*Advanced Nonlinear Studies* **8** (2008), 663–678

# Compactness Property of a Singular Quasilinear Elliptic Equation

**Jianqing Chen***

*Department of Mathematics*
*Fujian Normal University, Fuzhou, 350007, P. R. China*
*e-mail: jqchen@fjnu.edu.cn*

Received in revised form 21 January 2008
*Communicated by Ireneo Peral*

**Abstract**

We characterize a compactness property for a quasilinear equation with critical growth and singular term. Some applications of the compactness property are also pointed out.

*2000 Mathematics Subject Classification.* 35J60, 35J20.
*Key words.* Compactness property, singular quasilinear equation, positive solution

## 1 Introduction

In this paper, we are concerned with the compactness property for the following quasilinear elliptic equation with singular term:

$$\begin{cases} -\Delta_p u - \frac{\lambda}{|x|^p}|u|^{p-2}u &= |u|^{p^*-2}u + \mu|u|^{p-2}u, \quad x \in \Omega\backslash\{0\} \\ u &= 0 \qquad x \in \partial\Omega, \end{cases} \qquad (Q(\lambda,\mu,p))$$

where $0 \in \Omega \subset \mathbb{R}^N (1 < p < N)$ is an open bounded domain with smooth boundary, $\Delta_p = div(|\nabla \cdot|^{p-2}\nabla\cdot)$ is the standard $p$-Laplacian operator, $0 \le \lambda < \Lambda_{N,p} = ((N-p)/p)^p$, $\mu > 0$ and $p^* = Np/(N-p)$ is the critical exponent in the Sobolev

*The author acknowledges the support of Youth Foundation of NSFC (10501006) and the program for NCETFJ.

embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, and $\mathcal{D}_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|^p = \int_\Omega |\nabla \cdot|^p$.

According to the Hardy inequality [10]

$$\int_\Omega \frac{|u|^p}{|x|^p} \leq \Lambda_{N,p}^{-1} \int_\Omega |\nabla u|^p, \qquad u \in \mathcal{D}_0^{1,p}(\Omega) \tag{1.1}$$

the following eigenvalue problem

$$-\Delta_p u - \frac{\lambda}{|x|^p}|u|^{p-2}u = \mu |u|^{p-2}u, \quad u \in \mathcal{D}_0^{1,p}(\Omega) \tag{1.2}$$

has the first eigenvalue $\mu_1(\lambda) > 0$. Combining (1.1) with Sobolev embedding theorem, we know that $(Q(\lambda, \mu, p))$ is variational. Define the functional

$$F_{\lambda,\mu,p}(u) = \frac{1}{p}\int \left( |\nabla u|^p - \lambda \frac{|u|^p}{|x|^p} - \mu |u|^p \right) - \frac{1}{p^*}\int |u|^{p^*}, \quad u \in \mathcal{D}_0^{1,p}(\Omega). \tag{1.3}$$

Then $F_{\lambda,\mu,p} \in C^1(\mathcal{D}_0^{1,p}(\Omega), \mathbb{R})$ and there is a one-to-one correspondence between the weak solutions of $(Q(\lambda,\mu,p))$ and the critical points of $F_{\lambda,\mu,p}$. Here and after we say that $u \in \mathcal{D}_0^{1,p}(\Omega)$ is a weak solution of $(Q(\lambda,\mu,p))$ if and only if for any $\psi \in \mathcal{D}_0^{1,p}(\Omega)$ there holds

$$\int \left( |\nabla u|^{p-2}\nabla u \nabla \psi - \frac{\lambda}{|x|^p}|u|^{p-2}u\psi - \mu |u|^{p-2}u\psi - |u|^{p^*-2}u\psi \right) = 0. \tag{1.4}$$

Since neither $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ nor $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(|x|^{-p}, \Omega)$ (the weighted Sobolev space) is compact, $F_{\lambda,\mu,p}$ does not satisfy Palais-Smale ($(PS)$ in short, see Definition 1.1) conditions. This creates additional difficulties in using variational methods. Recalling that in the case of $\lambda = 0$ and $p = 2$, Brezis-Nirenberg [5] firstly proved that $F_{0,\mu,2}$ satisfies $(PS)_c$ conditions for all $c < \frac{1}{N}S_{0,2}^{N/2}$ (see (1.6) for the definition of $S_{0,2}$) and $(Q(0,\mu,2))$ has at least one positive solution for $0 < \mu < \mu_1(0)$ and $N \geq 4$. From then on, problems of this kind have been studied extensively for $\lambda = 0$. But only few results exist in the literature for $\lambda \neq 0$ due to the additional difficulties created by the singular term. To our best knowledge, there are two kinds of contributions to this type problem. One is to characterize the so called global compactness property of the energy functional either for $\lambda = 0$, $p \geq 2$ or $\lambda \neq 0$, $p = 2$, see e.g. [16, 18, 15, 6]. The other is to study the existence and multiplicity of solutions for $\lambda = 0$, $p \geq 2$ or $\lambda \neq 0$, $p = 2$ and for various $\mu$ and $N$, see e.g. [13, 9, 17] and the references therein. The main purpose of the present paper is to extend these results to the very general cases. Introducing the limiting problem

$$-\Delta_p u - \frac{\lambda}{|x|^p}|u|^{p-2}u = |u|^{p^*-2}u, \qquad x \in \mathbb{R}^N, \tag{$Q_\lambda$}$$

we denote the corresponding functional by

$$E_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N} \left( |\nabla u|^p - \lambda \frac{|u|^p}{|x|^p} \right) - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*}, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N). \tag{1.5}$$

The related minimization problem is

$$S_{\lambda,p} = \inf\{\int_{\mathbb{R}^N} \Big(|\nabla u|^p - \lambda\frac{|u|^p}{|x|^p}\Big);\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{p^*} = 1\}. \tag{1.6}$$

Note that $S_{0,p}$ is nothing but the best Sobolev constant in the embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and can be achieved only for $\Omega = \mathbb{R}^N$. Furthermore, Guedda and Veron [11] proved that $(Q_0)$ has a unique positive radial solution

$$U_\varepsilon(x) = \Big(N\varepsilon(\frac{N-p}{p-1})^{p-1}\Big)^{(N-p)/p^2}\Big(\varepsilon + |x|^{p/(p-1)}\Big)^{(p-N)/p},\quad \varepsilon > 0. \tag{1.7}$$

Recently, it is also known [1] that $S_{\lambda,p}$ $(0 \le \lambda < \Lambda_{N,p})$ is achieved by a unique positive radial function $u$ which defines a positive solution of $(Q_\lambda)$ up to a Lagrange multiplier. More precisely, Abdellaoui-Felli-Peral [1, Theorem 3.13 in Page 20] proved that all positive radial solutions of $(Q_\lambda)$ are

$$W_\sigma(\cdot) = \sigma^{(p-N)/p} W_0(\frac{\cdot}{\sigma}),\quad \sigma > 0, \tag{1.8}$$

where $W_0$ achieves $S_{\lambda,p}$ and $W_0$ is the unique positive radial solution in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ up to a scaling. Moreover, there exist constants $K_1,\ K_2 > 0$ such that

$$0 < K_1 \le \frac{W_0(x)}{(|x|^{\xi_1/\delta} + |x|^{\xi_2/\delta})^{-\delta}} \le K_2,$$

where $0 < \xi_1 < \xi_2$ are the only two roots of $a(\xi) := (p-1)\xi^p - (N-p)\xi^{p-1} + \lambda = 0$ and $\delta = (N-p)/p$.

