

## Existence of Nonnegative Solutions For a Class of p-Laplacian Equations in $\mathbb{R}^N$

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### Abstract

We consider the question of existence of nonnegative solutions of a class of p-Laplacian elliptic problems in the whole space  $\mathbb{R}^N$  having a sign-changing nonlinearity of the form  $a(x)g(u)$ .

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# 1 Introduction

In this paper, we are concerned with the existence of nonnegative solutions of the Schrödinger equation

$$-\Delta_p u - \lambda h(x)|u|^{p-2}u = a(x)g(u) \quad \text{in } \mathbb{R}^N, \tag{P_\lambda}$$

where  $\lambda \in \mathbb{R}$  is a parameter,  $1 < p < N$ ,  $\Delta_p := \text{div}(|\nabla u|^{p-2}\nabla u)$  denotes the p-Laplacian operator and  $h(x) \geq 0$  is a continuous function on  $\mathbb{R}^N$ . The function  $a(x)$  is continuous and changes sign in  $\mathbb{R}^N$ .

In the case  $p = 2$ , the problem has been extensively studied in bounded domains. In particular, it was discussed in [1], [2] and [3]. Among other things, a "thickness" condition on  $a(x)$  is required in order to verify the Palais-Smale condition, namely, it is required that  $\bar{\Omega}^+ \cap \bar{\Omega}^- = \emptyset$ , where  $\Omega^+ = \{x \in \Omega : a(x) > 0\}$ ,  $\Omega^- = \{x \in \Omega : a(x) < 0\}$ . The "thickness" condition was removed in [15]. In general, the functional corresponding to problem  $(P_\lambda)$  does not satisfy geometrical conditions in critical point theory due to the presence of the negative part  $-a^-$  of  $a$ . In [15], the authors first truncate the nonlinear term related to  $a^-$  and solve a modified problem; secondly, they obtain a uniform  $L^\infty$  bound for solutions  $u_n$  of modified problems, and then they show the limit function  $u$  of  $u_n$  is a solution of the original problem. In the establishment of  $L^\infty$  bound, an estimate of Morse index of critical points is involved in blow up argument. This work was generalized by [8], [9], [10], [11] and [12] etc.

It is known that the Morse theory for functional defined in a Banach space is not sufficient to be used, so in the case  $p \neq 2$ , problem  $(P_\lambda)$  can not be dealt with in the same routine as in [15]. Recently, J.Chabrowski and D.G.Costa [7] generalized the results to p-Laplacian problem. Their technique is to separate the Nehari manifold and analyse each part of the manifold carefully. But they require  $g(u) = |u|^{q-2}u$  for some  $p < q < p^* = \frac{Np}{(N-p)}$ . Their argument is difficult to extend to problems with a general  $g(u)$ .

On the other hand, Costa and Tehrani [10] proved in the case  $p = 2$  that problem  $(P_\lambda)$  possesses at least a positive solution if  $0 < \lambda < \lambda_1$  and two positive solutions if  $\lambda_1 < \lambda < \lambda_1 + \tilde{\alpha}$  for some  $\tilde{\alpha} > 0$ , where  $\lambda_1$  is the first eigenvalue of the problem

$$-\Delta_p u = \lambda h(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1}$$

The results in [10] are obtained by directly applying the mountain pass theorem. But the problem is that setting in  $\mathbb{R}^N$ , the Palais-Smale condition is usually not valid. They use the concentration-compactness principle to show that  $(PS)_c$  condition holds for all  $c$  under certain conditions. In present work, we shall establish similar results for problem  $(P_\lambda)$  with  $p \neq 2$ .

It is well-known that under our assumptions on  $h(x)$  (see below) problem (1) has a sequence of eigenvalues  $0 < \lambda_1(h) < \lambda_2(h) \leq \dots \leq \lambda_n(h) \leq \dots$ , which are obtained variationally through the Krasnoselski genus. Moreover, It is shown in [4] that the principal positive eigenvalue  $\lambda_1(h)$  is simple, isolated and possesses positive eigenfunctions  $\varphi_1(x)$ .

Our hypotheses are as follows.

(H1)  $0 \leq h(x) \in L^{N/p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ,  $h \not\equiv 0$ .

(G1)  $\lim_{s \rightarrow 0} \frac{g(s)}{s^{\alpha-1}} = 1$ .

$$(G2) \lim_{s \rightarrow \infty} \frac{g(s)}{s^{\beta-1}} = 1, \quad p < \alpha < \beta < p^* = \frac{Np}{N-p}, 1 < p < N.$$

$$(G3) g(s) > 0, \quad \forall s > 0.$$

$$(A1) a \in C(\mathbb{R}^N) \text{ changes sign in } \mathbb{R}^N.$$

$$(A2) \lim_{|x| \rightarrow \infty} a(x) = a_\infty < 0.$$

$$(A3) \int_{\mathbb{R}^N} a(x) \varphi_1^\alpha(x) dx < 0.$$

Denote  $\Omega^0 = \{x \in \mathbb{R}^N \mid a(x) = 0\}$ . We say  $a(x)$  has a thick zero set if  $\bar{\Omega}^+ \cap \bar{\Omega}^- = \emptyset$ . Moreover, since we are looking for nonnegative solutions, throughout this paper, we shall assume, without loss of generality, that  $g(s)$  is defined for all  $s \in \mathbb{R}$  as an odd function.

Because of assumptions (H1) and (G1) – (G3), we have

$$\int_{\mathbb{R}^N} h(x)|u|^p dx \leq \left( \int_{\mathbb{R}^N} h^{\frac{N}{p}} \right)^{\frac{p}{N}} \left( \int_{\mathbb{R}^N} |u|^{p^*} \right)^{1-\frac{p}{N}} = |h|_{\frac{N}{p}} |u|_{p^*}^p \leq C |h|_{\frac{N}{p}} |\nabla u|_p^p, \quad (2)$$

$$|g(s)| \leq C_1 |s|^{\alpha-1} + C_2 |s|^{\beta-1}, \quad |G(s)| \leq C_1 |s|^\alpha + C_2 |s|^\beta \quad \forall s \in \mathbb{R} \quad (3)$$

and for  $\delta > 0$  small,

$$\begin{aligned} C_0 |s|^\alpha &\leq G(s), & C_0 |s|^\alpha &\leq g(s)s, & \text{if } |s| &\leq \delta; \\ C_0 |s|^\beta &\leq G(s), & C_0 |s|^\beta &\leq g(s)s, & \text{if } |s| &\geq \delta. \end{aligned} \quad (4)$$

Our main results are as follows.

