\theta \in \partial B(0, 1)$) such that $f_{i,\theta}$ has precisely one critical point in $(0, \sigma)$ for *every* $\theta \in \partial B(0, 1)$, for large i.

We observe that our definition(when considered on $\mathbb{R}^n$) does not imply the one given by Li in [13] and vice versa; however it will be clear from the following sections that one can obtain the a priori estimates of [13] using Definition 2.2. Also, it can be seen that Definition 2.2 gives a notion of isolated simple blow up points on the more general nilpotent, stratified Lie groups.

For the Riemannian sphere, it was proved in [13] that the isolated blow up point has to be a critical point of the function $K(\xi) = \lim_{i\to\infty} K_i(\xi)$. We shall prove the same for CR sphere in Proposition 6.1. Hence, it is interesting to see how the flatness properties of K_i's and K affect the blow up behaviour of $\{u_i\}$. Also, this forces us to put conditions on the (usual) gradient ∇K_i instead of the Heisenberg gradient $\nabla_{\mathbb{H}^{2n+1}} K_i$. Thus, the following condition is similar to the one introduced in [13]:

Definition 2.3 For any real number $\beta \geq 1$, we say that a sequence of functions $\{K_i\}$ satisfies condition $(*)_\beta$ for some sequences of constants $\{L_1(\beta, i)\}$ and $\{L_2(\beta, i)\}$ in some region Ω_i if $\{K_i\} \in C^{[\beta]-1,1}(\Omega_i)$ satisfies

$$||\nabla K_i||_{C^0(\Omega_i)} \leq L_1(\beta, i)$$

and if $\beta \geq 2$, that

$$|\nabla^s K_i(\xi)| \leq L_2(\beta, i)|\nabla K_i(\xi)|^{(\beta-s)/(\beta-1)}$$

for all $2 \leq s \leq [\beta]$, $\xi \in \Omega_i$, $\nabla K_i(\xi) \neq 0$.

Please refer to [13] for examples and discussions of such functions K_i. We also refer to [3] for simpler conditions on K_i.

We now state the main theorem.

Theorem 2.1 *Suppose that $\{K_i\} \in \Gamma_{2+\alpha}(S^{2n+1})$, $0 < \alpha < 1$ and $n \geq 1$ with uniform C^1 modulo of continuity and satisfies, for some positive constant A_1, that*

$$K_i(\xi) \geq 1/A_1, \quad \text{for all} \quad \xi \in S^{2n+1}.$$

For a constant $d > 0$, let

$$\Omega_{d,i} = \{q \in S^{2n+1} : \ |\nabla K_i(q)| < d\}$$

and suppose there exists some constant $d > 0$, such that $\{K_i\}$ satisfies $(*)_\beta$, $\beta \geq (Q-2)$ for some constants $L_1(\beta)$ and $L_2(\beta)$ independent of i in $\Omega_{d,i}$. Consider solutions u_i of (1.2), such that

$$\mathcal{E}(u_i) \leq C_0, \tag{2.9}$$

for some constant C_0 (independent of i).

Then, after passing to a subsequence, either

(i) $\{u_i\}$ stays bounded in $L^\infty(S^{2n+1})$ and hence in $C^{2,\alpha}(S^{2n+1})$;

or

(ii) $\{u_i\}$ has finitely many isolated simple blow-up points and the distance between any two blow-up points is bounded below by some positive constant depending only on n, A_1, L_1, L_2, d and the uniform modulo of continuity of ∇K_i.

In fact, if $\beta > Q-2$ or $\{K_i\}$ satisfies $(*)_{(Q-2)}$ for some sequences $L_1(i), L_2(i) = \circ(1)$ in $\Omega_{d,i}$, then $\{u_i\}$ has precisely **one** *isolated simple blow up point.*

Here C^1 is the usual space of functions which have continuous first order derivatives, whereas $\Gamma_{2+\alpha}$ and S_1^2 are nonisotropic Sobolev spaces which are defined in preliminaries below.

3 Preliminaries

To denote the elements of H^{2n+1} we shall either use the notation $(z,t) \in C^n \times \mathbb{R}$ or $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ where $z = x+iy, x = (x_1,...,x_n), y = (y_1,...,y_n)$. The Heisenberg group H^{2n+1} is the space $\mathbb{R}^{2n+1}$ (or $C^n \times \mathbb{R}$) endowed with the group action $\circ$ defined by

$$\xi_0 \circ \xi = (x + x_0, y + y_0, t + t_0 + 2\sum_{i=1}^{n}(x_i y_{0_i} - y_i x_{0_i})). \tag{3.1}$$

We denote by δ_λ the parabolic dilation in H^{2n+1} where

$$\delta_\lambda(\xi) = \left(\lambda x, \lambda y, \lambda^2 t\right), \tag{3.2}$$

which satisfies $\delta_\lambda(\xi_o \circ \xi) = \delta_\lambda(\xi_o) \circ \delta_\lambda(\xi)$. The norm in H^{2n+1} is given by

$$d(\xi,0) = \| \xi \| := \left[\left(\sum_{i=1}^{n}(x_i^2 + y_i^2)\right)^2 + t^2\right]^{\frac{1}{4}} \equiv |t + i|z|^2|^{\frac{1}{2}}. \tag{3.3}$$

It is homogeneous of degree one with respect to the dilation δ_λ defined above. The associated distance between two points $\xi, \eta \in H^{2n+1}$ is defined accordingly by

$$d(\xi,\eta) = \| \eta^{-1} \circ \xi \| \tag{3.4}$$

where η^{-1} denotes the inverse of η with respect to $\circ$; in fact, it can be seen that $\eta^{-1} = -\eta$. We shall denote the Euclidean norm by $|\cdot|$ and the usual inner product by $\cdot$.

The vector fields $\{X_1,...,X_n,Y_1,...,Y_n,T\}$ defined by

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}, \quad \text{for } i = 1,...,n, \\ Y_i &= X_{n+i} = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t}, \quad \text{for } i = 1,...,n, \\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

form a basis of the Lie Algebra of vector fields which are left invariant with respect to the Heisenberg group action $\circ$. Observe that Z_α described in the previous section is the vector field $X_\alpha + iY_\alpha$, $\alpha = 1,\ldots,n$. The horizontal space $T_{1,0} = span\{Z_1,\ldots,Z_n\}$ gives a left-invariant CR structure on H^{2n+1} and for any function f on H^{2n+1}, the derivatives along the integral curves of these vector fields are referred to as *horizontal derivatives.* Thus, the Heisenberg gradient (or a horizontal gradient) of a function ϕ is defined as

$$\nabla_{H^{2n+1}}\phi = (X_1\phi, \dots, X_n\phi, Y_1\phi, \dots, Y_n\phi). \tag{3.5}$$

The real 1-form

$$\theta_0 = dt + \sum_{j=1}^{n}(iz_j d\overline{z}_j - i\overline{z}_j dz_j)$$

is left invariant and homogeneous of degree 2. Note that θ_0 annihilates $T_{1,0}$ and we take it to be the contact form for the CR structure on H^{2n+1}. The Levi form is given by

$$L_{\theta_0}(Z_j, \overline{Z}_k) := < -2id\theta_0, Z_j \wedge \overline{Z}_k > = 2\delta_{jk}$$

and for $u \in C^1(H^{2n+1})$,

$$du = (\frac{\partial u}{\partial t})\theta_0 + \sum_{j=1}^{n}\left(Z_j u dz_j + \overline{Z}_j u d\overline{z}_j\right).$$

Therefore, if u is real valued, we have

$$|du|^2_{\theta_0} = \sum_{j=1}^{n}|Z_j u|^2.$$

The *scalar curvature* of H^{2n+1} with pseudohermitian structure θ_0 is identically zero.

The subLaplacian operator $\Delta_{H^{2n+1}}$ associated to the contact form θ_0 is defined by

$$\begin{aligned}\Delta_{H^{2n+1}} &:= \sum\nolimits_{i=1}^{n} X_i^2 + Y_i^2 \\ &= \sum\nolimits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i\frac{\partial^2}{\partial x_i \partial t} - 4x_i\frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2)\frac{\partial^2}{\partial t^2}.\end{aligned}$$

Note that $\Delta_{H^{2n+1}}$ is a degenerate operator, but it is easy to check that X_i and Y_i satisfy

$$[X_i, Y_j] = -4T\delta_{i,j}, \ \ [X_i, X_j] = [Y_i, Y_j] = 0$$

for any $i, j \in \{1, ..., n\}$. Therefore, the vector fields X_i, Y_i $(i = 1, ..., n)$ and their first order commutators span the whole Lie Algebra. Hence, $\Delta_{H^{2n+1}}$ satisfies the Hormander rank condition, see [8]. In particular, this implies that Δ_H is hypoelliptic (i.e. if $\Delta_{H^{2n+1}}u \in C^\infty$ then $u \in C^\infty$) and it satisfies Bony's maximum principle (see [1]).

Alternatively, we can also express $\Delta_{H^{2n+1}}$ as

$$\Delta_{H^{2n+1}} = div(A\nabla) \tag{3.6}$$

where ∇ is the usual gradient and $A = (a_{i,j})$ is the symmetric matrix defined as

$$\left.\begin{array}{rcl} a_{i,j} & = & \delta_{ij} \text{ for } 1 \le i,j \le 2n \\ a_{i,2n+1} & = & 2y_i \text{ for } 1 \le i \le n \\ a_{i,2n+1} & = & -2x_i \text{ for } n+1 \le i \le 2n \\ a_{2n+1,2n+1} & = & 4|z|^2. \end{array}\right\} \tag{3.7}$$

The measure on H^{2n+1} is defined using the contact from θ_0 as

$$\begin{aligned} \theta_0 \wedge d\theta_0^n &= n!(2i)^n dt \wedge dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge \overline{z}_n \\ &= n! 2^{2n} dz dt, \end{aligned} \tag{3.8}$$

i.e., constant times the usual Lebesgue measure on $\mathbb{R}^{2n+1}$. The open ball of radius R centered at ξ_0 is the set:

$$B_{H^{2n+1}}(\xi_0, R) = \{\eta \in H^{2n+1} : d(\eta, \xi_0) < R\}.$$

It is important to note that

$$meas(B_{H^{2n+1}}(\xi_0, R)) = meas(B_{H^{2n+1}}(0, R)) = meas(B_{H^{2n+1}}(0,1))R^Q$$

where $Q = 2n+2$ and by *meas* we mean the Lebesgue measure. The even integer Q is called the homogeneous dimension of H^{2n+1}. Observe that for $R > 1$, if $B(0,R)$ is the Euclidean ball of radius R centered at the origin, then

$$B(0,R) \subset B_{H^{2n+1}}(0,R) \subset B(0,R^2).$$

Let $\Omega \subset H^{2n+1}$ be an open set. Using the notations of Folland-Stein [7], we define the non-isotropic Sobolev spaces S_k^p and Lipschitz spaces Γ_β as follows:

For $1 \le p \le \infty$ and $k = 0,1,2,\ldots$, we say that a function f belongs to the nonisotropic Sobolev space S_k^p, if f and its distributional derivatives $X^I f$, with $|I| \le k$ all belong to L^p. Here $X^I = X_1^{i_1}, X_2^{i_2}, \ldots, X_{2n}^{i_{2n}}$ where $I = (i_1, \ldots, i_{2n})$, $0 \le i_j \le 2n$ is a $2n$-tuple such that $|I| = i_1 + \cdots + i_{2n}$.

S_k^p is a Banach space under the norm

$$||f||_{S_k^p} = ||f||_{k,p} = \sum_{|I| \le k} ||X^I f||_p$$

and C_0^∞ is dense in S_k^p for $p < \infty$. Also, note that

$$||f||_{0,p} = ||f||_p = \left(\int |f|^p \, dx \right)^{1/p}.$$

The nonisotropic Lipschitz spaces and their corresponding Hölder seminorms are defined as follows: for $0 < \beta < 1$,

$$\begin{aligned} \Gamma_\beta &= \{f \in L^\infty \cap C^0 : \sup_{\xi,\eta \in \mathrm{H}^{2n+1}} \frac{|f(\xi) - f(\xi \circ \eta)|}{||\eta||^\beta} < \infty\}, \\ \Gamma_\beta(f) &= \sup_{\xi,\eta \in \mathrm{H}^{2n+1}} \frac{|f(\xi) - f(\xi \circ \eta)|}{||\eta||^\beta}; \\ \Gamma_1 &= \{f \in L^\infty \cap C^0 : \sup_{\xi,\eta \in \mathrm{H}^{2n+1}} \frac{|f(\xi \circ \eta) - 2f(\xi) + f(\xi \circ \eta^{-1})|}{||\eta||} < \infty\}, \\ \Gamma_1(f) &= \sup_{\xi,\eta \in \mathrm{H}^{2n+1}} \frac{|f(\xi \circ \eta) - 2f(\xi) + f(\xi \circ \eta^{-1})|}{||\eta||}; \end{aligned}$$

and for $\beta = k + \beta'$ where k is a positive integer and $0 < \beta' \leq 1$,

$$\begin{aligned} \Gamma_\beta &= \{f \in L^\infty \cap C^0 : X^J f \in \Gamma_{\beta'} \text{ for } |J| = k\} \\ \Gamma_\beta(f) &= \sup_{|J|=k} \Gamma_{\beta'}(X^J f). \end{aligned}$$

Note that we can also identify H^{2n+1} with its Lie algebra which is Euclidean space $\mathbb{R}^{2n+1}$ with the Euclidean norm $|\cdot|$ and the linear coordinates x_j via the exponential map. Hence, we can talk of the usual smooth spaces C^k for $0 \leq k \leq \infty$. Please refer to [7] and [6] for relations between the Lipschitz spaces Γ defined above and the usual Hölder spaces Λ. In particular, we have

$$\Lambda_\beta \subset \Gamma_\beta \subset \Lambda_{\beta/2} \text{ for all } \beta > 0. \tag{3.9}$$

For H^{2n+1}, the following Sobolev's theorem holds:(see Corollary IV.7.4 and IV.7.5 in [14]; also see [6], [7])

Theorem 3.1 *Let $1 \leq p < +\infty$, $\alpha \in N^* = N \setminus 1$ and $Q = 2n + 2$ be the homogenoeus dimension of H^{2n+1}. Then,*
(i) if $\alpha p < Q$, $||f||_{pQ/(Q-\alpha p)} \leq C||f||_{\alpha,p}$, for all $f \in C_0^\infty(\mathrm{H}^{2n+1})$.

(ii) if $\alpha p > Q$, $\Gamma_{(\alpha - Q/p)}(f) \leq C||f||_{\alpha,p}$, for all $f \in C_0^\infty(\mathrm{H}^{2n+1})$.

Furthermore, for $m \geq Q$, we have
(i) if $\alpha p < m$, $||f||_{pm/(m-\alpha p)} \leq C\sum_{\beta=0}^{\alpha} ||f||_{\beta,p}$, for all $f \in C_0^\infty(\mathrm{H}^{2n+1})$.

