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Asymptotic Behavior of The Unique Solution to a Singular Elliptic Problem With Nonlinear Convection Term And Singular Weight

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Abstract

By Karamata regular variation theory, we first derived the exact asymptotic behavior of the local solution to the problem $-\varphi''(s) = g(\varphi(s))$, $\varphi(s) > 0$, $s \in (0, a)$ and $\varphi(0) = 0$. Then, by a perturbation method and constructing comparison functions, we derived the exact asymptotic behavior of the unique classical solution near the boundary to a singular Dirichlet problem $-\Delta u = b(x)g(u) + \lambda|\nabla u|^q,\ u > 0,\ x \in \Omega$, $u|_{\partial\Omega} = 0$, where Ω is a bounded domain with smooth boundary, $\lambda \in \mathbb{R}$, $q \in [0, 2]$; $g \in C^1((0,\infty),(0,\infty))$, is decreasing in $(0,\infty)$ with $\lim_{s\to 0^+} g(s) = +\infty$; the weight b is positive in Ω and singular on the boundary.

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1 Introduction and the main results

The purpose of this paper is to investigate the exact asymptotic behavior of the unique classical solution near the boundary to the following model problem

$$-\triangle u = b(x)g(u) + \lambda|\nabla u|^q,\ \ u > 0,\ \ x \in \Omega,\quad u|_{\partial\Omega} = 0, \tag{1.1}$$

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where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N (N \geq 1)$, $\lambda \in \mathbb{R}$, $q \in [0,2]$; g, b satisfy

(g_1) g is decreasing in $(0,\infty)$ with $\lim_{s\to 0^+} g(s) = +\infty$,

(b_1) $b \in C^{\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$, is positive in Ω.

This problem arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials (see [3], [6], [9], [17], [21]).

For $\lambda = 0$, i.e., problem (1.1) becomes:

$$-\Delta u = b(x)g(u),\ u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0. \tag{1.2}$$

The problem was discussed in a number of works; see, for instance, [2], [3], [6], [7], [11]-[14], [20]-[24]. For $b \equiv 1$ on Ω: when g satisfies (g_1), Fulks and Maybee [6], Stuart [21], Crandall, Rabinowitz and Tartar [3] showed that problem (1.2) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$. Moreover, Crandall, Rabinowitz and Tartar ([3], Theorems 2.2 and 2.5) showed that if $\varphi \in C[0,a] \cap C^2(0,a]$ is the local solution to the problem

$$-\varphi''(s) = g(\varphi(s)),\ \varphi(s) > 0,\ 0 < s < a,\ \varphi(0) = 0, \tag{1.3}$$

then there exist positive constants C_1 and C_2 such that

(I) $C_1\varphi(d(x)) \leq u(x) \leq C_2\varphi(d(x))$ near $\partial\Omega$, where $d(x) = \mathrm{dist}(x, \partial\Omega)$.

In particular, when $g(u) = u^{-\gamma}$, $\gamma > 1$, u has the property:

(I_1) $C_1[d(x)]^{2/(1+\gamma)} \leq u(x) \leq C_2[d(x)]^{2/(1+\gamma)}$ near $\partial\Omega$.

In [13], by constructing global subsolution and supersolution, Lazer and McKenna showed that (I_1) continues to hold on $\bar{\Omega}$. Then $u \in H^1_0(\Omega)$ if and only if $\gamma < 3$. This is a basic character of problem (1.2).

Now, let $g \in C^{\alpha}_{\mathrm{loc}}(0,\infty)$ be non-negative and satisfy (g_1) and

(g_2) there exist positive constants C_0, η_0 and $\gamma \in (0,1)$ such that $g(s) \leq C_0 s^{-\gamma}$, $\forall s \in (0, \eta_0)$;

(g_3) there exist $\theta > 0$ and $t_0 \geq 1$ such that $g(\xi t) \geq \xi^{-\theta} g(t)$ for all $\xi \in (0,1)$ and $0 < t \leq t_0\xi$;

(g_4) the mapping $\xi \in (0,\infty) \to T(\xi) = \lim_{t\to 0^+} \frac{g(\xi t)}{\xi g(t)}$ is a continuous function;

and $b \in C^{\alpha}(\bar{\Omega})$ and satisfy the following assumptions: there exist $\delta_0 > 0$ and a positive non-decreasing function $k \in C(0, \delta_0)$ such that

(b_2) $\lim_{d(x)\to 0^+} \frac{b(x)}{k(d(x))} = c_0$;

(b_3) $\lim_{t\to 0^+} k(t)g(t) = +\infty$.

