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Multiple Positive Solutions For Nonautonomous Quasicritical Elliptic Problems in Unbounded Domains

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Abstract

The problem $-\Delta u + a_\varepsilon(x)u = u^{\frac{N+2}{N-2}-\varepsilon}$, $\varepsilon > 0$, with boundary Dirichlet zero data is considered in an exterior domain $\Omega \subset \mathbb{R}^N$. Assuming that, as $\varepsilon \to 0$, a_ε concentrates and blows up at a point of Ω, namely $a_\varepsilon(x) = a_0 + \frac{1}{\varepsilon^{2\alpha}} a\left(\frac{x-x_0}{\varepsilon^\alpha}\right)$, $\alpha \in \mathbb{R}^+ \setminus \{0\}$, $x_0 \in \Omega$, the existence of at least 2 distinct positive solutions is proved, if $|a|_{L^{N/2}}$ is suitably small. Furthermore, if $a_\varepsilon(x)$ has a suitable behaviour at infinity, the existence of another positive solution is shown.

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1 Introduction

This paper deals with the problem

$$(P_\varepsilon) \qquad \begin{cases} -\Delta u + a_\varepsilon(x)u &= u^{p-\varepsilon} \quad \text{in } \Omega \\ u &> 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where $\Omega = \mathbb{R}^N \setminus \overline{\omega}$, ω being a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $p = \frac{N+2}{N-2} = 2^* - 1$, 2^* being the critical Sobolev exponent. Also, ε is a positive real parameter and the potential a_ε is a positive function that, as ε goes to 0, concentrates and blows up at a point of Ω, namely a_ε has the form

$$a_\varepsilon(x) = a_0 + \frac{1}{\varepsilon^{2\alpha}} a\left(\frac{x - x_0}{\varepsilon^\alpha}\right), \qquad \alpha \in \mathbb{R}^+ \setminus \{0\},\ x_0 \in \Omega. \tag{1.1}$$

Problems like (P_ε) are variational in nature, but, as is well known, the lack of compactness, due to the unboundedness of the domain, does not permit the use of standard tools in a straight way. We refer the reader to [4, 9, 11], and the references therein, for a wide discussion of this topic and a survey of the most remarkable results. On the other hand, a considerable feature of (P_ε) is the nonlinear term behaviour, that although subcritical, becomes closer and closer to the critical growth, as ε approaches 0. Indeed, it has been observed that this fact gives rise to concentration phenomena of the solutions and, moreover, allows, in many cases, to obtain multiplicity results. This effect has been mainly investigated when the domain Ω is bounded and (P_ε) is an autonomuos equation, i.e. $a_\varepsilon(x) \equiv a_0 \geq 0$, see for instance [2, 6, 7].

In this paper we are concerned with unbounded exterior domains. In such a case the question of the existence of a solution to (P_ε) has been positively solved under suitable assumptions on the summability and the decay of $a_\varepsilon(x)$ in [3] (see also [4, 12], respectively, for existence and multiplicity results in the autonomous case and [9] for multiplicity of changing sign solutions in the nonautonomous case). Here, motivated by previous researches on singularly perturbed problems ([10, 11, 13]), that emphasize the role played by a concentrating potential to obtain multiplicity of solutions for problems exhibiting concentration phenomena, we investigate (P_ε) to this end. The results we obtain are contained in the following theorems:

Theorem 1.1 *Let a_ε be as in (1.1) and satisfying*

$$A_1) \qquad a_0, \alpha \in \mathbb{R}^+ \setminus \{0\},\ x_0 \in \Omega,\ a(x) \geq 0,\ a \in L^{N/2}(\mathbb{R}^N),\ |a|_{L^{N/2}(\mathbb{R}^N)} \neq 0.$$

Then there exists $k > 0$ such that, if $0 < |a|_{L^{N/2}} < k$, Problem (P_ε) has at least two distinct solutions, for all small ε.

Theorem 1.2 *Let a_ε be as in (1.1) and satisfying the assumption (A_1) and*

$$A_2) \qquad \int_{R^N} a(x) e^{2|x|}(1 + |x|^{\frac{N-1}{2}\sigma})dx < +\infty \quad \text{for some } \sigma \in (1, 2].$$

Then there exists $k > 0$ such that, if $0 < |a|_{L^{N/2}} < k$, Problem (P_ε) has at least three distinct solutions, for all small ε.

The paper is organized as follows: Section 2 is devoted to introducing some notations, recalling some results and proving some useful relations, in Section 3 some useful tools are introduced and some basic asymptotic estimates are proved, Section 4 contains the proofs of the Theorems 1.1 and 1.2.

2 Notations and preliminary results

Throughout the paper we use the following notations:

- $L^q(\mathcal{O})$, $1 \le q < +\infty$, $\mathcal{O} \subseteq \mathbb{R}^N$, denotes the Lebesgue space, the norm in $L^q(\mathcal{O})$ is denoted by $|\cdot|_{q,\mathcal{O}}$; when $\mathcal{O} = \mathbb{R}^N$ we simply and write $|\cdot|_q$.
- $H_0^1(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^N$ and $H^1(\mathbb{R}^N)$ denote the Sobolev spaces obtained, rspectively, as the closure of $\mathcal{C}_0^\infty(\mathcal{O})$ and $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norms
$$\|u\|_{\mathcal{O}} := \left(\int_{\mathcal{O}} (|\nabla u|^2 + a_0 u^2) dx\right)^{\frac{1}{2}}; \qquad \|u\| := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + a_0 u^2) dx\right)^{\frac{1}{2}}.$$
- $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm
$$|u| := \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$
- If $\mathcal{O}_1 \subset \mathcal{O}_2 \subseteq \mathbb{R}^N$ and $u \in H_0^1(\mathcal{O}_1)$, we also denote by u its extension to $\mathcal{O}_2$ obtained by setting $u \equiv 0$ outside $\mathcal{O}_1$.
- $B(y, \rho)$ denotes the open ball of $\mathbb{R}^N$ centered at y and having radius ρ.
- For every $A, B \in \mathbb{R}^N$ we let $[A, B] = \{tA + (1-t)B \ : \ t \in [0,1]\}$ and $(A, B) = \{tA + (1-t)B \ : \ t \in [0,1]\}$.

In what follows, without any loss of generality, **we assume $a_0 = 1$ and $x_0 = 0$. We denote by $d := \mathrm{dist}(0, \partial\Omega)$ and by $\bar{d} = d/10$.** The solutions of (P_ε) can be found looking for critical points of the functional $E_\varepsilon : H_0^1(\Omega) \to \mathbb{R}$
$$E_\varepsilon(u) := \int_{\mathbb{R}^N} \left[|\nabla u|^2 + \left(1 + \frac{1}{\varepsilon^{2\alpha}} a\left(\frac{x}{\varepsilon^\alpha}\right)\right) u^2\right] dx$$
constrained to lie on the manifold
$$\mathcal{M}_\varepsilon := \{u \in H_0^1(\Omega) \ : \ |u|_{2^*-\varepsilon,\Omega} = 1\}.$$
However, in spite of the variational structure of (P_ε), the natural minimization method does not provide any solution. Indeed, defining
$$m_\varepsilon := \inf\{E_\varepsilon(u) \ : \ u \in \mathcal{M}_\varepsilon\} \tag{2.1}$$
it is known that the following facts are true (see f.i. [11], §2):

a) the equality
$$m_\varepsilon = \inf\{\|u\|^2 \ : \ u \in H^1(\mathbb{R}^N), \ |u|_{2^*-\varepsilon} = 1\} \tag{2.2}$$
holds; the infimum in (2.1) is not attained;

b) the infimum in (2.2) is achieved by a positive function w_ε, that is unique, modulo translation, radially symmetric about the origin, decreasing when the radial coordinate increases and decaying exponentially at infinity;

c) setting
$$w_{\varepsilon,y_n}(x) = \frac{\phi(x)w_\varepsilon(x-y_n)}{|\phi(x)w_\varepsilon(x-y_n)|_{2^*-\varepsilon,\Omega}},$$
with $y_n \in \Omega$, $\lim_{n\to+\infty}|y_n| = +\infty$, and $\phi \in \mathcal{C}^\infty(\mathbb{R}^N,[0,1])$ cut-off function such that $\phi(x) = 0$ on ω and $\mathrm{supp}(1-\phi)$ is compact, the sequence $(w_{\varepsilon,y_n})_n$ is not relatively compact and approaches (2.1).

As usual, we denote by S the best Sobolev constant, i.e.
$$S := \inf\left\{\int_{\mathbb{R}^N}|\nabla u(x)|^2 dx \ : \ u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ |u|_{2^*} = 1\right\}. \tag{2.3}$$

The infimum in (2.3) is attained by the function
$$U(x) = \frac{C}{(1+|x|^2)^{\frac{N-2}{2}}} \qquad x \in \mathbb{R}^N$$
where C is a normalization constant (depending only on N), and by any of the functions
$$U_{y,\delta}(x) = C\frac{\delta^{\frac{N-2}{2}}}{(\delta^2+|x-y|^2)^{\frac{N-2}{2}}}; \qquad \delta > 0,\ y \in \mathbb{R}^N \tag{2.4}$$
obtained from U by translation and scaling. Moreover, the functions of the family (2.4) are the only functions realizing (2.3) (see [1, 14, 17]). It is easy to verify that for all $q \in (\frac{N}{N-2}, 2^*]$ we have
$$|U_{y,\delta}|_q^q = \delta^{N-q\frac{N-2}{2}}|U|_q^q \tag{2.5}$$
hence $\forall q \in (\frac{N}{N-2}, 2^*)$
$$\begin{array}{ll} (a) & |U_{y,\delta}|_q \underset{\delta\to 0}{\longrightarrow} 0, \quad (b)\ |U_{y,\delta}|_q \underset{\delta\to+\infty}{\longrightarrow} +\infty, \\ (c) & |U_{y,\delta}|_{2^*}^{2^*} = |U|_{2^*}^{2^*} = |U_{0,1}|_{2^*}^{2^*}. \end{array} \tag{2.6}$$

Furthermore, for any fixed $\rho > 0$ and $y \in \mathbb{R}^N$,
$$\int_{\mathbb{R}^N\setminus B(y,\rho)}|\nabla U_{y,\delta}(x)|^2 dx \longrightarrow 0 \qquad \text{as } \delta \to 0. \tag{2.7}$$

We denote, for all $\rho \in \mathbb{R}$, $\rho > 0$, by $\xi_\rho(t)$ a cut-off function, i.e. a monotone nonincreasing function belonging to $\mathcal{C}_0^\infty(\mathbb{R}^+,[0,1])$ such that $\xi_\rho(t) = 1$ when $t \in [0,\rho]$, and $\xi_\rho(t) = 0$ when $t \in [2\rho,+\infty)$.