Before stating the main theorem, we introduce the standard definition.

**Definition 1.1** Let $c \in \mathbb{R}$ and $H$ be a Banach space and $I \in C^1(H,\mathbb{R})$. We say that $\{u_n\} \subset H$ is a $(PS)_c$ sequence of $I$ if $I(u_n) \to c$ and $I'(u_n) \to 0$ in $H^*$(the dual space of $H$). We say that $I$ satisfies $(PS)_c$ *condition* if every $(PS)_c$ sequence has a convergent subsequence in $H$. If $I$ satisfies $(PS)_c$ condition for every $c \in \mathbb{R}$, then we say that $I$ satisfies $(PS)$ *condition.*

We are now in a position to state the main result:

**Theorem 1.1** *Suppose that $2 \le p < N$, $0 \le \lambda < \Lambda_{N,p}$, $0 < \mu < \mu_1(\lambda)$, $\{u_n\} \subset \mathcal{D}_0^{1,p}(\Omega)$, $u_n \ge 0$ is a $(PS)_d$ sequence of $F_{\lambda,\mu,p}$. Then there are $u \in \mathcal{D}_0^{1,p}(\Omega)$, $k,\ l \in \mathbb{N}$, $l$ sequences $(R_n^j) \subset \mathbb{R}_+$ $(1 \le j \le l)$, $k$ sequences $(r_n^j) \subset \mathbb{R}_+$ and $(y_n^j) \subset \Omega$ such that up to a subsequence*

- $u_n \rightharpoonup u$ *in* $\mathcal{D}_0^{1,p}(\Omega)$ *and* $F'_{\lambda,\mu,p}(u) = 0$;
- $y_n^j \to y_0^j \in \bar{\Omega},\quad r_n^j/|y_n^j| \to 0\quad (1 \le j \le k)$;
- $R_n^j \to 0\quad (1 \le j \le l)$;

- $d = F_{\lambda,\mu,p}(u) + \Sigma_{j=1}^{l} E_\lambda(W^j) + \Sigma_{j=1}^{k} E_0(U^j) + o(1)$;
- $u_n = u + \Sigma_{j=1}^{l} (R_n^j)^{\frac{p-N}{p}} W^j(\frac{x}{R_n^j}) + \Sigma_{j=1}^{k} (r_n^j)^{\frac{p-N}{p}} U^j(\frac{x-y_n^j}{r_n^j}) + \omega_n$,

*where* $\|\omega_n\| \to 0$ *and* $U^j$ *(*$1 \le j \le k$*) solves* $(Q_0)$ *and* $W^j$ *(*$1 \le j \le l$*) solves* $(Q_\lambda)$*.*

A direct consequence of Theorem 1.1 is

**Corollary 1.1** *If* $u_n \ge 0$ *is a* $(PS)_d$ *sequence of* $F_{\lambda,\mu,p}$ *with* $d < \frac{1}{N} S_{\lambda,p}^{N/p}$, *then* $\{u_n\}$ *has a convergent subsequence in* $\mathcal{D}_0^{1,p}(\Omega)$*.*

The proof of Theorem 1.1 is based on a scaling argument, which has been used in [16, 18, 15, 6]. We will prove it in Section 2. In Section 3, we give some remarks on the singular quasilinear equations and some applications of Theorem 1.1.

Let us end this introduction by some notations. Throughout this paper, $K$, $K_i$ are generic positive constants whose value can vary from line to line, $|\cdot|_q$ denotes the norm in $L^q(\Omega)$, $o(1)$ denotes infinitesimal as $n \to \infty$, $\to$ denotes the strong convergence, and $\rightharpoonup$ the weak convergence. $B(x,r)$ denotes a ball centered at $x$ with radius $r$, and $B(0,r)$ is simply denoted by $B_r$. All integrals are taken over $\Omega$ unless stated otherwise.

## 2 Compactness result

In this section, we prove Theorem 1.1 by a series of Lemmas. From now on, we assume $0 < \mu < \mu_1(\lambda)$, $2 \le p < N$, we simply write $F_{\lambda,\mu,p}$, $S_{\lambda,p}$ as $F_{\lambda,\mu}$ $S_\lambda$, and $F_{\lambda,0}$ is simply denoted by $F_\lambda$.

**Lemma 2.1** *Let* $\{u_n\} \subset \mathcal{D}_0^{1,p}(\Omega)$, $u_n \ge 0$ *be a* $(PS)_d$ *sequence of* $F_{\lambda,\mu}$*. Then* $\{u_n\}$ *is bounded in* $\mathcal{D}_0^{1,p}(\Omega)$*.*

*Proof.* Since $\{u_n\}$ is a $(PS)_d$ sequence, for $n$ large enough, we have that

$$\begin{aligned} d + 1 + o(\|u_n\|) &= F_{\lambda,\mu}(u_n) - \tfrac{1}{p^*}\langle F'_{\lambda,\mu}(u_n), u_n\rangle \\ &= \frac{1}{N}\int \left(|\nabla u_n|^p - \lambda\frac{|u_n|^p}{|x|^p} - \mu|u_n|^p\right) \\ &\ge \frac{1}{N}\left(1 - \frac{\mu}{\mu_1(\lambda)}\right)\int \left(|\nabla u_n|^p - \lambda\frac{|u_n|^p}{|x|^p}\right) \\ &\ge K\|u_n\|^p. \end{aligned}$$

Therefore $\{u_n\}$ is bounded in $\mathcal{D}_0^{1,p}(\Omega)$. ■

**Lemma 2.2** *Let* $\{u_n\}$ *be the sequence as in Lemma 2.1. Assume that* $u_n \rightharpoonup u$ *in* $\mathcal{D}_0^{1,p}(\Omega)$ *and* $u_n \to u$ *a.e. in* $\Omega$*. Then* $F'_{\lambda,\mu}(u) = 0$ *and* $v_n = u_n - u$ *is a* $(PS)_{d-F_{\lambda,\mu}(u)}$ *sequence of* $F_\lambda$*.*

**Remark 2.1** When $p \ne 2$, we are facing a nonlinear operator $-\Delta_p$. The weak convergence $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$ usually does **not** imply $\nabla u_n \to \nabla u$ a.e. in $\Omega$.