Ghergu and Rădulescu [7] showed that problem (1.2) has a unique solution $u \in C^{1,1-\alpha}(\bar{\Omega}) \cap C^2(\Omega)$ satisfying

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(d(x))} = \xi_0, \tag{1.4}$$

where $T(\xi_0) = c_0^{-1}$, and $\psi \in C^1[0,a] \cap C^2(0,a] (a \in (0, \delta_0))$ is the local solution to the problem

$$-\psi''(s) = k(s)g(\psi(s)),\ \psi(s) > 0,\ 0 < s < a,\ \psi(0) = 0. \tag{1.5}$$

When $\lambda = \pm 1$, $0 < q < 2$, $b(x) \equiv 1$ on Ω and the function $g : (0, \infty) \to (0, \infty)$ is locally Lipschitz continuous and decreasing, Giarrusso and Porru [10] showed that if g satisfies the following conditions:

(g_5) $\int_0^1 g(s)ds = \infty, \int_1^\infty g(s)ds < \infty$;

(g_6) Let $G(t) = \int_t^\infty g(s)ds$. There exist positive constants δ and M with $M > 1$ such that $G(t) < MG(2t), \forall t \in (0, \delta)$;

then the unique solution u has the properties:

(II_1) $|u(x) - \varphi(d(x))| < C_0 d(x), \quad \forall x \in \Omega$ for $0 < q \leq 1$;

(II_2) $|u(x) - \varphi(d(x))| < C_0 d(x)[G(\varphi(d(x)))]^{(q-1)/2}$, $\forall x \in \Omega$ for $1 < q < 2$;

where C_0 is a suitable positive constant and $\varphi \in C[0, \infty) \cap C^2(0, \infty)$ is uniquely determined by

$$\int_0^{\varphi(t)} \frac{ds}{\sqrt{2G(s)}} = t, \ \forall t \in (0, \infty). \tag{1.6}$$

These imply that

$$\lim_{d(x) \to 0} \frac{u(x)}{\varphi(d(x))} = 1. \tag{1.7}$$

In particular, if $g(u) = u^{-\gamma}$, $\gamma > 1$, then $\varphi(s) = cs^{2/(1+\gamma)}$, $c = \left[\frac{(1+\gamma)^2}{2(\gamma-1))}\right]^{1/(1+\gamma)}$, u satisfies

$$\lim_{d(x) \to 0} \frac{u(x)}{[d(x)]^{2/(1+\gamma)}} = \left[\frac{(1+\gamma)^2}{2(\gamma-1)}\right]^{1/(1+\gamma)}. \tag{1.8}$$

For other works, see [4], [5], [8], [9], [25], [26], [28] and the references therein.

In this paper, by Karamata regular variation theory and constructing comparison functions, we show the exact asymptotic behaviour of the unique solution near the boundary to problem (1.1) for the more general weight b which is singular on the boundary.

First we recall a basic definition in Karamata regular variation theory, which was first introduced and established by Karamata in 1930 and is a basic tool in stochastic process, see [15], [18], [19], and has been applied to study the exact asymptotic behavior of solutions near the boundary blow-up elliptic problems, see [1], [16], [27].

Definition 1.1 A positive measurable function g defined on some neighborhood $(0, a)$, for some $a > 0$, is called *regularly varying at zero with index* $-\beta$, written $g \in RVZ_{-\beta}$, if for each $\xi > 0$ and some $\beta \in \mathbb{R}$,

$$\lim_{t \to 0^+} \frac{g(\xi t)}{g(t)} = \xi^{-\beta}. \tag{1.9}$$

In particular, when $\beta = 0$, we say that g *is slowly varying at zero.*

It follows by Definitions that if $g \in RVZ_{-\beta}$ then there exists a function H which is slowly varying at zero such that

$$g(t) = u^{-\beta} H(t). \tag{1.10}$$

Let Λ denote the set of all positive non-increasing functions k satisfying $k \in L^1(0,\delta_0)\cap C^1(0,\delta_0)$ and

$$\lim_{t\to 0^+}\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right)=C_k\in(0,\infty), \tag{1.11}$$

where

$$K(t)=\int_0^t k(s)ds. \tag{1.12}$$

We note that for each $k\in\Lambda$, $\lim_{t\to 0^+}\frac{K(t)}{k(t)}=0$ and $C_k\geq 1$.