**Theorem 1** *Assume the conditions (H1), (A1)–(A3), (G1)–(G3) and  $\lambda \notin \sigma(-\Delta_p, \Omega^0, h(x))$ ,  
 (i) If  $\lambda \leq \lambda_1(h)$  then problem  $(P_\lambda)$  has at least one nonnegative solution  $u \in E$ .  
 (ii) There exists  $\bar{\alpha} > 0$  such that for  $\lambda_1(h) < \lambda < \lambda_1(h) + \bar{\alpha}$ , problem  $(P_\lambda)$  has at least two nonnegative solutions.*

To prove the result, we need to verify  $(PS)_c$  condition, this will be done in section 2 by the concentration-compactness principle. Next, we shall show that the corresponding functional has the mountain-pass structure in section 3. Theorem 1 is proved in section 4.

## 2 The proof of (PS) condition

Let  $G(u) = \int_0^u g(s) ds$ . We consider the functional associated to  $(P_\lambda)$ :

$$I_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{\lambda}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx - \int_{\mathbb{R}^N} a(x)G(u) dx,$$

which is defined on Banach space  $E = D^{1,p}(\mathbb{R}^N) \cap L^\alpha(\mathbb{R}^N)$  with the norm

$$\|u\| = \left[ \int_{\mathbb{R}^N} |\nabla u|^p dx + \left( \int_{\mathbb{R}^N} |u|^\alpha dx \right)^{\frac{p}{\alpha}} \right]^{\frac{1}{p}}.$$

It is known that  $I_\lambda \in C^1$  and its critical points are the weak solutions of  $(P_\lambda)$ . We denote by  $|\cdot|_p$  the  $L^p$  norm.

In this section, we shall verify the (PS) condition. A sequence  $\{u_n\}$  is said to be a  $(PS)_c$  sequence if  $\{u_n\}$  satisfies  $I_\lambda(u_n) \rightarrow c$  and  $\|I'(u_n)\| \rightarrow 0$ ; if any  $(PS)_c$  sequence of  $I$  contains a strongly convergent subsequence, we say  $I$  satisfies  $(PS)_c$  condition.

**Lemma 1** *Assume (H1), (A1), (A2), (G1) – (G3). Then, there exists  $C > 0$  such that for any  $u \in E$  we have  $|u|_\alpha^\alpha \leq C(|\nabla u|_p^p + |\nabla u|_p^{p^*} + I_\lambda(u))$ .*

*Proof.* Let us assume  $a_\infty < -1$  for simplicity. By (A2) we know that there exists  $R_0 > 0$  such that  $a(x) < -1$  if  $|x| \geq R_0$ . By Sobolev embedding,

$$\begin{aligned} & \int_{\{|x| < R_0\}} |u|^\alpha dx \\ & \leq C(R_0) \left( \int_{\{|x| < R_0\}} |u|^{p^*} dx \right)^{\frac{\alpha}{p^*}} \\ & \leq C(R_0) \left( \int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{\frac{\alpha}{p^*}} \leq C(R_0) |\nabla u|_p^\alpha. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_{\mathbb{R}^N} |u|^\alpha dx \\ & = \int_{\{|x| < R_0\}} |u|^\alpha dx + \int_{\{|x| \geq R_0\}} |u|^\alpha dx \\ & \leq C(R_0) |\nabla u|_p^\alpha + \int_{\{|x| \geq R_0, |u| \leq \delta\}} |u|^\alpha dx + \int_{\{|x| \geq R_0, |u| > \delta\}} |u|^\alpha dx. \end{aligned} \quad (5)$$

By (2),(3),(4),

$$\begin{aligned} & \int_{\mathbb{R}^N} |u|^\alpha dx \\ & \leq C |\nabla u|_p^\alpha + \int_{\{|x| \geq R_0, |u| \leq \delta\}} \frac{1}{C_0} G(u) dx + C_\delta \int_{\{|x| \geq R_0, |u| > \delta\}} |u|^{p^*} dx \\ & \leq C |\nabla u|_p^\alpha - \frac{1}{C_0} \int_{\{|x| \geq R_0\}} a(x) G(u) dx + C_\delta \int_{\{|x| \geq R_0, |u| > \delta\}} |u|^{p^*} dx \\ & \leq C |\nabla u|_p^\alpha - \frac{1}{C_0} \int_{\mathbb{R}^N} a(x) G(u) dx + \frac{1}{C_0} \int_{\{|x| < R_0\}} a(x) G(u) dx + C_\delta |\nabla u|_p^{p^*} \\ & \leq C (|\nabla u|_p^p + |\nabla u|_p^{p^*} - \int_{\mathbb{R}^N} a(x) G(u) dx) \\ & \leq C (|\nabla u|_p^p + |\nabla u|_p^{p^*} + I_\lambda(u) - \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p + \frac{\lambda}{p} C |h|_{\frac{N}{p}} |\nabla u|_p^p) \\ & \leq C (|\nabla u|_p^p + |\nabla u|_p^{p^*} + I_\lambda(u)). \end{aligned} \quad (6)$$

The first step to verify  $(PS)_c$  condition is to show that the  $(PS)_c$  sequence  $\{u_n\}$  is bounded. To this end, by Lemma 1, we need only to show that  $|\nabla u_n|_p$  is bounded. Let us consider the closed subspace  $D^{1,p}(\Omega^0)$  of  $D^{1,p}(\mathbb{R}^N)$  :

$$D^{1,p}(\Omega^0) = \{u \in D^{1,p}(\mathbb{R}^N) \mid u(x) = 0, a.e \text{ in } \mathbb{R}^N \setminus \Omega^0\}.$$

We say  $\lambda \in \sigma(-\Delta_p, \Omega^0, h(x))$  if there is a nontrivial solution  $v \in D^{1,p}(\Omega^0)$  of the problem

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \varphi dx - \lambda \int_{\mathbb{R}^N} h(x) |v|^{p-2} v \varphi dx = 0, \forall \varphi \in D^{1,p}(\Omega^0). \quad (7)$$