*Proof of Lemma 2.2.* The proof is divided into two steps.
**Step 1.** We prove that for some finite points $\{x_1, \cdots, x_m\}$, there holds

$$u_n \to u \quad \text{in} \quad \mathcal{D}^{1,p}_{0,loc}(\Omega \backslash \{0, x_1, \cdots, x_m\}). \tag{2.1}$$

Firstly, from $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}_0(\Omega)$, $u_n \to u$ a.e. in $\Omega$, $\nabla u_n \in (L^p(\Omega))^N$ and $\{|\nabla u_n|^{p-2}\nabla u_n\}$ is bounded in $(L^{p'}(\Omega))^N$ with $p' = p/(p-1)$. We can assume that there is $T \in (L^{p'}(\Omega))^N$ such that

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup T \quad \text{in} \quad (L^{p'}(\Omega))^N.$$

Clearly by letting $n \to +\infty$, one gets

$$\int T\nabla\varphi = \int \Big(|u|^{p^*-2}u\varphi + \frac{\lambda}{|x|^p}|u|^{p-2}u\varphi + \mu|u|^{p-2}u\varphi\Big), \ \forall\ \varphi \in \mathcal{D}^{1,p}_0(\Omega). \tag{2.2}$$

Since $\{u_n\}$ is bounded in $\mathcal{D}^{1,p}_0(\Omega)$, Concentration Compactness Principle [14] implies that there is an at most countable set $J$ such that

$$\begin{cases} (1) & |\nabla u_n|^p \rightharpoonup d\alpha \geq |\nabla u|^p + \sum_{j\in J}\alpha_j\delta_{x_j} + \alpha_0\delta_0, \\ (2) & |u_n|^{p^*} \rightharpoonup d\beta = |u|^{p^*} + \sum_{j\in J}\beta_j\delta_{x_j} + \beta_0\delta_0, \\ (3) & \alpha_j \geq S_0\beta_j^{p/p^*}, \\ (4) & \frac{|u_n|^p}{|x|^p} \rightharpoonup d\gamma = \frac{|u|^p}{|x|^p} + \gamma_0\delta_0, \\ (5) & \Lambda_{N,p}\gamma_0 \leq \alpha_0. \end{cases} \tag{$CCP$}$$

Secondly, we claim that $J$ is finite. Indeed from $\langle F'_{\lambda,\mu}(u_n), \varphi\rangle = o(1)\|\varphi\|$ and choosing $\varphi = \phi u_n$, we get that

$$\begin{aligned} &\int \Big(|\nabla u_n|^p\phi + u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi\Big) \\ &= \int \Big(|u_n|^{p^*}\phi + \frac{\lambda}{|x|^p}|u_n|^p\phi + \mu|u_n|^p\phi\Big) + o(1). \end{aligned} \tag{2.3}$$

It follows, by letting $n \to +\infty$, that

$$\int \phi d\alpha + \int uT\nabla\phi = \int \phi d\beta + \int \lambda\phi d\gamma + \int \mu|u|^p\phi. \tag{2.4}$$

Substituting $\varphi$ by $\phi u$ in (2.2), we have that

$$\int \Big(uT\nabla\phi + \phi T\nabla u\Big) = \int \Big(|u|^{p^*}\phi + \frac{\lambda}{|x|^p}|u|^p\phi + \mu|u|^p\phi\Big). \tag{2.5}$$

Combining (2.4), (2.5) and concentrating $\phi$ at $x_j$ (here and in the sequel, a function $\phi$ is called concentrated at $x_j$ always means $\phi \in C^1_0(\Omega)$ and $\phi(x) = 1$ for $|x - x_j| \leq r$ and $\phi(x) = 0$ for $|x - x_j| \geq 2r$ and $|\nabla\phi| \leq \frac{4}{r}$ and $r$ small), we get that $\alpha_j \leq \beta_j$. It follows from (3) in $(CCP)$ that

$$\text{either} \quad \beta_j = 0 \quad \text{or} \quad \beta_j \geq S_0^{N/p}. \tag{2.6}$$

Taking concentrated function $\phi$ at $x_0 = 0$ in (2.4) and (2.5), we have that

$$\alpha_0 - \lambda\gamma_0 \le \beta_0. \tag{2.7}$$

On the other hand from

$$\int \left(|\nabla u|^p - \frac{\lambda}{|x|^p}|u|^p\right) \ge S_\lambda\left(\int |u|^{p^*}\right)^{p/p^*}, \qquad u \in \mathcal{D}_0^{1,p}(\Omega),$$

we get that

$$\int \left(|\nabla(\phi u_n)|^p - \lambda\frac{|\phi u_n|^p}{|x|^p}\right) \ge S_\lambda\left(\int |\phi u_n|^{p^*}\right)^{p/p^*}.$$

Therefore

$$\int \left(|u_n\nabla\phi + \phi\nabla u_n|^p\right) \ge \lambda\int \frac{|\phi u_n|^p}{|x|^p} + S_\lambda\left(\int |\phi u_n|^{p^*}\right)^{p/p^*}.$$

Using the elementary inequality

$$\Big||X+Y|^p - |X|^p\Big| \le K(|X|^{p-1}|Y| + |Y|^p) \quad \forall\ X,\ Y \in \mathbb{R}^N,$$

we obtain that

$$\int \left(|\phi\nabla u_n + u_n\nabla\phi|^p - |\phi\nabla u_n|^p\right) \le K\int \left(|\phi\nabla u_n|^{p-1}|u_n\nabla\phi| + |u_n\nabla\phi|^p\right).$$

Hölder inequality implies that

$$\begin{aligned}
&\int |\phi\nabla u_n|^{p-1}|u_n\nabla\phi| \\
&\le \left(\int_{r\le|x|\le 2r} |\nabla\phi|^p|u_n|^p\right)^{\frac{1}{p}}\left(\int_{r\le|x|\le 2r} |\nabla u_n|^p\right)^{\frac{p-1}{p}} \\
&\le \left(\int_{r\le|x|\le 2r} |\nabla u_n|^p\right)^{\frac{p-1}{p}}\left[\frac{K_1}{r^p}\left(\int_{r\le|x|\le 2r} 1dx\right)^{\frac{p}{N}}\left(\int_{r\le|x|\le 2r} |u_n|^{p^*}\right)^{\frac{N-p}{N}}\right]^{\frac{1}{p}} \\
&\le K_2\left(\int_{r\le|x|\le 2r} |\nabla u_n|^p\right)^{\frac{p-1}{p}}\left(\int_{r\le|x|\le 2r} |u_n|^{p^*}\right)^{\frac{N-p}{N}}.
\end{aligned}$$