The set Λ was first introduced for non-decreasing functions by Cîrstea and Rădulescu [1] and by Mohammed [16] for non-increasing functions for studying the exact asymptotic behaviour of solutions near the boundary to boundary blow-up elliptic problems, see also [27].

Some basic examples of the non-increasing functions in Λ are

(i) $k\equiv C_0>0$, $K(t)=C_0t$, $C_k=1$;
(ii) $k(t)=t^{-\sigma/2}$ with $\sigma\in(0,2)$, $K(t)=\frac{2t^{(2-\sigma)/2}}{2-\sigma}$, $C_k=\frac{2}{2-\sigma}>1$;
(iii) $k(t)=-\ln t$, $K(t)=t(1-\ln t)$, $C_k=1$;
(iv) $k(t)=\frac{-\ln t}{t^\sigma}$ with $\sigma\in(0,1)$, $K(t)=\frac{t^{1-\sigma}}{(1-\sigma)^2}\left(1-(1-\sigma)\ln t\right)$, $C_k=\frac{1}{1-\sigma}>1$;
(v) $k(t)=c_0t^{-\sigma/2}\exp\left(\int_t^{\delta_0}\frac{y_1(s)}{s}ds\right)$, $0<t<\delta_0$, where $c_0>0$, $\sigma>0$, $y_1\in C[0,\delta_0]$ is non-negative with $y_1(0)=0$, $C_k=\frac{2}{2-\sigma}>1$.

Our main results are summarized in the following theorem.

Theorem 1.1 *Let $g\in C^1((0,\infty),(0,\infty))$ be decreasing in $(0,\infty)$, $g'\in RVZ_{-\gamma-1}$ with $\gamma>1$, and $b\in C^\alpha_{\mathrm{loc}}(\Omega)$ be positive in Ω with $\lim_{d(x)\to 0}b(x)=\infty$. If there exist non-increasing function $k\in\Lambda$ and a positive constant c_0 such that*

(b_4) $\lim_{d(x)\to 0}\frac{b(x)}{k^2(d(x))}=c_0$,

then the unique solution $u_\lambda\in C(\bar{\Omega})\cap C^2(\Omega)$ to problem (1.1) satisfies

$$\lim_{d(x)\to 0}\frac{u_\lambda(x)}{\varphi(K(d(x)))}=\left(\frac{c_0(\gamma-1)}{C_k(\gamma+1)-2}\right)^{1/(1+\gamma)}, \tag{1.13}$$

where $\varphi\in C[0,a]\cap C^2(0,a]$ is uniquely determined by

$$\int_0^{\varphi(t)}\frac{ds}{\sqrt{2G(s)}}=t,\ G(t)=\int_t^b g(s)ds,\ \forall b>0,\ t\in(0,b). \tag{1.14}$$

Moreover, $\varphi\in RVZ_{2/(1+\gamma)}$ and there exists $H\in RVZ_0$ such that

$$\varphi(t)=t^{2/(1+\gamma)}H(t). \tag{1.15}$$

Remark 1.1 By (1.14), we see that the asymptotic behavior (1.13) of u_λ is independent of $\lambda|\nabla u_\lambda|^q$.

Remark 1.2 Some basic examples of the functions which satisfy the conditions in Theorem 1.1 are:

(i) $g(u) = u^{-\gamma}$, where $\gamma > 1$;
(ii) $g(u) = u^{-\gamma}\arctan(u^{-1})$, where $\gamma > 1$;
(iii) $g(u) = u^{-\gamma_1}(\ln(1+u))^{-\gamma_2}$, where $\gamma_1 > 0,\ \gamma_2 > 0$ and $\gamma_1 + \gamma_2 > 1$;
(iv) $g(u) = u^{-\gamma_1}(e^u - 1)^{-\gamma_2}$, where $\gamma_1 > 0,\ \gamma_2 > 0$ and $\gamma_1 + \gamma_2 > 1$;
(v) $g(u) = u^{-\gamma}(-\ln u)^{\sigma},\ 0 < u < \delta_0 < 1$, where $\gamma > 1,\ \sigma > 0$;
(vi) $g(u) = c_0 u^{-\gamma}\exp\left(\int_u^{\delta_0}\frac{y_2(s)}{s}ds\right),\quad 0 < u < \delta_0$, where $c_0 > 0,\quad \gamma > 1$, $y_2 \in C[0,\delta_0]$ is non-negative with $y_2(0) = 0$.