(ii) if $\alpha p > m$, $\Gamma_{(\alpha - m/p)}(f) \leq C\sum_{\beta=0}^{\alpha} ||f||_{\beta,p}$, for all $f \in C_0^\infty(\mathrm{H}^{2n+1})$.

If $u \in L^p_{loc}(U)$ is such that $\varphi u \in S_k^p(U)$ for every $\varphi \in C_0^k(U)$, then u is said to belong to $S_k^p(U, loc)$. The space $\Gamma_\beta(U, loc)$ are defined similarly. We also recall here the Lipschitz regularity theorem(Theorem 10.13) as stated by Folland-Stein [7]:

Theorem 3.2 *Suppose F, G are distributions satisfying $\Delta_{\mathrm{H}^{2n+1}} F = G$ on $U \subset \mathrm{H}^{2n+1}$. If $G \in \Gamma_\beta(U, loc)$ with $0 < \beta < \infty$, then $F \in \Gamma_{\beta+2}(U, loc)$. If $G \in L^p(U, loc)$ and $\beta = 2 - (Q/p) > 0$, then $F \in \Gamma_\beta(U, loc)$.*

The distance function on CR sphere S^{2n+1}: As in the Riemannian case, the distance function on the CR sphere S^{2n+1} can be obtained by the restriction of the distance function (3.4) defined on H^{2n+3}. We consider S^{2n+1} to be embedded in H^{2n+3} and for z, $w \in S^{2n+1}$, we have the corresponding points $(z, 0)$, $(w, 0) \in H^{2n+3}$. Then

$$\begin{aligned} dist((z,0),\,(w,0))^4 &= ||(w,0)^{-1} \circ (z,0)||^4 \\ &= |z-w|^4 + 4(Im(z \cdot w))^2 \\ &= (|z|^2 - 2Re(z \cdot w) + |w|^2)^2 + 4(Im(z \cdot w))^2 \\ &= 4(1 - Re(z \cdot w))^2 + 4(Im(z \cdot w))^2 \\ &= 4\{1 - 2Re(z \cdot w)) + Re(z \cdot w)^2 + (Im(z \cdot w))^2\} \\ &= 4\{1 - 2Re(z \cdot w) + |z \cdot w|^2\} \\ &= 4|1 - z \cdot w|^2. \end{aligned}$$

Therefore,

$$dist((z,0),\,(w,0)) = \sqrt{2}|1 - z \cdot w|^{1/2}. \tag{3.10}$$

Here $(z \cdot w)$ denotes the usual inner product in C^n.

It was shown in [11] that this distance (3.10) is equivalent to the distance function obtained on S^{2n+1} by pulling back the distance on H^{2n+1}(i.e. (3.4)) to S^{2n+1} via the Cayley transform. Thus it makes sense to talk of distance between two points on the CR sphere.

4 A Pohozaev identity

Let $D \subset \mathrm{H}^{2n+1}$ be open and $\mathcal{S}^2(\overline{D})$ denote the space of all continuous functions $u : \overline{D} \to \mathbb{R}$ such that $X_j u$, $Y_j u$, $X_j^2 u$, $Y_j^2 u$ are continuous functions in D which can be extended to $\overline{D}$. Furthermore, let

$$\mathcal{X} = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} + 2t \frac{\partial}{\partial t}.$$

Observe that $\mathcal{X}$ is the generator for the one parameter family of dilations in H^{2n+1} centered at the origin. Using this vector field, we can derive a Pohozaev type integral identity which is stated below (see [4] for the proof).

Theorem 4.1 *Let $D \subset \mathrm{H}^{2n+1}$ be a bounded, piecewise C^1 open set and let $u \in \mathcal{S}^2(\overline{D})$. Then*

$$2\int_{\partial D} (A\nabla u \cdot N)\mathcal{X} u dH_{Q-2} - \int_{\partial D} |\nabla_{H^n} u|^2 \mathcal{X} \cdot N\, dH_{Q-2}$$

$$= (2-Q)\int_D |\nabla_{\mathbb{H}^{2n+1}}u|^2 dzdt + 2\int_D \mathcal{X}u\Delta_{\mathbb{H}^{2n+1}}u dzdt \tag{4.1}$$

where N denotes the outer unit normal to ∂D and dH_{Q-2} denotes the $(Q-2)$-dimensional Hausdorff measure on $\mathbb{H}^{2n+1}$.

Here A, $\nabla_{\mathbb{H}^{2n+1}}$ and Q are as defined in the preliminaries, see (3.7) and (3.5).

Now let $B_\sigma = \{\xi \in \mathbb{H}^{2n+1} : ||\xi|| < \sigma\}$ denote a ball of radius σ in $\mathbb{H}^{2n+1}$ and u be a C^2 positive solution of

$$-\Delta_{\mathbb{H}^{2n+1}}u = c(n)K(\xi)u^p \quad \text{in} \quad B_\sigma. \tag{4.2}$$

Then multiplying this equation by u and integrating by parts we have

$$-\int_{\partial B_\sigma} A\nabla u\cdot N u\, dH_{Q-2} + \int_{B_\sigma} |\nabla_{\mathbb{H}^{2n+1}}u|^2 dzdt = \int_{B_\sigma} c(n)Ku^{p+1}. \tag{4.3}$$

Comparing equations (4.3) and (4.1) we have

$$\begin{aligned} &2\int_{\partial B_\sigma}(A\nabla u\cdot N)\mathcal{X}u dH_{Q-2} - \int_{\partial B_\sigma}|\nabla_{\mathbb{H}^{2n+1}}u|^2\mathcal{X}\cdot N\, dH_{Q-2}\\ &= (2-Q)\int_{\partial B_\sigma}(A\nabla u\cdot N)u\, dH_{Q-2} + c(n)(2-Q)\int_{B_\sigma} Ku^{p+1}\, dzdt\\ &\qquad -2\int_{B_\sigma}\mathcal{X}uKu^{p+1}\, dzdt. \end{aligned}$$

Using the definition of $\mathcal{X}$ and integrating the last term above by parts, we have

$$\begin{aligned} &\frac{Q-2}{2}\int_{\partial B_\sigma}(A\nabla u\cdot N)\, u dH_{Q-2} - \frac{1}{2}\int_{\partial B_\sigma}|\nabla_{\mathbb{H}^{2n+1}}u|^2\mathcal{X}\cdot N\, dH_{Q-2} +\\ &\int_{\partial B_\sigma}(A\nabla u\cdot N)\mathcal{X}u dH_{Q-2}\\ &= \frac{c(n)}{p+1}\int_{B_\sigma}\mathcal{X}(K)u^{p+1}\, dzdt + c(n)\left(\frac{Q}{p+1} - \frac{(Q-2)}{2}\right)\int_{B_\sigma} Ku^{p+1}\, dzdt\\ &\quad -\frac{c(n)}{p+1}\int_{\partial B_\sigma} Ku^{p+1}\mathcal{X}\cdot N\, dH_{Q-2}. \end{aligned} \tag{4.4}$$

Let us denote the boundary terms on the l.h.s of (4.4) by

$$\begin{aligned} \mathcal{B}(\sigma,\xi,u,\nabla_{\mathbb{H}^{2n+1}}u) &= \frac{Q-2}{2}(A\nabla u\cdot N)u - \frac{1}{2}|\nabla_{\mathbb{H}^{2n+1}}u|^2\mathcal{X}\cdot N\\ &\quad + (A\nabla u\cdot N)\mathcal{X}u. \end{aligned} \tag{4.5}$$

Thus we have shown

Corollary 4.2 *If u is a S^2, positive solution of (4.2) then*

$$\int_{\partial B_\sigma} \mathcal{B}(\sigma, \xi, u, \nabla_{\mathbb{H}^{2n+1}} u)\, dH_{Q-2} = c(n)\left(\frac{Q}{p+1} - \frac{(Q-2)}{2}\right)\int_{B_\sigma} K u^{p+1}\, dzdt +$$
$$+\frac{c(n)}{p+1}\int_{B_\sigma} \mathcal{X}(K) u^{p+1}\, dzdt - \frac{c(n)}{p+1}\int_{\partial B_\sigma} K u^{p+1} \mathcal{X}\cdot N\, dH_{Q-2}. \tag{4.6}$$

Proposition 4.3 *(i) For $u(\xi) = ||\xi||^{2-Q}$, where $Q = 2n+2$ the homogeneous dimension, and $r > 0$*

$$B(r, \xi, u, \nabla_{\mathbb{H}^{2n+1}} u) = 0 \quad \text{for all} \quad \xi \in \partial B_\sigma.$$

(ii) If $u(\xi) = ||\xi||^{2-Q} + C + h(\xi)$, where $C > 0$ is a positive constant and $h(\xi)$ is some function differentiable near the origin with $h(0) = 0$, then there exists $\sigma_0 > 0$ such that for any $0 < \sigma < \sigma_0$, we have

$$\int_{\partial B_\sigma} B(\sigma, \xi, u, \nabla_{\mathbb{H}^{2n+1}} u) < 0.$$

Furthermore,

$$\lim_{\sigma\to 0}\int_{\partial B_\sigma} B(\sigma, \xi, u, \nabla_{\mathbb{H}^{2n+1}} u) = -(Q-2)^2 C |S^{2n-1}| \int_0^{\pi/2} \cos^n \alpha d\alpha \tag{4.7}$$

where $|S^{2n-1}|$ denotes the surface measure of the unit sphere in $\mathbb{R}^{2n}$.

Proof. For B_σ, the normal to the boundary is given by $N = \frac{\nabla\rho}{|\nabla\rho|}$ where $\rho = ||\xi|| = (|z|^4 + t^2)^{1/4}$ is the distance function given by (3.3). Therefore $\nabla\rho = \frac{1}{2\rho^3}(2|z|^2 x, 2|z|^2 y, t)$, x, $y \in \mathbb{R}^n$ and

$$\mathcal{X}\cdot N = \mathcal{X}\cdot\frac{\nabla\rho}{|\nabla\rho|} = \frac{\mathcal{X}\rho}{|\nabla\rho|} = \frac{\rho}{|\nabla\rho|} = \frac{2\rho^4}{(4|z|^6 + t^2)^{1/2}}$$

since ρ is homogeneous of degree 1. Moreover,

$$A\nabla\rho\cdot N = \frac{2\rho|z|^2}{(4|z|^6 + t^2)^{1/2}}.$$

Substituting these values in (4.5), by direct computation one can see that

$$B(\sigma, \xi, u, \nabla_{\mathbb{H}^{2n+1}} u) = 0$$

when $u = \rho^{2-Q}$.

Proof of (ii). For $u(\xi) = \rho^{2-Q} + C + h(\xi)$ with $h(0) = 0$ and $C > 0$, we have

$$\int_{\partial B_\sigma} B(\sigma, \xi, u, \nabla_{\mathbb{H}^{2n+1}} u)$$

$$\begin{aligned}
&= -(Q-2)^2 \int_{\partial B_\sigma} \rho^{2-Q} \frac{|z|^2}{(4|z|^6+t^2)^{1/2}} (C+h)\, dH_{Q-2} \\
&+ \int_{\partial B_\sigma} A\nabla h \cdot N \left(-\frac{(Q-2)}{2}\rho^{2-Q} + \frac{(Q-2)}{2}(C+h) + \mathcal{X} h \right) dH_{Q-2} \\
&- 2(Q-2) \int_{\partial B_\sigma} \rho^{2-Q} \frac{|z|^2}{(4|z|^6+t^2)^{1/2}} \mathcal{X} h\, dH_{Q-2} \\
&+ (Q-2) \int_{\partial B_\sigma} \rho^{1-Q} \nabla_{\mathbb{H}^{2n+1}}\rho \cdot \nabla_{\mathbb{H}^{2n+1}} h \frac{2\rho^4}{(4|z|^6+t^2)^{1/2}}\, dH_{Q-2} \\
&- 1/2 \int_{\partial B_\sigma} |\nabla_{\mathbb{H}^{2n+1}} h|^2 \frac{2\rho^4}{(4|z|^6+t^2)^{1/2}}\, dH_{Q-2}. \qquad (4.8)
\end{aligned}$$

We observe that the volume element of the hypersurface

$$(x,y,t) = (z,t) \in \mathbb{R}^{2n} \times \mathbb{R} : |z|^4 + t^2 = \sigma^4, t > 0$$

is given by $\frac{(4|z|^6+t^2)^{1/2}}{t} dxdy$. Therefore,

$$\begin{aligned}
\int_{\partial B_\sigma} \frac{|z|^2}{(4|z|^6+t^2)^{1/2}}\, dH_{Q-2} &= \int_{|z|\le\sigma} \frac{|z|^2}{(\sigma^4-|z|^4)^{1/2}}\, dz \\
&= \sigma^{2n} |S^{2n-1}| \int_0^{\pi/2} \cos^n \alpha\, d\alpha \qquad (4.9)
\end{aligned}$$

using polar coordinates in R^{2n}. Therefore, the first term on the r.h.s. of(4.8) will be

$$-(Q-2)(C+h)|S^{2n-1}| \int_0^{\pi/2} \cos^n \alpha\, d\alpha.$$

The second term of (4.8) is dominated by

$$\int_{\partial B_\sigma} |A\nabla h| \frac{(Q-2)}{2} \rho^{2-Q}\, dH_{Q-2} \le c(h) \int_{\partial B_\sigma} \frac{(Q-2)}{2} \rho^{2-Q}\, dH_{Q-2}$$

where $c(h)$ is a small constant depending on the function h.

The third term of (4.8) is again $\le 2(Q-2)c(h) \int_{\partial B_\sigma} \rho^{2-Q} \frac{|z|^2}{(4|z|^6+t^2)^{1/2}}\, dH_{Q-2}$ whereas the fourth term

$$\begin{aligned}
& (Q-2) \int_{\partial B_\sigma} \rho^{1-Q} \nabla_{\mathbb{H}^{2n+1}}\rho \cdot \nabla_{\mathbb{H}^{2n+1}} h \frac{2\rho^4}{(4|z|^6+t^2)^{1/2}}\, dH_{Q-2} \\
\le\; & (Q-2) \int_{\partial B_\sigma} \rho^{1-Q} |\nabla_{\mathbb{H}^{2n+1}}\rho| |\nabla_{\mathbb{H}^{2n+1}} h| \frac{2\rho^4}{(4|z|^6+t^2)^{1/2}}\, dH_{Q-2} \\
\le\; & 2(Q-2)c(h) \int_{\partial B_\sigma} \rho^{4-Q} \frac{|z|}{(4|z|^6+t^2)^{1/2}}\, dH_{Q-2} \\
=\; & 2(Q-2)c(h)\sigma^{4-Q} \int_{\partial B_\sigma} \frac{|z|}{|t|}\, dz \\
=\; & 2(Q-2)c(h)\sigma |S^{2n-1}| \int_0^{\pi/2} \cos^{(2n-1)/2} \alpha\, d\alpha
\end{aligned}$$

which goes to 0 as $\sigma \to 0$.