**Lemma 2** *Suppose  $\{u_n\}$  is a  $(PS)_c$ -sequence and bounded in  $D^{1,p}(\mathbb{R}^N)$ . If  $u_n \rightharpoonup u$  in  $D^{1,p}(\mathbb{R}^N)$ , then  $\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla v dx \rightarrow \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla v dx$  for  $v \in D^{1,p}(\mathbb{R}^N)$ .*

*Proof.* Let  $\{u_n\}$  be a  $(PS)_c$ -sequence for  $I_\lambda$ , that is:

$$I_\lambda(u_n) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p dx - \frac{\lambda}{p} \int_{\mathbb{R}^N} h(x) |u_n|^p dx - \int_{\mathbb{R}^N} a(x) G(u_n) dx \rightarrow c, \quad (8)$$

$$\langle I'_\lambda(u_n), \varphi \rangle = o(1) \|\varphi\|. \quad (9)$$

Since  $|\nabla u_n|^{p-2} \nabla u_n$  is bounded in  $L^{\frac{p}{p-1}}(\mathbb{R}^N)$ , we have, up to a subsequence, that

$$|\nabla u_n|^{p-2} \nabla u_n \rightharpoonup Q \text{ in } L^{\frac{p}{p-1}}(\mathbb{R}^N). \quad (10)$$

We claim that  $\nabla u_n \rightarrow \nabla u$  a.e. In fact, let  $k > 0$ , we define a cut-off function  $T_k : \mathbb{R} \rightarrow \mathbb{R}$

$$T_k(s) = s, \quad \text{if } |s| \leq k; \quad T_k = ks/|s| \quad \text{if } |s| \geq k.$$

It is sufficient to prove that there exists a subsequence such that  $\nabla u_n \rightarrow \nabla u$  a.e in any ball  $B_{r/2}$ . Let  $\varphi \in C_0^\infty(B_r)$  such that  $0 \leq \varphi \leq 1$  in  $B_r$ , and  $\varphi = 1$  in  $B_{r/2}$ . We get from (9) that

$$\begin{aligned} & \int_{B_r} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi T_\eta(u_n - u) dx + \int_{B_r} |\nabla u_n|^{p-2} \nabla u_n \nabla [T_\eta(u_n - u)] \varphi dx \\ & - \lambda \int_{B_r} h(x) |u_n|^{p-2} u_n \varphi T_\eta(u_n - u) dx - \int_{B_r} a(x) g(u_n) \varphi T_\eta(u_n - u) dx \rightarrow 0. \end{aligned} \quad (11)$$

Therefore,

$$\begin{aligned} & \int_{B_r} \varphi (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \nabla [T_\eta(u_n - u)] dx = \\ & - \int_{B_r} \varphi |\nabla u|^{p-2} \nabla u \nabla [T_\eta(u_n - u)] dx - \int_{B_r} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi T_\eta(u_n - u) dx \\ & + \lambda \int_{B_r} h(x) |u_n|^{p-2} u_n \varphi T_\eta(u_n - u) dx + \int_{B_r} a(x) g(u_n) \varphi T_\eta(u_n - u) dx + o(1). \end{aligned} \quad (12)$$

Since  $T_\eta(u_n - u) \rightarrow 0$  in  $D^{1,p}(B_r)$  and  $T_\eta(u_n - u) \rightarrow 0$  in  $L^p(B_r)$ , the first term in the right hand side of (12) tends to zero as  $n \rightarrow \infty$ . Moreover

$$\begin{aligned} & \left| \int_{B_r} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \cdot T_\eta(u_n - u) dx \right| \\ & \leq \left( \int_{B_r} |T_\eta(u_n - u)|^p dx \right)^{1/p} \left( \int_{B_r} (|\nabla u_n|^{p-1} |\nabla \varphi|)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ & \leq C \left( \int_{B_r} |T_\eta(u_n - u)|^p dx \right)^{1/p} \rightarrow 0, \end{aligned}$$

So the second term of (12) on the right hand side also tends to zero as  $n \rightarrow \infty$ . On the other hand, for any  $\psi \in D^{1,p}(\mathbb{R}^N)$ ,

$$\begin{aligned} & \langle g_n, \psi \rangle : \\ & = \int_{B_r} \lambda h(x) |u_n|^{p-2} u_n \psi dx + \int_{B_r} a(x) g(u_n) \psi dx \\ & \leq C \left( \int_{B_r} |u_n|^p dx \right)^{\frac{p-1}{p}} \left( \int_{B_r} |\psi|^p dx \right)^{1/p} + C \left( \int_{B_r} |\psi|^p dx \right)^{1/p} \left( \int_{B_r} g(u_n)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ & \leq C \|\psi\|_{\infty, B_r}, \end{aligned}$$

it yields

$$\langle g_n, \varphi T_\eta(u_n - u) \rangle \leq C \|\varphi T_\eta(u_n - u)\|_{\infty, B_r} \leq \eta.$$

Consequently

$$\limsup_{n \rightarrow \infty} \int_{B_r} \varphi (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \nabla [T_\eta(u_n - u)] dx \leq C \eta. \quad (13)$$

Let

$$e_n = (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \nabla (u_n - u).$$

Fix  $\theta$  such that  $0 < \theta < 1$ , and denote

$$S_n^\eta = \{x \in B_r : |u_n - u| \leq \eta\}, \quad G_n^\eta = \{x \in B_r : |u_n - u| > \eta\}.$$

By Hölder inequality,

$$\begin{aligned} & \int_{B_{r/2}} |e_n|^\theta dx \\ & \leq \int_{S_n^\eta} |e_n|^\theta dx + \int_{G_n^\eta} |e_n|^\theta dx \\ & \leq (\int_{S_n^\eta} |e_n| dx)^\theta |S_n^\eta|^{1-\theta} + (\int_{G_n^\eta} |e_n| dx)^\theta |G_n^\eta|^{1-\theta}. \end{aligned}$$

For a fixed  $\eta$ ,  $|G_n^\eta| \rightarrow 0$  as  $n \rightarrow \infty$ , and by (13),  $e_n$  is bounded in  $L^1(B_{r/2})$ , we may assume  $e_n \rightarrow 0$  a.e. in  $B_{r/2}$ . As a result,

$$(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \nabla (u_n - u) \rightarrow 0 \text{ a.e. in } B_{r/2}.$$

The inequality

$$(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \nabla (u_n - u) \geq C_m |\nabla u_n - \nabla u|^p,$$

now implies  $\nabla u_n \rightarrow \nabla u$  a.e. in  $B_{r/2}$ . This, together with (10), gives the conclusion.