Remark 1.3 For the existence and uniqueness of classical solutions to problem (1.1), see [26].

Remark 1.4 We'll see in the following that $g' \in RVZ_{-\gamma-1}$, with $\gamma > 1$, implies that $\lim_{u\to 0^+} g(u) = \infty$.

2 Proof of Theorem 1.1

Let's continue to recall some basic properties to Karamata regular variation theory (see[15], [18], [19]).

Some basic examples of slowly varying functions are:

(i) every measurable function on $(0,a)$ which has a positive limit at zero;
(ii) $H(t) = \prod_{m=2}^{m=n}(\log_m(t^{-1}))^{\alpha_m}$, $\alpha_m \in \mathbb{R}$;
(iii) $H(t) = e^{(\prod_{m=2}^{m=n}(\log_m(t^{-1}))^{\alpha_m})}$, $0 < \alpha_m < 1$;
(iv) $H(t) = t\displaystyle\int_b^{t^{-1}}\frac{ds}{\ln s}$, $b > a^{-1}$;
(v) $H(t) = e^{((\ln t^{-1})^{1/3}\cos((\ln t^{-1})^{1/3}))}$, $\lim_{t\to 0^+}\inf L(t) = 0$, $\lim_{t\to 0^+}\sup L(t) = +\infty$.

Lemma 2.1 *If a functions H is slowly varying at zero, then for $t \to 0$,*

$$\int_0^t s^{\beta}H(s)ds \cong (\beta+1)^{-1}t^{1+\beta}H(t),\ \text{for } \beta > -1; \tag{2.1}$$

$$\int_t^a s^{\beta}H(s)ds \cong (-\beta-1)^{-1}t^{1+\beta}H(t),\ \text{for } \beta < -1. \tag{2.2}$$

Lemma 2.2 *If a function H is slowly varying at zero, then for every $\theta > 0$ and $t \to 0^+$,*

$$t^{-\theta}H(t) \to \infty,\quad t^{\theta}H(t) \to 0. \tag{2.3}$$

By Lemma 2.1, we can directly show the following result.

Lemma 2.3 *If g satisfies (g_1) and $g' \in RVZ_{-\gamma-1}$ with $\gamma > 1$, $g'(t) = t^{-\gamma-1}H(t)$, where H is slowly varying at zero, then*

$$\int_0^1 g(t)dt = \infty,\ G(t) < \infty,\ \forall t \in (0, b)\ and\ G(0) = \infty; \tag{2.4}$$

$$\lim_{t\to 0^+} \frac{G(t)}{g(t)} = 0 = \lim_{t\to 0^+} \frac{\sqrt{G(t)}}{g(t)}; \tag{2.5}$$

$$\lim_{t\to 0^+} \frac{tg'(t)}{g(t)} = -\gamma, \quad \lim_{t\to 0^+} \frac{tg(t)}{G(t)} = \gamma - 1. \tag{2.6}$$

Lemma 2.4 *Let g, k and φ be in Theorem 1.1. Then*

(i) $\lim_{t\to 0^+} \frac{k'(t)K(t)}{k^2(t)} = 1 - C_k$;

(ii) $\lim_{t\to 0^+} \frac{\varphi'(t)}{t\varphi''(t)} = -\frac{\gamma+1}{\gamma-1}$.

Proof.

(i) $\lim_{t\to 0^+} \frac{k'(t)K(t)}{k^2(t)} = 1 - \lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) = 1 - C_k$.