Therefore

$$\lim_{r\to 0}\limsup_{n\to\infty}\int |\phi\nabla u_n|^{p-1}|u_n\nabla\phi| = 0. \tag{2.8}$$

Similarly

$$\int |u_n\nabla\phi|^p \le \frac{K_3}{r^p}\left(\int_{r\le|x|\le 2r} 1dx\right)^{\frac{p}{N}}\left(\int_{r\le|x|\le 2r} |u_n|^{p^*}\right)^{\frac{N-p}{N}} \to 0.$$

It is deduced that

$$\lim_{r\to 0}\limsup_{n\to\infty}\int\Big(|\phi\nabla u_n+u_n\nabla\phi|^p-|\phi\nabla u_n|^p\Big)=0$$

and hence

$$\alpha_0-\lambda\gamma_0\geq S_\lambda\beta_0^{p/p^*}. \tag{2.9}$$

Combining this with (2.7), we obtain that

$$\text{either}\quad \beta_0=0\quad\text{or}\quad \beta_0\geq S_\lambda^{N/p}. \tag{2.10}$$

The claim easily follows.

Next, taking $\psi\in C^1(\bar\Omega)$ satisfying $\psi\geq 0$, $\psi(x_j)=0$ for $j\in J\cup\{0\}$, we have

$$\int|\psi u_n|^{p^*}\to\int|\psi u|^{p^*}+\sum_{j\in J}\beta_j\psi^{p^*}(x_j)+\beta_0\psi^{p^*}(0)=\int|\psi u|^{p^*}. \tag{2.11}$$

It follows from the uniform convex of $L^{p^*}(\Omega)$ that

$$\psi u_n\to\psi u\quad\text{in}\quad L^{p^*}(\Omega). \tag{2.12}$$

Similarly

$$\psi u_n\to\psi u\quad\text{in}\quad L^p(|x|^{-p},\Omega). \tag{2.13}$$

From $\langle F'_{\lambda,\mu}(u_n),\varphi\rangle=o(1)\|\varphi\|$ and choosing $\varphi=\psi(u_n-u)$, we obtain that

$$\begin{aligned}&\int\psi(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)(\nabla u_n-\nabla u)\\&=\int\psi\Big(|u_n|^{p^*-2}u_n+\frac{\lambda}{|x|^p}|u_n|^{p-2}u_n+\mu|u_n|^{p-2}u_n\Big)(u_n-u)+o(1).\end{aligned}$$

It follows that

$$\int\psi\Big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\Big)\Big(u_n-u\Big)\to 0\quad(n\to\infty).$$

Therefore an elementary inequality easily implies (2.1).

**Step 2.** Firstly, from (2.1) and $\langle F'_{\lambda,\mu}(u_n),\varphi\rangle=o(1)\|\varphi\|$, we know that $F'_{\lambda,\mu}(u)=0$. Brezis-Lieb Lemma [4] implies that (see e.g. [12])

$$\|u_n\|^p=\|u_n-u\|^p+\|u\|^p+o(1),$$

$$\int|u_n|^p|x|^{-p}=\int|u_n-u|^p|x|^{-p}+\int|u|^p|x|^{-p}+o(1)\qquad\text{and}$$

$$\int|u_n|^{p^*}=\int|u_n-u|^{p^*}+\int|u|^{p^*}+o(1).$$

Therefore

$$F_{\lambda,\mu}(u_n)=F_{\lambda,\mu}(u)+F_\lambda(u_n-u)+o(1). \tag{2.14}$$

Secondly, we prove that for $v_n = u_n - u$,

$$F'_\lambda(v_n) \to 0 \quad \text{in} \quad (\mathcal{D}_0^{1,p}(\Omega))^*. \tag{2.15}$$

This is equivalent to proving that for any $\varphi \in \mathcal{D}_0^{1,p}(\Omega)$,

$$\begin{aligned}\langle F'_\lambda(v_n), \varphi\rangle &= \int \left(|\nabla v_n|^{p-2}\nabla v_n \nabla\varphi - \frac{\lambda}{|x|^p}|v_n|^{p-2}v_n\varphi - |v_n|^{p^*-2}v_n\varphi\right)\\ &= o(1)\|\varphi\|.\end{aligned} \tag{2.16}$$

To get (2.16), we first prove that

$$\begin{aligned}\Big|\int \Big(|\nabla u_n|^{p-2}\nabla u_n \quad &-|\nabla u|^{p-2}\nabla u - |\nabla v_n|^{p-2}\nabla v_n\Big)\nabla\varphi\Big|\\ &:= |\int A\nabla\varphi| = o(1)\|\varphi\|;\end{aligned} \tag{2.17}$$

$$\Big|\int |x|^{-p}\Big(|u_n|^{p-2}u_n - |u|^{p-2}u - |v_n|^{p-2}v_n\Big)\varphi\Big| := |\int B\varphi| = o(1)\|\varphi\|; \tag{2.18}$$

$$\Big|\int \Big(|u_n|^{p^*-2}u_n - |u|^{p^*-2}u - |v_n|^{p^*-2}v_n\Big)\varphi\Big| := |\int D\varphi| = o(1)\|\varphi\|. \tag{2.19}$$

Indeed, denote $\Omega_r = \cup B(x_j, r)$ for $j \in J \cup \{0\}$ and $r$ small. We have from mean value theorem that

$$|A| \leq K(|\nabla u_n|^{p-2} + |\nabla u|^{p-2})|\nabla u|. \tag{2.20}$$

It is deduced from (2.1) that

$$\begin{aligned}&\left(\int |A|^{p/(p-1)}\right)^{(p-1)/p}\\ &\leq o(1) + \left(\int_{\Omega_r} |A|^{p/(p-1)}\right)^{(p-1)/p}\\ &\leq o(1) + \Bigg[\left(\int_{\Omega_r} |\nabla u_n|^p\right)^{(p-2)/(p-1)}\left(\int_{\Omega_r}|\nabla u|^p\right)^{1/(p-1)}\\ &\qquad + \int_{\Omega_r}|\nabla u|^p\Bigg]^{(p-1)/p}\\ &\to 0 \quad \text{as} \quad n\to\infty, \quad r \to 0.\end{aligned} \tag{2.21}$$

Thus Hölder inequality implies that

$$|\int A\nabla\phi| \leq \left(\int |A|^{p/(p-1)}\right)^{(p-1)/p}\|\varphi\| = o(1)\|\varphi\|$$

and (2.17) holds. Using similar arguments, we arrive at (2.18) and (2.19).