Now we show any  $(PS)_c$  sequence of  $I_\lambda$  is bounded.

**Lemma 3** Assume (H1), (A1), (A2), (G1) – (G3) and  $\lambda \notin \sigma(-\Delta_p, \Omega^0, h(x))$ , then any  $(PS)_c$ -sequence of  $I_\lambda$  is bounded in  $E$  provided that either

(i)  $g(s)$  satisfies the stronger form  $(G'_2)$  of  $(G_2)$  given by

$$g(s) = |s|^{\beta-2} s + O(|s|^\gamma) \text{ as } |s| \rightarrow \infty, \text{ for some } 0 \leq \gamma < 1; \text{ or}$$

(ii)  $a(x)$  has a thick zero set.

*Proof.* Let  $\{u_n\}$  be a  $(PS)_c$ -sequence of  $I_\lambda$ , that is,  $\{u_n\}$  satisfies (8), (9). We use the arguments in [11] to show that  $|\nabla u_n|_p$  is bounded. Suppose  $|\nabla u_n|_p = t_n \rightarrow \infty$ , let  $v_n = \frac{u_n}{t_n}$ , then  $|\nabla v_n|_p = 1$  and for each bounded domain  $\Omega \subset \mathbb{R}^N$

$$\begin{aligned} v_n & \rightharpoonup v \quad \text{in } D^{1,p}(\mathbb{R}^N), \\ v_n & \rightarrow v \quad \text{in } L^t(\Omega) \quad 1 \leq t < p^*, \\ v_n & \rightarrow v \quad \text{a.e. in } \mathbb{R}^N, \\ |v_n(x)| & \leq w(x; t) \text{ for some } w(\cdot; t) \in L^t(\Omega). \end{aligned} \tag{14}$$

We claim  $v(x) = 0$ . In fact, dividing (9) by  $t_n^{p-1}$ , we get

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla v_n|^{p-2} \nabla v \nabla \varphi dx - \lambda \int_{\mathbb{R}^N} h(x) |v_n|^{p-2} v_n \varphi - \int_{\mathbb{R}^N} a(x) \frac{g(t_n v_n)}{t_n^{p-1}} \varphi dx \\ & = o(1) \|\varphi\|. \end{aligned} \tag{15}$$

Let

$$|t_n|^{\beta-p} A_n(\varphi) := |t_n|^{\beta-p} \int_{\mathbb{R}^N} a(x) |v_n|^{\beta-p} v_n^{p-1} \frac{g(t_n v_n)}{|t_n v_n|^{\beta-p} t_n^{p-1} v_n^{p-1}} \varphi \, dx$$

and  $h_n(x)$  denote the integrand of  $A_n(\varphi)$ . Then, on the set  $\{x \mid v(x) \neq 0\}$ , we have  $|t_n v_n(x)| \rightarrow +\infty$ , and by (G2)

$$\begin{aligned} h_n(x) &:= a(x) |v_n|^{\beta-p} v_n^{p-1} \frac{g(t_n v_n)}{|t_n v_n|^{\beta-p} (t_n v_n)^{p-1}} \varphi(x) \\ &\rightarrow a(x) |v(x)|^{\beta-p} v(x)^{p-1} \varphi(x) \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

On the set  $\{x \mid v(x) = 0\}$ , we have  $v_n(x) \rightarrow 0$ . By (G1), (G2) and (14),

$$|h_n(x)| \leq \frac{C(1 + |t_n|^{\beta-1} |v_n|^{\beta-1})}{t_n^{\beta-1}} \rightarrow 0, \quad |h_n(x)| \leq C(1 + |w(x)|^{\beta-1}) \in L^1(\Omega),$$

where  $\Omega = \text{supp}(\varphi)$ . We conclude by Lebesgue dominated convergence theorem that

$$A_n(\varphi) \rightarrow A(\varphi) := \int_{\mathbb{R}^N} a(x) |v|^{\beta-p} v^{p-1} \varphi \, dx.$$

It follows by passing to the limit in (15) necessarily  $A(\varphi) = \int_{\mathbb{R}^N} a(x) |v|^{\beta-p} v^{p-1} \varphi \, dx = 0$ . Because  $\varphi \in E$  is arbitrary, we obtain that  $v(x) = 0$  a.e if  $x \in \Omega^+ \cup \Omega^-$ . Similarly, we obtain from (15) that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \varphi \, dx - \lambda \int_{\mathbb{R}^N} h(x) |v|^{p-2} v \varphi \, dx = 0$$

for  $\varphi \in D^{1,p}(\Omega^0)$ . Since  $\lambda \notin \sigma(-\Delta_p, \Omega^0, h(x))$ , it yields  $v = 0$  a.e in  $\Omega^0$ . Hence  $v = 0$  a.e in  $\mathbb{R}^N$ .

Next, dividing (8) by  $t_n^p$  and (9) by  $t_n^{p-1}$ , we obtain

$$\frac{1}{p} \int_{\mathbb{R}^N} |\nabla v_n|^p \, dx - \frac{\lambda}{p} \int_{\mathbb{R}^N} h(x) |v_n|^p \, dx - \int_{\mathbb{R}^N} a(x) \frac{G(u_n)}{t_n^p} \, dx = o(1). \quad (16)$$

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v_n|^{p-2} \nabla v_n \nabla \varphi \, dx - \lambda \int_{\mathbb{R}^N} h(x) |v_n|^{p-2} v_n \varphi \, dx - \int_{\mathbb{R}^N} a(x) \frac{g(t_n v_n)}{t_n^{p-1}} \varphi \, dx \\ &= \frac{o(1)}{t_n^{p-1}} \|\varphi\|. \end{aligned} \quad (17)$$

Since  $h(x) \in L^{\frac{N}{p}}(\mathbb{R}^N)$ ,  $v \equiv 0$ ,

$$\int_{\mathbb{R}^N} h(x) |v_n|^p \, dx \rightarrow 0.$$

Passing to the limit in (16), we obtain

$$\frac{1}{p} = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} a(x) \frac{G(u_n)}{t_n^p} \, dx. \quad (18)$$

