Advanced Nonlinear Studies **8** (2008), 413–428

Orbital Instability of Standing Waves for the Klein-Gordon-Zakharov System

Zaihui Gan *

College of Mathematics and Software Science,
Sichuan Normal University, *Chengdu 610066, P.R.China*
e-mail: ganzaihui2008cn@yahoo.com.cn

Received in revised form 26 October 2007
Communicated by Yiming Long

Abstract

This paper deals with the instability of the ground state solitary wave solution to the Klein-Gordon-Zakharov system in three space dimensions with $c \geq 1$, which is a model to describe the Langmuir turbulence in plasma. First we construct a suitable constrained variational problem and obtain the existence of the standing waves with ground state by using variational calculus and scaling argument. Then by defining invariant sets and applying some priori estimates, we prove the orbital instability of the ground state in the following sense: in each neighborhood of it, there exists a solution whose energy diverges in finite or infinite time.

1991 Mathematics Subject Classification. 35L05,35A15.

Key words. Orbital instability, Standing wave, Klein-Gordon-Zakharov system, Invariant set

1 Introduction

In this paper we consider the orbital instability of standing waves for the Klein-Gordon-Zakharov system with different propagation speeds in three space dimensions

*This work is supported by National Natural Science Foundation of P.R.China(10771151, 10726034), Sichuan Youth Sciences and Technology Foundation(07ZQ026-009) and Morningside Center

$$\phi_{tt} - \Delta\phi + \phi = -\psi\phi, \quad t > 0, \quad x \in R^3, \tag{1.1}$$

$$\psi_t + \nabla \cdot V = 0, \quad t > 0, \quad x \in R^3, \tag{1.2}$$

$$V_t + c^2\nabla\psi + \nabla|\phi|^2 = 0, \quad t > 0, \quad x \in R^3, \tag{1.3}$$

where $c \geq 1$. The system (1.1)-(1.3) describes the interaction of the Langmuir wave and the ion acoustic wave in a plasma . The function ϕ denotes the fast time scale component of electric field raised by electrons, the function ψ denotes the deviation of ion density from its equilibrium, and the function V denotes the electron fluid velocity (see [2, 3, 25]). In the present paper, we take the function $\phi = \phi(t, x)$ as complex scalar-valued, the function $V = V(t, x)$ as real vector-valued and the function $\psi = \psi(t, x)$ as real scalar-valued.

We impose the following initial conditions on (1.1)-(1.3)

$$\begin{cases} \phi(0, x) = \phi_0(x) \in H^1(R^3), \quad \phi_t(0, x) = \phi_1(x) \in L^2(R^3), \\ \psi(0, x) = \psi_0(x) \in L^2(R^3), \quad V(0, x) = V_0(x) \in L^2(R^3). \end{cases} \tag{1.4}$$

In the theoretical investigations of the dynamics of strong Langmuir turbulence in plasma physics, various types of Zakharov equations take a very important role (see [9, 13, 22]). When $c = 1$, Ozawa, Tsutaya and Tsutsumi [13] obtained the global existence and asymptotic behaviour of solutions to (1.1)-(1.4), and Tsutaya [23] proved that there exist global solutions to (1.1)-(1.4) for small data by using the invariant Sobolev space but without applying the null condition technique and improved the regularity requirements on the initial data. When $0 < c < 1$, Ozawa, Tsutaya, and Tsutsumi [14] studied the well-posedness of the Cauchy problem (1.1)-(1.4) in the energy space $H^1(R^3) \times L^2(R^3) \times L^2(R^3) \times L^2(R^3)$. In the case of $c > 1$, using the ideas of the papers of Ozawa, Tsutaya, and Tsutsumi [13, 14] as well as Tsutaya [23], the well-posedness of the Cauchy problem (1.1)-(1.4) in the energy space $H^1(R^3) \times L^2(R^3) \times L^2(R^3) \times L^2(R^3)$ can be proved, which is also mentioned in Ohta and Todorova [12]. That is, for $c \geq 1$, the following result is true: for any $(\phi_0, \phi_1, \psi_0, V_0) \in H^1(R^3) \times L^2(R^3) \times L^2(R^3) \times L^2(R^3)$, there exists a unique solution $(\phi, \phi_t, \psi, V) \in C([0, T_{max}); H^1(R^3) \times L^2(R^3) \times L^2(R^3) \times L^2(R^3))$ of (1.1)-(1.4) with initial data $(\phi_0, \phi_1, \psi_0, V_0)$ satisfying the conservation law of the energy

$$\mathcal{E}\left(\phi(t), \phi_t(t), \psi(t), V(t)\right) = \mathcal{E}\left(\phi_0, \phi_1, \psi_0, V_0\right) \tag{1.5}$$

for all $t \in [0, T_{max})$, where

$$\mathcal{E}\left(\phi(t), \phi_t(t), \psi(t), V(t)\right) = \|\phi_t\|^2_{L^2(R^3)} + \|\nabla\phi\|^2_{L^2(R^3)} + \|\phi\|^2_{L^2(R^3)} + \frac{c^2}{2}\|\psi|^2_{L^2(R^3)}$$

$$+\frac{1}{2}\|V\|^2_{L^2(R^3)} + \int_{R^3} \psi|\phi|^2 dx. \tag{1.6}$$

Furthermore, Gan and Zhang [4, 5] established a sharp condition of global existence and obtained the instability of standing waves for (1.1)-(1.3). More generally, in two and three space dimensions, for any $c > 0$, Ohta and Todorova [12] proved strong instability of some

particular standing wave for the system (1.1)-(1.3). Moreover, as far as two space dimensions are concerned, via the so-called continuous method and a delicate priori estimates, Guo and Yuan [10] proved the existence and uniqueness of the global smooth solution for the Cauchy problem (1.1)-(1.4) with the presence of the term $-|\phi|^2\phi$ in the right hand side of (1.1).