(ii) We see by (1.14) and a direct calculation that

$$\varphi'(t) = \sqrt{2G(\varphi(t))},\ -\varphi''(t) = g(\varphi(t)),\ 0 < t < b.$$

It follows by Lemma 2.3 and l'Hospital's rule that

$$\lim_{t\to 0^+} \frac{\varphi'(t)}{t\varphi''(t)} = -\lim_{t\to 0^+} \frac{\sqrt{2G(\varphi(t))}}{tg(\varphi(t))} = -\lim_{u\to 0^+} \frac{\sqrt{2G(u)}/g(u)}{\int_0^u \frac{ds}{\sqrt{2G(s)}}}$$

$$= -\left(1 + 2\lim_{u\to 0^+} \frac{g'(u)G(u)}{g^2(u)}\right) = -\left(1 + 2\lim_{u\to 0^+} \frac{ug'(u)}{g(u)} \lim_{u\to 0^+} \frac{G(u)}{ug(u)}\right) = -\frac{\gamma+1}{\gamma-1}.$$

The proof is finished.

Lemma 2.5. *Under the assumption in Theorem 1.1, $\varphi \in RVZ_{2/(1+\gamma)}$.*

Proof. Let $f_1(t) = \int_0^t \frac{ds}{\sqrt{2G(s)}}$ $\forall t \in (0, \tau)$. By l'Hospital's rule and Proposition 0.8 in [18], we can easily see that $f_1 \in RVZ_{(1+\gamma)/2}$ and $\varphi = f_1^{-1} \in RVZ_{2/(1+\gamma)}$.

Proof of Theorem 1.1. Set

$$\tau_0 = 1 + \frac{(C_k - 1)(\gamma + 1)}{\gamma - 1} = \frac{C_k(\gamma + 1) - 2}{\gamma - 1},\ \xi_0 = (c_0/\tau_0)^{1/(1+\gamma)}.$$

For arbitrary $\varepsilon \in (0, \tau_0/4)$, let

$$\xi_{1\varepsilon} = \left(\frac{c_0}{\tau_0 - 2\epsilon}\right)^{1/(1+\gamma)},\ \xi_{2\varepsilon} = \left(\frac{c_0}{\tau_0 + 2\epsilon}\right)^{1/(1+\gamma)}.$$

We see that

$$(2/3)^{1/(1+\gamma)}\xi_0 < \xi_{2\varepsilon} < \xi_0 < \xi_{1\varepsilon} < 2\xi_0.$$

Since $\partial\Omega \in C^2$, there exists a constant $\delta \in (0, \delta_0/2)$ which only depends on Ω such that

(i) $d(x) \in C^2(\bar{\Omega}_\delta)$ and $|\nabla d| \equiv 1$ on $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$.

By the assumption on b and Lemma 2.4, we see that corresponding to ε, there is $\delta_\varepsilon \in (0, \delta)$ sufficiently small such that

(ii) for $i = 1, 2$,

$$|\frac{k'(d(x))K(d(x))}{k^2(d(x))}\frac{\varphi'(s)}{s\varphi''(s)} - (\tau_0 - 1) + \frac{K(d(x))}{k(d(x))}\frac{\varphi'(s)}{s\varphi''(s)}\Delta d(x) + \frac{\lambda\xi_{i\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))}\frac{(\varphi'(s))^q}{\varphi''(s)}| < \varepsilon,\ \forall (x, s) \in \Omega_{\delta_\varepsilon} \times (0, \delta_\varepsilon);$$

(iii) $\frac{\xi_{2\varepsilon}k^2(d(x))g(\varphi(K(d(x))))}{g(\xi_{2\varepsilon}\varphi(K(d(x))))}(\tau_0 + \varepsilon) < b(x) < \frac{\xi_{1\varepsilon}k^2(d(x))g(\varphi(K(d(x))))}{g(\xi_{1\varepsilon}\varphi(K(d(x))))}(\tau_0 - \varepsilon)$ in $\Omega_{\delta_\varepsilon}$.

For any $x \in \Omega_{\delta_\varepsilon}$, define $\bar{u}_\varepsilon = \xi_{1\varepsilon}\varphi(K(d(x)))$, and $\underline{u}_\varepsilon = \xi_{2\varepsilon}\varphi(K(d(x)))$. We see that