Taking  $\varphi = v_n$  in (17), we get

$$\int_{\mathbb{R}^N} |\nabla v_n|^p dx - \lambda \int_{\mathbb{R}^N} h(x)|v_n|^p dx - \int_{\mathbb{R}^N} a(x) \frac{g(t_n v_n)}{t_n^{p-1}} v_n dx = \frac{o(1)}{t_n^{p-1}} \|v_n\|.$$

Similarly,

$$1 - \int_{\mathbb{R}^N} a(x) \frac{g(u_n)}{t_n^p} u_n dx = \frac{o(1)}{t_n^p} \|u_n\| + o(1).$$

Moreover, if we take  $\varphi = v_n \xi$  in (17) with  $\xi \in C_0^\infty(\mathbb{R}^N)$ , we see that

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla v_n|^p \xi dx - \int_{\mathbb{R}^N} a(x) \frac{g(t_n v_n)}{t_n^p} u_n \xi dx \\ &= \lambda \int_{\mathbb{R}^N} h(x)|v_n|^p \xi dx - \int_{\mathbb{R}^N} |\nabla v_n|^{p-2} \nabla v_n \cdot v_n \nabla \xi dx + \frac{o(1)}{t_n^p} \|v_n \xi\| \\ &= \text{I} + \text{II} + \text{III} \end{aligned}$$

By Lemma 2, I and II tend to zero. Now let us consider III. There holds

$$\begin{aligned} \|u_n \xi\| &= (|\nabla(u_n \xi)|_p^p + |u_n \xi|_\alpha^p)^{\frac{1}{p}} \\ &\leq C|\nabla u_n|_p + C\left(\int_{\mathbb{R}^N} |u_n \nabla \xi|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N} |u_n \xi|^\alpha dx\right)^{\frac{1}{\alpha}} \\ &\leq C(|\nabla u_n|_p + |u_n|_{p^*}) \leq C|\nabla u_n|_p = Ct_n, \end{aligned}$$

which implies that III tends to zero. Consequently,

$$\int_{\mathbb{R}^N} |\nabla v_n|^p \xi dx - \int_{\mathbb{R}^N} a(x) \frac{g(u_n)}{t_n^p} u_n \xi dx = o(1). \tag{19}$$

Next, we consider the cases (i) and (ii) separately. In the case (i),

$$\int_{\mathbb{R}^N} a(x) \frac{\beta G(u_n)}{t_n^p} \xi dx = \int_{\mathbb{R}^N} a(x) \frac{g(u_n)}{t_n^p} u_n \xi dx + o(1). \tag{20}$$

Indeed, since  $|\beta G(s) - g(s)s| \leq C$  for  $|s| \leq M$  and  $t_n \rightarrow \infty$ , obviously,

$$\int_{\{|u_n| \leq M\}} |a(x)| \frac{|\beta G(u_n) - g(u_n)u_n|}{t_n^p} |\xi| dx = o(1).$$

On the other hand, as (i) implies  $|\beta G(s) - g(s)s| = O(|s|^{\gamma+1})$  as  $|s| \rightarrow \infty$ , where  $1 \leq \gamma + 1 < 2$ , we have

$$\int_{\{|u_n| \geq M\}} |a(x)| \frac{|\beta G(u_n) - g(u_n)u_n|}{t_n^p} |\xi| dx \leq C \int_{\text{supp}(\xi)} \frac{|u_n|^{\gamma+1}}{t_n^p} dx = o(1),$$

So, (20) holds. Finally, choose  $\xi \in C_0^\infty(\mathbb{R}^N)$  such that  $0 \leq \xi \leq 1$  and  $\xi(x) = 1$  if  $|x| \leq R_0$ , with  $R_0 > 0$  as in the proof of Lemma 1, we know from (18), (19), and (20) that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} a(x) \frac{G(u_n)}{t_n^p} (1 - \xi) dx \\ &= \frac{1}{p} - \frac{1}{\beta} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} a(x) \frac{g(u_n)u_n}{t_n^p} \xi dx \\ &= \frac{1}{p} - \frac{1}{\beta} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla v_n|^p \xi dx, \\ &\geq \frac{1}{p} - \frac{1}{\beta} > 0, \end{aligned}$$

contradicting with the choice of  $R_0$ . The assertion follows.

In the Case (ii), by the thickness condition, we may choose  $\xi \in C_0^\infty(\mathbb{R}^N)$  such that  $0 \leq \xi \leq 1$  with  $\xi(x) = 1$ , if  $x \in \Omega^+$  and  $\xi(x) = 0$ , if  $x \in \Omega^-$ . By (G2), we have  $0 < rG(s) \leq g(s)s$  if  $|s| \geq M$  for some  $r > p$  and  $M > 0$ . Hence, we deduce from (19) that

$$\begin{aligned} \int_{\mathbb{R}^N} a(x) \frac{G(u_n)}{t_n^p} \xi \, dx &= \int_{\{|u_n| < M\}} a(x) \frac{G(u_n)}{t_n^p} \xi \, dx + \int_{\{|u_n| \geq M\}} a(x) \frac{G(u_n)}{t_n^p} \xi \, dx \\ &\leq o(1) + \frac{1}{r} \int_{\{|u_n| \geq M\}} a(x) \frac{g(u_n)u_n}{t_n^p} \xi \, dx \\ &\leq \frac{1}{r} \int_{\mathbb{R}^N} |\nabla v_n|^p \xi \, dx + o(1) \leq \frac{1}{r} + o(1) \end{aligned}$$

contradicting to (18).

**Lemma 4** Suppose  $\{u_n\}$  is a bounded  $(PS)_c$ -sequence of  $I_\lambda$  in  $E$ . If  $u_n \rightharpoonup u$  in  $D^{1,p}(\mathbb{R}^N)$ , then  $\int_{\mathbb{R}^N} |\nabla u_n|^p \rightarrow \int_{\mathbb{R}^N} |\nabla u|^p$  up to a subsequence.