In the present paper, we are devoted to studying the orbital instability of standing waves for the Klein-Gordon-Zakharove system (1.1)-(1.3). Many results concerning the standing waves for the single nonlinear wave equation have been obtained . Strauss [19] established the existence of solitary waves by introducing a compactness lemma. Berestycki and Cazenave [1], Levine [11] and Pagne [15] got the instability of the solitary waves by showing the blowup properties. Shatah[16, 17], Shatah and Strauss [18], Strauss [20] and Weinstein [24] also studied the stability or instability of the single nonlinear wave equation. Soon after, these results were generalized in Grillakis,Shatah and Strauss [7, 8] as well as Souganidis and Strauss [21]. Gan and Zhang [6] studied the instability of standing waves for inhomogeneous Klein-Gordon equations with critical exponent. Motivated by the results of single nonlinear wave equation, in this paper we will investigate the orbital instability of the system (1.1)-(1.3) and obtain:

Theorem 1.1 *Consider solutions of (1.1)-(1.4) with Cauchy data* $[(\phi_0, \psi_0), (\phi_1, V_0)] \in R_2$. *Then either*

(i) the solutions exist only locally in time $[0, T_0)$, $T_0 < \infty$, *and for some sequence* (t_m) ,

$$\|\phi(t_m)\|_{H^1(R^3)} + \|\psi(t_m)\|_{L^2(R^3)} \to \infty \quad as \quad t_m \to T_0$$

or

(ii) the solutions exist globally on $[0, \infty)$ *and* $\exists$ *a sequence* t_m *such that*

$$\|\phi(t_m)\|_{H^1(R^3)} + \|\psi(t_m)\|_{L^2(R^3)} \to \infty \quad as \quad t_m \to \infty.$$

In either case the standing waves are orbitally unstable.

Here, R_2 will be defined in Section 3. It should be pointed out that the orbital instability result in the present paper is different from that in Gan and Zhang [5] as well as Ohta and Todorova [12]. In [5], Gan and Zhang proved the instability of standing wave of (1.1)-(1.3) with finite time blow up for $0 < c < 1$, while in [12], Ohta and Todorova showed the strong instability of some particular standing wave of (1.1)-(1.3) for $c > 0$. But in the present paper, we introduce a new estimate to show the orbital instability of the ground state without blow up for $c \geq 1$, which generalizes the result in [12] for the case $c \geq 1$. Furthermore, the condition $c \geq 1$ is necessary for the orbital instability of ground state in the present paper.

For simplicity, throughout this paper we denote $\int_{R^3} \cdot dx$ by $\int \cdot dx$.

2 Existence of the standing waves

If a triplet

$$(u, v, n) = (u(x), v(x), n(x)), \quad x \in R^3$$

verifies the semilinear system

$$\begin{cases} -\triangle u + u = -vu, \quad x \in R^3, \\ \nabla \cdot n = 0, \quad x \in R^3, \\ -c^2 \nabla v = \nabla |u|^2, \quad x \in R^3, \end{cases} \tag{2.1}$$

and

$$(u, v) \in H^1(R^3) \times L^2(R^3) \setminus \{(0, 0)\}, n \in L^2(R^3),$$

then

$$\phi(t, x) = u(x), \quad \psi(t, x) = v(x), V(t, x) = n(x), \quad t \geq 0, \quad x \in R^3$$

verify (1.1)-(1.3), which is the standing wave solution of (1.1)-(1.3).

From the physical viewpoint, an important role is played by the ground state solution of (2.1). We recall that a solution (u, v, n) of (2.1) is termed as a ground state if it has some minimal action among all solutions of (2.1).

Now we define the action of the solution (u, v, n) of (2.1) as follows:

$$J(u, v, n) := \int |\nabla u|^2 dx + \int |u|^2 dx + \frac{c^2}{2} \int |v|^2 dx + \int v|u|^2 dx. \tag{2.2}$$

In addition, we define the functional $K(u, v, n)$ by

$$K(u, v, n) := \int |\nabla u|^2 dx + 3 \int |u|^2 dx + \frac{3c^2}{2} \int |v|^2 dx + 3 \int v|u|^2 dx, \tag{2.3}$$

and the manifold M by

$$M := \left\{(u, v) \in H^1(R^3) \times L^2(R^3) \setminus \{(0, 0)\}, n \in L^2(R^3), K(u, v, n) = 0\right\}. \tag{2.4}$$

Now we consider the constrained variational problem

$$\inf_{(u,v,n) \in M} J(u, v, n) = d. \tag{2.5}$$

Firstly about the standing waves of (1.1)-(1.3), we have the following existence theorem of the standing waves associated with the ground state.

Theorem 2.1 *There exists $(w, h, q) \in M$ such that*

1. $J(w, h, q) = \inf\limits_{(u,v,n) \in M} J(u, v, n) = d$;
2. (w, h, q) *is a ground state solution of (2.1).*

Before we begin to prove Theorem 2.1, we first consider the following four Lemmas and two Propositions.

Lemma 2.1 *Let $u \in H_r^1(R^N)$. Then*

$$|u(x)| \le C_N |x|^{(1-N)/2} \, \|u\|_{H_r^1(R^N)} \quad a.e.,$$

where C_N depends only on spatial dimension $N(N \ge 2)$, and the inclusion map

$$H_r^1(R^N) \hookrightarrow L^p(R^N), 2 < p < 2 + 4/(N-2)$$

is compact.

Proof. See Strauss [13].

Lemma 2.2 *If $(u, v, n) \in H^1(R^3) \times L^2(R^3) \times L^2(R^3)$ is a solution of (2.1), then $K(u, v, n) = 0$.*

Proof. Let $u_\beta(x) = u(x/\beta), v_\beta(x) = v(x/\beta)$ and $n_\beta(x) = n(x/\beta)$. Then

$$J(u_\beta, v_\beta, n_\beta) = \int |\nabla u_\beta|^2 dx + \int |u_\beta|^2 dx + \frac{c^2}{2} \int |v_\beta|^2 dx + \int v_\beta |u_\beta|^2 dx$$

$$= \beta \int |\nabla u|^2 dx + \beta^3 \int |u|^2 dx + \frac{c^2 \beta^3}{2} \int |v|^2 dx + \beta^3 \int v|u|^2 dx.$$

Since (u, v, n) is a solution of (2.1), then $\delta_u J(u, v, n) = 0$, $\delta_v J(u, v, n) = 0$ and $\delta_n J(u, v, n) = 0$, or $\frac{d(J(u_\beta, v_\beta, n_\beta))}{d\beta}\Big|_{\beta=1} = 0$. But

$$\frac{d(J(u_\beta, v_\beta, n_\beta))}{d\beta}\Big|_{\beta=1} = \int |\nabla u|^2 dx + 3 \int |u|^2 dx + \frac{3c^2}{2} \int |v|^2 dx + 3 \int v|u|^2 dx,$$

then $K(u, v, n) = 0$.