Lemma 2.1 *Let m_ε be as defined in (2.1). Then*
$$\lim_{\varepsilon\to 0} m_\varepsilon = S. \tag{2.8}$$

Proof. From Lemma 4.1 of [6], the relation $\lim_{\varepsilon\to 0} m_\varepsilon \le S$ follows. To prove (2.8), we argue by contradiction and we assume that the strict inequality

$$\lim_{\varepsilon\to 0} m_\varepsilon < S \tag{2.9}$$

holds. Then there exist $h < S$ and $u_\varepsilon \in \mathcal{M}_\varepsilon$ such that $E_\varepsilon(u_\varepsilon) < h$, for all $\varepsilon > 0$ small enough. By interpolation we deduce

$$1 = |u_\varepsilon|_{2^*-\varepsilon} \le |u_\varepsilon|_2^{\sigma_\varepsilon} |u_\varepsilon|_{2^*}^{1-\sigma_\varepsilon},$$

where $\sigma_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$. Hence, being $|u_\varepsilon|_2^2 \le E_\varepsilon(u_\varepsilon) < h$, we obtain $\liminf_{\varepsilon\to 0} |u_\varepsilon|_{2^*} \ge 1$, that with $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \le \lim_{\varepsilon\to 0} m_\varepsilon < S$ contradicts the definition of S. q.e.d.

Lemma 2.2 *If $(u_\varepsilon)_\varepsilon$ is a family of functions such that $u_\varepsilon \in \mathcal{M}_\varepsilon$ and*

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = S, \tag{2.10}$$

then there exist a family $(\delta_\varepsilon)_\varepsilon$, $\delta_\varepsilon \in \mathbb{R}^+ \setminus \{0\}$, a family $(y_\varepsilon)_\varepsilon$, $y_\varepsilon \in \mathbb{R}^N$, and a family $(\Phi_\varepsilon)_\varepsilon$, $\Phi_\varepsilon \in H^1(\mathbb{R}^N)$, such that, for all $\rho > 0$,

$$u_\varepsilon(x) = \xi_\rho(|x-y_\varepsilon|)U_{y_\varepsilon,\delta_\varepsilon}(x) + \Phi_\varepsilon(x), \quad \textit{with } \delta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \textit{and } \Phi_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \textit{ in } H^1(\mathbb{R}^n). \tag{2.11}$$

Proof. Arguing as in Lemma 2.1, we obtain

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} |\nabla u_\varepsilon(x)|^2 dx \le S$$

and

$$\liminf_{\varepsilon\to 0} |u_\varepsilon|_{2^*} \ge 1,$$

which imply, because of the minimality of S,

$$\lim_{\varepsilon\to 0} |u_\varepsilon|_{2^*} = 1. \tag{2.12}$$

Setting now $z_\varepsilon(x) = \frac{u_\varepsilon(x)}{|u_\varepsilon|_{2^*}}$, by (2.10), (2.12) and (2.3) we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} |\nabla z_\varepsilon(x)|^2 dx = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} (|\nabla z_\varepsilon(x)|^2 + z_\varepsilon^2(x)) = S, \tag{2.13}$$

so, by the uniqueness of the family (2.4) realizing S,

$$z_\varepsilon(x) = U_{y_\varepsilon,\delta_\varepsilon}(x) + \varphi(x), \text{ where } \varphi_\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N). \tag{2.14}$$

Now, we can write

$$z_\varepsilon(x) = \xi_\rho(|x-y_\varepsilon|)(U_{y_\varepsilon,\delta_\varepsilon}(x)+\varphi_\varepsilon(x)) + (1-\xi_\rho(|x-y_\varepsilon|))(U_{y_\varepsilon,\delta_\varepsilon}(x)+\varphi_\varepsilon(x)) \tag{2.15}$$

and observe that

$$\int_{\mathbb{R}^N} |U_{y,\delta}|^2 dx = \begin{cases} +\infty & \text{if } N = 3, 4, \\ K_{N,\delta} & \text{if } N \geq 5, \end{cases} \tag{2.16}$$

with $K_{N,\delta} > 0$ and such that $K_{N,\delta} \to +\infty$, as $\delta \to +\infty$, $\forall N \geq 5$. Hence, it has to be $\delta_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0$, otherwise, by (2.14), (2.3) and (2.16), we would infer $\|z_\varepsilon\|^2 \geq \text{const} > S$ for small ε, contrary to (2.13). Since $\delta_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0$, taking into account (2.7), we get $\|\xi_\rho(\cdot - y_\varepsilon)U_{y_\varepsilon,\delta_\varepsilon}\|^2 \underset{\varepsilon \to 0}{\longrightarrow} S$, which, together (2.13) and (2.14), implies

$$\xi_\varepsilon(|x - y_\varepsilon|)\varphi_\varepsilon(x) + (1 - \xi_\rho(|x - y_\varepsilon|))[U_{y_\varepsilon,\delta_\varepsilon}(x) + \varphi_\varepsilon(x)] \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ in } H^1(\mathbb{R}^N),$$

and completes the proof. q.e.d.

Corollary 2.1 *Let the family* $(u_\varepsilon)_\varepsilon$ *be as in Lemma 2.2. Then*

$$u_\varepsilon(x) = U_{y_\varepsilon,\delta_\varepsilon}(x) + \varphi_\varepsilon(x),$$

where y_ε *and* δ_ε *are as in Lemma 2.2,* $\varphi_\varepsilon(x) \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *and* $\varphi_\varepsilon(x) \underset{\varepsilon \to 0}{\longrightarrow} 0$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

The functional E_ε constrained on $\mathcal{M}_\varepsilon$ does not verify globally the Palais-Smale condition, however, the compactness is preserved in some energy range and, moreover, the critical points whose energy values lie in that interval are constant sign functions:

Lemma 2.3 *Let* $\varepsilon \in (0, p-1)$ *be fixed and let* $(u_n)_n$ *be a Palais-Smale sequence for* E_ε *constrained on* $\mathcal{M}_\varepsilon$*, i.e.* $u_n \in \mathcal{M}_\varepsilon$ *and*

$$\begin{cases} \lim\limits_{n \to +\infty} E_\varepsilon(u_n) = c \\ \lim\limits_{n \to +\infty} \nabla E_{\varepsilon|\mathcal{M}_\varepsilon}(u_n) = 0. \end{cases}$$

If $c \in (m_\varepsilon, 2^{1-\frac{2}{2^*-\varepsilon}} m_\varepsilon)$*, then* $(u_n)_n$ *is relatively compact.*

Lemma 2.4 *Let* $\varepsilon \in (0, p-1)$ *be fixed and let* u *be a critical point of* E_ε *constrained on* $\mathcal{M}_\varepsilon$*. If* $u^+ \not\equiv 0$ *and* $u^- \not\equiv 0$*, then* $E_\varepsilon(u) > 2^{1-\frac{2}{2^*-\varepsilon}} m_\varepsilon$.

The proof of Lemma 2.3 has been given in [4] while the proof of Lemma 2.4 is contained in the proof of Theorem A of [6].

3 Tools, an existence result, basic estimates

For what follows we need to introduce some barycenter type functions. Let $P \in \omega$ be a fixed point. We define

$$\beta : L^{2^*}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$$

by

$$\beta(u) = \frac{1}{|u|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \frac{x - P}{1 + |x - P|} \, |u(x)|^{2^*} dx.$$

We also define, for every $\varepsilon > 0$, another map $\beta_\varepsilon : L^{2^*}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$\beta_\varepsilon(u) = \frac{1}{|u|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \chi_\varepsilon(x) \, |u(x)|^{2^*} dx,$$

where

$$\chi_\varepsilon(x) = \begin{cases} L \, \frac{\frac{x}{\varepsilon^\alpha}}{1+\left|\frac{x}{\varepsilon^\alpha}\right|} & \text{if } |x| \in \left[0, \frac{d}{2}\right] \\ L \, \frac{x}{|x|} \left[\frac{\frac{d}{2\varepsilon^\alpha}}{1+\frac{d}{2\varepsilon^\alpha}} + \frac{|x|-\frac{d}{2}}{\frac{d}{4}} \left(1 - \frac{\frac{d}{2\varepsilon^\alpha}}{1+\frac{d}{2\varepsilon^\alpha}}\right)\right] & \text{if } |x| \in \left[\frac{d}{2}, \frac{3}{4}d\right] \\ \frac{x}{1+|x|} & \text{if } |x| \geq \frac{3}{4}d, \end{cases}$$

and $L := \frac{\frac{3}{4}d}{1+\frac{3}{4}d}$. We remark that both the maps, β and β_ε, are well defined and continuous in $L^{2^*}(\mathbb{R}^N) \setminus \{0\}$. Moreover, $\beta(U_{P,\delta}) = 0$ and $\chi_\varepsilon(P) = \frac{P}{1+|P|}$. For every $\varepsilon > 0$, we introduce the continuous map

$$\gamma_\varepsilon : L^{2^*}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R},$$

defined by

$$\gamma_\varepsilon(u) = \frac{1}{|u|_{2^*}^{2^*}} \int_{\mathbb{R}^N} |\chi_\varepsilon(x) - \beta_\varepsilon(u)| \, |u(x)|^{2^*} dx,$$

in order to evaluate how much a function u is concentrated around its barycenter. We, then, denote by θ_ε the map defined in $L^{2^*}(\mathbb{R}^N) \setminus \{0\}$ by

$$\theta_\varepsilon(u) = (\beta_\varepsilon(u), \gamma_\varepsilon(u)) \,. \tag{3.1}$$

We remark that, for all $\varepsilon > 0$, θ_ε is a continuous map. For all $\varepsilon > 0$, we put

$$\mathcal{B}_{\varepsilon,P} := \inf\{E_\varepsilon(u) \ : \ u \in \mathcal{M}_\varepsilon, \ \beta(u) = 0\} \tag{3.2}$$

$$\mathcal{B}_{\theta_\varepsilon,0} := \inf\left\{E_\varepsilon(u) \ : \ u \in \mathcal{M}_\varepsilon, \ \theta_\varepsilon(u) = \left(0, \frac{L}{2}\right)\right\} \tag{3.3}$$

$$\mathcal{B}_{\theta_\varepsilon,P} := \inf\left\{E_\varepsilon(u) \ : \ u \in \mathcal{M}_\varepsilon, \ \theta_\varepsilon(u) = \left(\frac{P}{1+|P|}, \frac{L}{2}\right)\right\} \tag{3.4}$$

$$\mathcal{B}_{\theta_\varepsilon,s} := \inf\left\{E_\varepsilon(u) \ : \ u \in \mathcal{M}_\varepsilon, \ \theta_\varepsilon(u) \in \left[0, \frac{P}{1+|P|}\right] \times \left\{\frac{L}{2}\right\}\right\}. \tag{3.5}$$

Proposition 3.1 *Let a_ε satisfy (A_1). Then there exists a number $b_\omega \in \mathbb{R}$, $b_\omega > S$, such that the relation*

$$\mathcal{B}_{\varepsilon,P} > b_\omega \tag{3.6}$$

holds for all ε small enough.