Hence it follows that

$$\int_{\partial B_\sigma} \rho^{2-Q} \frac{|z|^2}{(4|z|^6 + t^2)^{1/2}} \, dH_{Q-2} < 0$$

for $0 < \sigma < \sigma_0$, σ_0 sufficiently small. Also, since $h(0) = 0$, taking limit as $\sigma \to 0$, we see from (4.9) that

$$\lim_{\sigma \to 0} \int_{\partial B_\sigma} B(\sigma, \xi, u, \nabla_{\mathbb{H}^{2n+1}} u) = -(Q-2)^2 C |S^{2n-1}| \int_0^{\pi/2} \cos^n \alpha \, d\alpha.$$

□

5 Estimates for isolated simple blow up points

Let Ω be a bounded domain in $\mathbb{H}^{2n+1}$, $\tau_i \geq 0$ satisfy $\lim_{i\to\infty} \tau_i = 0$, $p_i = \frac{Q+2}{Q-2} - \tau_i$ and $\{K_i\} \in \Gamma_{2+\alpha}(\Omega)$ satisfies (2.6). We consider the solutions $\{u_i\}$'s of

$$\left.\begin{array}{rcl} -\Delta_{\mathbb{H}^{2n+1}} u_i & = & c(n) K_i(\xi) u_i^{p_i} \quad \text{in} \quad \Omega \\ u_i & > & 0. \end{array}\right\} \tag{5.1}$$

with *finite energy*, i.e.

$$\mathcal{E}(u_i) := \int_\Omega |\nabla_{\mathbb{H}^{2n+1}} u_i|^2 \, dx dt \leq C_0 \tag{5.2}$$

for a constant C_0 independent of i.

In this section, we shall derive estimates for isolated simple blow up points of $\{u_i\}$, similar to the ones derived in section 2 of [13]. We state the results below, giving a sketch of the proof wherever required. The necessity of imposing the finite energy assumption (5.2) will be clear the in proof of Proposition 5.2 below.

Lemma 5.1 *(A Harnack inequality) Let $\{K_i\}$ satisfy (2.6), $\{u_i\}$ satisfy (5.1) and $\xi_i \to \bar{\xi}$ be an isolated blow up point. Then for any $0 < r < 1/3\sigma$, we have*

$$\max_{(1/2)r < d(\xi,\xi_i) < 2r} u_i(\xi) \leq C \min_{(1/2)r < d(\xi,\xi_i) < 2r} u_i(\xi) \tag{5.3}$$

where $C = C(Q, sup_i ||K_i||_{L^\infty(B(\xi_i,\sigma))})$.

Proof. Define

$$v(\xi) = r^{2/(p_i - 1)} u_i(\xi_i \circ r\xi) \quad \text{for} \quad \xi \in B(0,3),$$

where $\circ$ is Heisenberg group operation as defined in the preliminaries (2.1). It satisfies

$$-\Delta_{\mathbb{H}^{2n+1}} v(\xi) = c(n) K_i(\xi_i \circ r\xi) v(\xi)^{p_i} \quad \text{in} \quad B(0,3). \tag{5.4}$$

Applying the Harnack inequality ([15], Theorem 2.1 of [5]) in the annulus $B_{9/4}\setminus B_{1/4} \subset B(0,3)$, we have

$$ess \sup_{B_{9/4}\setminus B_{1/4}} v \le C ess \inf_{B_{9/4}\setminus B_{1/4}} v. \tag{5.5}$$

where $C = C(Q, \sup_i ||K_i||_{L^\infty(B(\xi_i,\sigma))}, B_{9/4} \setminus B_{1/4}, B(0,3))$. Translating equation (5.5) in terms of u_i, we get (5.3) for all i. Note that the constant C of (5.5) is independent of i and r. □

Proposition 5.2 *Suppose $\{K_i\}$ is bounded in $\Gamma_{2+\alpha}(\Omega, loc)$, $0 < \alpha < 1$, satisfies (2.6) and $\{u_i\}$ satisfies (5.1) and (5.2). Let $\xi_i \to \bar{\xi}$ be an isolated blow up point. Then for any $R_i \to \infty$, $\varepsilon_i \to 0^+$, we have, after passing to a subsequence that*

$$\left.\begin{array}{rcl} ||u_i(\xi_i)^{-1}u_i(\xi_i \circ u_i(\xi_i)^{-(p_i-1)/2}\xi) - \Lambda_i(\xi)||_{C^2(B(0,2R_i))} & \le & \varepsilon_i \\ R_i u_i(\xi_i)^{-(p_i-1)/2} \to 0 \quad \text{as} \quad i \to \infty & & \end{array}\right\} \tag{5.6}$$

where $\Lambda_i(\xi) = \Lambda_0(K_i(\xi_i)^{1/2}\xi)$, Λ_0 is as defined in (2.4).

Proof. Let σ be as in Definition 2.1, $\alpha_i = \sigma u_i(\xi_i)^{(p_i-1)/2}$ and set

$$w_i(\xi) = u_i(\xi_i)^{-1}u_i(\xi_i \circ u_i(\xi_i)^{-(p_i-1)/2}\xi) \ \text{ for } \ ||\xi|| < \alpha_i.$$

It satisfies the equation

$$\begin{aligned} -\Delta_{\mathbb{H}^{2n+1}} w_i &= c(n)K_i(\xi_i \circ u_i(\xi_i)^{-(p_i-1)/2}\xi)w_i^{p_i} \ \text{ for } \ ||\xi|| < \alpha_i &(5.7)\\ w_i(0) &= 1 &(5.8)\\ \nabla w_i(0) &= 0 &(5.9)\\ 0 < w_i(\xi) &< \overline{C}||\xi||^{-2/(p_i-1)} \ \text{ for } \ ||\xi|| < \alpha_i &(5.10) \end{aligned}$$

where (5.10) follows from the definition of isolated blow up point. Also, it can be seen by change of variables that the energy

$$\begin{aligned} \mathcal{E}(w_i) &:= \int_{B(0,\alpha_i)} |\nabla_{\mathbb{H}^{2n+1}} w_i|^2 d\xi \\ &= \int_{B(0,\alpha_i)} u_i(\xi_i)^{-p_i-1}|\nabla_{\mathbb{H}^{2n+1}} u_i(\xi_i \circ u_i(\xi_i)^{-(p_i-1)/2}\xi)\, d\xi \\ &= \int_{B(\xi_i,\sigma)} u_i(\xi_i)^{(2-Q)\tau_i/2}|\nabla_{\mathbb{H}^{2n+1}} u_i^{p_i+1}(\eta)u_i(\xi_i)^{Q(p_i-1)/2-p_i-1} d\eta \\ &\le \mathcal{E}(u_i) \\ &\le C_0. \end{aligned} \tag{5.11}$$

Clearly, (5.10) implies that w_i are bounded outside the unit ball. Next we prove that w_i's are also bounded in the unit ball: Suppose by contradiction,

$$\lim_{i\to\infty} w_i(\nu_i) = \infty, \tag{5.12}$$

where $w_i(\nu_i) = \max_{\overline{B(0,1)}} w_i(\xi)$. From (5.10) we conclude that $\|\nu_i\| \neq 1$ for all large i. Also, from (5.8) it follows that ν_i cannot converge to the origin. Hence we should have $\|\nu_i\| > \delta$ for all large i, for some $\delta > 0$. Let $\nu_i \to \tilde{\nu}$. Then $\|\tilde{\nu}\| \geq \delta > 0$.

It follows from Lemma 5.1 that for all $0 < r < 1$

$$\begin{aligned} \max_{\xi \in \partial B(0,r)} w_i(\xi) &\leq \max_{1/2r < d(\xi,\xi_i) < 2r} w_i(\xi) \\ &\leq C \min_{1/2r < d(\xi,\xi_i) < 2r} w_i(\xi) \\ &\leq C \min_{\xi \in \partial B(0,r)} w_i(\xi). \end{aligned} \tag{5.13}$$

Thus,

$$\begin{aligned} w_i(\nu_i) &= \max_{\xi \in \partial B(0,\|\nu_i\|)} w_i(\xi) \\ &\leq C \min_{\xi \in \partial B(0,\|\nu_i\|)} w_i(\xi) \\ &\leq C\overline{C}\|\nu_i\|^{-2/(p_i-1)} \\ &\leq C\delta^{-2/(p_i-1)} \end{aligned}$$

a contradiction. The third inequality above follows from (5.10). Thus w_i's are bounded in $\overline{B(0,1)}$.

Claim 5.3 *There exists a subsequence of $\{w_i\}$ which converges in $C^2_{loc}(H^{2n+1})$ to a positive function w.*

Proof of the Claim: Fix a compact set $U \subset \mathrm{H}^{2n+1}$. For large i, $U \subset B(0, \alpha_i)$ since $\alpha_i \to \infty$ as $i \to \infty$. By above discussion, $\{w_i\}$ is uniformly bounded where as $\{K_i\}$ is uniformly bounded by assumption. Therefore, $w_i^{p_i+1} K_i \in L^\infty_{loc}$ and hence in L^p_{loc} for all $1 \leq p \leq \infty$.

Now, from Theorem 6.1 of [6] we have, for $1 < p < \infty$ and $k = 0, 1, 2, \ldots$,

$$||w_i||_{p,k+2} \leq C_{p,k} \left(||\Delta_{\mathrm{H}^{2n+1}} w_i||_{p,k} + ||w_i||_p\right). \tag{5.14}$$

Using the fact that w_i satisfies the differential equation (5.7) in (5.14), it follows that $w_i \in S^p_2(loc)$ for all p. Furthermore, from (ii) of Sobolev embedding Theorem 3.1 given in the preliminaries, we have locally

$$\Gamma_{(2-Q/p)}(w_i) \leq C||w_i||_{p,2} \quad \text{for} \quad 2p > Q. \tag{5.15}$$

Choose p large such that $1 < \alpha = 2 - Q/p < 2$. Then $\{w_i\}$ are uniformly bounded in $\Gamma_\alpha(U, loc)$. Hence, by Arzela-Ascoli theorem, there exists a subsequence of $\{w_i\}$ converging in C^0 to a positive function w in U.

It is easy to see that w satisfies

$$\begin{aligned} \Delta_{\mathrm{H}^{2n+1}} w + c(n) \lim_{\imath \to \infty} K_i(\xi_i \circ u_i(\xi_i)^{-(p_i-1)/2}\xi) w^{Q+2/Q-2} &= 0 \\ w \geq 0, \quad w(0) = 1. \end{aligned} \tag{5.16}$$

in H^{2n+1} and is bounded in the whole space.

From (5.15), $w_i \in \Gamma_{1+\beta}(U, loc)$, $\beta = 1 - (Q/p) > 0$ for large p. Therefore, we have $K_i w_i^{p_i+1} \in \Gamma_{1+\beta}$ since $K_i \in \Gamma^{2+\alpha}$. Hence, from Theorem 3.2 we can further conclude that $w_i, w \in \Gamma_{\beta+2}(U, loc)$, with $\beta + 2 > 3$. Again, using the fact that $K_i w_i^{p_i+1} \in \Gamma_{2+\alpha}$ and Theorem 3.2 we have $w_i, w \in \Gamma_{4+\alpha}(U, loc)$ and hence in C^2_{loc}. Hence the claim is proved. Moreover, (5.11) implies that

$$\int_{\mathrm{H}^{2n+1}} w^{2Q/Q-2}\theta_0 \wedge d\theta_0 \le C_0 \tag{5.17}$$

Hence it follows from [10] that

$$w(\xi) = \Lambda_0(k^{1/2}\xi)$$

where $k = \lim_{i\to\infty} k_i$, $k_i = K_i(\xi_i)$. Furthermore, if $\Lambda_i(\xi) = \Lambda_0(k_i^{1/2}\xi)$ then it can be seen that Λ_i is close to Λ_0 for large i. Therefore, given $\varepsilon > 0$, for all large i we have

$$||u_i(\xi_i)^{-1}u_i(\xi_i \circ u_i(\xi_i)^{-(p_i-1)/2}\xi) - \Lambda_i(\xi)||_{C^2(B(0,a_i)))} \le \varepsilon \tag{5.18}$$

Now, given $R_i \to \infty$ and $\varepsilon_i \to 0^+$, one can always choose a subsequence of $\{w_i\}$, such that (5.6) holds and the proposition follows. □

Remark 5.4 For application of Proposition 5.2, we will choose R_i and ε_i(depending on R_i) as follows:

Given $R_i \to \infty$ satisfying (5.6)), we first choose an ε_i such that ξ_i is the only critical point of u_i in $B(\xi_i, R_i u_i(\xi_i)^{-(p_i-1)/2})$ and for and $\varepsilon > 0$, we have

$$||u_i(\xi_i)^{-1}u_i(\xi_i \circ u_i(\xi_i)^{-(p_i-1)/2}\xi) - \Lambda_i(\xi)||_{C^2(B(0,2R_i))} < \varepsilon_i. \tag{5.19}$$

Furthermore, for a fixed $\theta = (z_0, t_0) \in \partial B(0,1)$, consider the function

$$\begin{aligned} g_{i,\theta}(s) &:= s^{(Q-2)/2}\Lambda_i(s\theta) \\ &= s^{(Q-2)/2}\left(1 + 2k_i s^2|z_0|^2 + k_i^2 s^4\right)^{(2-Q)/4} \end{aligned}$$

where $|z_0|$ is euclidean norm in $I\!R^{2n}$. It can be seen that $g_{i,\theta}(0) = 0$, $g_{i,\theta}(s) \ge 0$ and has precisely one critical point which is a point of maximum at $s_i(\theta) = k_i^{-1/2}$. Therefore, for τ_i small, the function $s^{2/(p_i-1)}\Lambda_i(s\theta)$ is close to $g_{i,\theta}$ and has similar properties. Thus, for each fixed $\theta \in \partial B(0,1)$, we may further modify ε_i so that

$$f_{w_i,\theta}(s) = s^{2/(p_i-1)}w_i(s\theta)$$

has unique critical point in $(0, R_i)$. Observe that, a priori this ε_i will depend on θ, but since θ varies over a compact set $\partial B(0,1)$, we can choose ε_i independent of θ such that $f_{w_i,\theta}$ has a unique critical point in $(0, R_i)$ for every $\theta \in \partial B(0,1)$. Thus, given $R_i \to \infty$, we can choose $\varepsilon_i \to 0^+$ such that

(i) u_i has a unique critical point in $B(\xi_i, R_i u_i(\xi_i)^{-(p_i-1)/2})$;

(ii) $f_{u_i,\theta}$ has a unique critical point in $(0, R_i u_i(\xi_i)^{-(p_i-1)/2})$ for every $\theta \in \partial B(\xi_i, 1)$.

Remark 5.5 Suppose further in Proposition 5.2 that 0 is a isolated simple blow up point. If σ is as in definition 2.2, then it follows from (ii) above that the function $f_{u_i,\theta}$ is strictly decreasing from $R_i u_i(\xi_i)^{-(p_i-1)/2}$ to σ.