Lemma 2.3 *M is a C^1 hypersurface in $H^1(R^3) \times L^2(R^3)$ bounded away from (0,0).*

Proof. Since

$$M = \Big\{ (u, v, n) : K(u, v, n) = \int |\nabla u|^2 dx +$$

$$3 \int |u|^2 dx + \frac{3c^2}{2} \int |v|^2 dx + 3 \int v|u|^2 dx = 0, (u, v) \ne (0, 0), n \in L^2(R^3) \Big\},$$

then $K(u, v, n)$ is a C^1 functional. Now consider $(u^*, v^*, n^*) \in H^1(R^3) \times L^2(R^3) \times L^2(R^3)$ such that $\delta_{u^*} K(u^*, v^*, n^*) = 0$ and $\delta_{v^*} K(u^*, v^*, n^*) = 0$, namely,

$$\begin{cases} -\triangle u^* + 3u^* + 3v^* u^* = 0, \\ c^2 v^* + |u^*|^2 = 0. \end{cases}$$

Using the method of Lemma 2.2, we get

$$\int |\nabla u^*|^2 dx + 9 \int |u^*|^2 dx + \frac{9c^2}{2} \int |v^*|^2 dx + 9 \int v^* |u^*|^2 dx = 0. \qquad (2.6)$$

Then if $K(u^*, v^*, n^*) = 0$, we have

$$\int |\nabla u^*|^2 dx - 3\int |\nabla u^*|^2 dx = 0,$$

which together with (2.6) implies that $(u^*, v^*) = (0, 0)$. Hence for $(u^*, v^*, n^*) \in M$, $\delta K(u^*, v^*, n^*) \neq 0$ and M is a C^1 hypersurface. Finally, by the inequality

$$\int -v|u|^2 dx \leq \frac{1}{2}\int |v|^2 dx + \frac{1}{2}\int |u|^4 dx,$$

we have

$$K(u, v, n) \geq \int |\nabla u|^2 dx + 3\int |u|^2 dx + \frac{3c^2}{2}\int |v|^2 dx - \frac{3}{2}\int |v|^2 dx - \frac{3}{2}\int |u|^4 dx$$

$$= \int |\nabla u|^2 dx + 3\int |u|^2 dx + \frac{3c^2 - 3}{2}\int |v|^2 dx - \frac{3}{2}\int |u|^4 dx.$$

From Sobolev embedding and $N = 3$ we have

$$K(u, v, n) \geq \int |\nabla u|^2 dx + 3\int |u|^2 dx + \frac{3c^2 - 3}{2}\int |v|^2 dx$$

$$-\frac{3}{2}(\int |\nabla u|^2 dx + \int |u|^2 dx)^2$$

$$\geq \int |\nabla u|^2 dx + \int |u|^2 dx + \frac{3c^2 - 3}{2}\int |v|^2 dx - \frac{3}{2}(\int |\nabla u|^2 dx + \int |u|^2 dx)^2$$

$$= \|u\|^2_{H^1(R^3)} + \frac{3c^2 - 3}{2}\|v\|^2_{L^2(R^3)} - \frac{3}{2}\|u\|^4_{H^1(R^3)}.$$

For $0 < \|u\|^2_{H^1(R^3)} + \|v\|^2_{L^2(R^3)} < \varepsilon$, ε small, we have $K(u, v, n) > 0$, which implies that M is bounded away from $(0, 0)$.

Lemma 2.4 *The minimization problem*

$$d \equiv \inf_{(u,v,n)\in M} J(u, v, n) = \inf\left\{\frac{2}{3}\int |\nabla u|^2 dx, K(u, v, n) = 0, (u, v) \neq (0, 0)\right\}$$

is equivalent to

$$d \equiv \inf\left\{\frac{2}{3}\int |\nabla u|^2 dx, K(u, v, n) \leq 0, (u, v) \neq (0, 0)\right\}.$$

Proof. For any $(u_0^*, v_0^*, n_0^*) \in H^1(R^3) \times L^2(R^3) \times L^2(R^3)$ such that

$$K(u_0^*, v_0^*, n_0^*) < 0.$$

let $u^*_\beta(x) = u^*_0(x/\beta), v^*_\beta(x) = v^*_0(x/\beta)$ and $n^*_\beta(x) = n^*_0(x/\beta)$. Then

$$K(u^*_\beta, v^*_\beta, n^*_\beta) = \int |\nabla u^*_\beta|^2 dx + 3\int |u^*_\beta|^2 dx + \frac{3c^2}{2}\int |v^*_\beta|^2 dx + 3\int v^*_\beta |u^*_\beta|^2 dx$$

$$= \beta \int |\nabla u^*_0|^2 dx + 3\beta^3 \int |u^*_0|^2 dx + \frac{3c^2}{2}\beta^2 \int |v^*_0|^2 dx + 3\beta^3 \int v^*_0 |u^*_0|^2 dx.$$

Now for $\beta = 1$, $K(u^*_1, v^*_1, n^*_1) = K(u^*_0, v^*_0, n^*_0) < 0$ and for β close to zero, we have $K(u^*_\beta, v^*_\beta, n^*_\beta) > 0$. Therefore there exists $\beta_0 \in (0,1)$ such that

$$K(u^*_{\beta_0}, v^*_{\beta_0}, n^*_{\beta_0}) = 0 (i.e. (u^*_{\beta_0}, v^*_{\beta_0}, n^*_{\beta_0}) \in M).$$

But

$$J(u^*_{\beta_0}, v^*_{\beta_0}, n^*_{\beta_0}) = \frac{2}{3}\int |\nabla u^*_{\beta_0}|^2 dx = \frac{2}{3}\beta_0 \int |\nabla u^*_0|^2 dx < \frac{2}{3}\int |\nabla u^*_0|^2 dx$$

since $\beta_0 \in (0,1)$. Consequently the two minimizations are equivalent.