*Proof.* Obviously,  $u$  is a weak solution of  $(P_\lambda)$ . Since  $\text{supp}^+$  is bounded, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx - \lambda \int_{\mathbb{R}^N} h(x)|u_n|^p \, dx + \int_{\mathbb{R}^N} a^-(x)g(u_n)u_n \\ &= \int_{\mathbb{R}^N} a^+(x)g(u_n)u_n + o(1) \\ &= \int_{\mathbb{R}^N} a^+(x)g(u)u + o(1) \\ &= \int_{\mathbb{R}^N} |\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N} h(x)|u|^p \, dx + \int_{\mathbb{R}^N} a^-(x)g(u)u + o(1). \end{aligned} \tag{21}$$

Let  $l := \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p$ . We may assume  $\int_{\mathbb{R}^N} |\nabla u_n|^p \rightarrow l$ , and then  $\int_{\mathbb{R}^N} |\nabla u|^p \leq l$ . By passing limit in (21), we obtain

$$\begin{aligned} &l - \lambda \int_{\mathbb{R}^N} h(x)|u|^p \, dx + \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} a^-(x)g(u_n)u_n \, dx \\ &= \int_{\mathbb{R}^N} |\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N} h(x)|u|^p \, dx + \int_{\mathbb{R}^N} a^-(x)g(u)u \, dx. \end{aligned}$$

Hence,

$$l + \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} a^-(x)g(u_n)u_n \leq \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} a^-(x)g(u)u \, dx.$$

By Fatou's Lemma:

$$l + \int_{\mathbb{R}^N} a^-(x)g(u)u \leq \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} a^-(x)g(u)u \, dx,$$

that is,  $l \leq \int_{\mathbb{R}^N} |\nabla u|^p$ . So  $l = \int_{\mathbb{R}^N} |\nabla u|^p$ .

Next, we verify the  $(PS)_c$  condition by the Concentration-Compactness principle due to P.L. Lions [13].

**Proposition 1** *Suppose the conditions in Lemma 3 hold, then the functional  $I_\lambda$  satisfies  $(PS)_c$  condition for all  $c \in \mathbb{R}$ .*

*Proof.* Let  $\{u_n\} \in E$  be a  $(PS)_c$ -sequence for  $I_\lambda$ . By Lemma 3, we know that  $\{u_n\}$  is bounded and without loss of generality, we may assume that

$$\|u_n\|^p = |\nabla u_n|_p^p + |u_n|_\alpha^p \geq C > 0.$$

In fact, if there exists a subsequence such that  $\|u_{n_k}\| \rightarrow 0$ , (PS) condition is then satisfied. Now, let

$$\rho_n = |\nabla u_n|^p + |u_n|^\alpha$$

be the concentration function. We may assume

$$\int_{\mathbb{R}^N} \rho_n \, dx \rightarrow \bar{\lambda} \geq 0.$$

By the Concentration-Compactness principle, to show that  $\{u_n\}$  satisfies  $(PS)_c$  condition, we need only to rule out vanishing and dichotomy.

First, we rule out the vanishing. Suppose by contradiction that vanishing occurs, we have

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} \rho_n(x) \, dx \rightarrow 0.$$

It follows from a result of P.L.Lions [13] that

$$u_n \rightarrow 0 \text{ in } L^\gamma(\mathbb{R}^N), \quad \alpha < \gamma < p^* := \frac{Np}{N-p}.$$

Since  $u_n$  is bounded in  $D^{1,p}(\mathbb{R}^N)$ , we may assume  $u_n \rightarrow 0$  in  $D^{1,p}(\mathbb{R}^N)$ . Thus

$$\int_{\mathbb{R}^N} h(x)|u_n|^p \, dx \rightarrow \int_{\mathbb{R}^N} h(x)|u|^p \, dx = 0.$$

We know that  $\{u_n\}$  satisfies

$$\frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx - \int_{\mathbb{R}^N} a(x)G(u_n) \, dx \rightarrow c, \tag{22}$$

$$\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx - \int_{\mathbb{R}^N} a(x)g(u_n)u_n \, dx = o(1). \tag{23}$$

By (3),

$$\left| \int_{\{|x| \leq R_0\}} a(x)g(u_n)u_n \, dx \right| \leq C \left( \int_{\{|x| \leq R_0\}} |u_n|^\alpha \, dx + \int_{\{|x| \leq R_0\}} |u_n|^\beta \, dx \right)$$

and  $u_n \rightarrow 0$  in  $L^t_{loc}(\mathbb{R}^N)$  for  $1 \leq t < p^*$ ,

$$\left| \int_{\{|x| \leq R_0\}} a(x)g(u_n)u_n \, dx \right| = o(1).$$

Thus, by (23),

$$\int_{\mathbb{R}^N} |\nabla u_n|^p dx - \int_{\{|x| \geq R_0\}} a(x)g(u_n)u_n dx = o(1).$$

Note that  $\int_{\mathbb{R}^N} |\nabla u_n|^p dx$  and  $-\int_{\{|x| \geq R_0\}} a(x)g(u_n)u_n dx$  are nonnegative, it follows

$$\int_{\mathbb{R}^N} |\nabla u_n|^p dx = o(1), \quad \int_{\mathbb{R}^N} a(x)g(u_n)u_n dx = o(1).$$

By Lemma 1, we obtain  $|u_n|_\alpha \rightarrow 0$  and then  $\|u_n\|^p = |\nabla u_n|^p + |u_n|_\alpha^p \rightarrow 0$ . This is a contradiction with the fact  $\|u_n\|^p \geq C > 0$ .

Next, we rule out dichotomy. If dichotomy occurs, then there would exist  $0 < \alpha < \bar{\lambda}$  such that for any  $\varepsilon > 0$  and  $\bar{R} \geq 1$ , there are sequences  $\{y_n\} \in \mathbb{R}^N$ ,  $\{R_n\} \subset \mathbb{R}_+$ , and  $\hat{R} > \bar{R}$  satisfying  $R_0 < \hat{R} < \frac{R_1}{2}$ ,  $R_n < R_{n+1} \rightarrow \infty$  and

$$\begin{aligned} \alpha - \varepsilon &< \int_{\{|x-y_n| \leq \frac{\hat{R}}{2}\}} |\nabla u_n|^p + |u_n|^\alpha dx < \alpha + \varepsilon, \\ &\int_{\{|x-y_n| \leq 3R_n\}} |\nabla u_n|^p + |u_n|^\alpha dx < \alpha + \varepsilon, \end{aligned} \tag{24}$$

$$\int_{\{|x-y_n| \geq 3R_n\}} |\nabla u_n|^p + |u_n|^\alpha dx > \bar{\lambda} - \alpha - \varepsilon. \tag{25}$$