Henceforth, we shall denote a ball in H^{2n+1} with centre origin and radius s by B_s, $s > 0$.

Proposition 5.6 *Suppose* $\{K_i\} \in \Gamma_{2+\alpha}(B_2, loc)$, $0 < \alpha < 1$ *satisfies (2.6) with* $\Omega = B_2$ *and*

$$| \nabla K_i(y) | \leq A_2, \quad \text{for all } y \in B_2 \tag{5.20}$$

for some positive constant A_2. *Let* u_i *satisfy (5.1) with* $\Omega = B_2$ *and* $\xi_i \to 0$ *be an isolated blow up point with,*

$$d(\xi, \xi_i)^{2/(p_i-1)} u_i(\xi) \leq A_3, \quad \text{for all } \xi \in B_2 \tag{5.21}$$

for some positive constant A_3. *Then there exists some positive constant* $C = C(n, A_1, A_2, A_3)$, *such that*

$$u_i(\xi) \geq C^{-1} u_i(\xi_i) \Lambda_0(k_i^{1/2} u_i(\xi_i)^{(p_i-1)/2} \xi) \quad \text{for all } d(\xi, \xi_i) \leq 1.$$

In particular, for any $e \in \mathrm{H}^{2n+1}$ *with* $||e|| = 1$, *we have*

$$u_i(\xi_i \circ e) \geq C^{-1} u_i(\xi_i)^{-1+((Q-2)/2)\tau_i}.$$

Proof. We observe that $||\xi||^{2-Q}$ is a fundamental solution of the sub Laplacian $\Delta_{\mathrm{H}^{2n+1}}$ and the proof of Proposition 2.2 given in [13] goes through with n replaced by the homogeneous dimension Q. □

Proposition 5.7 *Let* $\{K_i\} \subset \Gamma_{2+\alpha}(B_2, loc)$, $0 < \alpha < 1$ *satisfy (2.6) with* $\Omega = B_2$ *and (5.20) for some positive constant* A_2. *Suppose also that* u_i *satisfies (5.1) with* $\Omega = B_2$ *and* $\xi_i \to 0$ *is an isolated simple blow up point with (5.21) for some positive constant* A_3. *Then there exists some positive constant* $C = C(n, A_1, A_2, A_3, \sigma)$, *such that,*

$$u_i(\xi) \leq C u_i(\xi_i)^{-1} d(\xi, \xi_i)^{2-Q}, \quad \text{for all } d(\xi, \xi_i) \leq 1, \tag{5.22}$$

where σ *is the constant in Definition 2.2.*

Furthermore, after passing to a subsequence, we have

$$u_i(\xi_i) u_i(\xi) \to u(\xi) = a||\xi||^{2-Q} + h(\xi) \quad \text{in } C^2_{loc}(B_1 \backslash \{0\}),$$

in B_1, *where* $a > 0$ *is a constant and* $\Delta_{\mathrm{H}^{2n+1}} h = 0$.

For proving Proposition 5.7 we need the following lemmas.

Lemma 5.8 *Under the hypothesis of Proposition 5.7, except for (5.20), there exists* $\delta_i > 0$, $\delta_i = 0(R_i^{-2+o(1)})$, *such that*

$$u_i(\xi) \leq C_1 u_i(\xi_i)^{-\lambda_i} d(\xi, \xi_i)^{2-Q+\delta_i} \quad \text{for all } R_i u_i(\xi_i)^{-(p_i-1)/2} \leq d(\xi, \xi_i) \leq 1, \tag{5.23}$$

where $\lambda_i = (Q - 2 - \delta_i)(p_i - 1)/2 - 1$ *and* C_1 *is some positive constant depending only on* Q, A_1, A_3, σ.

Proof. Let $r_i = R_i u_i(\xi_i)^{-(p_i-1)/2} \to 0$ be chosen as in Remark 5.4. Since

$$\Lambda_i(\xi) = 2^{(Q-2)/2}\left(1+2k_i|z|^2) + k_i^2(|z|^4+t^2)\right)^{(2-Q)/4} \le C||\xi||^{2-Q},$$

from Proposition 5.2, it follows that

$$\begin{aligned} u_i(\xi) &\le Cu_i(\xi_i)\Lambda_i(u_i(\xi_i)^{(p_i-1)/2}(\xi_i^{-1}\circ\xi)) \\ &\le Cu_i(\xi_i)(u_i(\xi_i)^{(p_i-1)/2}r_i)^{2-Q} \\ &= Cu_i(\xi_i)R_i^{2-Q} \end{aligned} \tag{5.24}$$

for $d(\xi,\xi_i) = r_i$. Since 0 is a isolated simple blow up point, there exists $\sigma > 0$ such that for every $\theta \in B(0,1)$, $f_{i,\theta}(s)$ is strictly decreasing for $r_i < s < \sigma$ (see Remark 5.5). Hence for all $r_i < s = d(\xi,\xi_i) < \sigma$ and for every $\theta \in B(0,1)$,

$$f_{i,\theta}(s) \le f_{i,\theta}(r_i).$$

Therefore, for all $r_i < s = d(\xi,\xi_i) < \sigma$, we have

$$\begin{aligned} d(\xi,\xi_i)^{2/(p_i-1)}u_i(\xi) &\le Cr_i^{2/(p_i-1)}u_i(\xi_i\circ r_i\frac{\xi}{||\xi||}) \\ &\le CR_i^{(Q-2)/2+o(1)} \end{aligned}$$

from (5.24). Hence

$$u_i(\xi)^{p_i-1} \le CR_i^{-2+o(1)}d(\xi,\xi_i)^{-2} \tag{5.25}$$

for all ξ such that $r_i < s = d(\xi,\xi_i) < \sigma$.

Now consider the operator

$$\mathcal{L}_i\varphi = \Delta_{\mathrm{H}^{2n+1}}\varphi + c(n)K_iu_i^{p_i-1}\varphi. \tag{5.26}$$

To obtain the estimate (5.23) in the annulus $A_i := \{\xi \in \mathrm{H}^{2n+1} : r_i < d(\xi,\xi_i) < \sigma\}$, we look for a supersolution φ_i of the operator $\mathcal{L}_i$ such that $\varphi_i \ge u_i$ on the boundary of the annulus A_i. ¿From direct computation we have,

$$\Delta_{\mathrm{H}^{2n+1}}(d(\xi,\xi_i)^{-\mu}) = -\mu(Q-2-\mu)|z-z_i|^2 d(\xi,\xi_i)^{-\mu-4}$$

where we have used the notation $\xi = (z,t)$ and $\xi_i = (z_i,t)$. Therefore,

$$\begin{aligned} \mathcal{L}_i(d(\xi,\xi_i)^{-\mu}) &= \Delta_{\mathrm{H}^{2n+1}}(d(\xi,\xi_i)^{-\mu}) + c(n)K_iu_i^{p_i-1}d(\xi,\xi_i)^{-\mu} \\ &\le -\mu(Q-2-\mu)|z-z_i|^2 d(\xi,\xi_i)^{-\mu-4} + CR_i^{-2+o(1)}d(\xi,\xi_i)^{-\mu-2} \end{aligned}$$

where the last inequality follows from (5.25). Note that $\frac{|z-z_i|^2}{d(\xi,\xi_i)^2} = O(1)$. Thus we can choose $\delta_i = O(R_i^{-2+o(1)}) \to 0^+$ such that we have

$$\left.\begin{aligned} \mathcal{L}_i(d(\xi,\xi_i)^{-\delta_i}) &\le 0 \text{ in } A_i \\ \mathcal{L}_i(d(\xi,\xi_i)^{2-Q-\delta_i}) &\le 0 \text{ in } A_i. \end{aligned}\right\} \tag{5.27}$$

Now set $M_i = \max_{\partial} B(\xi_i, \sigma) u_i$, $\lambda_i = 1/2(Q-2-\delta_i)(p_i-1)-1$ and

$$\varphi_i(\xi) = M_i \sigma^{\delta_i} d(\xi, \xi_i)^{-\delta_i} + B u_i(\xi_i)^{-\lambda_i} d(\xi, \xi_i)^{2-Q-\delta_i} \text{ in } A_i$$

where $B > 1$ is a constant to be chosen later. By (5.27), it follows that φ_i is a supersolution of $\mathcal{L}_i$ in A_i. Furthermore,

$$\varphi_i(\xi) \geq M_i \geq u_i(\xi) \text{ for } d(\xi, \xi_i) = \sigma. \tag{5.28}$$

Also,

$$\begin{aligned} \varphi_i(\xi) &\geq B u_i(\xi_i)^{-\lambda_i} d(\xi, \xi_i)^{2-Q-\delta_i} \\ &\geq B u_i(\xi_i) R_i^{2-Q} \text{ for } d(\xi, \xi_i) = r_i. \end{aligned} \tag{5.29}$$

Comparing (5.29) with (5.24), we choose B large such that $B \geq C$ occuring in equation (5.24). With this choice of B, we have

$$\varphi_i(\xi) \geq u_i(\xi) \text{ for } d(\xi, \xi_i) = r_i. \tag{5.30}$$

From (5.28), (5.29) and the maximum principle, it follows that

$$u_i(\xi) \leq \varphi_i(\xi) \text{ for all } r_i \leq d(\xi, \xi_i) \leq \sigma. \tag{5.31}$$

From Lemma 5.1, for any $\theta \in \partial B(0,1)$, we have

$$\begin{aligned} f_{i,\theta}(\sigma) &= \sigma^{(p_i-1)/2} u_i(\xi_i \circ \sigma\theta) \\ &\geq \sigma^{(p_i-1)/2} \min_{d(\xi,\xi_i)=\sigma} u_i(\xi) \\ &\geq C^{-1} \sigma^{(p_i-1)/2} \max_{d(\xi,\xi_i)=\sigma} u_i(\xi) \\ &= C^{-1} \sigma^{(p_i-1)/2} M_i \end{aligned}$$

(see definition of M_i). Since $f_{i,\theta}$ is decreasing in the interval (r_i, σ), we have that for any s, $r_i < s < \sigma$ and $\theta \in \partial B(0,1)$

$$\sigma^{(p_i-1)/2} M_i \leq C f_{i,\theta}(\sigma) \leq C f_{i,\theta}(s).$$

Since $u_i \leq \varphi_i$,

$$\begin{aligned} \sigma^{(p_i-1)/2} M_i &\leq s^{(p_i-1)/2} u_i(\xi_i \circ s\theta) \\ &\leq C s^{(p_i-1)/2} [M_i \sigma^{\delta_i} s^{-\delta_i} + B_i u_i(\xi_i)^{-\lambda_i} s^{2-Q+\delta_i}]. \end{aligned}$$

Choose $s_0 = s_0(\sigma, Q, A_2, A_3) > 0$(note that it is independent of θ) small such that

$$C s_0^{(p_i-1)/2} \sigma^{\delta_i} s_0^{-\delta_i} < \sigma^{(p_i-1)/2}/2.$$

Hence

$$M_i \leq C u_i(\xi_i)^{-\lambda_i}.$$

Thus from (5.31) we have the required estimate

$$u_i(\xi) \le C u_i(\xi_i)^{-1} d(\xi,\xi_i)^{2-Q}, \quad \text{for all } r_i \le d(\xi,\xi_i) \le \sigma. \tag{5.32}$$

For ξ with $\sigma \le d(\xi,\xi_i) \le 1$, we have

$$d(\xi,\xi_i)^{Q-2-\delta_i} u_i(\xi) \le u_i(\xi)$$

and from Lemma 5.1

$$\begin{aligned} u_i(\xi) &\le \max_{r_i \le d(\xi,\xi_i) \le 1} u_i(\xi) \\ &\le C \min_{r_i \le d(\xi,\xi_i) \le 1} u_i(\xi) \\ &\le C \min_{d(\xi,\xi_i)=\sigma} u_i(\xi) \\ &\le C u_i(\xi_i)^{-\lambda_i} \sigma^{2-Q+\delta_i} \end{aligned}$$

where the last inequality follows from (5.32). Choose $C_1 > C\sigma^{2-Q+\delta_i}$. This completes the proof of the lemma.

□

Next, we note that the vector field

$$\mathcal{X} = \sum_{i=1}^{n} (x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}) + 2t \frac{\partial}{\partial t} \tag{5.33}$$

on the Heisenberg group differs from the Euclidean Killing field(on $\mathbb{R}^{2n+1}$)

$$\sum_{i=1}^{n} (x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}) + t \frac{\partial}{\partial t} \tag{5.34}$$

by the factor 2 in the last coordinate. Also, we can write $\mathcal{X}$ in a simplified way as

$$\mathcal{X}(\xi) = \nu(\xi) \cdot \nabla \tag{5.35}$$

where $\nu(\xi) = \nu(x,y,t)$ is the vector $(x,y,2t) \in \mathbb{R}^{2n+1}$, $\cdot$ denotes the Euclidean scalar product and ∇ is the usual gradiant.

Lemma 5.9 *Under the hypothesis of Proposition 5.7, we have*

$$\tau_i = O(u_i(\xi_i)^{-2/(Q-2)+o(1)}).$$

and therefore

$$u_i(\xi_i)^{\tau_i} = 1 + o(1).$$

This lemma can be proved exactly as Lemma 2.3 of [13], using the expression (5.35) of $\mathcal{X}$.

Proof of Proposition 5.7. Without loss of generality, we assume that σ occuring in the definition 2.2 is less than 1/2. The inequality (5.22) follows from Proposition 5.2 and lemma 5.9 for $d(\xi,\xi_i) < r_i$.