Proposition 2.1 *J is bounded below on M.*

Proof. From (2.2)-(2.4), on M one has

$$J(u,v,n) = \frac{2}{3}\int |\nabla u|^2 dx.$$

It follows that

$$J(u,v,n) \geq 0 \quad on \quad M.$$

So J is bounded below on M.

Proposition 2.2 *If $(u_0, v_0, n_0) \in H^1(R^3) \times L^2(R^3) \times L^2(R^3)$ is a solution of (2.1), let $\lambda > 0$ and $u_\lambda(x) = u_0(x/\lambda)$, $v_\lambda(x) = v_0(x/\lambda)$ and $n_\lambda(x) = n_0(x/\lambda)$. Then there exists a unique $\mu > 0$ (depending on (u_0, v_0, n_0)) such that $K(u_\mu, v_\mu, n_\mu) = 0$. Moreover, $K(u_\lambda, v_\lambda, n_\lambda) > 0$ for $\lambda \in (0, \mu)$, $K(u_\lambda, v_\lambda, n_\lambda) < 0$ for $\lambda \in (\mu, \infty)$, and $J(u_\mu, v_\mu, n_\mu) \geq J(u_\lambda, v_\lambda, n_\lambda)$ for $\forall\, \lambda > 0$.*

Proof. By (2.2) and (2.3), we have

$$K(u_\lambda, v_\lambda, n_\lambda)$$

$$= \lambda \int |\nabla u_0|^2 dx + 3\lambda^3 \int |u_0|^2 dx + \frac{3c^2}{2}\lambda^3 \int |v_0|^2 dx + 3\lambda^3 \int v_0 |u_0|^2 dx,$$

$$J(u_\lambda, v_\lambda, n_\lambda) = \lambda \int |\nabla u_0|^2 dx + \lambda^3 \int |u_0|^2 dx + \frac{c^2}{2}\lambda^3 \int |v_0|^2 dx + \lambda^3 \int v_0 |u_0|^2 dx.$$

Since (u_0, v_0, n_0) is a solution of (2.1), then there exists a unique $\mu > 0$ (depending on (u_0, v_0)) such that

$$K(u_\mu, v_\mu, n_\mu) = 0.$$

Moreover,

$$K(u_\lambda, v_\lambda, n_\lambda) > 0 \quad for \quad \lambda \in (0, \mu),$$
$$K(u_\lambda, v_\lambda, n_\lambda) < 0 \quad for \quad \lambda \in (\mu, \infty).$$

Because

$$\frac{d}{d\lambda} J(u_\lambda, v_\lambda, n_\lambda) = \lambda^{-1} K(u_\lambda, v_\lambda, n_\lambda),$$

noting that $K(u_\mu, v_\mu, n_\mu) = 0$, it follows that

$$J(u_\mu, v_\mu, n_\mu) \geq J(u_\lambda, v_\lambda, n_\lambda) \quad \forall \lambda > 0.$$

This completes the proof of Proposition 2.2.

Proof of Theorem 2.1. Due to Proposition 2.1, we may let

$$\{(u_m, v_m, n_m), \quad m \in N\} \subset M$$

be a minimizing sequence for (2.5), that is

$$J(u_m, v_m, n_m) \to \inf_{(u,v,n)\in M} J(u, v, n) \quad (n \to \infty). \tag{2.7}$$

Let u^*, v^*, and n^* denote the Schwartz spherical rearrangement of functions u, v and n, respectively. We recall that u^*, v^* and n^* are spherically symmetric, non-increasing (with respect to $|x|$) functions. The symmetrization has the following properties:

$$\int |\nabla u^*|^2 dx \leq \int |\nabla u|^2 dx, \quad \int |\nabla v^*|^2 dx \leq \int |\nabla v|^2 dx, \tag{2.8}$$

$$\int |\nabla n^*|^2 dx \leq \int |\nabla n|^2 dx, \tag{2.8^a}$$

$$\int |u^*|^\sigma dx = \int |u|^\sigma dx, \quad \int |v^*|^\sigma dx = \int |v|^\sigma dx, \tag{2.9}$$

$$\int |n^*|^\sigma dx = \int |n|^\sigma dx, \quad for \quad \sigma > 1. \tag{2.9^a}$$

Furthermore, it is straightforward to check that

$$(u_\lambda)^* = (u^*)_\lambda, \quad (v_\lambda)^* = (v^*)_\lambda, \quad (n_\lambda)^* = (n^*)_\lambda, \tag{2.10}$$

where as in proposition 2.2,

$$u_\lambda(x) = u_0(x/\lambda), v_\lambda(x) = v_0(x/\lambda) \quad and \quad n_\lambda(x) = n_0(x/\lambda).$$

Now for the minimizing sequence $\{(u_m, v_m, n_m), m \in N\}$, we let

$$w_n = (u_m^*)_{\mu_m}, \quad h_m = (v_m^*)_{\mu_m}, \quad n_m = (n_m^*)_{\mu_m},$$

where $\mu_m > 0$ is uniquely determined by

$$K(w_m, h_m, q_m) = K[(u_m^*)_{\mu_m}, (v_m^*)_{\mu_m}, (n_m^*)_{\mu_m}] = 0. \tag{2.11}$$

In view of (2.8), (2.8^a), (2.9),(2.9^a) and (2.10), one has

$$w_m = [(u_m)_{\mu_m}]^*, \quad h_m = [(v_m)_{\mu_m}]^*, \quad q_m = [(n_m)_{\mu_m}]^*$$

and

$$J(w_m, h_m, q_m) \le J[(u_m)_{\mu_m}, (v_m)_{\mu_m}, (n_m)_{\mu_m}] \le J(u_m, v_m, n_m). \tag{2.12}$$

The right-hand side inequality in (2.12) is a consequence of Proposition 2.2, since we have $K(u_m, v_m, n_m) = 0$ (note that $\mu = 1$ in this case). Thus one has

$$\{(w_m, h_m, q_m), m \in N\} \subset M$$

and by (2.12),

$$J(w_m, h_m, q_m) \le J(u_m, v_m, n_m).$$

Therefore $\{(w_m, h_m, q_m), m \in N\}$ is also a minimizing sequence for (2.5).