$$\begin{aligned}
& \triangle\bar{u}_\varepsilon(x) + b(x)g(\bar{u}_\varepsilon(x)) + \lambda|\nabla\bar{u}_\varepsilon(x)|^q \\
= {} & \xi_{1\varepsilon}\varphi''(K(d(x)))k^2(d(x)) + \xi_{1\varepsilon}\varphi'(K(d(x)))k'(d(x)) \\
& + \xi_{1\varepsilon}\varphi'(K(d(x)))k'(d(x))\Delta d(x) + b(x)g(\xi_1\varphi(K(d(x)))) \\
& + \lambda\xi_{1\varepsilon}^q(\varphi'(K(d(x))))^q k^q(d(x)) \\
= {} & \xi_{1\varepsilon}g(\varphi(K(d(x))))k^2(d(x))\Bigg[\frac{b(x)g(\xi_{1\varepsilon}\varphi(K(d(x))))}{\xi_{1\varepsilon}k^2(d(x))g(\varphi(K(d(x))))} - 1 \\
& - \frac{k'(d(x))}{k^2(d(x))}\frac{\varphi'(K(d(x)))}{\varphi''(K(d(x)))} - \frac{\varphi'(K(d(x)))}{k(d(x))\varphi''(K(d(x)))}\Delta d(x) \\
& - \frac{\lambda\xi_{1\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))}\frac{(\varphi'(K(d(x))))^q}{\varphi''(K(d(x)))}\Bigg] \\
= {} & \xi_{1\varepsilon}g(\varphi(K(d(x))))k^2(d(x))\Bigg[\frac{b(x)g(\xi_{1\varepsilon}\varphi(K(d(x))))}{\xi_{1\varepsilon}k^2(d(x))g(\varphi(K(d(x))))} - \tau_0 \\
& - \left(\frac{k'(d(x))K(d(x))}{k^2(d(x))}\frac{\varphi'(K(d(x)))}{K(d(x))\varphi''(K(d(x)))} - (\tau_0 - 1)\right) \\
& - \frac{K(d(x))}{k(d(x))}\frac{\varphi'(K(d(x)))}{K(d(x))\varphi''(K(d(x)))}\Delta d(x) - \frac{\lambda\xi_{1\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))}\frac{(\varphi'(K(d(x))))^q}{\varphi''(K(d(x)))}\Bigg] \\
\leq {} & \xi_{1\varepsilon}g(\varphi(K(d(x))))k^2(d(x))\Bigg[(\tau_0 - \varepsilon) - \tau_0 \\
& - \left(\frac{k'(d(x))K(d(x))}{k^2(d(x))}\frac{\varphi'(K(d(x)))}{K(d(x))\varphi''(K(d(x)))} - (\tau_0 - 1)\right) \\
& - \frac{K(d(x))}{k(d(x))}\frac{\varphi'(K(d(x)))}{K(d(x))\varphi''(K(d(x)))}\Delta d(x) - \frac{\lambda\xi_{1\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))}\frac{(\varphi'(K(d(x))))^q}{\varphi''(K(d(x)))}\Bigg]
\end{aligned}$$

$$\leq \quad 0;$$

and

$$\begin{aligned}
& \triangle \underline{u}_\varepsilon(x) + b(x)g(\underline{u}_\varepsilon(x)) + \lambda|\nabla \underline{u}_\varepsilon(x)|^q \\
= \; & \xi_{2\varepsilon}\varphi''(K(d(x)))k^2(d(x)) + \xi_{2\varepsilon}\varphi'(K(d(x)))k'(d(x)) \\
& + \xi_{2\varepsilon}\varphi'(K(d(x)))k'(d(x))\Delta d(x) + b(x)g(\xi_{2\varepsilon}\varphi(K(d(x)))) \\
& + \lambda\xi_{2\varepsilon}^q k^q(d(x))(\varphi'(K(d(x))))^q \\
= \; & \xi_{2\varepsilon} g(\varphi(K(d(x))))k^2(d(x)) \left[\frac{b(x)g(\xi_{2\varepsilon}\varphi(K(d(x))))}{\xi_{2\varepsilon}k^2(d(x))g(\varphi(K(d(x))))} - 1 \right. \\
& - \frac{k'(d(x))}{k^2(d(x))} \frac{\varphi'(K(d(x)))}{\varphi''(K(d(x)))} - \frac{\varphi'(K(d(x)))}{k(d(x))\varphi''(K(d(x)))} \Delta d(x) \\
& \left. - \frac{\lambda\xi_{2\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))} \frac{(\varphi'(K(d(x))))^q}{\varphi''(K(d(x)))} \right] \\
= \; & \xi_{2\varepsilon} g(\varphi(K(d(x))))k^2(d(x)) \left[\frac{b(x)g(\xi_{2\varepsilon}\varphi(K(d(x))))}{\xi_{2\varepsilon}k^2(d(x))g(\varphi(K(d(x))))} - \tau_0 \right. \\
& - \left(\frac{k'(d(x))K(d(x))}{k^2(d(x))} \frac{\varphi'(K(d(x)))}{K(d(x))\varphi''(K(d(x)))} - (\tau_0 - 1) \right) \\
& \left. - \frac{K(d(x))}{k(d(x))} \frac{\varphi'(K(d(x)))}{K(d(x))\varphi''(K(d(x)))} \Delta d(x) - \frac{\lambda\xi_{2\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))} \frac{(\varphi'(K(d(x))))^q}{\varphi''(K(d(x)))} \right] \\
\geq \; & \xi_{2\varepsilon} g(\varphi(K(d(x))))k^2(d(x)) \left[(\tau_0 + \varepsilon) - \tau_0 \right. \\
& - \left(\frac{k'(d(x))K(d(x))}{k^2(d(x))} \frac{\varphi'(K(d(x)))}{K(d(x))\varphi''(K(d(x)))} - (\tau_0 - 1) \right) \\
& \left. - \frac{K(d(x))}{k(d(x))} \frac{\varphi'(K(d(x)))}{K(d(x))\varphi''(K(d(x)))} \Delta d(x) - \frac{\lambda\xi_{2\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))} \frac{(\varphi'(K(d(x))))^q}{\varphi''(K(d(x)))} \right] \\
\geq \; & 0.
\end{aligned}$$