Fix $\theta_0 \in \mathrm{H}^{2n+1}$ with $||\theta_0|| = 1$ and set $v_i(\xi) = u_i(\xi_i \circ \theta_0)^{-1}u_i(\xi)$. Then v_i satisfies

$$-\Delta_{\mathrm{H}^{2n+1}} v_i = c(n)u_i(\xi_i \circ \theta_0)^{p_i-1}K_i(\xi)v_i^{p_i} \quad \text{in} \quad B_2 \tag{5.36}$$

From Lemma 5.1 and arguing as in Claim 5.3, after passing to a subsequence, $\{v_i\}$ converges in $C^2_{loc}(B_2\backslash\{0\})$ to a positive function $v \in C^2(B_2\backslash\{0\})$. Since from Lemma 5.8, $u_i(\xi_i \circ \theta_0) \to 0$, taking limit as $i \to \infty$ in equation (5.36), we see that v satisfies

$$\Delta_{\mathrm{H}^{2n+1}} v = 0 \quad \text{in} \quad B_2 \setminus \{0\}.$$

Moreover, for any θ with $||\theta|| = 1$,

$$f_{v_i,\theta} = u_i(\xi_i \circ \theta_0)^{-1} f_{u_i,\theta} \to f_{v,\theta} = s^{(Q-2)/2}v(s\theta).$$

Since 0 is isolated simple blow up point, it follows from remark 5.5 that $f_{v_i,\theta}$ is strictly decreasing from r_i to σ. Hence $f_{v,\theta}$ is non increasing near the origin for every $\theta \in \partial B_1$ which gives a contradiction if v is regular near 0. Hence v must be singular at 0 and hence we can write

$$v(\xi) = a||\xi||^{2-Q} + h(\xi)$$

where a is a positive constant and

$$\Delta_{\mathrm{H}^{2n+1}} h \equiv 0 \quad \text{in} \quad B_2. \tag{5.37}$$

We first prove the inequality (5.22) for $||\xi|| = 1$, i.e.,

$$u_i(\xi_i \circ \theta_0) \leq C u_i(\xi_i)^{-1}. \tag{5.38}$$

On the contrary, suppose

$$u_i(\xi_i)u_i(\xi_i \circ \theta_0) \to \infty \quad \text{as} \quad i \to \infty.$$

Consider the function $v_i(\xi) - h(\xi)$ in the ball B_1. From (5.36) and (5.37), it follows that

$$\Delta_{\mathrm{H}^{2n+1}}(v_i(\xi) - h(\xi)) = c(n)u_i(\xi_i \circ \theta_0)^{-1}K_i(\xi)u_i^{p_i}$$

in B_2. Integrating by parts on B_1, we have

$$-\int_{\partial B_1} A\nabla(v_i - h)\cdot N\, dH_{Q-2} = \int_{B_1} c(n)u_i(\xi_i \circ \theta_0)^{-1}K_i(\xi)u_i^{p_i}. \tag{5.39}$$

Now

$$\lim_{i\to\infty}\int_{\partial B_1} A\nabla(v_i - h)\cdot N\, dH_{Q-2} = \int_{\partial B_1} A\nabla(a||\xi||^{2-Q})\cdot N dH_{Q-2} < 0 \tag{5.40}$$

since using (4.9) we have

$$\begin{aligned}\int_{\partial B_1} A\nabla(a||\xi||^{2-Q})\cdot N\,dH_{Q-2} &= a(2-Q)\int_{\partial B_1} A\nabla||\xi||\cdot\frac{\nabla(||\xi||)}{|\nabla(||\xi||)|}\,dH_{Q-2}\\ &= 2a(2-Q)\int_{\partial B_1}\frac{|z|^2}{(4|z|^6+t^2)^{1/2}}dH_{Q-2}\\ &= 2a(2-Q)|S^{2n-1}|\int_0^{\pi/2}\cos^n\alpha\,d\alpha\\ &< 0. \qquad (5.41)\end{aligned}$$

From Proposition 5.2 and Lemma 5.9,

$$\int_{d(\xi,\xi_i)\le r_i} K_i u_i^{p_i} \le C u_i(\xi_i)^{-1}. \qquad (5.42)$$

and from Lemma 5.8 and Lemma 5.9,

$$\int_{r_i\le d(\xi,\xi_i)\le 1} K_i u_i^{p_i} \le o(1) u_i(\xi_i)^{-1} \qquad (5.43)$$

as in inequality (2.18) of [13]. We can now proceed as in proof of Prop. 2.3 of [13] to complete the proof of (5.22).

□

Lemma 5.10 *Under the hypothesis of Proposition 5.7, we have*

$$\int_{d(\xi,\xi_i)\le r_i} d(\xi,\xi_i)^s u_i(\xi)^{p_i+1}$$

$$= \begin{cases} u_i(\xi_i)^{-2s/(Q-2)}\{\int_{\mathbb{R}^{2n+1}} \rho^s(1+2k_i\mid z\mid^2+k_i^2\rho^4)^{-Q/2}dz+o(1)\} & -Q<s<Q,\\ O(u_i(\xi_i)^{-2Q/(Q-2)}\log u_i(\xi_i)) & s=Q,\\ o(u_i(\xi_i)^{-2Q/(Q-2)}) & s>Q.\end{cases}$$

Here we have used the notation $\rho = ||\xi|| = (|z|^4+t^2)^{1/4}$ *for* $\xi=(z,t)\in \mathrm{H}^{2n+1}$.

$$\int_{r_i\le d(\xi,\xi_i)\le 1} d(\xi,\xi_i)^s u_i(\xi)^{p_i+1} \le \begin{cases} o(u_i(\xi_i)^{-2s/(Q-2)}), & -Q<s<Q,\\ O(u_i(\xi_i)^{-2Q/(Q-2)}\log u_i(\xi_i)), & s=Q,\\ O(u_i(\xi_i)^{-2n/(n-2)}), & s>Q\end{cases}$$

where $k_i = K_i(\xi_i)$.

The above inequalities can be obtained by direct computation.

□

For our future analysis, we need to consider the equation of the type

$$\left.\begin{aligned} -\Delta_{\mathrm{H}^{2n+1}} u_i &= c(n)K_i(\xi)H_i(\xi)^{\tau_i}u_i^{p_i} \quad \text{in} \quad B(0,2)\\ u_i &> 0\end{aligned}\right\} \qquad (5.44)$$

where recall that $\tau_i \to 0$ and $p_i = \frac{Q+2}{Q-2} - \tau_i$.

Lemma 5.11 *Suppose* $\{K_i\} \in \Gamma_{2+\alpha}(B_2, loc)$, $0 < \alpha < 1$ *satisfies (2.6), (5.20) for some positive constants* A_1, A_2 *and* $(*)_\beta$ *for* $\{L_1(\beta,\ i)\}$, $\{L_2(\beta,\ i)\}$ *in* B_2 *for some* $2 \leq \beta < Q$, $\{H_i\} \in C^1_{loc}(B_2)$ *satisfying*

$$A_4^{-1} \leq H_i(0) \leq A_4, \quad | \nabla H_i(\xi) | \leq A_5 \text{ for all } \xi \in B_2, \tag{5.45}$$

for some positive constants A_4, A_5. *Suppose also that* u_i *satisfies (5.44)-(5.2) and* $\xi_i \to 0$ *is a isolated simple blow up point with (5.21) for some positive constant* A_3. *Then we have*

$$\begin{aligned}\tau_i \leq \ & Cu_i(\xi_i)^{-2} + C \mid \nabla K_i(\xi_i) \mid u_i(\xi_i)^{-2/(Q-2)} + C(L_2(\beta, i) + L_2(\beta, i)^{\beta-1} \\ & + L_2(\beta, i)^{(\beta-[\beta])/(\beta-1)+1} L_1(\beta, i)^{(\beta-2)(\beta-[\beta])/(\beta-1)^2}) \\ & \times u_i(\xi_i)^{(-2/(Q-2))([\beta]+(\beta-[\beta])/(\beta-1)))},\end{aligned}$$

where $C = C(Q,\ C_0,\ A_1,\ A_2,\ A_3,\ A_4,\ A_5, \sigma,\ \beta)$.

Proof. Observe that the generator of one parameter family of dilations around the point $\xi_i = (x_0^i, y_0^i, t_0^i)$ is given by

$$\begin{aligned}\mathcal{X}_i &= \sum_{j=1}^{n} ((x - x_0^i)_j \frac{\partial}{\partial x_j} + (y - y_0^i)_j \frac{\partial}{\partial y_j}) + 2(t - t_0^i + 2(x_0^i \cdot y - y_0^i \cdot x)) \frac{\partial}{\partial t} \\ &= \nu_i(\xi) \cdot \nabla \end{aligned} \tag{5.46}$$

where $\nu_i(\xi) = \nu(\xi_i^{-1} \circ \xi) = (x - x_0^i, y - y_0^i, 2(t - t_0^i + 2(x_0^i \cdot y - y_0^i \cdot x)))$, and the corresponding Pohozaev identity for $B(\xi_i, 1)$ is

$$\begin{aligned}&\int_{\partial B(\xi_i,1)} \mathcal{B}(1, \xi_i^{-1} \circ \xi, u_i, \nabla_{\mathbb{H}^{2n+1}} u_i)\, dH_{Q-2} = \frac{c(n)}{p_i + 1} \int_{B(\xi_i,1)} \mathcal{X}_i(K_i H_i^{\tau_i}) u^{p_i+1}\, dzdt \\ &+ c(n) \left(\frac{Q}{p_i + 1} - \frac{(Q-2)}{2} \right) \int_{B(\xi_i,1)} K_i H_i^{\tau_i} u^{p_i+1}\, dzdt \\ &- \frac{c(n)}{p_i + 1} \int_{\partial B(\xi_i,1)} K_i H_i^{\tau_i} u^{p_i+1} \mathcal{X}_i \cdot N\, dH_{Q-2}.\end{aligned} \tag{5.47}$$

Also, observe that

$$\begin{aligned}|\nu_i(\xi)|^2 &= |x - x_0^i|^2 + |y - y_0^i|^2 + 4(t - t_0^i + 2(x_0^i \cdot y - y_0^i \cdot x))^2 \\ &\leq 4\left(|x - x_0^i|^2 + |y - y_0^i|^2 + (t - t_0^i + 2(x_0^i \cdot y - y_0^i \cdot x))^2\right) \\ &\leq 4||\xi_i^{-1} \circ \xi||^2 = 4d(\xi, \xi_i)^2. \end{aligned} \tag{5.48}$$

The proof can be now completed as that of Lemma 2.5 of [13].

□

Lemma 5.12 *Under the hypothesis of Lemma 5.11 we have,*

$$
\begin{aligned}
|\nabla K_i(\xi_i)| \leq\ & Cu_i(\xi_i)^{-2} + C(L_2(\beta,i) + L_2(\beta,i)^{\beta-1} \\
& + L_2(\beta,i)^{(\beta-[\beta])/(\beta-1)+1} L_1(\beta,i)^{(\beta-2)(\beta-[\beta]/(\beta-1)^2}) \\
& \times u_i(\xi_i)^{-(2/(Q-2))([\beta]-1+(\beta-[\beta])/(\beta-1))},
\end{aligned}
$$

where C depends only on Q, C_0 A_1, A_2, A_3, A_4, A_5, σ, and β.

Proof. Let $\eta \in C_c^\infty(B_{1/2})$ be a cut off function such that

$$
\begin{aligned}
\eta(\xi) &= 1 \quad \text{for} \quad ||\xi|| \leq 1/4 \\
&= 0 \quad \text{for} \quad ||\xi|| \geq 1/2.
\end{aligned}
$$

Multiply (5.44) by $(\partial_t u)\eta$ and integrate by parts. We observe that the matrix A defined in preliminaries is independent of t variable. Hence, proceeding exactly as in the proof of Lemma 2.6 of [13] we have

$$
\begin{aligned}
\frac{c(n)}{(p_i+1)} \int_{B_1} \partial_t K_i H_i^{\tau_i} u_i^{p_i+1} \eta &= \frac{1}{2} \int_{B_1} (A\nabla u_i \cdot \nabla u_i) \partial_t \eta - \int_{B_1} (A \nabla u_i \cdot \nabla \eta) \partial_t u_i \\
& \quad - \frac{c(n)}{(p_i+1)} \int_{B_1} K_i \tau_i H_i^{\tau_i - 1} \partial_t H_i u_i^{p_i}.
\end{aligned}
$$

From Proposition (5.2) and (5.7) we have

$$
\left| \int_{B_1} \partial_t K_i H_i^{\tau_i} u_i^{p_i+1} \right| \leq C \int_{B_{1/2} \setminus B_{1/4}} |\nabla_{\mathbb{H}^{2n+1}} u_i|^2 + Cu_i(\xi_i)^{-p_i-1} + C\tau_i. \tag{5.49}
$$

As in the proof of Proposition (5.7), it follows that for any $\theta \in \partial B_{(0,1)}$ fixed, the function $u_i(\xi_i \circ \theta)^{-1} u_i(\xi)$ converges in $C^2_{loc}(B_2 \setminus \{0\})$ to a positive function $v(\xi) = a_1 ||\xi||^{2-\theta} + h(\xi)$. Therefore,

$$
\begin{aligned}
\int_{B_{1/2} \setminus B_{1/4}} (A \nabla u_i \cdot \nabla u_i) = \int_{B_{1/2} \setminus B_{1/4}} |\nabla_{\mathbb{H}^{2n+1}} u_i|^2 &\leq Cu_i(\xi \circ \theta)^2 \\
&\leq Cu_i(\xi)^{-2} \qquad (5.50)
\end{aligned}
$$

where the last inequality follows from Proposition (5.7). We deduce from (5.49) and (5.50) that

$$
\left| \int_{B_1} \partial_t K_i H_i^{\tau_i} u_i^{p_i+1} \right| \leq Cu_i(\xi_i)^{-2} + C\tau_i.
$$

Using the condition $(*)_\beta$, we get

$$\left|\partial_t K_i(\xi_i)\int_{B_1} H_i^{\tau_i} u_i^{p_i+1}\right| - Cu_i(\xi_i)^{-2} - C\tau_i \le \left|\int_{B_1}(\partial_t K_i(\xi_i) - \partial_t K_i(\xi))H_i^{\tau_i}u_i^{p_i+1}\right|$$

$$\begin{aligned}\le\ & CL_2(\beta,i)\int_{B_1}\left\{\sum_{s=2}^{[\beta]}|\nabla K_i(\xi_i)|^{(\beta-s)/(\beta-1)}|\xi-\xi_i|^{s-1}\right.\\ & +L_2(\beta,i)^{(\beta-[\beta])/(\beta-1)}L_1(\beta,i)^{(\beta-2)(\beta-[\beta])/(\beta-1)^2}\\ & \left.\times d(\xi,\xi_i)^{[\beta]-1+(\beta-[\beta])/(\beta-1)}\right\}u_i^{p_i+1}. \end{aligned} \tag{5.51}$$

Note that $\int_{B_1} H_i^{\tau_i}u_i^{p_i+1} \ge C(\min_{B_1} u_i)^{p_i+1}$ since H_i is positive. Thus

$$\begin{aligned}|\partial_t K_i(\xi_i)| \le\ & Cu_i(\xi_i)^{-2} + C\tau_i\\ & +CL_2(\beta,i)\int_{B_1}\left\{\sum_{s=2}^{[\beta]}|\nabla K_i(\xi_i)|^{(\beta-s)/(\beta-1)}||\xi-\xi_i||^{s-1}\right.\\ & +L_2(\beta,i)^{(\beta-[\beta])/(\beta-1)}L_1(\beta,i)^{(\beta-2)(\beta-[\beta])/(\beta-1)^2}\\ & \left.\times\ ||\xi-\xi_i||^{[\beta]-1+(\beta-[\beta])/(\beta-1)}\right\}u_i^{p_i+1} \end{aligned} \tag{5.52}$$

Define the vector fields

$$\overline{X_j} = \frac{\partial}{\partial x_j} - 2y_j\frac{\partial}{\partial t} \quad \text{for} \quad 1 \le j \le n \tag{5.53}$$

$$\overline{Y_j} = \frac{\partial}{\partial y_j} + 2x_j\frac{\partial}{\partial t} \quad \text{for} \quad 1 \le j \le n. \tag{5.54}$$

Then, as for $\partial_t K_i$, we can estimate

$$\left|\int_{B_1}(\overline{X}_j K_i)H_i^{\tau_i}u_i^{p_i+1}\right| \le Cu_i(\xi_i)^{-2} + C\tau_i$$

and

$$\left|\int_{B_1}(\overline{Y}_j K_i)H_i^{\tau_i}u_i^{p_i+1}\right| \le Cu_i(\xi_i)^{-2} + C\tau_i.$$

We can now use the fact that for $1 \le j \le n$,

$$\left|\int_{B_1}\partial_j K_i(\xi_i)H_i^{\tau_i}u_i^{p_i+1}\right| \le \left|\int_{B_1}\overline{X}_j K_i(\xi_i)H_i^{\tau_i}u_i^{p_i+1}\right| + \left|\int_{B_1}2y_j\partial_t K_i(\xi_i)H_i^{\tau_i}u_i^{p_i+1}\right|$$

$$\left|\int_{B_1} \partial_{n+j} K_i(\xi_i) H_i^{\tau_i} u_i^{p_i+1}\right| \leq \left|\int_{B_1} \overline{Y}_j K_i(\xi_i) H_i^{\tau_i} u_i^{p_i+1}\right| + \left|\int_{B_1} 2x_j \partial_t K_i(\xi_i) H_i^{\tau_i} u_i^{p_i+1}\right|$$

to obtain the estimate (5.52) for $\partial_j K_i(\xi_i)$, $1 \leq j \leq 2n$. The proof can be now completed as that of Lemma 2.6 of [13].