Now from (2.17), (2.18), (2.19) and $v_n \to 0$ in $L^p(\Omega)$, we get that

$$
\begin{aligned}
&\Big|\langle F'_{\lambda,\mu}(u_n) - F'_{\lambda,\mu}(u) - F'_\lambda(v_n), \varphi\rangle\Big| \\
&= \Big|\int \Big(A\nabla\varphi + B\varphi + D\varphi\Big) - \mu\int \Big(|u_n|^{p-2}u_n\varphi - |u|^{p-2}u\varphi\Big)\Big| \\
&= o(1)\|\varphi\|.
\end{aligned}
$$

Combining this with $\langle F'_{\lambda,\mu}(u_n), \varphi\rangle = o(1)\|\varphi\|$ and $F'_{\lambda,\mu}(u) = 0$, we get (2.16). The proof is complete. ∎

**Lemma 2.3** *Let $(v_n) \subset \mathcal{D}_0^{1,p}(\Omega)$, $v_n \geq 0$ be such that*

$$F_\lambda(v_n) \to c, \quad F'_\lambda(v_n) \to 0. \tag{2.22}$$

*If there is a sequence $R_n \to 0$ such that*

$$\tilde{v}_n(x) := \begin{cases} R_n^{\frac{N-p}{p}} v_n(R_n x), & x \in \Omega_n = \{x \in \mathbb{R}^N;\ R_n x \in \Omega\}, \\ 0, & x \notin \Omega_n \end{cases}$$

*with*

$$\tilde{v}_n(x) \rightharpoonup V_1 \quad \text{in } \mathcal{D}^{1,p}(\mathbb{R}^N), \quad \tilde{v}_n(x) \to V_1 \quad \text{a.e. in } \mathbb{R}^N, \tag{2.23}$$

*then $V_1$ solves $(Q_\lambda)$ and the sequence $w_n(x) = v_n(x) - R_n^{\frac{p-N}{p}} V_1(\frac{x}{R_n}) + o(1)$ (see (2.31))is a $(PS)$ sequence of $F_\lambda$ at level $c - E_\lambda(V_1)$.*

*Proof.* For any $\varphi \in C_0^\infty(\Omega_n)$, there holds

$$\langle E'_\lambda(\tilde{v}_n), \varphi\rangle = \int_{\mathbb{R}^N} \Big(|\nabla\tilde{v}_n|^{p-2}\nabla\tilde{v}_n\nabla\varphi - \frac{\lambda}{|x|^p}|\tilde{v}_n|^{p-2}\tilde{v}_n\varphi - |\tilde{v}_n|^{p^*-2}\tilde{v}_n\varphi\Big). \tag{2.24}$$

Therefore, using the notation $\varphi_n^*(x) = R_n^{\frac{p-N}{p}}\varphi(\frac{x}{R_n})$, we have that

$$\langle E'_\lambda(\tilde{v}_n), \varphi\rangle = \langle F'_\lambda(v_n), \varphi_n^*\rangle = o(1)\|\varphi_n^*\| = o(1)\|\varphi\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}. \tag{2.25}$$

Choosing $\varphi = \phi\tilde{v}_n$ in (2.24), we can get from the same argument as those in the proof of Lemma 2.2 but with $u_n$ being replaced by $\tilde{v}_n$ and $u$ replaced by $V_1$ that for some finite set $J = \{1, \cdots, m_1\}$,

$$\text{either} \quad \beta_j = 0, \quad j \in J \cup \{0\} \quad \text{or} \quad \beta_j \geq S_0^{N/p},\ \beta_0 \geq S_\lambda^{N/p}. \tag{2.26}$$

Moreover, the same proofs imply that

$$\nabla\tilde{v}_n \to \nabla V_1 \quad \text{in} \quad L^p_{loc}(\mathbb{R}^N \backslash \{0, x_1, \cdots, x_{m_1}\}). \tag{2.27}$$

Combining this with (2.23) and (2.25), we know that $V_1$ solves $(Q_\lambda)$.

Now let $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfy $0 \le \varphi \le 1$, $|\nabla \varphi| \le 2$ in $\mathbb{R}^N$, $\varphi(x) = 1$ in $B(0,1)$ and $\varphi(x) = 0$ outside $B(0,2)$ and let

$$w_n(x) = v_n(x) - R_n^{\frac{p-N}{p}} V_1(\frac{x}{R_n})\varphi(\frac{x}{\bar{R}_n}), \tag{2.28}$$

where the sequence $\bar{R}_n$ is chosen such that

$$\tilde{R}_n = \frac{R_n}{\bar{R}_n} \to 0, \qquad \frac{dist(0, \partial\Omega_n)}{\bar{R}_n} \to \infty. \tag{2.29}$$

Therefore

$$\tilde{w}_n(x) = \tilde{v}_n(x) - V_1(x)\varphi(\tilde{R}_n x). \tag{2.30}$$

Set $\varphi_n(x) = \varphi(\tilde{R}_n x)$. Then $\varphi_n \equiv 1$ for $|\tilde{R}_n x| \le 1$ and $\varphi_n \equiv 0$ for $|\tilde{R}_n x| \ge 2$. Noting that $|\nabla V_1| \in L^p(\mathbb{R}^N)$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(V_1(\varphi_n - 1))|^p \\
&\le K(\int_{\mathbb{R}^N} \Big(|\nabla V_1|^p |\varphi_n - 1|^p + |V_1|^p |\nabla(\varphi_n - 1)|^p\Big) \\
&\le K\Big(\int_{|x|\ge 1/\tilde{R}_n} |\nabla V_1|^p + (2\tilde{R}_n)^p \int_{1/\tilde{R}_n \le |x| \le 2/\tilde{R}_n} |V_1|^p\Big) \\
&\le o(1) + (2\tilde{R}_n)^p \Big(\int_{1/\tilde{R}_n \le |x| \le 2/\tilde{R}_n} |V_1|^{p^*}\Big)^{p/p^*} \Big(\int_{1/\tilde{R}_n \le |x| \le 2/\tilde{R}_n} 1dx\Big)^{p/N} \\
&= o(1).
\end{aligned}$$