Proof. We argue by contradiction. So, we assume that a sequence $(\varepsilon_n)_n$, $0<\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$, and a sequence $(u_n)_n$, $u_n\in\mathcal{M}_{\varepsilon_n}$, exist such that

$$\begin{cases} a) & \lim_{n\to+\infty} E_{\varepsilon_n}(u_n)=S, \\ b) & \beta(u_n)=0 \quad \forall n\in\mathbb{N}. \end{cases} \tag{3.7}$$

Then, according to Corollary 2.1,

$$u_n(x)=U_{y_n,\delta_n}(x)+\varphi_n(x) \qquad \forall x\in\mathbb{R}^N$$

with $y_n\in\mathbb{R}^N$, $\delta_n\in\mathbb{R}^+\setminus\{0\}$, $\delta_n \underset{n\to\infty}{\longrightarrow} 0$, $\varphi_n \underset{n\to\infty}{\longrightarrow} 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Thus it is easy to see that

$$\lim_{n\to+\infty} |\beta(u_n)-\beta(U_{y_n,\delta_n})|=0.$$

So, in view of (3.7) (b), we deduce, for all $r>0$

$$\begin{aligned} \frac{|y_n-P|}{1+|y_n-P|} &= \left|\beta(U_{y_n,\delta_n})-\frac{y_n-P}{1+|y_n-P|}\right|+o(1) \\ &\le \sup_{B(y_n,r)}\left|\frac{x-P}{1+|x-P|}-\frac{y_n-P}{1+|y_n-P|}\right| |U_{y_n,\delta_n}|^{2^*}_{2^*,B(y_n,r)} \\ &\quad +2\int_{\mathbb{R}^N\setminus B(y_n,r)} U^{2^*}_{y_n,\delta_n}(x)\,dx+o(1) \end{aligned}$$

which gives $y_n \underset{n\to\infty}{\longrightarrow} P$, because $\sup_{B(y_n,r)}\left|\frac{x-P}{1+|x-P|}-\frac{y_n-P}{1+|y_n-P|}\right| \underset{r\to 0}{\longrightarrow} 0$ and $\delta_n\to 0$ imply

$$\int_{\mathbb{R}^N\setminus B(y_n,r)} U^{2^*}_{y_n,\delta_n}(x)dx=\int_{\mathbb{R}^N\setminus B(0,r)} U^{2^*}_{0,\delta_n}(x)dx \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, since $P\in\omega$, $\delta_n \underset{n\to\infty}{\longrightarrow} 0$ and $\varphi_n \underset{n\to\infty}{\longrightarrow} 0$, we get, for large n,

$$|u_n|_{2^*,\omega}=|U_{y_n,\delta_n}+\varphi_n|_{2^*,\omega}\ge |U_{y_n,\delta_n}|_{2^*,\omega}+o(1)>0$$

contradicting $u_n(x)=0$ on ω. q.e.d.

We now state an existence result, that holds for problem (P_ε), for all $\varepsilon\in(0,p-1)$; it can be considered a consequence of Theorem I.1 in [3] (see also [4, 12, 15]).

Theorem 3.1 *Let a_ε satisfy $(A_1),(A_2)$. Let $\varepsilon\in(0,p-1)$. Then $\mathcal{B}_{\varepsilon,P}>m_\varepsilon$ and the functional E_ε constrained on $\mathcal{M}_\varepsilon$ has a critical value c_ε such that*

$$\mathcal{B}_{\varepsilon,P}\le c_\varepsilon<2^{1-\frac{2}{2^*-\varepsilon}}m_\varepsilon.$$

The following Lemmas 3.1, 3.2 and 3.3 give some asymptotic basic estimates.

Lemma 3.1 *Let (a_ε) satisfy (A_1). Then there exists a number $b_a>S$ such that the relation*

$$\mathcal{B}_{\theta_\varepsilon,0}>b_a \tag{3.8}$$

holds for all ε small enough.

Proof. We argue by contradiction and we suppose that a sequence $(\varepsilon_n)_n$, $0 < \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$, and a sequence $(u_n)_n$, $u_n \in \mathcal{M}_{\varepsilon_n}$ exist so that

$$\begin{cases} a) \quad \beta_{\varepsilon_n}(u_n) = 0 \quad b) \ \gamma_{\varepsilon_n}(u_n) = \frac{L}{2} \\ c) \quad \lim_{n\to+\infty} E_{\varepsilon_n}(u_n) = S. \end{cases} \tag{3.9}$$

Then, according to Corollary 2.1,

$$\begin{cases} a) \quad \lim_{n\to+\infty} |u_n|_{2^*} = 1 \\ b) \quad u_n(x) = U_{y_n,\delta_n}(x) + \varphi_n(x) \end{cases} \tag{3.10}$$

with $\delta_n \underset{n\to\infty}{\longrightarrow} 0$, $y_n \in \mathbb{R}^N$, $|\varphi_n| \underset{n\to\infty}{\longrightarrow} 0$. Let us first observe that

$$a) \quad \lim_{n\to+\infty} \beta_{\varepsilon_n}(U_{y_n,\delta_n}) = 0 \qquad b) \quad \lim_{n\to+\infty} \gamma_{\varepsilon_n}(U_{y_n,\delta_n}) = \frac{L}{2}. \tag{3.11}$$

Indeed, in view of (3.9) (a), (3.10) and (2.6) (a),

$$\begin{aligned} |\beta_{\varepsilon_n}(U_{y_n,\delta_n})| &= |\beta_{\varepsilon_n}(U_{y_n,\delta_n}) - \beta_{\varepsilon_n}(u_n)| \\ &= \left| \int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x)[|U_{y_n,\delta_n}|^{2^*} - |U_{y_n,\delta_n} + \varphi_n|^{2^*}]dx + o(1) \right| \\ &\le \int_{\mathbb{R}^N} |\chi_{\varepsilon_n}(x)| 2^* (U_{y_n,\delta_n} + |\varphi_n|)^{2^*-1} |\varphi_n| dx + o(1) \\ &\le k(|U_{y_n,\delta_n}|_{2^*}^{2^*-1} + |\varphi_n|_{2^*}^{2^*-1})|\varphi_n|_{2^*} + o(1) = o(1), \end{aligned}$$

so (3.11) (a) holds true. Now, using (3.9) (b) and (3.11) (a), we deduce

$$\begin{aligned} \left|\gamma_{\varepsilon_n}(U_{y_n,\delta_n}) - \frac{L}{2}\right| &= |\gamma_{\varepsilon_n}(U_{y_n,\delta_n}) - \gamma_{\varepsilon_n}(u_n)| \\ &= \bigg| \int_{\mathbb{R}^N} [|\chi_{\varepsilon_n}(x) - \beta_{\varepsilon_n}(U_{y_n,\delta_n})|\, |U_{y_n,\delta_n}|^{2^*} \\ &\quad - |\chi_{\varepsilon_n}(x)|\, |U_{y_n,\delta_n} + \varphi_n|^{2^*}]dx + o(1) \bigg| \\ &\le \int_{\mathbb{R}^N} |\chi_{\varepsilon_n}(x)| (U_{y_n,\delta_n}^{2^*} - |U_{y_n,\delta_n} + \varphi_n|^{2^*})dx + o(1). \end{aligned}$$

Thus, arguing as in proving (3.11) (a), we obtain (3.11) (b). Now let us put $\sigma_n = \frac{\delta_n}{\varepsilon_n^\alpha}$, $z_n = \frac{y_n}{\varepsilon_n^\alpha}$. We claim that, up to subsequences,

$$a) \quad \lim_{n\to+\infty} \sigma_n = \bar{\sigma} > 0; \qquad b) \quad \lim_{n\to+\infty} z_n = \bar{z}. \tag{3.12}$$

Let us first show that σ_n is bounded. In fact if for some subsequence (still denoted by σ_n) $\lim_{n\to+\infty} \sigma_n = +\infty$ occurs, then for all $r > 0$

$$\begin{aligned} \lim_{n\to+\infty} \int_{B(0,\varepsilon_n^\alpha r)} |u_n(x)|^{2^*} dx &= \lim_{n\to+\infty} \int_{B(0,\varepsilon_n^\alpha r)} |U_{y_n,\delta_n}(x)|^{2^*} dx \\ &= \lim_{n\to+\infty} \int_{B\left(0,\frac{r}{\sigma_n}\right)} \left| U_{0,1}\left(x - \frac{z_n}{\sigma_n}\right)\right|^{2^*} dx = 0. \end{aligned} \tag{3.13}$$

Now, using (3.9) (a) and (3.10) (a) we obtain, for any fixed $r > 0$

$$\begin{aligned}\gamma_{\varepsilon_n}(u_n) &= \frac{1}{|u_n|_{2^*}^{2^*}} \int_{\mathbb{R}^N} |\chi_{\varepsilon_n}(x)|\,|u_n(x)|^{2^*} dx \\ &\geq (1+o(1)) \int_{\mathbb{R}^N \setminus B(0,\varepsilon_n^\alpha r)} |\chi_{\varepsilon_n}(x)|\,|u_n(x)|^{2^*} dx.\end{aligned}$$