□

The following lemma follows from Lemma 5.11 and Lemma 5.12:

Lemma 5.13 *Under the hypothesis of Lemma 5.11 we have*

$$\begin{aligned}\tau_i \leq\ & C u_i(\xi_i)^{-2} + C(L_2(\beta, i) + L_2(\beta, i)^{\beta-1} \\ & + L_2(\beta, i)^{(\beta-[\beta])/(\beta-1)+1} L_1(\beta, i)^{(\beta-2)(\beta-[\beta]/(\beta-1)^2})\\ & \times u_i(\xi_i)^{-(2/(Q-2))[\beta]+(\beta-[\beta])/(\beta-1))},\end{aligned}$$

where $C = C(Q,\ C_0\ A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ \rho,\ \beta)$.

Lemma 5.14 *Under the hypothesis of Lemma 5.11, for any* $0 < s < 1$, *we have, for* $\beta = 2$, *that*

$$\left|\int_{B_s(\xi_i)} \nu_i(\xi) \cdot \nabla(K_i H_i^{\tau_i}) u_i^{p_i+1}\right| \leq C\tau_i u_i(\xi_i)^{-2/(Q-2)} + \circ(\mid \nabla K_i(\xi_i) \mid u_i(\xi_i)^{-2/(Q-2)})$$

$$+ C L_2(\beta, i) u_i(\xi_i)^{-4/(Q-2)}$$

and for $\beta > 2$, *we have*

$$\begin{aligned}&\left|\int_{B_s(\xi_i)} \nu_i(\xi) \nabla(K_i H_i^{\tau_i}) u_i^{p_i+1}\right| \\ \leq\ & C\tau_i u_i(\xi_i)^{-2/(Q-2)} + C \mid \nabla K_i(\xi_i) \mid u_i(\xi_i)^{-2/(Q-2)} \\ +\ & C\left\{L_2(\beta, i) + L_2(\beta, i)^{\beta-1} + L_2(\beta, i)^{\beta-[\beta])/(\beta-1)+1} L_1(\beta, i)^{(\beta-2)(\beta-[\beta])/(\beta-1)^2})\right\} \\ \times\ & u_i(\xi_i)^{-(2/(Q-2))([\beta]+(\beta-1))}\end{aligned}$$

where $C = C(Q,\ A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ s,\ \beta,\ \sigma)$.

The proof is similar to that of Lemma 2.7 of [13].

Corollary 5.15 *In addition to the hypothesis of Lemma 5.11, we further assume that either* $\beta = Q - 2$ *and* $L_1(\beta, i)$, $L_2(\beta, i) = \circ(1)$ *or* $\beta > Q - 2$ *and* $L_1(\beta, i)$, $L_2(\beta, i) = O(1)$. *Then for any* $0 < s < 1$ *we have*

$$\lim_{i\to\infty} u_i(\xi_i)^2 \int_{B_s(\xi_i)} \nu_i(\xi) \cdot \nabla(K_i H_i^{\tau_i}) u_i^{p_i+1} = 0.$$

Proof. The corollary follows from Lemma 5.14, Lemma 5.13 and Lemma 5.12.

□

6 Local results

In this section we prove some local results regarding isolated blow up points, namely that an isolated blow up point is a critical point for the function $K = \lim_{i\to\infty} K_i$ and we give sufficient conditions for an isolated blow up point to be a simple isolated blow up point.

Proposition 6.1 *Suppose that $\{K_i\} \in \Gamma_{2+\alpha}(\Omega)$ with uniform $C^1(B_2)$ modulo of continuity and satisfies*

$$K_i(\xi) \geq 1/A_1, \quad \xi \in B_2,$$

for some positive constant A_1 and (5.20) for some positive constant A_2. Suppose also that

$$A_4^{-1} \leq H_i(0) \leq A_4, \quad |\bigtriangledown H_i(y)| \leq A_5 \quad \textit{for all} \quad \xi \in B_2,$$

for some positive constants A_4, A_5. Let u_i satisfy (5.44)-(5.2) and $\xi_i \to 0$ be an isolated blow up point with (5.21) for some positive constant A_3. Then $|\bigtriangledown K_i(y_i)| \to 0$.

Proof. Suppose $|\nabla K_i(\xi_i)| \to d > 0$. Without loss of generality we assume that $\xi_i = 0$ for all i.
*Case (i)*If 0 is an isolated simple blow up point, then arguing as in Lemma 5.12, we can get

$$|\int_{B_1} \nabla K_i H_i^{\tau_i} u_i^{p_i+1}| = o(1).$$

Then by the uniform continuity of $|\nabla K_i|$ and Lemma 5.10, we have

$$|\nabla K_i(0)| \leq c \int_{B_1} |\nabla K_i(\xi) - \nabla K_i(0)| H_i^{\tau_i} u_i^{p_i+1} + o(1) = o(1).$$

which is a contradiction.
Case (ii) 0 is not a isolated simple blow up point: Recall, from Remark 5.4, we know that for every $\theta \in \partial B(0,1)$, there exists $r_i = R_i u_i(\xi_i)^{-(p_i-1)/2}$ such that $f_{i,\theta} = s^{(p_i-1)/2} u_i(\xi_i \circ s\theta)$ has a unique critical point in $(0, r_i)$. Let $\mu_i(\theta)$ denote second critical point of $f_{i,\theta}$. Then

$$\mu_i(\theta) \geq r_i. \tag{6.1}$$

Define,

$$\mu_i = \inf_{||\theta||=1} \mu_i(\theta) \tag{6.2}$$

Since 0 is not a isolated simple blow up point, we have

$$\lim_{1\to\infty} \mu_i = 0. \tag{6.3}$$

Consider the function

$$w_i(\xi) = \mu_i^{2/(p_i-1)} u_i(\mu_i \xi) \text{ for } ||\xi|| < 1/\mu_i.$$

From (5.44) and (6.3), it follows that w_i satisfies

$$\begin{aligned} -\Delta_{\mathbb{H}^{2n+1}} w_i &= c(n)\tilde{K}_i(\xi)\tilde{H}_i(\xi)^{\tau_i} w_i^{p_i} \quad \text{for} \quad ||\xi|| < 1/\mu_i \\ ||\xi||^{2/(p_i-1)} w_i(\xi) &\leq A_3, \ ||\xi|| < 1/\mu_i \\ \lim_{i\to\infty} w_i(0) &= \infty \end{aligned} \tag{6.4}$$

where $\tilde{K}_i(\xi) = K_i(\mu_i\xi)$ and $\tilde{H}_i(\xi) = H_i(\mu_i\xi)$. Moreover, for θ with $||\theta|| = 1$,

$$s^{2/(p_i-1)} f_{w_i,\theta} = s^{2/(p_i-1)} \mu_i^{2/(p_i-1)} u_i(s\mu_i\theta)$$

has precisely one critical point in $(0,1)$ since for $0 < s < 1$ we have $0 < s\mu_i < \mu_i \leq \mu_i(\theta)$. Hence, origin is a isolated simple blow up point for $\{w_i\}$.

Applying Lemma 5.1, Proposition 5.7 and subelliptic estimates(see Claim 5.3), after passing to subsequence we have

$$\lim_{i\to\infty} w_i(0)w_i(\xi) = w(\xi) = a||\xi||^{2-Q} + h(\xi) \tag{6.5}$$

where a is a positive constant and h satisfies

$$\Delta_{\mathbb{H}^{2n+1}} h \equiv 0.$$

Since w is positive, $\liminf_{||\xi||\to\infty} h(\xi) \geq 0$. The Harnack inequality implies that $h \equiv C$, a nonnegative constant.

We claim that the constant a occuring in (6.5) is positive: Let $\theta_i \in \partial B(0,1)$ be such that $\mu_i(\theta_i) = \mu_i$ then

$$\frac{d}{ds}\{s^{2/(p_i-1)} f_{w_i,\theta_i}\} \,|_{s=1} = 0. \tag{6.6}$$

Multiplying (6.6) by $w_i(0)$ we still have

$$\frac{d}{ds} w_i(0)\{s^{2/(p_i-1)} f_{w_i,\theta_i}\} \,|_{s=1} = 0.$$

Taking limit as $i \to \infty$ we see that there exists $\theta_0 \in \partial B(0,1)$ such that

$$\frac{d}{ds}\{s^{(Q-2)/2} f_{w,\theta_0}\} \,|_{s=1} = 0. \tag{6.7}$$

Substituting the expression for w from (6.5), we have

$$\begin{aligned} 0 &= \frac{d}{ds}|_{s=1} \left\{ a||s\theta_0||^{2-Q} s^{(Q-2)/2} + C s^{(Q-2)/2} \right\} \\ &= a(2-Q)/2 + C(Q-2)/2. \end{aligned} \tag{6.8}$$

and hence $a = C > 0$.

Now applying Corollary 4.2 to the equation (6.11) and using Proposition 5.7 to estimate w_i, we have for any $0 < \sigma < 1$, that

$$\begin{aligned}&\int_{\partial B(0,\sigma)} \mathcal{B}(\sigma, \xi, w_i, \nabla_{\mathrm{H}^{2n+1}} w_i)\\ \geq\ & \frac{c(n)}{p_i+1}\int_{B_\sigma} \nu(\xi)\cdot\nabla(\tilde{K}_i\tilde{H}_i)v_i^{p_i+1} - \frac{\sigma c(n)}{p_i+1}\int_{\partial B_\sigma} \tilde{K}_i\tilde{H}_i w_i^{p_i+1}\\ \geq\ & \frac{c(n)}{p_i+1}\int_{B_\sigma} \nu(\xi)\cdot\nabla(\tilde{K}_i\tilde{H}_i)w_i^{p_i+1} - O(w_i(0)^{-p_i-1}).\end{aligned}$$

We can now complete the proof as in [13].

□

We will now show that under suitable hypothesis, the isolated blow up points are isolated simple blow up points.

Proposition 6.2 *Suppose $\{K_i\} \in \Gamma_{2+\alpha}$ satisfies (2.6) in the ball $B(0,2) \in H^{2n+1}$ for some positive constant A_1 and $(*)_{Q-2}$ for some constants L_1, L_2 in $B(0,2)$, $\{H_i\} \in C^1(B(0,2))$ satisfies (5.45) for some positive constants A_4, A_5. Suppose also that u_i satisfies (5.44)-(5.2) and $\xi_i \to 0$ is an isolated blow up point with (5.21) for some positive constant A_3. Then it has to be a isolated simple blow up point.*

Proof: By contradiction, suppose that 0 is not an isolated simple blow up point. Also, witout loss of generality assume that $\xi_i = 0$ for all i. We then proceed as in Case(ii) in the proof of Proposition(6.1) to choose $\mu_i(\theta)$, a second critical point of $f_{i,\theta}$ with

$$\mu_i(\theta) \geq r_i. \tag{6.9}$$

Again let

$$\mu_i = \inf_{||\theta||=1} \mu_i(\theta).$$

Since 0 is not a isolated simple blow up point, we have

$$\lim_{\imath\to\infty} \mu_i = 0. \tag{6.10}$$

Define

$$w_i(\xi) = \mu_i^{2/(p_i-1)} u_i(\mu_i\xi), \quad ||\xi|| < 1/\mu_i.$$

As before, it can be verified that w_i satisfies the equation

$$\begin{aligned} -\Delta_{\mathrm{H}^{2n+1}} w_i &= c(n)\tilde{K}_i(\xi)\tilde{H}_i(\xi)^{\tau_i} w_i^{p_i} \quad \text{in} \quad B(0, 1/s_i)\\ ||\xi||^{2/(p_i-1)} w_i(\xi) &\leq A_3, \quad \text{for} \quad ||\xi|| < 1/s_i\\ \lim_{i\to\infty} w_i(0) &= \infty \end{aligned} \tag{6.11}$$

where $\tilde{K}_i(\xi) = K_i(s_i\xi)$ and $\tilde{H}_i(\xi) = H_i(s_i\xi)$. Furthermore, note that for any $\theta \in \mathrm{H}^{2n+1}$, $||\theta|| = 1$,

$$f_{w_i,\theta} \text{ has precisely one critical point in } 0 < s < 1. \tag{6.12}$$

In other words, 0 is a isolated simple blow up point for the sequence $\{w_i\}$. Applying Lemma 5.1, Proposition 5.7 and subelliptic estimates as in Claim 5.3, we conclude that there exists a constant $a > 0$ and a function h such that

$$w_i(0)w_i(\xi) \to w(\xi) = a||\xi||^{2-Q} + h(\xi) \tag{6.13}$$

where

$$\Delta_{\mathbb{H}^{2n+1}} h(\xi) = 0. \tag{6.14}$$

Since w is positive, $\liminf_{||\xi||\to\infty} h(\xi) \geq 0$, the maximum principle implies that h is a nonnegative function. Applying the Harnack inequality, we further conclude that $h(\xi) \equiv constant = C \geq 0$. Moreover, as in (6.8) we have $C = a > 0$.