It follows that $\tilde{w}_n = \tilde{v}_n - V_1 + o(1)$ with $o(1) \to 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Hence

$$w_n(x) = v_n(x) - R_n^{\frac{p-N}{p}} V_1(\frac{x}{R_n}) + o(1). \tag{2.31}$$

Using (2.23), (2.24), (2.27) and Brezis-Lieb Lemma, we obtain that

$$F_\lambda(w_n) = c - E_\lambda(V_1) + o(1). \tag{2.32}$$

It remains to prove that $F'_\lambda(w_n) \to 0$. Noticing $F'_\lambda(v_n) \to 0$ and $E'_\lambda(V_1) = 0$ and the proof of (2.16), we only need to prove that

$$\begin{aligned}
&|\langle F'_\lambda(v_n), \varphi\rangle - \langle F'_\lambda(w_n), \varphi\rangle - \langle E'_\lambda(V_1), \varphi_n^*\rangle| \\
&:= |\int_{\mathbb{R}^N} \Big(A_1 \nabla\varphi_n^* - \lambda B_1 \varphi_n^* - D_1 \varphi_n^*\Big)| = o(1)\|\varphi_n^*\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} = o(1)\|\varphi\|,
\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}
A_1 &= |\nabla\tilde{v}_n|^{p-2}\nabla\tilde{v}_n - |\nabla\tilde{w}_n|^{p-2}\nabla\tilde{w}_n - |\nabla V_1|^{p-2}\nabla V_1, \\
B_1 &= |x|^{-p}(|\tilde{v}_n|^{p-2}\tilde{v}_n - |\tilde{w}_n|^{p-2}\tilde{w}_n - |V_1|^{p-2}V_1), \\
D_1 &= |\tilde{v}_n|^{p^*-2}\tilde{v}_n - |\tilde{w}_n|^{p^*-2}\tilde{w}_n - |V_1|^{p^*-2}V_1.
\end{aligned}$$

Hölder inequality implies that

$$|\int_{\mathbb{R}^N} A_1 \nabla \varphi_n^*| \le \Big(\int_{\mathbb{R}^N} |A_1|^{p/(p-1)}\Big)^{(p-1)/p} \|\varphi_n^*\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}.$$

For $R$ large enough

$$\int_{\mathbb{R}^N} |A_1|^{p/(p-1)} \le \int_{B(0,R)} |A_1|^{p/(p-1)} + \int_{|x|\ge R} |A_1|^{p/(p-1)}. \tag{2.34}$$

The estimate for the first integrand in the right hand side of (2.34) is similar to those in Lemma 2.2. For the second integrand in the right hand side of (2.34), we use mean value theorem to obtain that

$$\begin{aligned}
&\int_{|x|\ge R} |A_1|^{p/(p-1)} \\
&\le K \int_{|x|\ge R} \Big((|\nabla \tilde{v}_n|^{p-2} + |\nabla V_1|^{p-2})|\nabla V_1|\Big)^{p/(p-1)} \\
&\le K \Big[\Big(\int_{|x|\ge R} |\nabla \tilde{v}_n|^p\Big)^{(p-2)/(p-1)} \Big(\int_{|x|\ge R} |\nabla V_1|^p\Big)^{1/(p-1)} \\
&\qquad + \int_{|x|\ge R} |\nabla V_1|^p\Big] \\
&\to 0 \quad \text{as} \quad R \to \infty.
\end{aligned} \tag{2.35}$$

It follows that

$$|\int_{\mathbb{R}^N} A_1 \nabla \varphi_n^*| = o(1) \|\varphi_n^*\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}. \tag{2.36}$$

Similarly, we have that

$$|\int_{\mathbb{R}^N} B_1 \varphi_n^*| = o(1) \|\varphi_n^*\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}, \quad |\int_{\mathbb{R}^N} D_1 \varphi_n^*| = o(1) \|\varphi_n^*\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}. \tag{2.37}$$

Therefore (2.33) holds. The proof is complete. ■

**Lemma 2.4** *Let $(v_n) \subset \mathcal{D}_0^{1,p}(\Omega)$, $v_n \ge 0$ be a $(PS)_c$ sequence of $F_\lambda$. Assume that there exist sequences $y_n \to y \in \bar{\Omega}$, $r_n \to 0$ such that*

$$\tilde{v}_n(x) := \begin{cases} r_n^{\frac{N-p}{p}} v_n(r_n x + y_n), & x \in \tilde{\Omega}_n = \{x \in \mathbb{R}^N;\ r_n x + y_n \in \Omega\}, \\ 0 & x \notin \tilde{\Omega}_n \end{cases}$$

*and for some finite points $\{x_1, \cdots, x_{m_2}\}$,*

$$\begin{cases} \tilde{v}_n(x) \rightharpoonup V_0 \ne 0 \ \textit{in}\ \mathcal{D}_0^{1,p}(\mathbb{R}^N), \quad \tilde{v}_n(x) \to V_0 \quad \textit{a.e. in}\ \mathbb{R}^N, \\ \tilde{v}_n(x) \to V_0 \quad \textit{in} \quad \mathcal{D}_{0,loc}^{1,p}(\mathbb{R}^N \backslash \{0, x_1, \cdots, x_{m_2}\}). \end{cases} \tag{2.38}$$

*If $r_n/|y_n| \to 0$, then $V_0$ solves $(Q_0)$ and the sequence*

$$w_n(x) = v_n(x) - r_n^{\frac{p-N}{p}} V_0(\frac{x - y_n}{r_n}) + o(1)$$

*is a $(PS)$ sequence of $F_\lambda$ at level $c - E_0(V_0)$.*

*Proof.* We distinguish two cases:

**(i)** $dist(y_n, \partial\Omega_n)/r_n \le K < \infty$. Then, up to a scaling transformation, $\tilde{\Omega}_n \to \mathbb{R}^N_+$;

**(ii)** $dist(y_n, \partial\Omega_n)/r_n \to \infty$. Then $\tilde{\Omega}_n \to \mathbb{R}^N$.

Noting that in any case, we have, for any $\varphi \in C_0^\infty(\tilde{\Omega}_n)$,

$$\int_{\tilde{\Omega}_n} |\tilde{v}_n|^{p-2}\tilde{v}_n\varphi \Big/ |x + \frac{y_n}{r_n}|^p \to 0.$$

It is deduced from the scaling invariance and $F'_\lambda(v_n) \to 0$ that for $n$ large,

$$\begin{aligned}
&\int_{\tilde{\Omega}_n} \left(|\nabla V_0|^{p-2}\nabla V_0 \nabla\varphi - |V_0|^{p^*-2}V_0\varphi\right) \\
&= \int_{\tilde{\Omega}_n} \left(|\nabla \tilde{v}_n|^{p-2}\nabla \tilde{v}_n \nabla\varphi - \lambda\frac{|\tilde{v}_n|^{p-2}\tilde{v}_n\varphi}{|x + \frac{y_n}{r_n}|^p} - |\tilde{v}_n|^{p^*-2}\tilde{v}_n\varphi\right) + o(1) \\
&= \int_{\Omega} \Big(|\nabla v_n|^{p-2}\nabla v_n \nabla\varphi^{**} - \lambda|v_n|^{p-2}v_n\varphi^{**}|x|^{-p} \\
&\qquad -|v_n|^{p^*-2}v_n\varphi^{**}\Big) + o(1) \\
&= o(1).
\end{aligned} \tag{2.39}$$