Now, from Lemma 2.4 we get

$$\begin{aligned} 0 &\ge K(w_m, h_m, q_m) \\ &= \int |\nabla w_m|^2 dx + 3\int |w_m|^2 dx + \frac{3c^2}{2}\int |h_m|^2 dx + 3\int h_m |w_m|^2 dx \\ &\ge \int |\nabla w_m|^2 dx + 3\int |w_m|^2 dx + \frac{3c^2}{2}\int |h_m|^2 dx - \\ &\quad \frac{3}{2}\int |h_m|^2 dx - \frac{3}{2}\int |w_m|^4 dx. \end{aligned} \tag{2.13}$$

By the Sobolev embedding and the boundedness of $\int |\nabla w_m|^2 dx$, we have that $\|w_m\|_{H^1(R^3)}$ and $\|h_m\|_{L^2(R^3)}$ are both bounded for $m \in N$. Thus there exists a subsequence also denoted by (w_m, h_m, q_m) such that

$$w_m \rightharpoonup w_\infty \quad weakly \quad in \quad H^1(R^3), \quad h_m \rightharpoonup h_\infty \quad weakly \quad in \quad L^2(R^3). \tag{2.14}$$

So by Lemma 2.1 we have

$$w_m \to w_\infty \quad strongly \quad in \quad L^4(R^3). \tag{2.15}$$

Now we assert that $(w_\infty, h_\infty) \ne (0, 0)$, which can be obtained by contradiction. Next, we take $w = (w_\infty)_\mu, \quad h = (h_\infty)_\mu$ and $q = (q_\infty)_\mu$, with $\mu > 0$ uniquely determined from the condition $K(w, h, q) = K[(w_\infty)_\mu, (h_\infty)_\mu, (q_\infty)_\mu] = 0$, and thus from (2.14) and (2.15), one gets

$$\begin{cases} (w_m)_\mu \to w \quad strongly \quad in \quad L^4(R^3), \\ (w_m)_\mu \rightharpoonup w \quad weakly \quad in \quad H^1(R^3), \\ (h_m)_\mu \rightharpoonup h \quad weakly \quad in \quad L^2(R^3). \end{cases} \tag{2.16}$$

Since $K(w_m, h_m, q_m) = 0$, Proposition 2.2 shows that

$$J[(w_m)_\mu, (h_m)_\mu, (q_m)_\mu] \le J(w_m, h_m, q_m). \tag{2.17}$$

Using (2.16) and (2.17), one has

$$J(w,h,q) \le \lim_{m\to\infty} J[(w_m)_\mu, (h_m)_\mu, (q_m)_\mu] \le \lim_{n\to\infty} J(w_m, h_m, q_m) = \inf_M J. \tag{2.18}$$

As $(w,h) \neq (0,0)$ and $K(w,h,q) = 0$, one has $(w,h,q) \in M$. Therefore from (2.18), (w,h,q) solves the minimization problem

$$J(w,h,q) = \min_{(u,v,n)\in M} J(u,v,n). \tag{2.19}$$

Thus we have proved 1 of Theorem 2.1. In the following, we prove 2 of Theorem 2.1.

Since (w,h,q) is a critical point of $J(u,v,n)$ restricted to M, by the Lagrange multiplier method we have

$$\delta J(w,h,q) + \eta \delta K(w,h,q) = 0, \quad \eta \in R,$$

or

$$\begin{cases} -\triangle w + w + hw + \eta(-\triangle w + 3w + 3hw) = 0, \\ c^2 h + |w|^2 + \eta(3c^2 h + 3|w|^2) = 0. \end{cases} \tag{2.20}$$

Thus by Lemma 2.2 we get

$$K(w,h,q) + \eta\Big(\int |\nabla w|^2 dx + 9\int |w|^2 dx + \frac{9}{2}c^2 \int |h|^2 dx + 9\int h|w|^2 dx\Big) = 0,$$

or

$$K(w,h,q) + \eta[-2\int |\nabla w|^2 dx + 3K(w,h,q)] = 0.$$

But $K(w,h,q) = 0$ since $(w,h,q) \in M$. Hence $\eta \int |\nabla w|^2 dx = 0$ and this implies that $\eta = 0$ and (w,h,q) is a solution of (2.1). Thus the proof of 2 of Theorem 2.1 is completed.

3 Orbital instability of the standing waves

In this section, we prove Theorem 1.1. First we rewrite the energy functional (1.5) as

$$\mathcal{E}[(\phi,\psi),(\phi^*,V^*)] = \int |\phi^*|^2 dx + \frac{1}{2}\int |V^*|^2 dx + J(\phi,\psi,V^*),$$

$$(\phi,\psi) \in H^1(R^3)\times L^2(R^3), \quad (\phi^*,V^*) \in L^2(R^3)\times L^2(R^3),$$

and define two regions R_1 and R_2 by

$$R_1 := \big\{(\phi,\psi) \in H^1(R^3)\times L^2(R^3), (\phi^*,V^*) \in L^2(R^3)\times L^2(R^3) :$$

$$\mathcal{E}[(\phi,\psi),(\phi^*,V^*)]<d,\quad K(\phi,\psi,V^*)>0\}\bigcup\{(0,0),(0,0)\}$$
$$=\Big\{(\phi,\psi)\in H^1(R^3)\times L^2(R^3),(\phi^*,V^*)\in L^2(R^3)\times L^2(R^3):$$
$$\mathcal{E}[(\phi,\psi),(\phi^*,\psi^*)]<d,\quad \frac{2}{3}\int|\nabla\phi|^2dx<d\Big\},$$
$$R_2=\Big\{(\phi,\psi)\in H^1(R^3)\times L^2(R^3),(\phi^*,V^*)\in L^2(R^3)\times L^2(R^3):$$
$$\mathcal{E}[(\phi,\psi),(\phi^*,V^*)]<d,\quad K(\phi,\psi,V^*)<0,(\phi,\psi)\neq(0,0)\}$$
$$=\Big\{(\phi,\psi)\in H^1(R^3)\times L^2(R^3),(\phi^*,V^*)\in L^2(R^3)\times L^2(R^3):$$
$$\mathcal{E}[(\phi,\psi),(\phi^*,\psi^*)]<d,\quad \frac{2}{3}\int|\nabla\phi|^2dx>d\Big\}.$$