Now applying Proposition 4.3 and Proposition 5.7 to the equation (6.11), we have for any $0 < \sigma < 1$, that

$$\begin{aligned} & \int_{\partial B(0,\sigma)} \mathcal{B}(\sigma, \xi, v_i, \nabla_{\mathbb{H}^{2n+1}} w_i) \\ \geq & \frac{c(n)}{p_i+1} \int_{\partial B_\sigma} \nu(\xi)\cdot \nabla(\tilde{K}_i \tilde{H}_i) w_i^{p_i+1} - \frac{\sigma c(n)}{p_i+1} \int_{\partial B_\sigma} \tilde{K}_i \tilde{H}_i w_i^{p_i+1} \\ \geq & \frac{c(n)}{p_i+1} \int_{\partial B_\sigma} \nu(\xi)\cdot \nabla(\tilde{K}_i \tilde{H}_i) w_i^{p_i+1} - O(w_i(0)^{-p_i-1}). \end{aligned}$$

Multiplying this equation by $w_i(0)^2$ and letting $i \to \infty$ we have

$$\begin{aligned} \int_{\partial B(0,\sigma)} \mathcal{B}(\sigma, \xi, w, \nabla_{\mathbb{H}^{2n+1}} w) & \geq \lim_{i\to\infty} w_i(0)^2 \int_{B(0,\sigma)} \mathcal{B}(\sigma, \xi, w_i, \nabla_{\mathbb{H}^{2n+1}} w_i) \\ & \geq \lim_{i\to\infty} w_i(0)^2 \frac{c(n)}{p_i+1} \int_{B_\sigma} \nu\cdot \nabla(\tilde{K}_i \tilde{H}_i) w_i^{p_i+1} \\ & = 0 \end{aligned} \tag{6.15}$$

where (6.15) follows from Corollary 5.15.

On the other hand, from (6.13) and (ii) of Proposition 4.3 we conclude that

$$\int_{\partial B(0,\sigma)} \mathcal{B}(\sigma, \xi, w, \nabla_{\mathbb{H}^{2n+1}} w) < 0$$

for $\sigma > 0$ sufficiently small, a contradiction to (6.15). This completes the proof. □

7 Proof of theorem 2.1

Observe that the equation (1.2) can be rewritten as

$$\Delta_b u - n(2n+1)u + c(n)\tilde{K} u^{Q+2/Q-2} = 0$$

where $\tilde{K}(\xi) = u^{p-Q+2/Q-2}(\xi)K(\xi)$ and $c(n) = \frac{2(2n+1)}{n+1}$. This allows us to use the transformation laws given in [9] and hence, $v = \Lambda_0 u$ satisfies

$$\Delta_{\mathrm{H}^{2n+1}} v + c(n)\tilde{K} v^{Q+2/Q-2} = 0 \text{ in } \mathrm{H}^{2n+1}. \tag{7.1}$$

The following proposition shows that under suitable conditions on the curvature functions, if a sequence of solutions for (1.2) with finite energy blows up, then the blow up points are necessarily isolated blow up points.

Proposition 7.1 *Suppose that $K \in \Gamma_{2+\alpha}(S^{2n+1})$ satisfies*

$$K(\xi) \geq 1/A_1 \text{ and } |\nabla K|_{L^\infty} \leq A_2$$

for some positive constants A_1 and A_2. Given R, $\varepsilon > 0$, there exist some positive constants $C_0^* = C_0^*(\varepsilon, R, n, A_1, A_2)$, $C_1^* = C_1^*(\varepsilon, R, n, A_1, A_2)$ such that if u_i is any solution of (1.2) with

$$\mathcal{E}(u_i) \leq C_0$$

where $\mathcal{E}(u_i)$ is defined in (1.3), and

$$max_{S^{2n+1}} u_i \geq C_0^*.$$

Then there are points $S(u_i) = \{P_1^{(i)}, \ldots, P_k^{(i)}\}$ $(1 \leq k = k(u_i) < \infty)$ which are local maxima of u_i such that
(i) $0 \leq \tau_i \leq \varepsilon$.
(ii) for each j, $1 \leq j \leq k$, and ξ a CR normal coordinates centered at P_j, we have

$$||v_i(0)^{-1} v_i(v_i(0)^{-2/(Q-2)}\xi) - \Lambda_i(\xi)||_{C^2(B(0,2R))} < \varepsilon \tag{7.2}$$

where $\Lambda_i(\xi) = \Lambda_0(K(P_i)^{1/2}\xi)$ and $v_i = \Lambda_0 u_i$. Recall that $\Lambda_0(\xi) = C((1+|z|^2)^2 + t^2)^{-(Q-2)/4}$.
(iii) $u_i(\xi) \leq C_1^*\{dist(\xi, S(u_i)\}^{-2/(p-1)}$ for all $\xi \in S^{2n+1}$. Here dist is the distance on the sphere described in the preliminaries.

Proof. Let $P_1^{(i)} \in S^{2n+1}$ be such that $u_i(P_1^{(i)}) = \max_{S^{2n+1}} u_i$. Since $\{u_i\}$ is a blow up sequence, we know that $u_i(P_1^{(i)}) \to \infty$. We do the following analysis for large i: Reduce the problem to H^{2n+1} using the Cayley transform such that the point $P_1^{(i)}$ is mapped to the origin. Let $v_i := \Lambda_0 u_i$. Then v_i satisfies (7.1). Define

$$w_i(\xi) = v_i(0)^{-1} v_i(v_i(0)^{-2/Q-2}\xi),$$

then w_i satisfies

$$\begin{aligned} \Delta_{\mathrm{H}^{2n+1}} w_i + c(n)\tilde{K}_i w_i^{Q+2/Q-2} &= 0 \text{ in } \mathrm{H}^{2n+1} \\ w_i(0) &= 1 \end{aligned}$$

$$\tilde{K}_i(\xi) = K_i(v_i(0)^{-2/Q-2}\xi) u^{-\tau_i}(v_i(0)^{-2/Q-2}\xi).$$

Also, using change of variables and the relation (2.2) in the preliminaries, it can be seen that

$$\begin{aligned}\int_{\mathbb{H}^{2n+1}} |\nabla_{\mathbb{H}^{2n+1}} w_i|^2 \theta_0 \wedge d\theta_0^n &= \int_{\mathbb{H}^{2n+1}} |\nabla_{\mathbb{H}^{2n+1}} v_i|^2 \theta_0 \wedge d\theta_0^n \\ &= \int_{S^{2n+1}} \left(L_{\theta_1}^*(du_i, du_i) + n(2n+1)u_i^2\right) \theta_1 \wedge d\theta_1^n \\ &\leq C_0.\end{aligned}$$

Using subelliptic estimates as in Claim 5.3, we can conclude that a subsequence of $\{w_i\}$ converges in C^2_{loc} to $\Lambda_1(\xi) = \Lambda_0(k^{1/2}\xi)$ which is the solution of

$$\Delta_{\mathbb{H}^{2n+1}}\Lambda_1 + c(n)k\Lambda_1^{Q+2/Q-2} = 0 \text{ in } \mathbb{H}^{2n+1} \tag{7.3}$$

where $k := \lim_{i\to\infty} K_i(P_1^{(i)})$. Let $\Lambda_i = \Lambda_0(K_i(P_1^{(i)})^{1/2}\xi)$, then $\lim_{i\to\infty}\Lambda_i = \Lambda_0$. Hence for given $\varepsilon > 0$, it follows that

$$||w_i(\xi) - \Lambda_i||_{B(0,2R)} < \varepsilon. \tag{7.4}$$

Choose $\varepsilon < min_{||\xi||=R}\Lambda_i(\xi)$. Then for $||\xi|| < R$, we have

$$w_i(\xi) = v_i(0)^{-1}v_i(v_i(0)^{-2/Q-2}\xi) < \varepsilon + \Lambda_i(\xi) \leq 2\Lambda_i(\xi)$$

i.e.,

$$(||\xi||v_i(0)^{-2/(Q-2)})^{(Q-2)/2}v_i(v_i(0)^{-2/Q-2}\xi) \leq C$$

for some constant C which implies that

$$u_i(\xi) \leq Cd(\xi, P_1^{(i)})^{-2/(p_i-1)}$$

in a neighbourhood of the point $P_1^{(i)}$.

Without loss of generality, if we further choose $\varepsilon < 1/2\lim_{i\to\infty}\min_{B(0,1)}\Lambda_i$. then from (2.3) and (ii) of Proposition 7.1 we have,

$$\begin{aligned}\int_{\overline{B}(P,u_i)} K_i u_i^{p_i+1}\theta_1 \wedge d\theta_1 &= \int_{\overline{B}(0,v_i)} K_i v_i^{p_i+1}\theta_0 \wedge d\theta_0 \\ &\geq \int_{\overline{B}(0,v_i)} K_i \left(\Lambda_i(v_i(0)^{2/(Q-2)}\xi) - \varepsilon\right)^{p_i+1} dzdt \\ &\geq C1/A_1\varepsilon^{p_i+1}v_i(0)^{-\tau_i}|B(0,1)| \\ &> 0,\end{aligned} \tag{7.5}$$

where $\overline{B}(P, u_i) := B(P, u_i(P)^{-2/(Q-2)})$.

If the inequality

$$u_i(\xi) \leq Cd(\xi, P_1^{(i)})^{-2/(p_i-1)}$$

does not hold for every $\xi \in S^{2n+1}$, then we may repeat the above argument by taking $P_2^{(i)}$ to be a maximum of the function $d(\xi, P_1^{(i)})^{2/(p_i-1)} u_i(\xi)$.

Observe that, for each fixed i the integral $\int K_i u_i^{p_i+1}$ is finite as the function u_i is smooth. Hence, there are only finitely many points in the set $\mathcal{S}(u_i)$ since from (7.5), each ball $B(P_k, u_i(P_k)^{-2/(p_i-1)})$ contributes a positive amount to the integral $\int K_i u_i^{p_i+1}$.

This completes the proof.

□

After passing to a subsequence, if $\{u_i\}$ stays bounded in $L^\infty(S^{2n+1})$, then subelliptic estimates (as in Claim 5.3) further imply that it remains bounded in $C^{2,\alpha}$, $0 < \alpha < 1$.

However, if the sequence $\{u_i\}$ blows up, then in view of Proposition 7.1, Proposition 6.1 and Proposition 6.2, under the assumptions of the Theorem 2.1 it follows that $\{u_i\}$ has only isolated simple blow up points. Depending on the assumptions on 'flatness' of K_i, we consider following two cases:

Case 1 of theorem 2.1. *K_i satisfies $(*)_\beta$, $\beta \geq Q-2$ in $\Omega_{d,i}$ with $L_1(\beta)$ and $L_2(\beta)$ constants independent of i:*

Under this assumption, we prove that the isolated simple blow up points are separated by a fixed, positive distance.

Theorem 7.2 *Suppose that $K \in C^1(S^{2n+1})$ satisfies, for some positive constant A_1, that*

$$K(q) \geq 1/A_1, \quad \text{for all} \quad q \in S^{2n+1}.$$

Suppose also that there exists some constant $d > 0$, such that, K satisfies $(*)_{(n-2)}$ for some constants L_1 and L_2 in $\Omega_d = \{q \in S^{2n+1} | \; |\nabla K(q)| < d\}$. Then for $\varepsilon > 0$ and $R > 1$, there exists some positive constant $\delta^* = \delta^*(n, \varepsilon, R, A_1, L_1, L_2, d$ the modulo of continuity of $\nabla K) > 0$ such that for any solution u of (1.2)-(2.9) with $\max_{S^{2n+1}} u > C_0^*$ we have

$$|q_j - q_l| \geq \delta^*, \quad \text{for all} \quad 1 \leq j \neq l \leq k,$$

where $q_j = q_j(u), q_l = q_l(u), k = k(u)$ are the ones defined in Proposition 7.1.

Proof: Suppose that there exists a sequence $\{u_i\}$ such that

$$\lim_{i \to \infty} \inf_{j \neq l} d(\xi_i^{(j)}, \xi_i^{(l)}) = 0.$$

Without loss of generality, let

$$d(\xi_i^{(1)}, \xi_i^{(2)}) = \inf_{j \neq l} d(\xi_i^{(j)}, \xi_i^{(l)}) \to 0. \tag{7.6}$$

From Proposition 7.1, it follows that the balls $B(\xi_i^{(1)}, R u_i^{-(p_i-1)/2})$ and $B(\xi_i^{(2)}, R u_i^{-(p_i-1)/2})$ are disjoint. Hence (7.6) implies that

$$\lim_{i \to \infty} u_i(\xi_i^{(1)}) = \infty = lim_{i \to \infty} u_i(\xi_i^{(2)}).$$

Take the Cayley transform from S^{2n+1} to $\mathbb{H}^{2n+1}$ such that $\xi_i^{(1)}$ is mapped onto the origin. We continue to denote the image of $\xi_i^{(2)}$ as $\xi_i^{(2)}$. Let $v_i(\xi) = \Lambda_0(\xi)u_i(\xi)$. From (7.1), it satisfies the equation

$$\Delta_{\mathbb{H}^{2n+1}} v_i + c(n)K_i\Lambda_0^{\tau_i}v_i^{p_i} = 0 \quad \text{in} \quad \mathbb{H}^{2n+1} \tag{7.7}$$

where $\tau_i = \frac{Q+2}{Q-2} - p_i$. Observe that this is similar to the equation (5.44) with $H_i(\xi) = \Lambda_0(\xi)$.

Let $\sigma_i = ||\xi_i^{(2)}|| \to 0$ and without loss of generality suppose that $\xi_i^{(2)}$ is a point of local maximum of u_i. From the blow up analysis, it is clear that there exists a constant $C(n)$ depending on n such that

$$\sigma_i > \frac{1}{C(n)} \max\{R_i u_i(0)^{-(p_i-1)/2}, R_i u_i(\xi_i^{(2)})^{-(p_i-1)/2}\}.$$

We rescale the function v_i by defining

$$w_i(\xi) = \sigma_i^{2/(p_i-1)} v_i(\sigma_i\xi) \quad \text{in} \quad ||\xi|| < 1/\sigma_i.$$

It satisfies the equation

$$\left.\begin{array}{rcl} -\Delta_{\mathbb{H}^{2n+1}} w_i &=& c(n)\tilde{K}_i\tilde{H}_i^{\tau_i} w_i^{p_i} \quad \text{in} \quad ||\xi|| < 1/\sigma_i \\ w_i(\xi) &>& 0 \quad \text{in} \quad ||\xi|| < 1/\sigma_i. \end{array}\right\} \tag{7.8}$$

where $\tilde{K}_i = K_i(\sigma_i\xi)$, $\tilde{H}_i = H_i(\sigma_i\xi)$. It follows from Prop. 7.1 that

$$\begin{array}{rcl} ||\xi||^{2/(p_i-1)} v_i(\xi) &\leq& C_1 \quad \text{in} \quad ||\xi|| < 1/2\sigma_i \\ ||\xi - \xi_i^2||^{2/(p_i-1)} v_i(\xi) &\leq& C_1 \quad \text{in} \quad ||\xi - \xi_i^2|| < 1/2\sigma_i. \end{array}$$

It follows that

$$\left.\begin{array}{rcl} \lim_{i\to\infty} w_i(0) = \infty, & & \lim_{i\to\infty} w_i(\sigma_i^{-1}\xi_i^2) = \infty \\ ||\xi||^{2/(p_i-1)} w_i(\xi) &\leq& C_1 \quad \text{in} \quad ||\xi|| < 1/2 \\ ||\xi - \sigma_i^{-1}\xi_i^2||^{2/(p_i-1)} w_i(\xi) &\leq& C_1 \quad \text{in} \quad ||\xi - \sigma_i^{-1}\xi_i^2|| < 1/2 \end{array}\right\} \tag{7.9}$$

i.e., both 0 and $\sigma_i^{-1}\xi_i^{(2)}$ are isolated blow up points for $\{w_i\}$. Infact, we claim that they are isolated simple blow up points. From Proposition 6.1, it follows that $|\nabla\tilde{K}_i(0)| \to 0$ and that $|\nabla\tilde{K}_i(\sigma_i^{-1}\xi_i^{(2)})| \to 0$. Since $\sigma_i \to 0$, we consider the following two cases:

Case (i): $|\nabla K_i(0)| \geq d$ for large i.