So, if n is large enough,

$$\gamma_{\varepsilon_n}(u_n) \geq (1+o(1))L\,\frac{r}{1+r} \int_{\mathbb{R}^N \setminus B(0,\varepsilon_n^\alpha r)} |u_n(x)|^{2^*} dx$$

and, in view of (3.13),

$$\liminf_{n\to+\infty} \gamma_{\varepsilon_n}(u_n) \geq L\,\frac{r}{1+r}$$

that, taking $r > 1$, contradicts (3.9) (b). Hence, up to a subsequence, $\lim_{n\to+\infty} \sigma_n = \bar\sigma \geq 0$. If $\bar\sigma = 0$ occurs, then for all $r > 0$

$$\begin{aligned}\lim_{n\to+\infty} \int_{\mathbb{R}^N \setminus B(y_n,\varepsilon_n^\alpha r)} |u_n(x)|^{2^*} dx &= \lim_{n\to+\infty} \int_{\mathbb{R}^N \setminus B(y_n,\varepsilon_n^\alpha r)} |U_{y_n,\delta_n}(x)|^{2^*} dx \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^N \setminus B\left(\frac{y_n}{\delta_n},\frac{r}{\sigma_n}\right)} |U_{0,1}\left(x - \frac{y_n}{\delta_n}\right)|^{2^*} dx = 0.\end{aligned}$$

Hence, for all $r > 0$

$$\begin{aligned}|\chi_{\varepsilon_n}(y_n)| &= |\chi_{\varepsilon_n}(y_n) - \beta_{\varepsilon_n}(u_n)| \\ &= \frac{1}{|u_n|_{2^*}^{2^*}} \left| \int_{\mathbb{R}^N} [\chi_{\varepsilon_n}(y_n) - \chi_{\varepsilon_n}(x)]\,|u_n(x)|^{2^*} dx \right| \\ &\leq (1+o(1)) \left[\int_{\mathbb{R}^N \setminus B(y_n,\varepsilon_n^\alpha r)} |\chi_{\varepsilon_n}(y_n) - \chi_{\varepsilon_n}(x)|\,|u_n(x)|^{2^*} dx \right. \\ &\qquad \left. + \int_{B(y_n,\varepsilon_n^\alpha r)} |\chi_{\varepsilon_n}(y_n) - \chi_{\varepsilon_n}(x)|\,|u_n(x)|^{2^*} dx \right] \\ &\leq (1+o(1))[o(1) + kr], \qquad k = \text{const},\end{aligned}$$

which implies $\lim_{n\to+\infty} \chi_{\varepsilon_n}(y_n) = 0$. Now, we obtain

$$\begin{aligned}\lim_{n\to+\infty} \gamma_{\varepsilon_n}(u_n) &= \lim_{n\to+\infty} \frac{1}{|u_n|_{2^*}^{2^*}} \int_{\mathbb{R}^N} |\chi_{\varepsilon_n}(x) - \beta_{\varepsilon_n}(u_n)|\,|u_n(x)|^{2^*} dx \\ &= \lim_{n\to+\infty} \frac{1}{|u_n|_{2^*}^{2^*}} \int_{\mathbb{R}^N} |\chi_{\varepsilon_n}(x) - \chi_{\varepsilon_n}(y_n)|\,|u_n(x)|^{2^*} dx = 0\end{aligned}$$

that contradicts (3.9) (b). Thus the first part of the claim is proved. Let us show that $z_n = \frac{y_n}{\varepsilon_n^\alpha}$ is bounded. Again, we argue by contradiction and we assume that, up to a subsequence, $\left|\frac{y_n}{\varepsilon_n^\alpha}\right| \underset{n\to\infty}{\longrightarrow} +\infty$, which implies $\liminf_{n\to+\infty} |\chi_{\varepsilon_n}(y_n)| > 0$. On the other hand, $\forall \eta > 0$ and $\forall R > 0$, if n is large enough we have

$$\left|\frac{x}{\varepsilon_n^\alpha} - \frac{y_n}{\varepsilon_n^\alpha}\right| < R \Rightarrow \left|\chi_{\varepsilon_n}\left(\frac{x}{\varepsilon_n^\alpha}\right) - \chi_{\varepsilon_n}\left(\frac{y_n}{\varepsilon_n^\alpha}\right)\right| = \left|\frac{x}{\varepsilon_n^\alpha + |x|} - \frac{y_n}{\varepsilon_n^\alpha + |y_n|}\right| < \eta \tag{3.14}$$

and, $\forall \eta > 0$, $\exists \bar{R}$ such that $\forall R > \bar{R}$

$$\int_{\mathbb{R}^N \setminus B(0,R)} |U_{0,\bar\sigma}(x)|^{2^*} dx < \eta. \tag{3.15}$$

Then, let $\eta > 0$ be arbitrarily chosen and let us fix $R > 0$ so that (3.15) is verified: if n is large enough, using also (3.14), we get

$$\begin{aligned}
|\beta_{\varepsilon_n}(u_n) - \chi_{\varepsilon_n}(y_n)| &= \frac{1}{|u_n|_{2^*}^{2^*}} \left|\int_{\mathbb{R}^N} (\chi_{\varepsilon_n}(x) - \chi_{\varepsilon_n}(y_n)) |u_n(x)|^{2^*} dx\right| \\
&\le (1 + o(1)) \left[\int_{\mathbb{R}^N \setminus B(y_n, \varepsilon_n^\alpha R)} |\chi_{\varepsilon_n}(x) - \chi_{\varepsilon_n}(y_n)| \, |U_{y_n, \varepsilon_n^\alpha \sigma_n}(x)|^{2^*} dx \right. \\
&\qquad \left. + \int_{B(y_n, \varepsilon_n^\alpha R)} |\chi_{\varepsilon_n}(x) - \chi_{\varepsilon_n}(y_n)| \, |U_{y_n, \varepsilon_n^\alpha \sigma_n}(x)|^{2^*} dx + o(1)\right] \\
&= (1 + o(1)) \left[\int_{\mathbb{R}^N \setminus B(y_n/\varepsilon_n^\alpha, R)} \left|\chi_{\varepsilon_n}\left(\frac{x}{\varepsilon_n^\alpha}\right) - \chi_{\varepsilon_n}\left(\frac{y_n}{\varepsilon_n^\alpha}\right)\right| \left|U_{0,\sigma_n}\left(x - \frac{y_n}{\varepsilon_n^\alpha}\right)\right|^{2^*} dx \right. \\
&\qquad \left. + \int_{B(y_n/\varepsilon_n^\alpha, R)} \left|\chi_{\varepsilon_n}\left(\frac{x}{\varepsilon_n^\alpha}\right) - \chi_{\varepsilon_n}\left(\frac{y_n}{\varepsilon_n^\alpha}\right)\right| \left|U_{0,\sigma_n}\left(x - \frac{y_n}{\varepsilon_n^\alpha}\right)\right|^{2^*} dx + o(1)\right] \\
&< (1 + o(1)) \left[k \int_{\mathbb{R}^N \setminus B(0,R)} |U_{0,\bar\sigma}(x)|^{2^*} dx + \eta \int_{B(0,R)} |U_{0,\bar\sigma}(x)|^{2^*} dx + o(1)\right] \\
&\le \bar{k}\eta + o(1).
\end{aligned}$$

Hence $\chi_{\varepsilon_n}(y_n) \underset{n\to\infty}{\longrightarrow} 0$, a contradiction. So, (3.13) (b) is proved. Let us now evaluate

$$\begin{aligned}
&\lim_{n\to+\infty} \int_{\mathbb{R}^N} \frac{1}{\varepsilon_n^{2\alpha}} a\left(\frac{x}{\varepsilon_n^\alpha}\right) u_n^2(x) dx \\
&\quad \ge \lim_{n\to+\infty} \left[\int_{\mathbb{R}^N} \frac{1}{\varepsilon_n^{2\alpha}} a\left(\frac{x}{\varepsilon_n^\alpha}\right) U_{y_n,\delta_n}^2(x) dx \right. \\
&\qquad \left. -2\left|\int_{\mathbb{R}^N} \frac{1}{\varepsilon_n^{2\alpha}} a\left(\frac{x}{\varepsilon_n^\alpha}\right) U_{y_n,\delta_n}(x)\varphi_n(x) dx\right| - \left|\int_{\mathbb{R}^N} \frac{1}{\varepsilon_n^{2\alpha}} a\left(\frac{x}{\varepsilon_n^\alpha}\right) \varphi_n^2(x) dx\right|\right] \\
&\quad \ge \lim_{n\to+\infty} \left[\int_{\mathbb{R}^N} a(x) U_{z_n,\sigma_n}^2(x) dx \right. \\
&\qquad \left. -2\left|\frac{1}{\varepsilon_n^{2\alpha}} a\left(\frac{x}{\varepsilon_n^\alpha}\right)\right|_{N/2} |U_{y_n,\delta_n}|_{2^*} |\varphi_n|_{2^*} - \left|\frac{1}{\varepsilon_n^{2\alpha}} a\left(\frac{x}{\varepsilon_n^\alpha}\right)\right|_{N/2} |\varphi_n|_{2^*}^2\right] \\
&\quad = \int_{\mathbb{R}^N} a(x) U_{\bar{z},\bar\sigma}^2(x) dx + o(1) \ge \hat{k} > 0, \qquad \hat{k} = \text{const}.
\end{aligned}$$

Hence, using (3.9) (c), (3.10) (a) and the above relation, we deduce

$$S = \lim_{n\to+\infty} E_{\varepsilon_n}(u_n) = \lim_{n\to+\infty} \|u_n\|^2 + \int_{\mathbb{R}^N} \frac{1}{\varepsilon_n^{2\alpha}} a\left(\frac{x}{\varepsilon_n^{\alpha}}\right) u_n^2(x)dx \geq S + \hat{k} > S$$

which is impossible, completing the proof. q.e.d.

Lemma 3.2 *Let a_ε satisfy (A_1). Then there exists a number $b_p > S$ such that the relation*

$$\inf\left\{E_\varepsilon(u) \;:\; u \in \mathcal{M}_\varepsilon,\; \beta_\varepsilon(u) = \frac{P}{1+|P|}\right\} > b_p$$

holds for all ε small enough.