Thus we have

Proposition 3.1 *R_1 and R_2 are invariant regions under the flow generated by the Cauchy problem (1.1),(1.2) and (3.1).*

Proof. We will show this by contradiction. Let $[(\phi_0,\psi_0),(\phi_1,V_0)]\in R_1$ and assume there is a t_0 such that $[(\phi(t_0),\psi(t_0)),(\phi_t(t_0),V(t_0))]\notin R_1$. By lower semicontinuity of $K(\phi(t),\psi(t))$ there is a minimal t_1 such that

$$[(\phi(t_1),\psi(t_1)),(\phi_t(t_1),V(t_1))]\notin R_1,$$

i.e. for $t\geq t_1$, $K(\phi(t_1),\psi(t_1),V(t_1))\leq 0$, and for $0\leq t<t_1$, $K(\phi(t),\psi(t),V(t))>0$. Now

$$\frac{2}{3}\int|\nabla\phi(t_1)|^2dx\leq\lim_{\substack{t\to t_1\\ t<t_1}}\frac{2}{3}\int|\nabla\phi(t)|^2dx$$
$$\leq\lim_{\substack{t\to t_1\\ t<t_1}}\left[\frac{2}{3}\int|\nabla\phi(t)|^2dx+\frac{1}{3}K(\phi(t),\psi(t),V(t))\right]$$

and therefore

$$\frac{2}{3}\int|\nabla\phi(t_1)|^2dx\leq\lim_{t\to t_1}J(\phi(t),\psi(t),V(t))\leq\lim_{t\to t_1}\mathcal{E}[(\phi(t),\psi(t)),(\phi_t(t),V(t))]<d.$$

But we also have $K(\phi(t_1),\psi(t_1),V(t_1))\leq 0$ and this contradicts the definition of d as the infimum of $\frac{2}{3}\int|\nabla\phi|^2dx$ restricted to $K(\phi,\psi,V)\leq 0$ and $(\phi,\psi)\neq(0,0)$. Therefore R_1 is an invariant region. Similarly we can show that R_2 is also invariant.

Let

$$F_m(r,t)=rH(m-r)+[r\ln(m)/\ln(r)]\,[H(r-m)-H(r-(t+2m))]$$
$$+[(t+2m)\ln(m)/\ln(t+2m)]\,H(r-(t+2m)),$$

where $H(s)=1$ for $s\geq 0$, $H(s)=0$ for $s<0$, ln is the natural log function and $\bar{\phi}_r=\frac{x}{r}.\nabla\bar{\phi}$. Moreover, from (1.2) and (1.3) we can get

$$\psi_{tt}-c^2\Delta\psi=\Delta|\phi|^2. \tag{3.1}$$

Now multiplying equation (1.1) by $2F_m(r,t)\bar{\phi}_r$, equation (3.1) by $F_m(r,t)\psi_r$ and taking the real part, we obtain

$$2Re\int \phi_{tt}F_m(r,t)\bar{\phi}_r dx - 2Re\int \triangle\phi F_m(r,t)\bar{\phi}_r dx + 2Re\int \phi F_m(r,t)\bar{\phi}_r dx$$

$$+2Re\int \psi\phi F_m(r,t)\bar{\phi}_r dx - \int \triangle^{-1}\psi_{tt}F_m(r,t)\psi_r dx + c^2\int \psi F_m(r,t)\psi_r dx$$

$$+\int |\phi|^2 F_m(r,t)\psi_r dx$$

$$= 2\frac{d}{dt}Re\int F_m(r,t)\bar{\phi}_r\phi_t dx - 2Re\int \phi_t\frac{d}{dt}(F_m(r,t)\bar{\phi}_r)dx -$$

$$2Re\int \triangle\phi F_m(r,t)\bar{\phi}_r dx + 2Re\int \phi F_m(r,t)\bar{\phi}_r dx +$$

$$2Re\int \psi\phi F_m(r,t)\bar{\phi}_r dx - \frac{d}{dt}\int F_m(r,t)\psi_r\triangle^{-1}\psi_t dx$$

$$+\int \triangle^{-1}\psi_t\frac{d}{dt}(F_m(r,t)\psi_r)dx + c^2\int \psi F_m(r,t)\psi_r dx + \int |\phi|^2 F_m(r,t)\psi_r dx$$

$$= 2\frac{d}{dt}\int ReF_m(r,t)\bar{\phi}_r\phi_t dx - \frac{d}{dt}\int F_m(r,t)\psi_r\triangle^{-1}\psi_t dx$$

$$+\int_{r<m}\left(3|\phi_t|^2 + 3|\nabla^{-1}\psi_t|^2 - |\nabla\phi|^2 - 3|\phi|^2 - \frac{3}{2}c^2|\psi|^2 - 3\psi|\phi|^2\right)dx$$

$$+\int_{m<r<2m+t}\frac{\ln m}{\ln r}\left(3|\phi_t|^2 + 3|\nabla^{-1}\psi_t|^2 - |\nabla\phi|^2 - 3|\phi|^2 - \frac{3}{2}c^2|\psi|^2 - 3\psi|\phi|^2\right)dx$$

$$-\int_{m<r<2m+t}\frac{\ln m}{\ln^2 r}\left(|\phi_t|^2 + |\nabla^{-1}\psi_t|^2 + |\nabla\phi|^2 - |\phi|^2 - \frac{1}{2}c^2|\psi|^2 - \psi|\phi|^2\right)dx$$

$$+\int_{2m+t<r}\Big[e_{1m}(r,t)\left(Re(2\phi_t\bar{\phi}_r) + \triangle^{-1}\psi_t\psi_r\right) + e_{2m}(r,t)\left(|\phi_t|^2 + |\nabla^{-1}\psi_t|^2\right.$$

$$\left.-2|\nabla\phi|^2 + 2|\phi|^2 + c^2|\psi|^2 + 2\psi|\phi|^2\right)\Big]dx = 0, \tag{3.2}$$

where $|e_{im}(r,t)| < C$ for $r > 2m + t$ and $i = 1, 2$. In accordance with (3.2), integrating with respect to t, we obtain