If case $(i)$ occurs, then we obtain from (2.39) that $V_0$ is a solution of the equation

$$-\Delta_p u = u^{p^*-1}, \quad u > 0,\ x \in \mathbb{R}^N_+,\ u = 0,\ x \in \partial\mathbb{R}^N_+.$$

Thus Pohozaev identity implies that $V_0 \equiv 0$, which contradicts to (2.38). So we have case (ii) and $V_0$ solves $(Q_0)$. Set

$$w_n(x) = v_n(x) - r_n^{\frac{p-N}{p}} V_0\Big(\frac{x - y_n}{r_n}\Big)\varphi\Big(\frac{x - y_n}{\bar{r}_n}\Big),$$

where $\bar{r}_n$ and $\varphi$ are defined similar to those in Lemma 2.3. Moreover we obtain from the same arguments as those in Lemma 2.3 that

$$w_n(x) = v_n(x) - r_n^{\frac{p-N}{p}} V_0\Big(\frac{x - y_n}{r_n}\Big) + o(1)$$

is a $(PS)$ sequence of $F_\lambda$ at level $c - E_0(V_0)$. The proof is complete. ■

*Proof of Theorem 1.1.* $(a)$ Since $(u_n) \subset \mathcal{D}_0^{1,p}(\Omega)$, $u_n \ge 0$ is a $(PS)_d$ sequence of $F_{\lambda,\mu}$, we obtain from Lemma 2.1 and Lemma 2.2 that up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in} \quad \mathcal{D}_0^{1,p}(\Omega), \qquad u_n \to u \quad \text{a.e. in} \quad \Omega \quad \text{and} \quad F'_{\lambda,\mu}(u) = 0.$$

Moreover $v_n = u_n - u$ is a $(PS)_{d-F_{\lambda,\mu}(u)}$ sequence of $F_\lambda$ and $v_n \rightharpoonup 0$ in $\mathcal{D}_0^{1,p}(\Omega)$.

$(b)$ If $v_n \to 0$ in $L^{p^*}(\Omega)$, then standard arguments imply that $v_n \to 0$ in $\mathcal{D}_0^{1,p}(\Omega)$ and we are done. If not, choosing $0 < \tau < S_0^{N/p}(1 - \lambda\Lambda_{N,p}^{-1})^{N/p}$ such that

$$\limsup_{n\to\infty} \int_\Omega |v_n|^{p^*} > \tau.$$

Up to a subsequence, let $R_n > 0$ be such that $\int_{B(0,R_n)} |v_n(x)|^{p^*} = \tau$ and $R_n$ be the minimal with this property. Define $\tilde{v}_n(x) = R_n^{\frac{N-p}{p}} v_n(R_n x)$. Clearly

$$\int_{B(0,1)} |\nabla \tilde{v}_n(x)|^p = \int_{B(0,R_n)} |\nabla v_n(x)|^p \quad \text{and} \quad \int_{B(0,1)} |\tilde{v}_n(x)|^{p^*} = \tau. \tag{2.40}$$

Denote $\Omega_n = \{x \in \mathbb{R}^N;\ R_n x \in \Omega\}$. Then $\tilde{v}_n \in \mathcal{D}_0^{1,p}(\Omega_n)$. By extending $\tilde{v}_n(x)$ to be zero for $x$ outside of $\Omega_n$, we can assume $\tilde{v}_n \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and up to a subsequence, there is $V_1 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that

$$\tilde{v}_n(x) \rightharpoonup V_1 \quad \text{in } \mathcal{D}^{1,p}(\mathbb{R}^N), \quad \tilde{v}_n(x) \to V_1 \quad a.e. \text{ in } \mathbb{R}^N. \tag{2.41}$$

We distinguish two cases:
(I) $V_1 \neq 0$. Since $v_n \rightharpoonup 0$ in $\mathcal{D}_0^{1,p}(\Omega)$, we have $R_n \to 0$ and $\Omega_n \to \mathbb{R}^N$. In this case, we obtain from Lemma 2.3 that $V_1$ solves $(Q_\lambda)$ and the sequence $w_n(x) = v_n(x) - R_n^{\frac{p-N}{p}} V_1(\frac{x}{R_n}) + o(1)$ is a $(PS)$ sequence of $F_\lambda$ at the level $d - F_{\lambda,\mu}(u) - E_\lambda(V_1)$. Moreover $w_n \rightharpoonup 0$ in $\mathcal{D}_0^{1,p}(\Omega)$.
(II) $V_1 \equiv 0$. Choosing a test function $h \in C_0^\infty(B(0,\rho))$ $(0 < \rho < 1)$ such that $h(x) = 1$ for $|x| < \rho/2$ and $h(x) = 0$ for $|x| > \rho$ and using the fact that $\tilde{v}_n(x) \rightharpoonup V_1 \equiv 0$ and hence $\tilde{v}_n(x) \to 0$ in $L^q(\Omega)$ for any $p \le q < p^*$, we obtain from direct computations that

$$\int_{\mathbb{R}^N} |\nabla(h\tilde{v}_n)|^p = \int_{\mathbb{R}^N} h^p |\nabla \tilde{v}_n|^p + o(1).$$

Noticing

$$\langle E_\lambda'(\tilde{v}_n), \varphi\rangle = \langle F_\lambda'(v_n), \varphi_n^*\rangle = o(1)\|\varphi_n^*\| = o(1)\|\varphi\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \tag{2.42}$$

and substituting $\varphi$ by $h^p \tilde{v}_n$ in $\langle E_\lambda'(\tilde{v}_n), \varphi\rangle$, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(h\tilde{v}_n)|^p \\
&= \int_{\mathbb{R}^N} h^p |\nabla \tilde{v}_n|^p + o(1) \\
&= \langle E_\lambda'(\tilde{v}_n, \varphi\rangle + \int_{\mathbb{R}^N} \left(\frac{\lambda}{|x|^p}|h\tilde{v}_n|^p + |\tilde{v}_n|^{p^*} h^p\right) + o(1) \\
&\le o(1)\|\varphi\| + \lambda\Lambda_{N,p}^{-1} \int_{\mathbb{R}^N} |\nabla(h\tilde{v}_n)|^p \\
&\qquad + \left(\textstyle\int_{\mathbb{R}^N} |h\tilde{v}_n|^{p^*}\right)^{\frac{p}{p^*}} \left(\textstyle\int_{\mathbb{R}^N} |\tilde{v}_n|^{p^*}\right)^{\frac{p}{N}} + o(1) \\
&\le o(1) + \left(\lambda\Lambda_{N,p}^{-1} + S_0^{-1}\left[\int_{B(0,1)} |\tilde{v}_n|^{p^*}\right]^{\frac{p}{N}}\right) \int_{\mathbb{R}^N} |\nabla(h\tilde{v}_n)|^p.
\end{aligned} \tag{2.43}$$