Proof. We argue by contradiction and we assume that a sequence $(\varepsilon_n)_n$, $0 < \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$, and a sequence $(u_n)_n$, $u_n \in \mathcal{M}_{\varepsilon_n}$, exist so that

$$\lim_{n\to+\infty} E_{\varepsilon_n}(u_n) = S, \qquad \beta_{\varepsilon_n}(u_n) = \chi_{\varepsilon_n}(P) = \frac{P}{1+|P|}.$$

Then, as in Lemma 3.1, we deduce

$$\begin{cases} u_n(x) = U_{y_n,\delta_n}(x) + \varphi_n(x) \\ \delta_n \underset{n\to\infty}{\longrightarrow} 0,\ \varphi_n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ and in } L^{2^*}(\mathbb{R}^N),\ y_n \in \mathbb{R}^N \end{cases} \tag{3.16}$$

and

$$|\beta_{\varepsilon_n}(u_n) - \beta_{\varepsilon_n}(U_{y_n,\delta_n})| \underset{n\to\infty}{\longrightarrow} 0$$

from which

$$\lim_{n\to+\infty} \beta_{\varepsilon_n}(U_{y_n,\delta_n}) = \frac{P}{1+|P|} \tag{3.17}$$

follows. Now, in order to achieve the thesis, we just need to show that

$$\lim_{n\to+\infty} y_n = P. \tag{3.18}$$

In fact, in this case, since $P \in \omega$ and (3.16) holds, we obtain for large n

$$|u_n|_{2^*,\omega} = |U_{y_n,\delta_n} + \varphi_n|_{2^*,\omega} \geq |U_{y_n,\delta_n}|_{2^*,\omega} - |\varphi_n|_{2^*,\omega} > 0$$

while $u_n(x) = 0$ for all $x \in \omega$. To prove (3.18), we start observing that $|y_n|$ must be bounded. In the opposite case, in fact, the relation

$$\liminf_{n\to+\infty} |\beta_{\varepsilon_n}(U_{y_n,\delta_n})| \geq 1,$$

incompatible with (3.17), would come out, considering that

$$|\beta_{\varepsilon_n}(U_{y_n,\delta_n})| \geq \left|\int_{B(y_n,1)} \chi_{\varepsilon_n}(x)(U_{y_n,\delta_n}(x))^{2^*}dx\right| - \int_{\mathbb{R}^N\setminus B(y_n,1)} |\chi_{\varepsilon_n}(x)|(U_{y_n,\delta_n}(x))^{2^*}dx$$

and $\lim_{n\to+\infty}|y_n| = +\infty$ with (3.16) imply

$$\int_{\mathbb{R}^N\setminus B(y_n,1)} |\chi_{\varepsilon_n}(x)|(U_{y_n,\delta_n}(x))^{2^*}\,dx \le \int_{\mathbb{R}^N\setminus B(y_n,1)} (U_{y_n,\delta_n}(x))^{2^*}\,dx \underset{n\to\infty}{\longrightarrow} 0, \quad (3.19)$$

$$\left|\int_{B(y_n,1)} [\chi_{\varepsilon_n}(x) - \chi_{\varepsilon_n}(y_n)](U_{y_n,\delta_n}(x))^{2^*}\,dx\right|$$
$$\le \left(\sup_{x\in B(y_n,1)}\left|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\right|\right)|U_{y_n,\delta_n}|_{2^*}^{2^*} \underset{n\to\infty}{\longrightarrow} 0, \quad (3.20)$$

$$\left|\int_{B(y_n,1)} \chi_{\varepsilon_n}(y_n)(U_{y_n,\delta_n}(x))^{2^*}\,dx\right|$$
$$= \frac{|y_n|}{1+|y_n|}\int_{B(y_n,1)} (U_{y_n,\delta_n}(x))^{2^*}\,dx \underset{n\to\infty}{\longrightarrow} 1. \quad (3.21)$$

Therefore, $y_0 \in \mathbb{R}^N$ exists so that, passing eventually to a subsequence,

$$\lim_{n\to+\infty} y_n = y_0. \quad (3.22)$$

Thus, in view of (3.17), to have (3.18), what is left to prove is

$$\lim_{n\to+\infty} \beta_{\varepsilon_n}(U_{y_n,\delta_n}) = \frac{y_0}{1+|y_0|}. \quad (3.23)$$

Now

$$|\beta_{\varepsilon_n}(U_{y_n,\delta_n})|$$
$$\le \int_{B(y_n,\frac{d}{8})} |\chi_{\varepsilon_n}(x)|(U_{y_n,\delta_n}(x))^{2^*}\,dx + \int_{\mathbb{R}^N\setminus B(y_n,\frac{d}{8})} |\chi_{\varepsilon_n}(x)|(U_{y_n,\delta_n}(x))^{2^*}\,dx, \quad (3.24)$$

so, $|y_n| < \frac{7}{8}d$ and (3.16) would imply

$$|\beta_{\varepsilon_n}(U_{y_n,\delta_n})| < \frac{d}{1+d}\int_{B(y_n,\frac{d}{8})} (U_{y_n,\delta_n}(x))^{2^*}\,dx + o(1) < \frac{d}{1+d} + o(1)$$

which contradicts (3.17), because, as $P \in \omega$, $|P| > d$. Hence $|y_n| \ge \frac{7}{8}d$, which, with (3.16) and (3.22), gives

$$\sup_{B(y_n,\sqrt{\delta_n})}\left|\chi_{\varepsilon_n}(x) - \frac{y_0}{1+|y_0|}\right| \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, we infer

$$\lim_{n\to+\infty}\left|\beta_{\varepsilon_n}(U_{y_n,\delta_n}) - \frac{y_0}{1+|y_0|}\right|$$

$$
\begin{aligned}
&\leq \lim_{n\to+\infty}\left[\int_{B(y_n,\sqrt{\delta_n})}\left|\chi_{\varepsilon_n}(x)-\frac{y_0}{1+|y_0|}\right|(U_{y_n,\delta_n}(x))^{2^*}\,dx\right.\\
&\qquad\left.+\int_{\mathbb{R}^N\setminus B(y_n,\sqrt{\delta_n})}\left|\chi_{\varepsilon_n}(x)-\frac{y_0}{1+|y_0|}\right|(U_{y_n,\delta_n}(x))^{2^*}\,dx\right]\\
&\leq \lim_{n\to+\infty}\left[\sup_{B(y_n,\sqrt{\delta_n})}\left|\chi_{\varepsilon_n}(x)-\frac{y_0}{1+|y_0|}\right|\int_{B(y_n,\sqrt{\delta_n})}(U_{y_n,\delta_n}(x))^{2^*}\,dx\right]\\
&\quad+\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B(y_n,\sqrt{\delta_n})}(U_{y_n,\delta_n}(x))^{2^*}\,dx\\
&= 0
\end{aligned}
$$

which gives (3.23), completing the proof. q.e.d.

Corollary 3.1 *Let a_ε satisfy (A_1). Let b_p the number whose existence is stated in Lemma 3.2. Then*

$$\mathcal{B}_{\theta_\varepsilon,P} > b_p > S \tag{3.25}$$

for all ε small enough.

Lemma 3.3 *Let a_ε satisfy (A_1). Then for all $\varepsilon \in (0,p-1)$ the relation*

$$\inf\{E_\varepsilon(u)\ :\ u\in\mathcal{M}_\varepsilon,\ \beta_\varepsilon(u)\in[0,\chi_\varepsilon(P)]\} > m_\varepsilon$$

holds.

Proof. Let us assume, by contradiction, that $\bar\varepsilon \in (0,p-1)$ and a sequence $(u_n)_n$ exist so that

$$
\begin{array}{ll}
a) & |u_n|_{2^*-\bar\varepsilon}=1,\quad \lim_{n\to+\infty}E_{\bar\varepsilon}(u_n)=m_{\bar\varepsilon}\\
b) & \beta_{\bar\varepsilon}(u_n)\in\left[0,\frac{P}{1+|P|}\right]\quad \forall n\in\mathbb{N}.
\end{array}
\tag{3.26}
$$

Hence, taking into account the uniqueness, up to translation, of the function $w_{\bar\varepsilon}$ realizing (2.2) and the fact that the infimum in (2.1) is not attained, we deduce the existence of sequences $(y_n)_n$, $y_n \in \mathbb{R}^N$, and $(\psi_n)_n$, $\psi_n \in H^1(\mathbb{R}^N)$, such that

$$
\begin{cases}
u_n(x)=w_{\bar\varepsilon}(x-y_n)+\psi_n(x)\\
\lim_{n\to+\infty}\psi_n(x)=0\ \text{ in } H^1(\mathbb{R}^N),\quad \lim_{n\to+\infty}|y_n|=+\infty.
\end{cases}
\tag{3.27}
$$

Now, from (3.27), we easily obtain

$$|\beta_{\bar\varepsilon}(u_n)-\beta_{\bar\varepsilon}(w_{\bar\varepsilon}(x-y_n))| \underset{n\to\infty}{\longrightarrow} 0$$

which, with (3.26) (b), gives

$$|\beta_{\bar\varepsilon}(w_{\bar\varepsilon}(x-y_n))| \leq \frac{|P|}{1+|P|}+o(1)<1. \tag{3.28}$$

On the other hand,

$$|\beta_{\bar\varepsilon}(w_{\bar\varepsilon}(\cdot - y_n))| \geq \frac{1}{|w_{\bar\varepsilon}|_{2^*}^{2^*}} \left[\left| \int_{B(y_n,\sqrt{|y_n|})} \chi_{\bar\varepsilon}(x)(w_{\bar\varepsilon}(x-y_n))^{2^*} dx \right| - \int_{\mathbb{R}^N \setminus B(y_n,\sqrt{|y_n|})} |\chi_{\bar\varepsilon}(x)|(w_{\bar\varepsilon}(x-y_n))^{2^*} dx \right] \tag{3.29}$$

and

$$\int_{\mathbb{R}^N \setminus B(y_n,\sqrt{|y_n|})} |\chi_{\bar\varepsilon}(x)|(w_{\bar\varepsilon}(x-y_n))^{2^*} dx \leq \int_{\mathbb{R}^N \setminus B(0,\sqrt{|y_n|})} (w_{\bar\varepsilon}(x))^{2^*} dx \xrightarrow[n\to\infty]{} 0. \tag{3.30}$$