Suppose 0 is not a isolated simple blow up point. Therefore, there exists $\mu_i \to 0$, chosen as in (6.2) i.e.,

$$\mu_i = \inf_{||\theta||=1} \mu_i(\theta)$$

such that $\mu_i(\theta)$ is a critical point of $f_{w_i,\theta}$, $\theta \in \partial B(0,1)$. Clearly, $\mu_i(\theta) \geq r_i = R_i u_i(0)^{-(p_i-1)/2}$.

Now consider the function $\tilde{w}_i(\xi) = s^{2/(p_i-1)} w_i(\mu_i\xi)$ in $||\xi|| < 1/\mu_i$. From (7.8) and (7.9), it can be verified that $\tilde{w}_i$ satisfies

$$\left.\begin{array}{rcl} -\Delta_{\mathbb{H}^{2n+1}} \tilde{w}\mu_i & = & c(n)\hat{K}_i\hat{H}_i^{\tau_i}\tilde{w}_i^{p_i} \quad \text{in} \quad ||\xi|| < 1/\mu_i \\ ||\xi||^{2/(p_i-1)}\tilde{w}_i(\xi) & \leq & C_1 \quad \text{in} \quad ||\xi|| < 1/si \\ \lim_{i\to\infty} \tilde{w}_i(0) & = & \infty \\ \frac{d}{ds}|_{s=1} f_{\tilde{w}_i,\theta_i} & = & 0, \end{array}\right\} \tag{7.10}$$

where $\hat{K}_i(\xi) = \tilde{K}_i(\mu_i\xi) = K_i(\mu_i\sigma_i\xi)$ and similarly, $\hat{H}_i(\xi) = H_i(\mu_i\sigma_i\xi)$. Moreover

$$f_{\tilde{w}_i,\theta} \text{ has precisely one critical point in the interval } (0,1) \text{ for every } \theta \in \partial B(0,1) \tag{7.11}$$

Thus 0 is a isolated simple blow up point for $\tilde{w}_i$ and from subelliptic theory (as in Claim 5.3), Proposition 5.7 and Lemma 5.1, we conclude that there exists a positive constant a and a function h with $\Delta_{\mathbb{H}^{2n+1}} h \equiv 0$ in $\mathbb{H}^{2n+1}$ such that

$$\tilde{w}_i(0)\tilde{w}_i(\xi) \to w(\xi) = a||\xi||^{2-Q} + h(\xi) \text{ in } C^2_{loc}(\mathbb{H}^{2n+1} \setminus \{0\}). \tag{7.12}$$

Since w is positive, $\liminf_{||\xi||\to\infty} h(\xi) \geq 0$. The maximum principle implies that h is non negative. We can further apply th Harnack inequality to conclude that $h(\xi) \equiv constant = C > 0$(say). Let θ_i be such that μ_i is a critical point of f_{w_i,θ_i} and $\lim_{i\to\infty}\theta_i = \tilde{\theta} \in \partial B(0,1)$. Then,

$$\frac{d}{ds}|_{s=1} f_{w,\tilde{\theta}} = \lim_{i\to\infty} \frac{d}{ds}|_{s=1} f_{\tilde{w}_i,\theta_i} = 0$$

and hence as in proof of Proposition 6.1 we get

$$C = a > 0.$$

Thus, for any $0 < \sigma < 1$, we have

$$\int_{\partial B(0,\sigma)} \mathcal{B}(\sigma, \xi, \tilde{w}_i, \nabla_{\mathbb{H}^{2n+1}}\tilde{w}_i). \tag{7.13}$$

On the other hand, applying Corollary 1.1 and Proposition 5.7 to (7.9), we have for $0 < \sigma < 1$,

$$\begin{aligned} &\int_{\partial B(0,\sigma)} \mathcal{B}(\sigma, \xi, \tilde{w}_i, \nabla_{\mathbb{H}^{2n+1}}\tilde{w}_i) \\ \geq\ & \frac{c(n)}{p+1}\int_{B(0,\sigma)} \nu(\xi)\cdot\nabla(\hat{K}_i\hat{H}_i^{\tau_i})\tilde{w}_i^{p_i+1} - \frac{c(n)}{p_i+1}\int_{B(0,\sigma)} \hat{K}_i\hat{H}_i^{\tau_i}\tilde{w}_i^{p_i+1} X\cdot N\, dzdt \\ \geq\ & \frac{c(n)}{p+1}\int_{B(0,\sigma)} \nu(\xi)\cdot\nabla(\hat{K}_i\hat{H}_i^{\tau_i})\tilde{w}_i^{p_i+1} - O(\tilde{w}_i(0)^{-p_i-1}). \end{aligned}$$

Multiplying by $\tilde{w}_i(0)^2$ and taking the limit as $i \to \infty$, we have

$$\begin{aligned}
&\int_{\partial B(0,\sigma)} \mathcal{B}(\sigma,\xi,w,\nabla_{\mathbb{H}^{2n+1}} w) \\
&= \lim_{i\to\infty} \tilde{w}_i(0)^2 \int_{\partial B(0,\sigma)} \mathcal{B}(\sigma,\xi,\tilde{w}_i,\nabla_{\mathbb{H}^{2n+1}} \tilde{w}_i) \\
&\geq \lim_{i\to\infty} \tilde{w}_i(0)^2 \frac{c(n)}{p_i+1} \int_{B(0,\sigma)} \nu(\xi)\cdot\nabla(\hat{K}_i \hat{H}_i^{\tau_i})\tilde{w}_i^{p_i+1} \qquad (7.14)
\end{aligned}$$

Now as in proof of Proposition 4.2 of [13], we can show that

$$\lim_{i\to\infty} \tilde{w}_i(0)^2 \frac{c(n)}{p_i+1} \int_{B(0,\sigma)} \nu(\xi)\cdot\nabla(\hat{K}_i \hat{H}_i^{\tau_i})\tilde{w}_i^{p_i+1} = 0$$

which contradicts (7.13). Hence 0 is an isolated simple blow up point of $\{w_i\}$.

We can similarly show that $\sigma_i^{-1}\xi_i^{(2)}$ is also an isolated simple blow up point.

Case (ii): If $|\nabla K_i(0)| < d$, then from the hypothesis, K_i satisfies the condition $(*)_{n-2}$ there and it can be seen that $\{\hat{K}_i\}$ satisfies $(*)_{n-2}$ for $L_1(i)$, $L_2(i) = o(1)$ in $B_2 \in \mathbb{H}^{2n+1}$. Therefore, Proposition 6.2 implies that 0 and $\sigma_i^{-1}\xi_i^{(2)}$ are isolated simple blow up points in this case too.

Because of property (iii) in Proposition 7.1, the set $\mathcal{S}$ of blow up points for $\{w_i\}_i$ is countable with distance between any two points atleast one due to the rescaling. Also, the Harnack inequality implies that the function

$$\begin{aligned}
w_i(0)w_i(\xi) &= w(\xi) \quad \text{in} \quad C^0_{loc}(\mathbb{H}^{2n+1}\setminus\mathcal{S}) \\
w(\xi) &> 0.
\end{aligned}$$

Since 0 and $\sigma_i^{-1}\xi_i^{(2)}$ are isolated simple blow up points, it follows that w is singular at 0 and $\xi^{(2)} = \lim_{i\to\infty}\sigma_i^{-1}\xi_i^{(2)}$. Also, observe that the points at which w is singular is contained in $\mathcal{S}$. Therefore, using subelliptic estimates and maximum principle as before, we can write

$$w(\xi) = a_1||\xi||^{2-Q} + a_2||\xi-\xi^{(2)}||^{2-Q} + h(\xi) \quad \text{in} \quad \mathbb{H}^{2n+1}\setminus\{\mathcal{S}\setminus\{0,\xi^{(2)}\}\}$$

where

$$\begin{aligned}
\Delta_{\mathbb{H}^{2n+1}} h(\xi) &= 0 \quad \text{in} \quad \mathbb{H}^{2n+1}\setminus\{\mathcal{S}\setminus\{0,\xi^{(2)}\}\} \\
h(\xi) &\geq 0 \quad \text{in} \quad \mathbb{H}^{2n+1}\setminus\{\mathcal{S}\setminus\{0,\xi^{(2)}\}\}.
\end{aligned}$$

Therefore, from Proposition 4.3 for $0<\sigma<1$ small we have

$$\int_{\partial B_\sigma} \mathcal{B}(\sigma,\xi,w,\nabla_{\mathbb{H}^{2n+1}} w) < 0. \qquad (7.15)$$

However, using (7.8) and Corollary 4.2 we have

$$\begin{aligned}
&\int_{\partial B(0,\sigma)} \mathcal{B}(\sigma,\xi,w,\nabla_{\mathrm{H}^{2n+1}}w)\\
&= \lim_{i\to\infty} w_i(0)^2 \int_{\partial B(0,\sigma)} \mathcal{B}(\sigma,\xi,w_i,\nabla_{\mathrm{H}^{2n+1}}w_i)\\
&\geq \lim_{i\to\infty} w_i(0)^2 \frac{c(n)}{p_i+1}\int_{B(0,\sigma)} \nu(\xi)\cdot\nabla(\tilde{K}_i\tilde{H}_i^{\tau_i})w_i^{p_i+1}\\
&= 0.
\end{aligned}\tag{7.16}$$

The last inequality follows from Corollary 5.15 and the fact that $\tilde{K}_i$ satisfies $(*)_{Q-2}$ for $L_1(i)$, $L_2(i)=o(1)$ in the Case (ii). Whereas, for Case (i) we conclude it from the direct estimates as in Proposition 4.2 of [13].

The equations (7.15) and (7.16) give a contradiction and the proof of Theorem 7.2 is complete.

□.

Case 2 of theorem 2.1. *Either K_i satisfies $(*)_\beta$, for $\beta > Q-2$ in $\Omega_{d,i}$ with $L_1(\beta)$ and $L_2(\beta)$ constants independent of i, or K_i satisfies $(*)_{Q-2}$ with $L_1(i)$, $L_2(i)=o(1)$ in $\Omega_{d,i}$:*

Claim: $\{u_i\}$ has precisely one isolated simple blow up point.

Proof: Suppose on the contrary that there exists points $\xi_i^{(1)}$, $\xi_i^{(2)} \in S^{2n+1}$, $\xi_i^{(1)} \neq \xi_i^{(2)}$ such that $\lim_{i\to\infty} u_i(\xi_i^{(1)}) = \infty = \lim_{i\to\infty} u_i(\xi_i^{(2)})$. Let $\xi_i^{(1)} \to \xi^{(1)}$, $\xi_i^{(2)} \to \xi^{(2)}$ and suppose that $\xi^{(1)} \neq \xi^{(2)}$.

Without loss of generality, we may also assume that $\xi^{(1)}$ and $\xi^{(2)}$ are not antipodal points. As in the previous proof, we reduce the problem to H^{2n+1} using the Cayley transform, using the same notations for the images of the points under this map such that $\xi^{(1)} \mapsto 0$ and $\xi^{(2)} \mapsto \xi^{(2)}$. We may further assume that $\xi_i^{(1)}$, $\xi_i^{(2)}$ are both local maxima of $v_i := \Lambda_0 u_i$.

The function v_i satisfies

$$\Delta_{\mathrm{H}^{2n+1}} v_i + c(n) K_i \Lambda_0^{\tau_i} v_i^{p_i} = 0 \quad \text{in} \quad \mathrm{H}^{2n+1}.$$

Moreover, in this new coordinates, K_i satisfies $(*)_\beta$ for some constants $L_1'(\beta)$, $L_2'(\beta)$ independent of i or the condition $(*)_{Q-2}$ for some constants $L_1'(i)$, $L_2'(i) = o(i)$ in some open set of H^{2n+1} containing 0 and $\xi^{(2)}$.

We conclude that 0 and $\xi_i^{(2)}$ are isolated simple blow up points from Theorem 7.2. Note that Theorem 7.2 infact implies that the number of blow up points for u_i is bounded by some constant independent of i. Therefore, there ex ists a finite set $\mathcal{F} \subset \mathrm{H}^{2n+1}$ such that 0, $\xi^{(2)} \in \mathcal{F}$, constants a_1, $a_2 > 0$ and functions $h(\xi) \in C^0(\mathrm{H}^{2n+1} \setminus \mathcal{F})$, $g(\xi) \in C^0(\mathrm{H}^{2n+1}\{\setminus \mathcal{F}\ \{0,\xi^{(2)}\}\})$, such that

$$\begin{aligned}
lim_{i\to\infty} v_i(0) v_i(\xi) &= h(\xi) \ \text{ in } \ C^0_{loc}(\mathrm{H}^{2n+1}\setminus\mathcal{F})\\
h(\xi) &= a_1\|\xi\|^{2-Q} + a_2\|\xi - \xi^{(2)}\|^{2-Q} + g(\xi) \ \text{ in } \ \mathrm{H}^{2n+1}\setminus\mathcal{F}\\
\Delta_{\mathrm{H}^{2n+1}} g(\xi) &= 0 \ \text{ in } \ \mathrm{H}^{2n+1}\setminus\{\mathcal{F}\setminus\{0,\xi^2\}\}
\end{aligned}$$

and h is singular near $\mathcal{F}$. Since h is positive, maximum principle implies that

$$g \geq 0 \text{ in } \mathrm{H}^{2n+1} \setminus \{\mathcal{F} \setminus \{0, \xi^2\}\}.$$

Therefore, there exists a constant $A > 0$ such that for ξ near 0 we have

$$h(\xi) = a_1 ||\xi||^{2-Q} + A + O(\xi).$$

Applying (ii) of Proposition 4.3 we conclude that for σ small

$$\int_{\partial B_\sigma} B(\sigma, \xi, v_i, \nabla_{\mathrm{H}^{2n+1}} v_i) < 0.$$

whereas, the Corollary 5.15 implies that

$$\lim_{i \to \infty} v_i(0)^2 \int_{B_\sigma} \nu(\xi) \cdot \nabla(K_i H_i^{\tau_i}) v_i^{p_i+1} = 0$$

a contradiction.

□

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