It follows from the choice of $\tau$ that

$$\int_{B(0,\rho)} |\nabla \tilde{v}_n|^p \to 0 \quad (n \to \infty). \tag{2.44}$$

Denote

$$L_n(r) = \sup_{x \in \Omega_n} \int_{B(0,r)} |\tilde{v}_n|^{p^*} := \sup_{x \in \Omega_n} \int_{B(0,r)} |z_n|^{p^*}.$$

Noticing that $\int_{B(0,1)} |z_n|^{p^*} = \tau$, we can find $\tau^*$ small enough such that

$$0 < \tau^* = \sup_{q \in \Omega_n} \int_{B(q,s_n)} |z_n|^{p^*} = \int_{B(q_n,s_n)} |z_n|^{p^*}. \tag{2.45}$$

From (2.44), $(q_n)$ and $(s_n)$ can be chosen such that $s_n \to 0$ and $|q_n| \geq \frac{1}{2}$. Denote $\tilde{z}_n(x) = s_n^{\frac{N-p}{p}} z_n(s_n x + q_n)$ and $\tilde{\Omega}_n = \{x \in \mathbb{R}^N;\ s_n x + q_n \in \Omega_n\}$. We assume that $\tilde{z}_n \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and

$$\tilde{z}_n(x) \rightharpoonup V_0 \quad \text{in } \mathcal{D}^{1,p}(\mathbb{R}^N), \quad \tilde{z}_n(x) \to V_0 \quad a.e. \text{ in } \mathbb{R}^N. \tag{2.46}$$

Similar to the proof of those in Lemma 2.3, we have that for some finite points

$$\tilde{z}_n(x) \to V_0 \quad \text{in} \quad \mathcal{D}^{1,p}_{loc}(\mathbb{R}^N \backslash \{0, x_1, \cdots, x_{m_2}\}). \tag{2.47}$$

(2.45) implies that $V_0 \neq 0$.

Noticing that $\tilde{z}_n(x) = s_n^{\frac{N-p}{p}} \tilde{v}_n(s_n x + q_n) = (R_n s_n)^{\frac{N-p}{p}} \tilde{v}_n(R_n s_n x + R_n q_n)$, we define $r_n = R_n s_n$, $y_n = R_n q_n$ and have that

$$\frac{r_n}{|y_n|} < 2 s_n \to 0 \qquad y_n \to y_0 \in \bar{\Omega}. \tag{2.48}$$

Combining (2.46), (2.47), (2.48) and using Lemma 2.4, we know that $V_0$ solves $(Q_0)$ and $w_n(x) = v_n(x) - r_n^{\frac{p-N}{p}} V_0(\frac{x-y_n}{r_n}) + o(1)$ is a $(PS)$ sequence of $F_\lambda$ at level $d - F_{\lambda,\mu}(u) - E_0(V_0)$. Moreover $w_n \rightharpoonup 0$ in $\mathcal{D}_0^{1,p}(\Omega)$.

In summation, from a $(PS)$ sequence $(v_n)$ we can single out another $(PS)$ sequence $(w_n)$ at a level strictly lower, with a fixed minimum amount of decrease. Arguing recursively, we infer that this process has to stop after finite steps. In the last step, the $(PS)$ sequence converges strongly to zero. The proof is complete. ■

# 3 Concluding remarks

We conclude this paper by some remarks on the existence of positive solution of $(Q(\lambda,\mu,p))$ and some other possible application of Theorem 1.1.

**Remark 3.1** When $p = 2$ and $0 < \mu < \mu_1(\lambda)$, Jannelli [13] proved that $(Q(\lambda,\mu,2))$ has at least one positive solution in $H_0^1(\Omega)$ under some further assumptions on $\lambda$ and $\mu$. For the general $2 \leq p < N$, under suitable conditions on $\lambda$ and $\mu$, one can easily get one positive solution of $(Q(\lambda,\mu,p))$ with the help of mountain pass theorem [2] and Corollary1.1. See also [7] for a general existence result.

**Remark 3.2** In the spirit of the well known result of Cerami et al [8] and the results of [12, 3], it is possible to use the compactness result (Theorem 1.1) to obtain sign changing solutions of $(Q(\lambda, \mu, p))$. This will be a problem for further study.

**Acknowledgement** The author thanks the unknown referee for many valuable comments.

# References

[1] B. Abdellaoui, V. Felli and I. Peral, *Existence and nonexistence results for quasilinear elliptic equations involving the p−Laplacian*, Boll.U.M.I. **8** (2006), 445-484.

[2] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349-381.

[3] H. Brezis and T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures et Appl. **58** (1979), 137-151.

[4] H. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983) 486-490.

[5] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical exponents*, Comm. Pure Appli. Math. **34** (1983), 437-477.

[6] D. Cao and S. Peng, *A global compactness result for singular elliptic problems involving critical Sobolev exponent*, Proc. Amer. Math. Soc. **131** (2003), 1857-1866.

[7] Y. Cao and D. Kang, *Solutions of a quasilinear elliptic problem involving a critical Sobolev exponent and multiple Hardy-type terms*, J. Math. Anal. Appl. **333** (2007), 889-903.

[8] G. Cerami, S. Solimini and M. Struwe, *Some existence results for superlinear elliptic boundary value problems involving critical exponents*, J. Funct. Anal. **69** (1986), 289-306.

[9] J. Chen, *Existence of solutions for a nonlinear PDE with an inverse square potential*, J. Differential Equations **195** (2003), 497-519.

[10] J. Garcia Azorero and I. Peral, *Hardy inequalities and critical elliptic and parabolic problems*, J. Differental Equations **144** (1998), 441-476.

[11] M. Guedda and L. Veron, *Local and global properties of solutions of quasilinear elliptic equations*, J. Differential Equations **76** (1988), 159-189.

[12] N. Ghoussoub and C. Yuan, *Multiple solutions for quasilinear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc. **352** (2000), 5703-5743.

[13] E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equation **156** (1999), 407-426.

[14] P.L. Lions, *The concentration compactness principle in the calculus of variations, The limit case, I, II*, Rev. Matematica Iberoamericana, **1** (1985), 145-201; 45-121.

[15] D. Smets, *Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities*, Trans. Amer. Math. Soc. **357** (2005), 2909-2938.

[16] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearity*, Math. Z. **187** (1984), 511-517.

[17] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.

[18] S. Yan, *A global compactness result for quasilinear elliptic boundary value problems involving limiting nonlinearity*, Chinese Ann. Math. **16A** (1995), 397-402.