Let us observe, now, that, since $|y_n| \to +\infty$,

$$\lim_{n\to+\infty} \frac{1}{|w_{\bar\varepsilon}|_{2^*}^{2^*}} \left| \int_{B(y_n,\sqrt{|y_n|})} \chi_{\bar\varepsilon}(y_n)(w_{\bar\varepsilon}(x-y_n))^{2^*} dx \right| = \lim_{n\to+\infty} \frac{|\chi_{\bar\varepsilon}(y_n)|}{|w_{\bar\varepsilon}|_{2^*}^{2^*}} \int_{B(0,\sqrt{|y_n|})} (w_{\bar\varepsilon}(x))^{2^*} dx = 1 \tag{3.31}$$

and

$$\lim_{n\to+\infty} \left| \int_{B(y_n,\sqrt{|y_n|})} [\chi_{\bar\varepsilon}(x)(w_{\bar\varepsilon}(x-y_n))^{2^*} - \chi_{\bar\varepsilon}(y_n)(w_{\bar\varepsilon}(x-y_n))^{2^*}] dx \right| \leq \lim_{n\to+\infty} \sup_{B(y_n,\sqrt{|y_n|})} |\chi_{\bar\varepsilon}(x) - \chi_{\bar\varepsilon}(y_n)| \int_{\mathbb{R}^N} (w_{\bar\varepsilon}(x))^{2^*} dx = 0. \tag{3.32}$$

So, using (3.30), (3.31) and (3.32), we deduce from (3.29) that

$$\lim_{n\to+\infty} |\beta_{\bar\varepsilon}(w_{\bar\varepsilon}(\cdot - y_n))| \geq 1,$$

contradicting (3.28). q.e.d.

Corollary 3.2 *Let a_ε satisfy (A_1). Then for all $\varepsilon \in (0, p-1)$ the relation*

$$\mathcal{B}_{\theta_\varepsilon, s} > m_\varepsilon$$

holds.

4 Proofs of the results

In what follows we put $\forall y \in \mathbb{R}^N$ and $\delta > 0$

$$v_{y,\delta}(x) = \xi_{\frac{\bar d}{2}}(|x-y|)U_{y,\delta}(x) \qquad x \in \mathbb{R}^N$$

where the cut-off function ξ and the constant $\bar{d}$ are those defined in Section 2. For any $\varepsilon > 0$, we define the operator

$$V_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \longrightarrow H^1(\mathbb{R}^N)$$

by

$$V_\varepsilon[y,\delta](x) = \frac{v_{y,\delta}(x)}{|v_{y,\delta}|_{2^*-\varepsilon}}.$$

Furthermore, for all $\varepsilon > 0$, we set

$$\begin{aligned}
&\mathcal{K}_\varepsilon := \left\{(y,\delta) \in \overline{B(0,d/4)} \times \mathbb{R}^+ \ : \ \varepsilon^{\alpha/2} \le \frac{\delta}{\varepsilon^\alpha} \le \varepsilon^{-\alpha/2}\right\}\\
&\mathcal{A}_\varepsilon := \max\{E_\varepsilon(V_\varepsilon[y,\delta]) \ : \ (y,\delta) \in \mathcal{K}_\varepsilon\} \qquad (4.1)\\
&\mathcal{F}_\varepsilon := \max\{E_\varepsilon(V_\varepsilon[y,\delta]) \ : \ (y,\delta) \in \partial\mathcal{K}_\varepsilon\}. \qquad (4.2)
\end{aligned}$$

For all $\varepsilon > 0$ and $c \in \mathbb{R}$, we use, also, the following notation:

$$E_\varepsilon^c = \{u \in \mathcal{M}_\varepsilon \ : \ E_\varepsilon(u) \le c\}.$$

Proposition 4.1 *Let a_ε satisfy (A_1). Let $\mathcal{A}_\varepsilon$ and $\mathcal{F}_\varepsilon$ be the numbers defined in (4.1) and (4.2) respectively. Let b_ω be the number whose existence is stated in Proposition 3.1. Then*

$$\left\{\begin{array}{ll} a) & \lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon = S \\ b) & \exists k > 0 \ \ \textit{such that } |a|_{N/2} < k \textit{ implies } \mathcal{A}_\varepsilon < b_\omega \\ & \textit{for all } \varepsilon > 0 \textit{ small enough.}\end{array}\right. \qquad (4.3)$$

Proof. First of all, let us observe that, in view of (2.3), (2.4) and the definitions of ξ_ρ and $V_\varepsilon[y,\rho]$,

$$\max_{\mathcal{K}_\varepsilon} \|V_\varepsilon[y,\delta]\|^2 \underset{\varepsilon\to 0}{\longrightarrow} S, \qquad (4.4)$$

because $\delta \longrightarrow 0$ uniformly as $\varepsilon \to 0$. Thus, in order to prove (4.3) (a) we must show that

$$\max_{\partial K_\varepsilon} \int_\Omega \frac{1}{\varepsilon^{2\alpha}} a\left(\frac{x}{\varepsilon^\alpha}\right) (V_\varepsilon[y,\delta](x))^2 dx \underset{\varepsilon\to 0}{\longrightarrow} 0. \qquad (4.5)$$

If $y \in \partial B(0,d/4)$, then $(\operatorname{supp} V_\varepsilon[y,\delta]) \cap B(0,\bar{d}) = \emptyset$ and

$$\begin{aligned}
&\int_\Omega \frac{1}{\varepsilon^{2\alpha}} a\left(\frac{x}{\varepsilon^\alpha}\right) (V_\varepsilon[y,\delta](x))^2 dx \\
&\qquad \le \left|\frac{1}{\varepsilon^{2\alpha}} a\left(\frac{\cdot}{\varepsilon^\alpha}\right)\right|_{N/2,\mathbb{R}^N\setminus B(0,\bar{d})} |V_\varepsilon[y,\delta]|^2_{2^*,B(y,\bar{d})} \\
&\qquad = (1+o(1))|a|_{N/2,\mathbb{R}^N\setminus B(0,\bar{d}/\varepsilon^\alpha)} \underset{\varepsilon\to 0}{\longrightarrow} 0. \qquad (4.6)
\end{aligned}$$

Now let $(y,\delta) \in \overline{B(0,d/4)} \times \{\varepsilon^{(3/2)\alpha}\}$. Fixing $\eta > 0$, then there exists R_η such that, for small ε

$$|a|_{N/2,B(y/\varepsilon^\alpha,R_\eta)} < \eta \quad \text{and } |U_{y/\varepsilon^\alpha,\varepsilon^{\alpha/2}}|^2_{2^*,\mathbb{R}^N\setminus B(y/\varepsilon^\alpha,R_\eta)} < \eta$$

so we deduce

$$\begin{aligned}
&\int_\Omega \frac{1}{\varepsilon^{2\alpha}} a\left(\frac{x}{\varepsilon^\alpha}\right)(V_\varepsilon[y,\varepsilon^{(3/2)\alpha}](x))^2 dx \\
&\quad \le k' \int_{\mathbb{R}^N} \frac{1}{\varepsilon^{2\alpha}} a\left(\frac{x}{\varepsilon^\alpha}\right)(U_{y,\varepsilon^{(3/2)\alpha}}(x))^2 dx = k' \int_{\mathbb{R}^N} a(x)(U_{y/\varepsilon^\alpha,\varepsilon^{\alpha/2}}(x))^2 dx \\
&\quad \le k'[|a|_{N/2,B(y/\varepsilon^\alpha,R_\eta)}|U_{y/\varepsilon^\alpha,\varepsilon^{\alpha/2}}|^2_{2^*} + |a|_{N/2}|U_{y/\varepsilon^\alpha,\varepsilon^{\alpha/2}}|^2_{2^*,\mathbb{R}^N\setminus B(y/\varepsilon^\alpha,R_\eta)}] \\
&\quad \le \bar{k}\eta, \qquad\qquad k', \bar{k} \text{ constants.}
\end{aligned} \tag{4.7}$$

Lastly, let us consider $(y,\delta) \in \overline{B(0,d/4)} \times \{\varepsilon^{\alpha/2}\}$. Fixing $\eta > 0$, let $\bar{R}_\eta$ be such that $|a|_{N/2,\mathbb{R}^N\setminus B(0,\bar{R}_\eta)} < \eta$ and let $\varepsilon > 0$ be so small that $|U_{y/\varepsilon^\alpha,\varepsilon^{-\alpha/2}}|^2_{2^*,B(0,\bar{R}_\eta)} < \eta$. Then

$$\begin{aligned}
&\int_\Omega \frac{1}{\varepsilon^{2\alpha}} a\left(\frac{x}{\varepsilon^\alpha}\right)(V_\varepsilon[y,\varepsilon^{\alpha/2}](x))^2 dx \\
&\quad \le \tilde{k} \int_{\mathbb{R}^N} a(x)(U_{y/\varepsilon^\alpha,\varepsilon^{-\alpha/2}}(x))^2 dx \\
&\quad \le \tilde{k}[|a|_{N/2}|U_{y/\varepsilon^\alpha,\varepsilon^{-\alpha/2}}|^2_{2^*,B(0,\bar{R}_\eta)} + |a|_{N/2,\mathbb{R}^N\setminus B(0,\bar{R}_\eta)}|U_{y/\varepsilon^\alpha,\varepsilon^{-\alpha/2}}|^2_{2^*}] \\
&\quad \le \hat{k}\eta, \qquad\qquad \tilde{k}, \hat{k} \text{ constants.}
\end{aligned} \tag{4.8}$$

Thus, (4.5) easily follows from (4.6), (4.7) and (4.8) completing the proof of (4.3) (a). Assertion (4.3) (b) can be easily deduced from (4.4), the inequality $b_\omega > S$ and the estimate

$$\begin{aligned}
&\int_\Omega \frac{1}{\varepsilon^{2\alpha}} a\left(\frac{x}{\varepsilon^\alpha}\right)(V_\varepsilon[y,\delta](x))^2 dx \\
&\quad \le \int_{\mathbb{R}^N} a(x)(U_{y/\varepsilon^\alpha,\delta/\varepsilon^\alpha}(x))^2 dx + o(1) \\
&\quad \le |a|_{N/2}|U_{0,1}|^2_{2^*} + o(1) = |a|_{N/2} + o(1).
\end{aligned} \tag{4.9}$$

q.e.d.