$$2Re \int F_m(r,t)\bar{\phi}_r\phi_t dx - \int F_m(r,t)\psi_r \triangle^{-1}\psi_t dx - 2Re \int F_m(r,0)\bar{\phi}_r(0)\phi_t(0)dx$$

$$+ \int F_m(r,0)\psi_r(0)\triangle^{-1}\psi_t(0)dx$$

$$= \int_0^t \int_{r<m} \left(-3|\phi_t|^2 - 3|V|^2 + |\nabla\phi|^2 + 3|\phi|^2 + \frac{3}{2}c^2|\psi|^2 + 3\psi|\phi|^2\right) dxds$$

$$+ \int_0^t \int_{m<r<2m+t} \frac{\ln m}{\ln r} \left(-3|\phi_t|^2 - 3|V|^2 + |\nabla\phi|^2 + 3|\phi|^2\right.$$

$$\left. + \frac{3}{2}c^2|\psi|^2 + 3\psi|\phi|^2\right) dxds$$

$$+ \int_0^t \int_{m<r<2m+t} \frac{\ln m}{\ln^2 r} \left(|\phi_t|^2 + |V|^2 + |\nabla\phi|^2 - |\phi|^2 - \frac{1}{2}c^2|\psi|^2 - \psi|\phi|^2\right) dxds$$

$$+ \int_0^t \int_{2m+t<r} \left[-e_{1m}(r,t)\left(Re(2\phi_t\bar{\phi}_r) + \triangle^{-1}\psi_t\psi_r\right) + e_{2m}(r,t)\left(-|\phi_t|^2 - |V|^2\right.\right.$$

$$\left.\left. + 2|\nabla\phi|^2 - 2|\phi|^2 - c^2|\psi|^2 - 2\psi|\phi|^2\right)\right] dxds. \tag{3.3}$$

Proof of Theorem 1.1. (i) is just a consequence of the local existence of the solution to the Cauchy problem (1.1)-(1.4). Now we show (ii) by contradiction. Assume that the solution $(\phi(t), \psi(t), V(t))$ remains bounded in $H^1(R^3) \times L^2(R^3) \times L^2(R^3)$. Then there exists an $\varepsilon > 0$ such that for $\forall t$, $K(\phi(t), \psi(t), V(t)) < -\varepsilon$. Otherwise, we would have a sequence (t_k) such that (since R_2 is invariant) $K(\phi(t_k), \psi(t_k), V(t_k)) \to 0$, and

$$\frac{2}{3}\int |\nabla\phi(t_k)|^2 dx = J(\phi(t_k), \psi(t_k, V(t_k))) - \frac{1}{3}K(\phi(t_k), \psi(t_k, V(t_k)))$$

$$< \mathcal{E}[(\phi_0, \psi_0), (\phi_1, V_0)] - \frac{1}{3}K(\phi(t_k), \psi(t_k), V(t_k)).$$

Now we extract a weakly convergent subsequence as in section 2:

$$(\phi(t_k), \psi(t_k), V(t_k))) \rightharpoonup (\phi^0, \psi^0, V^0) \quad weakly \quad in \quad H^1(R^3) \times L^2(R^3) \times L^2(R^3),$$

and

$$\phi(t_k) \to \phi^0 \in L^p(R^3), \quad 2 < p < 6,$$

thus we get

$$\frac{2}{3}\int |\nabla\phi^0|^2dx \le E[(\phi_0,\psi_0),(\phi_1,V_0)] < d,$$

since $[(\phi_0,\psi_0),(\phi_1,V_0)] \in R_2$ and $K(\phi^0,\psi^0,V^0) \le 0$. But this contradicts the definition of d. Therefore such an ε exists.

From Lemma 2.1, for r large, the right-hand side of (3.3) can be bounded as follows:

$$\int_{m<r<2m+t}\left(3\frac{\ln m}{\ln r}-\frac{\ln m}{\ln^2 r}\right)\psi|\phi|^2dx < 3\int_{m<r<2m+t}\psi|\phi|^2dx,$$

$$\int_{m<r<2m+t}\left(\frac{\ln m}{\ln r}+\frac{\ln m}{\ln^2 r}\right)|\nabla\phi|^2dx$$

$$< \int_{m<r<2m+t}|\nabla\phi|^2dx+\frac{1}{2\ln m}\int_{m<r<2m+t}|\nabla\phi|^2dx,$$

$$\int_{m<r<2m+t}\left(3\frac{\ln m}{\ln r}-\frac{\ln m}{\ln^2 r}\right)|\phi|^2dx < 3\int_{m<r<2m+t}|\phi|^2dx,$$

$$\int_{m<r<2m+t}\left(\frac{3}{2}c^2\frac{\ln m}{\ln r}-\frac{1}{2}c^2\frac{\ln m}{\ln^2 r}\right)|\psi|^2dx < \frac{3}{2}c^2\int_{m<r<2m+t}|\psi|^2dx,$$

$$\int_{r>2m+t}\left(2|\bar\phi_r\phi_t|+|\psi_r\triangle^{-1}\psi_t|+|\phi_t|^2+|V|^2\right.$$

$$\left.+2|\nabla\phi|^2+2|\phi|^2+c^2|\psi|^2+2|\psi||\phi|^2\right)dx$$

$$\le C_0\int_{2m<r}\left(|\nabla\phi_0|^2+|\phi_0|^2+|\phi_1|^2+|\psi_0|^2+|V_0|^2+|\psi_0||\phi_0|^2\right)dx.$$

So by (3.3), we get

$$2\int ReF_m(r,t)\bar\phi_r\phi_t dx-\int F_m(r,t)\psi_r\triangle^{-1}\psi_t dx-2\int ReF_m(r,0)\bar\phi_r(0)\phi_t(0)dx$$

$$+\int F_m(r,0)\psi_r(0)\triangle^{-1}\psi_t(0)dx$$

$$\le\int_0^t\left[K\left(\phi(s),\psi(s),V(s)\right)+\frac{1}{2\ln m}\int_{m<r}|\nabla\phi|^2dx+\right.$$

$$\left.C_1\int_{2m<r}\left(|\nabla\phi_0|^2+|\phi_0|^2+|\phi_1|^2+|\psi_0|^2+|V_0|^2+|\psi_0||\phi_0|^2\right)dx\right]ds. \tag{3.4}$$