In particular,

$$\begin{aligned} &\int_{\{\frac{\hat{R}}{2} \leq |x-y_n| \leq 3R_n\}} |\nabla u_n|^p + |u_n|^\alpha dx \\ &= \int_{\{|x-y_n| \leq 3R_n\}} |\nabla u_n|^p + |u_n|^\alpha dx - \int_{\{|x-y_n| < \frac{\hat{R}}{2}\}} |\nabla u_n|^p + |u_n|^\alpha dx \\ &< \alpha + \varepsilon - (\alpha - \varepsilon) = 2\varepsilon. \end{aligned} \tag{26}$$

We claim that

$$\int_{\{\hat{R} \leq |x-y_n| \leq 2R_n\}} |u_n|^{p^*} dx \leq C(N)\varepsilon^{\frac{p^*}{\alpha}}. \tag{27}$$

Indeed, let  $\eta_n \in C_0^\infty(\mathbb{R}^N)$  be a function such that  $\eta_n(x) = 0$  if  $|x - y_n| \leq \frac{\hat{R}}{2}$ , or  $|x - y_n| \geq 3R_n$ ,  $\eta_n(x) = 1$  if  $\hat{R} \leq |x - y_n| \leq 2R_n$  and  $|\nabla \eta_n(x)| \leq \frac{C}{(R_n - \hat{R})}$ . By Sobolev and Hölder's inequalities

$$\begin{aligned} &\int_{\{\hat{R} \leq |x-y_n| \leq 2R_n\}} |u_n|^{p^*} dx \\ &\leq \int_{\mathbb{R}^N} |\eta_n u_n|^{p^*} dx \\ &\leq C(N) \left( \int_{\mathbb{R}^N} |\nabla(\eta_n u_n)|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
&\leq C(N) \left[ \int_{\{\frac{\hat{R}}{2} \leq |x-y_n| \leq 3R_n\}} |\nabla \eta_n|^p |u_n|^p dx + \int_{\{\frac{\hat{R}}{2} \leq |x-y_n| \leq 3R_n\}} |\nabla u_n|^p dx \right]^{\frac{p^*}{p}} \\
&\leq C(N) \left[ \varepsilon + \left( \int_{\{\frac{\hat{R}}{2} \leq |x-y_n| \leq 3R_n\}} |\nabla \eta_n|^{\frac{\alpha p}{\alpha-p}} \right)^{\frac{\alpha-p}{\alpha}} \left( \int_{\{\frac{\hat{R}}{2} \leq |x-y_n| \leq 3R_n\}} |u_n|^\alpha \right)^{\frac{p}{\alpha}} \right]^{\frac{p^*}{p}} \\
&\leq C(N) \left[ \varepsilon + \bar{R}^{N(\frac{\alpha-p}{\alpha})-p} \left( \int_{\{\frac{\hat{R}}{2} \leq |x-y_n| \leq 3R_n\}} |u_n|^\alpha \right)^{\frac{p}{\alpha}} \right]^{\frac{p^*}{p}}.
\end{aligned}$$

Since  $N(\frac{\alpha-p}{\alpha}) - p < 0$ , we get (27) from (26).

Next, let us take  $\xi \in C_0^\infty(\mathbb{R}^N)$  such that  $\xi(x) = 1$ , if  $|x| \leq 1$ ,  $\xi(x) = 0$  if  $|x| \geq 2$ , as well as  $\varphi(x) = 1 - \xi(x)$ . Set

$$u_n^1 := \xi\left(\frac{\cdot - y_n}{\hat{R}}\right) u_n := \xi_n u_n, \quad u_n^2 := \varphi\left(\frac{\cdot - y_n}{R_n}\right) u_n := \varphi_n u_n.$$

If  $\{y_n\}$  is bounded, then we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}^N} h(x) |u_n|^{p-2} u_n^2 (u_n^2 - u_n) dx \right| \\
& \leq \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} h(x) |u_n|^{p-2} |\varphi_n u_n| |\xi_n u_n| \\
& \leq C \varepsilon^{\frac{1}{\alpha}}.
\end{aligned}$$

In particular, letting  $\mu(\varepsilon)$  denote a function which goes to zero as  $\varepsilon$  goes to zero, we have

$$\int_{\mathbb{R}^N} h(x) |u_n|^{p-2} u_n u_n^2 dx = \int_{\mathbb{R}^N} h(x) |u_n|^{p-2} |u_n^2|^2 dx + \mu(\varepsilon). \quad (28)$$

Similarly

$$\begin{aligned}
& \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n^2 (\nabla u_n^2 - \nabla u_n) dx \right| \\
& \leq \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla u_n|^{p-2} |u_n \nabla \varphi_n + \varphi_n \nabla u_n| |u_n \nabla \varphi_n - \xi_n \nabla u_n| dx \\
& \leq \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla u_n|^p dx + C \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla u_n|^{p-2} u_n^2 |\nabla \varphi_n|^2 dx \\
& + C \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla u_n|^{p-2} u_n^2 |\varphi_n|^2 dx + C \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla u_n|^p |\varphi_n|^2 dx \\
& + \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla u_n|^{p-2} u_n^2 |\nabla \varphi_n|^2 dx \\
& \leq C\varepsilon + C \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla u_n|^{p-2} u_n^2 |\nabla \varphi_n|^2 dx \\
& \leq C\varepsilon + C \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla u_n|^p dx + C \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |u_n|^p |\nabla \varphi_n|^p dx \\
& \leq C\varepsilon + C \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |u_n|^p |\nabla \varphi_n|^p dx.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} & \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |u_n|^p |\nabla \varphi_n|^p \\ & \leq \left( \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |u_n|^\alpha \right)^{\frac{p}{\alpha}} \left( \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla \varphi_n|^{\frac{\alpha p}{\alpha-p}} \right)^{1-\frac{p}{\alpha}}. \end{aligned}$$

Since  $|\nabla \varphi_n| \leq \frac{C}{R_n}$ ,

$$\left( \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |\nabla \varphi_n|^{\frac{\alpha p}{\alpha-p}} \right)^{1-\frac{p}{\alpha}} \leq \left( \frac{C}{R_n} \right)^p C(R_n^N)^{\frac{\alpha-p}{\alpha}} = CR_n^{\frac{N(\alpha-p)}{\alpha}-p}.$$