Corollary 4.1 *Let $\mathcal{A}_\varepsilon$ and a_ε be as in Proposition 4.1. Then there exists $k^* > 0$ such that $|a|_{N/2} < k^*$ implies*

$$\mathcal{A}_\varepsilon < 2^{1-\frac{2}{2^*-\varepsilon}} m_\varepsilon \tag{4.10}$$

for all $\varepsilon > 0$ small enough.

Proof. It is an immediate consequence of the estimates (4.4) and (4.9). q.e.d.

Corollary 4.2 *Let $\mathcal{F}_\varepsilon$ and a_ε be as in Proposition 4.1. Let b_a and b_p be the numbers whose existence is stated, respectively, in Lemma 3.1 and Lemma 3.2. Then*

$$\mathcal{F}_\varepsilon < \min(b_a, b_p)$$

for all $\varepsilon > 0$ small enough.

Proposition 4.2 *Let a_ε satisfy (A_1). Let $\mathcal{B}_{\theta_\varepsilon,0}$ and $\mathcal{B}_{\theta_\varepsilon,s}$ be respectively the numbers defined in (3.3) and (3.5). Then for all small $\varepsilon > 0$ the following relations hold:*

$$\mathcal{B}_{\theta_\varepsilon,0} \le \mathcal{A}_\varepsilon, \tag{4.11}$$

$$\mathcal{B}_{\theta_\varepsilon,s} \le \mathcal{F}_\varepsilon. \tag{4.12}$$

Proof. Set

$$\Theta_\varepsilon(y,\delta) := \theta_\varepsilon(V_\varepsilon[y,\delta]), \qquad \Theta_\varepsilon : \mathcal{K}_\varepsilon \to \mathbb{R}^N \times \mathbb{R}^+. \tag{4.13}$$

To obtain (4.11) it is sufficient to prove, for all $\varepsilon > 0$ small enough, the existence of $(\bar{y}_\varepsilon, \bar{\delta}_\varepsilon) \in \operatorname{int}(\mathcal{K}_\varepsilon)$ such that

$$\Theta_\varepsilon(\bar{y}_\varepsilon, \bar{\delta}_\varepsilon) = \theta_\varepsilon(V_\varepsilon[\bar{y}_\varepsilon, \bar{\delta}_\varepsilon]) = (0, L/2). \tag{4.14}$$

So, let us consider the continuous map $g_\varepsilon : \mathcal{K}_\varepsilon \to \mathbb{R}^N \times \mathbb{R}^+$ defined by

$$g_\varepsilon(y,\delta) = (y, \delta/\varepsilon^\alpha) \tag{4.15}$$

and the homotopy map $\mathcal{H}_\varepsilon : [0,1] \times \mathcal{K}_\varepsilon \to \mathbb{R}^N \times \mathbb{R}^+$

$$\mathcal{H}_\varepsilon(y,\delta,t) = (1-t) g_\varepsilon(y,\delta) + t\Theta_\varepsilon(y,\delta). \tag{4.16}$$

Clearly, for ε suitably small, the equation $g_\varepsilon(y,\delta) = (0, L/2)$ has a solution in $\operatorname{int}(\mathcal{K}_\varepsilon)$; and moreover, $\mathcal{H}_\varepsilon$ is a continuous map. Hence, to prove (4.14) we just need to show that

$$\mathcal{H}_\varepsilon(y,\delta,t) \ne (0, L/2) \qquad \forall (y,\delta) \in \partial\mathcal{K}_\varepsilon, \ \forall t \in [0,1]. \tag{4.17}$$

Indeed, observing that, by the symmetry of $U_{0,1}(x)$ and the definition of $V_\varepsilon[y,\delta]$

$$\beta_\varepsilon(V_\varepsilon[y,\delta]) = \lambda_{\varepsilon,\delta}(|y|)\frac{y}{|y|} \qquad y \ne 0$$

$\lambda_{\varepsilon,\delta} : \mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$ being a positive strictly increasing function, (4.17) follows straightforward for all $(y,\delta) \in \partial\mathcal{K}_\varepsilon \setminus \{(0, \varepsilon^{\alpha/2}), (0, \varepsilon^{(3/2)\alpha})\}$. On the other hand,

$$\beta_\varepsilon(V_\varepsilon[0, \varepsilon^{\alpha/2}]) \equiv 0, \qquad |V_\varepsilon[0, \varepsilon^{\alpha/2}]|_{2^*} \underset{\varepsilon \to 0}{\longrightarrow} 1,$$

thus, we deduce for all $r > 0$, for small ε

$$\begin{aligned}
&\gamma_\varepsilon(V_\varepsilon[0, \varepsilon^{\alpha/2}]) \\
&\quad = \frac{1}{|V_\varepsilon[0, \varepsilon^{\alpha/2}]|_{2^*}^{2^*}} \int_{\mathbb{R}^N} |\chi_\varepsilon(x) - \beta_\varepsilon(V_\varepsilon[0, \varepsilon^{\alpha/2}])|\, |V_\varepsilon[0, \varepsilon^{\alpha/2}](x)|^{2^*} dx \\
&\quad = (1 + o(1)) \left[\int_{\mathbb{R}^N} |\chi_\varepsilon(x)|\, |V_\varepsilon[0, \varepsilon^{\alpha/2}](x)|^{2^*} dx \right] \\
&\quad \ge (1 + o(1)) \int_{\mathbb{R}^N \setminus B(0, \varepsilon^\alpha r)} |\chi_\varepsilon(x)|\, |V_\varepsilon[0, \varepsilon^{\alpha/2}](x)|^{2^*} dx
\end{aligned}$$

$$
\begin{aligned}
&\geq (1+o(1))\left[L\,\frac{r}{1+r}\int_{\mathbb{R}^N\setminus B(0,\varepsilon^\alpha r)}|V_\varepsilon[0,\varepsilon^{\alpha/2}](x)|^{2^*}dx\right]\\
&\geq (1+o(1))\left[L\,\frac{r}{1+r}\int_{\mathbb{R}^N\setminus B(0,\varepsilon^{\alpha/2} r)}|V_\varepsilon[0,1](x)|^{2^*}dx+o(1)\right]\\
&\geq L\,\frac{r}{1+r}+o(1)
\end{aligned}
$$

which implies

$$\liminf_{\varepsilon\to 0}\gamma_\varepsilon(V_\varepsilon[0,\varepsilon^{\alpha/2}])\geq L$$

allowing to obtain (4.17) when $(y,\delta)=(0,\varepsilon^{\alpha/2})$. Finally, let us prove (4.17) when $(y,\delta)=(0,\varepsilon^{(3/2)\alpha})$. As in the previous case, first let us remark that

$$\beta_\varepsilon(V_\varepsilon[0,\varepsilon^{(3/2)\alpha}])\equiv 0,\qquad |V_\varepsilon[0,\varepsilon^{(3/2)\alpha}]|_{2^*}\underset{\varepsilon\to 0}{\longrightarrow}1.$$

Hence we infer, for all $r>0$,

$$
\begin{aligned}
0&\leq\gamma_\varepsilon(V_\varepsilon[0,\varepsilon^{(3/2)\alpha}])\\
&=\frac{1}{|V_\varepsilon[0,\varepsilon^{(3/2)\alpha}]|_{2^*}^{2^*}}\int_{\mathbb{R}^N}|\chi_\varepsilon(x)-\beta_\varepsilon(V_\varepsilon[0,\varepsilon^{(3/2)\alpha}])|\,|V_\varepsilon[0,\varepsilon^{(3/2)\alpha}](x)|^{2^*}dx\\
&=(1+o(1))\int_{\mathbb{R}^N}|\chi_\varepsilon(x)|\,|V_\varepsilon[0,\varepsilon^{(3/2)\alpha}](x)|^{2^*}dx\\
&=(1+o(1))\left[\int_{\mathbb{R}^N\setminus B(0,\varepsilon^\alpha r)}|\chi_\varepsilon(x)|\,|V_\varepsilon[0,\varepsilon^{(3/2)\alpha}](x)|^{2^*}dx\right.\\
&\qquad\left.+\int_{B(0,\varepsilon^\alpha r)}|\chi_\varepsilon(x)|\,|V_\varepsilon[0,\varepsilon^{(3/2)\alpha}](x)|^{2^*}dx\right]\\
&\leq(1+o(1))\left[\int_{\mathbb{R}^N\setminus B(0,r/\varepsilon^{\alpha/2})}|V_\varepsilon[0,1](x)|^{2^*}dx+kr\int_{B(0,r/\varepsilon^{\alpha/2})}|V_\varepsilon[0,1](x)|^{2^*}dx+o(1)\right]\\
&=(1+o(1))(o(1)+kr)
\end{aligned}
$$

which implies

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon(V_\varepsilon[0,\varepsilon^{(3/2)\alpha}])=0$$

concluding the proof of (4.17) and, then, of (4.11). In order to show that (4.12) holds, let us prove that, for all $\varepsilon>0$ small enough, $(\hat{y}_\varepsilon,\hat{\delta}_\varepsilon)\in\partial\mathcal{K}_\varepsilon$ exists such that

$$\Theta_\varepsilon(\hat{y}_\varepsilon,\hat{\delta}_\varepsilon)\in\left(0,\frac{P}{1+|P|}\right)\times\left\{\frac{L}{2}\right\}. \tag{4.18}$$

As for proving (4.14), we consider for all ε the maps g_ε and $\mathcal{H}_\varepsilon$. Clearly, for ε suitably small, being $P\in\omega=\mathbb{R}^N\setminus\Omega$, the relation $g_\varepsilon(y,\delta)=\left(\tau\frac{P}{1+|P|},\frac{L}{2}\right)$ is verified for some

$(y,\delta)\in\partial\mathcal{K}_\varepsilon$ and $\tau\in(0,1)$. Hence, taking into account (4.17), (4.18) follows, considering that $\forall\varepsilon>0\ \forall(y,\delta)\in\mathcal{K}_\varepsilon$

$$\begin{aligned}|\beta_\varepsilon(V_\varepsilon[y,\delta])| &\le \frac{1}{|V_\varepsilon[y,\delta]|_{2^*}^{2^*}}\int_{\mathbb{R}^N}|\chi_\varepsilon(x)|\,(V_\varepsilon[y,\delta](x))^{2^*}dx\\ &= \frac{1}{|V_\varepsilon[y,\delta]|_{2^*}^{2^*}}\int_{B(0,(3/4)d)}|\chi_\varepsilon(x)|\,(V_\varepsilon[y,\delta](x))^{2^*}dx\\ &< \frac{d}{1+d}.\end{aligned}$$

q.e.d.