Choose m large enough so that

$$\frac{1}{2\ln m}\int_{m<r}|\nabla\phi|^2dx < \frac{\varepsilon}{4},$$

$$C_1 \int_{2m<r} \left(|\nabla\phi_0|^2 + |\phi_0|^2 + |\phi_1|^2 + |\psi_0|^2 + |V_0|^2 + |\psi_0||\phi_0|^2\right) dx < \frac{\varepsilon}{4}.$$

Since $K(\phi(t), \psi(t), V(t)) < -\varepsilon$, (3.4) becomes

$$2\int ReF_m(r,t)\bar{\phi}_r\phi_t dx - \int F_m(r,t)\psi_r\triangle^{-1}\psi_t dx - 2\int ReF_m(r,0)\bar{\phi}_{0r}\phi_1 dx$$

$$+ \int F_m(r,0)\psi_{0r}\triangle^{-1}\psi_1 dx \leq -\frac{\varepsilon t}{2}, \tag{3.5}$$

but $|F_m(r,t)| \leq \frac{Ct}{\ln t}$ for t large. Then (3.4) implies that

$$4\left(\int |\nabla\phi(t)|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int |\phi_t(t)|^2 dx\right)^{\frac{1}{2}} + 2\left(\int |\psi(t)|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int |V|^2 dx\right)^{\frac{1}{2}}$$

$$> C(\varepsilon)\ln t$$

for t large.

This contradicts the assumption that $(\phi(t), \psi(t), V(t))$ is bounded in $H^1(R^3) \times L^2(R^3) \times L^2(R^3)$. Therefore there is a sequence (t_m) such that

$$\|\phi(t_m)\|_{H^1(R^3)} + \|\psi(t_m)\|_{L^2(R^3)} \to \infty \quad as \quad t_m \to \infty. \tag{3.6}$$

At last, we take

$$\phi_0(x) = w(x/\lambda), \quad \phi_1(x) = 0, \quad \psi_0(x) = h(x/\lambda), \quad \psi_1(x) = 0, \lambda > 1.$$

Then $[(\phi_0, \psi_0), (\phi_1, \psi_1)] \in R_2$ and is close to the standing waves. From Proposition 3.1, (3.5) implies the orbital instability of standing waves for the Klein-Gordon-Zakharov system (1.1)-(1.3).

References

[1] H. Berestycki, T. Cazenave, *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéairees*, C. R. Acad. Sci. Paris **293**(1981),489-492.

[2] L. Bergé, B. Bidégaray, and T. Colin, *A perturbative analysis of the time-envelope approximation in strong Langmuir turbulence*, Phys. D **95**(1996), 351-379.

[3] R.O. Dendy, Plasma Dynamics, *Oxford University Press, Oxford, 1990.*

[4] J. Zhang, Z. Gan, *Sharp conditions'of global existence for Klein-Gordon-Zakharov equations in three space dimensions*, Adv. Math. **34**(2)(2005),241-244.

[5] Z. Gan, J. Zhang, *Instability of standing waves for Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions*, J. Math. Anal. Appl. **307**(2005), 219-231.

[6] Z. Gan, J. Zhang, *Standing waves of the inhomogeneous Klein-Gordon equations with critical exponent*, Acta Math. Sinica **22**(2)(2006), 357-366.

[7] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry*, Part I, J. Funct. Anal. **74**(1987),160-197.

[8] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry*, Part II, J. Funct. Anal. **94**(1990),308-348.

[9] B. Guo, *Global smooth solution for the system of Zakharov equations in nonhomogeneous medium*, Northeastern Math. J.**6**(1990), 379-390.

[10] B. Guo, G. Yuan, *Global smooth solution for the Klein-Gordon-Zakharov equations*, J. Math. Phys. **36**(8)(1995), 4119-4124.

[11] H.A. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form* $pu_{tt} = -Au - F(u)$. Transactions of the American Mathematical Society, **92**(1974),1-21.

[12] M. Ohta, G. Todorova, *Strong instability of standing waves for the nonlinear Klein-Gordon equation and the Klein-Gordon-Zakharov system*, SIAM J. Math. Anal. **38**(6)(2007), 1912-1931.

[13] T. Ozawa, K. Tsutaya, and Y. Tsutsumi, *Normal form and global solutions for the Klein-Gordon-Zakharov equations*,Ann.Inst.Henri Poincaré, Analyse non linéaire,**12**(1995),459-503.

[14] T. Ozawa, K. Tsutaya, and Y. Tsutsumi, *Well-posedness in energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions*, Math. Ann., **313**(1999),127-140.

[15] L.E. Pagne, D.H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations.* Israel Journal of Mathematics, **22**(3-4)(1975),273-303.

[16] J. Shatah, *Stable standing waves of nonlinear Klein-Gordon equations*, Commun. Math. Phys. **91**(1983),313-327.

[17] J. Shatah, *Unstable ground state of nonlinear Klein-Gordon equations*, Transactions of the American Mathematical Society**290**(2)(1985),701-710.

[18] J. Shatah, W. Strauss, *Instability of nonlinear bound states*, Comm. Math. Phys.**100**(1985),173-190.

[19] W.A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55**(1977),149-162.

[20] W. Strauss, *Stability of solitary waves*, Contemporary Math. **107**(1990),123-129.

[21] P. E. Souganidis, W. A. Strauss, *Instability of a class of dissipersive solitary waves*, Proc. Roy. Soc. Edin.**114A**(1990),195-212.

[22] S. G. Thornhill, D.ter Haar, *Langmuir turbulence and modulational instability*, Phys. Reports(Sect. C of Phys. Lett.) **43**(1978),43-99.

[23] K. Tsutaya, *Global existence of small amplitude solutions for the Klein-Gordon-Zakharov equations*, Nonlinear Anal. TMA **27**(1996),1373-1380.

[24] M. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39**(1986),51-68.

[25] V. E. Zakharov, *Collapse of Langmuir waves*, Sov. Phys.JETP **35**(1972),908-914.