Let $u_\lambda \in C(\bar{\Omega}) \cap C^{2+\alpha}(\Omega)$ be the unique solution to problem (1.1). We assert

$$\xi_{2\varepsilon}\varphi(K(d(x))) = \underline{u}_\varepsilon(x) \leq u_\lambda(x) \leq \bar{u}_\varepsilon(x) = \xi_{1\varepsilon}\varphi(K(d(x))), \; \forall x \in \Omega_{\delta_\varepsilon}.$$

In fact, denote $\Omega_{\delta_\varepsilon} = \Omega_{\delta+} \cup \Omega_{\delta-}$, where $\Omega_{\delta+} = \{x \in \Omega_{\delta_\varepsilon} : u_\lambda(x) \geq \underline{u}_\varepsilon(x)\}$ and $\Omega_{\delta-} = \{x \in \Omega_{\delta_\varepsilon} : u_\lambda(x) < \underline{u}_\varepsilon(x)\}$. We need to show $\Omega_{\delta-} = \emptyset$. Assume the contrary. Then we see that there exists $x_0 \in \Omega_{\delta-}$ (note that $\underline{u}_\varepsilon(x) = u_\lambda(x)$, $\forall x \in \partial\Omega_{\delta-}$) such that

$$0 < \underline{u}_\varepsilon(x_0) - u_\lambda(x_0) = \max_{x \in \bar{\Omega}_{\delta-}} (\underline{u}_\varepsilon(x) - u_\lambda(x)),$$

and

$$\nabla \underline{u}_\varepsilon(x_0) = \nabla u_\lambda(x_0), \quad \triangle(\underline{u}_\varepsilon(x_0) - u_\lambda(x_0)) \leq 0.$$

On the other hand, we see by (g_1) and (b_1) that

$$-\Delta(u_\lambda - \underline{u}_{\varepsilon)}(x_0) = b(x_0)(g(\underline{u}_\varepsilon(x_0)) - g(u_\lambda(x_0))) < 0,$$

which is a contradiction. Hence $\Omega_{\delta-} = \emptyset$, i.e., $u_\lambda(x) \geq \underline{u}_\varepsilon(x)$ in Ω_δ. The same way, we can see that $u_\lambda(x) \leq \bar{u}_\varepsilon(x)\ \forall x \in \Omega_\delta$. It follows that

$$\xi_{2\varepsilon} \leq \lim_{d(x)\to 0} \inf \frac{u_\lambda(x)}{\varphi(K(d(x)))} \leq \lim_{d(x)\to 0} \sup \frac{u_\lambda(x)}{\varphi(K(d(x)))} \leq \xi_{1\varepsilon}.$$

Thus let $\varepsilon \to 0$, we derive that

$$\frac{u_\lambda(x)}{\varphi(K(d(x)))} \to \xi_0.$$

By Lemma 2.5, the proof is finished.

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