Proof of Theorem 1.1. To prove the theorem, we show that if a_ε is a function satisfying (A_1) and $0<|a_\varepsilon|_{N/2}<k^*$, k^* being the number whose existence is stated in Corollary 4.1, then, for small ε, E_ε has on $\mathcal{M}_\varepsilon$ two distinct critical values, lying in the energy range $(m_\varepsilon, 2^{1-\frac{2}{2^*-\varepsilon}}m_\varepsilon)$, to which there correspond at least two distinct solutions of (P_ε) that are positive by Lemma 2.4. So let us consider $\varepsilon>0$ chosen small enough to guarantee Lemma 3.1, Corollaries 3.1, 4.1, 4.2, 3.2 and Proposition 4.2 hold; and, furthermore, $\varepsilon^{\alpha/2}<\frac{L}{2}<\varepsilon^{-\alpha/2}$. Then

$$m_\varepsilon<\mathcal{B}_{\theta_\varepsilon,s}\le\mathcal{F}_\varepsilon<\min(b_a,b_p)<\mathcal{B}_{\theta_\varepsilon,0}\le\mathcal{A}_\varepsilon<2^{1-\frac{2}{2^*-\varepsilon}}m_\varepsilon. \tag{4.19}$$

Our aim is to show that there exist a critical value $c_{a,\varepsilon}\in[\mathcal{B}_{\theta_\varepsilon,0},\mathcal{A}_\varepsilon]$ and another critical value $c_{\omega,\varepsilon}\in[\mathcal{B}_{\theta_\varepsilon,s},\mathcal{F}_\varepsilon]$. Since, by (4.19), $m_\varepsilon<c_{a,\varepsilon}<c_{\omega,\varepsilon}<2^{1-\frac{2}{2^*-\varepsilon}}m_\varepsilon$, we then have at least 2 positive distinct solutions of (P_ε). Step 1 **Existence of a critical value in the interval** $[\mathcal{B}_{\theta_\varepsilon,0},\mathcal{A}_\varepsilon]$**.** We assume, by contradiction, that

$$\{u\in\mathcal{M}_\varepsilon\ :\ \mathcal{B}_{\theta_\varepsilon,0}\le E_\varepsilon(u)\le\mathcal{A}_\varepsilon,\ \nabla E_{\varepsilon|\mathcal{M}_\varepsilon}(u)=0\}=\emptyset.$$

Since, by Lemma 2.3, the functional E_ε constrained on $\mathcal{M}_\varepsilon$ satisfies the Palais-Smale condition in $(m_\varepsilon, 2^{1-\frac{2}{2^*-\varepsilon}}m_\varepsilon)$, we can use, by (4.19), a well known deformation Lemma (see f.i. [16]) to find a positive number η and a continuous map

$$\mathcal{I}: E_\varepsilon^{\mathcal{A}_\varepsilon}\longrightarrow E_\varepsilon^{\mathcal{B}_{\theta_\varepsilon,0}-\eta} \tag{4.20}$$

such that

$$\mathcal{I}_{|E_\varepsilon^{\mathcal{B}_{\theta_\varepsilon,0}-\eta}}=\mathrm{id} \tag{4.21}$$

and

$$\mathcal{F}_\varepsilon<\mathcal{B}_{\theta_\varepsilon,0}-\eta. \tag{4.22}$$

Then we define the continuous map $\mathcal{G}_\varepsilon:\mathcal{K}_\varepsilon\to\mathbb{R}^N\times\mathbb{R}^+$ by

$$\mathcal{G}_\varepsilon(y,\delta)=\theta_\varepsilon(\mathcal{I}(V_\varepsilon[y,\delta]))=(\,\beta_\varepsilon(\mathcal{I}(V_\varepsilon[y,\delta])),\gamma_\varepsilon(\mathcal{I}(V_\varepsilon[y,\delta]))\,)$$

and we remark that (4.21), (4.22) imply

$$\mathcal{G}_\varepsilon(y,\delta)=\theta_\varepsilon(V_\varepsilon[y,\delta])=\Theta_\varepsilon(y,\delta)\qquad\forall(y,\delta)\in\partial\mathcal{K}_\varepsilon, \tag{4.23}$$

Θ_ε being the map defined in (4.13). Clearly $\mathcal{G}_\varepsilon$ is homotopically equivalent to Θ_ε by the homotopy $t\Theta_\varepsilon + (1-t)\mathcal{G}_\varepsilon$, so, taking into account (4.23) and (4.14), we infer the existence of $(\tilde{y}_\varepsilon, \tilde{\delta}_\varepsilon) \in \mathrm{int}(\mathcal{K}_\varepsilon)$ for which $\mathcal{G}_\varepsilon(\tilde{y}_\varepsilon, \tilde{\delta}_\varepsilon) = (0, L/2)$ and, hence, $E_\varepsilon(\mathcal{I}(V_\varepsilon[\tilde{y}_\varepsilon, \tilde{\delta}_\varepsilon])) \geq \mathcal{B}_{\theta_\varepsilon,0}$, contradicting (4.20). Step 2 **Existence of a critical value in the interval $[\mathcal{B}_{\theta_\varepsilon,s}, \mathcal{F}_\varepsilon]$.** As in Step 1, we argue by contradiction and we assume

$$\{u \in \mathcal{M}_\varepsilon \ : \ \mathcal{B}_{\theta_\varepsilon,s} \leq E_\varepsilon(u) \leq \mathcal{F}_\varepsilon, \ \nabla E_{\varepsilon|\mathcal{M}_\varepsilon}(u) = 0\} = \emptyset.$$

So, considering (4.19) and the fact that E_ε, constrained on $\mathcal{M}_\varepsilon$, satisfies the Palais-Smale condition on $(m_\varepsilon, 2^{1-\frac{2}{2^*-\varepsilon}} m_\varepsilon)$, again using standard arguments, we find a number $\sigma > 0$ and a continuous map

$$\mathcal{T} : E_\varepsilon^{\mathcal{F}_\varepsilon} \times [0,1] \longrightarrow E_\varepsilon^{\mathcal{F}_\varepsilon}$$

such that

$$\begin{aligned} &\mathcal{T}(\cdot, 0) = \mathrm{id} \\ &\mathcal{T}(E_\varepsilon^{\mathcal{F}_\varepsilon}, 1) \subset E_\varepsilon^{\mathcal{B}_{\theta_\varepsilon,s} - \sigma}. \end{aligned} \tag{4.24}$$

Let us consider now the homotopy map $\mathcal{S} : \partial\mathcal{K}_\varepsilon \times [0,1] \to \mathbb{R}^N \times \mathbb{R}^+$ defined by

$$\mathcal{S}(y, \delta, t) = \theta_\varepsilon(\mathcal{T}(V_\varepsilon[y, \delta], t))$$

and remark that, being $\mathcal{S}(y, \delta, 0) = \Theta_\varepsilon([y, \delta])$ by (4.18), $\mathcal{S}(\partial\mathcal{K}_\varepsilon, 0) \cap \{[0, \chi_\varepsilon(P)] \times \{L/2\}\} = \Theta_\varepsilon(\partial\mathcal{K}_\varepsilon) \cap \{[0, \chi_\varepsilon(P)] \times \{L/2\}\} \neq \emptyset$ and, moreover, $\mathcal{S}(y, \delta, t) \cap \{(0, L/2), (\chi_\varepsilon(P), L/2)\} = \emptyset \ \forall (y, \delta) \in \partial\mathcal{K}_\varepsilon$, $\forall t \in [0,1]$, because, by (4.2) and (4.19), we have $E_\varepsilon(\mathcal{T}(V_\varepsilon[y, \delta], t)) < \mathcal{F}_\varepsilon < \min(b_a, b_p)$. Hence $\mathcal{S}(\partial\mathcal{K}_\varepsilon, 1) \cap \{[0, \chi_\varepsilon(P)] \times \{L/2\}\}$ must not be empty, contradicting (4.24). q.e.d.

Proof of Theorem 1.2. Let us choose $\varepsilon > 0$ so small that $\varepsilon^{\alpha/2} < \frac{L}{2} < \varepsilon^{-\alpha/2}$ and that Propositions 3.1, 4.1 and 4.2, Lemma 3.1 and Corollaries 4.2, 3.1, 3.2 hold; let k be the number whose existence is stated in Lemma 4.1. Therefore we have

$$m_\varepsilon \leq \mathcal{B}_{\theta_\varepsilon,s} \leq \mathcal{F}_\varepsilon < \min(b_a, b_p) < \mathcal{B}_{\theta_\varepsilon,0} \leq \mathcal{A}_\varepsilon < b_\omega < \mathcal{B}_{\varepsilon,P} \tag{4.25}$$

and, since the assumptions of Theorem 3.1 are fulfilled, we also know that a critical value c_ε exists so that

$$\mathcal{B}_{\varepsilon,P} \leq c_\varepsilon < 2^{1-\frac{2}{2^*-\varepsilon}} m_\varepsilon. \tag{4.26}$$

Then, arguing exactly as in Step 1 and in Step 2 of the proof of Theorem 1.1, we prove the existence of two more critical levels $c_{a,\varepsilon} \in [\mathcal{B}_{\theta_\varepsilon,0}, \mathcal{A}_\varepsilon]$ and $c_{\omega,\varepsilon} \in [\mathcal{B}_{\theta_\varepsilon,s}, \mathcal{F}_\varepsilon]$. Finally, since (4.25), (4.26) give

$$m_\varepsilon < c_{\omega,\varepsilon} < c_{a,\varepsilon} < c_\varepsilon < 2^{1-\frac{2}{2^*-\varepsilon}} m_\varepsilon,$$

we conclude, using also Lemma 2.4, that at least three positive solutions of (P_ε) exist. q.e.d.

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