We know  $\frac{N(\alpha-p)}{\alpha} - p < 0$  and  $R_n > 2\bar{R}$ , then

$$\int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |u_n|^p |\nabla \varphi_n|^p \leq C \left( \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |u_n|^\alpha \right)^{\frac{p}{\alpha}} \leq C\varepsilon.$$

Consequently,

$$\left| \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n^2 (\nabla u_n^2 - \nabla u_n) dx \right| \leq \mu(\varepsilon) \tag{29}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla u_n^2 dx = \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} |\nabla u_n^2|^2 dx + \mu(\varepsilon). \tag{30}$$

On the other hand, in view of (26), (27) and (3),

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} a(x)(g(u_n) - g(u_n^2))u_n^2 dx \right| \\ & \leq C_1 \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |u_n|^\alpha dx + C_2 \int_{\{R_n \leq |x-y_n| \leq 2R_n\}} |u_n|^\beta dx \\ & \leq \mu(\varepsilon). \end{aligned}$$

Since  $\{u_n\}$  is a bounded  $(PS)_c$ -sequence,

$$\begin{aligned} o(1) & = \langle I_\lambda'(u_n), u_n^2 \rangle \\ & = \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} |\nabla u_n^2|^2 dx - \lambda \int_{\mathbb{R}^N} h(x) |u_n|^{p-2} |u_n^2|^2 dx \\ & \quad - \int_{\mathbb{R}^N} a(x)g(u_n^2)u_n^2 dx + \mu(\varepsilon). \end{aligned}$$

If  $\{y_n\}$  is bounded, then  $u_n^2 \rightarrow 0$  a.e in  $\mathbb{R}^N$  and

$$\int_{\mathbb{R}^N} h(x) |u_n|^{p-2} |u_n^2|^2 dx = \int_{\mathbb{R}^N} h(x) |u_n^2|^p dx \rightarrow 0.$$

Therefore,

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} |\nabla u_n^2|^2 dx - \int_{\mathbb{R}^N} a(x)g(u_n^2)u_n^2 dx = o(1) + \mu(\varepsilon),$$

$$\int_{\mathbb{R}^N} |\nabla u_n^2|^p dx - \int_{\mathbb{R}^N} a(x)g(u_n^2)u_n^2 dx = o(1) + \mu(\varepsilon).$$

Both  $\int_{\mathbb{R}^N} |\nabla u_n^2|^p dx$  and  $-\int_{\mathbb{R}^N} a(x)g(u_n^2)u_n^2 dx$  are non-negative, thus

$$\int_{\mathbb{R}^N} |\nabla u_n^2|^p dx = o(1) + \mu(\varepsilon), \quad -\int_{\mathbb{R}^N} a(x)g(u_n^2)u_n^2 dx = o(1) + \mu(\varepsilon).$$

By Lemma 1

$$|u_n^2|^\alpha \leq C(|\nabla u_n^2|^p + |\nabla u_n^2|_p^{p^*} - \int_{\mathbb{R}^N} a(x)g(u_n^2)u_n^2 dx),$$

it results from (25) and (26) that

$$\int_{\mathbb{R}^N} |\nabla u_n^2|^p dx + \int_{\mathbb{R}^N} |u_n^2|^\alpha dx = o(1) + \mu(\varepsilon).$$

However,

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u_n^2|^p dx + \int_{\mathbb{R}^N} |u_n^2|^\alpha dx \\ & \geq \int_{\{|x-y_n| \geq 3R_n\}} (|\nabla u_n^2|^p + |u_n^2|^\alpha) dx \\ & = \int_{\{|x-y_n| \geq 3R_n\}} (|\nabla u_n|^p + |u_n|^\alpha) dx + o(1) + \mu(\varepsilon) \\ & \geq \bar{\lambda} - \alpha - \varepsilon > 0, \end{aligned}$$

which is a contradiction.

If  $\{y_n\}$  is unbounded, we may also get  $u_n^1 \rightarrow 0$  a.e in  $\mathbb{R}^N$ . Applying the same arguments above to  $u_n^1$ , we may obtain a contradiction.

Finally, we show the compactness. We have ruled out vanishing and dichotomy, the case tight necessarily happens, i.e for any  $\varepsilon > 0$ , there exists  $R > 0$  such that

$$\int_{\{y_n + B_R\}} (|\nabla u_n|^p + |u_n|^\alpha) dx \geq \bar{\lambda} - \varepsilon.$$

Especially,

$$\int_{\{|x-y_n| \geq R\}} (|\nabla u_n|^p + |u_n|^\alpha) dx < \varepsilon. \quad (31)$$

First of all, we claim that  $\{y_n\}$  remains bounded. In fact, if  $\{y_n\}$  is not bounded, by (31) we obtain

$$u_n \rightharpoonup 0 \text{ in } D^{1,p}(\mathbb{R}^N).$$

Hence

$$o(1) = \langle I'(u_n), u_n \rangle = \int_{\mathbb{R}^N} |\nabla u_n|^p dx - \int_{\mathbb{R}^N} a(x)g(u_n)u_n dx + o(1).$$

Since

$$\begin{aligned} & \left| \int_{\{|x-y_n| \geq R\}} a(x)g(u_n)u_n \, dx \right| \\ & \leq C_1 \left| \int_{\{|x-y_n| \geq R\}} |u_n|^\alpha \, dx \right| + C_2 \left| \int_{\{|x-y_n| \geq R\}} |u_n|^{p^*} \, dx \right| \\ & \leq C_1 \varepsilon + C_3 \left( \int_{\{|x-y_n| \geq R\}} |\nabla u_n|^p \right)^{\frac{p^*}{p}} \\ & \leq \mu(\varepsilon), \end{aligned}$$

we have

$$o(1) = \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx - \int_{\{|x-y_n| \leq R\}} a(x)g(u_n)u_n \, dx + o(1) + \mu(\varepsilon).$$

However, for  $n$  large,  $a(x) < -1$  if  $|x - y_n| \leq R$ , thus

$$\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx = o(1) + \mu(\varepsilon), \quad - \int_{\mathbb{R}^N} a(x)g(u_n)u_n \, dx = o(1) + \mu(\varepsilon).$$

By Lemma 1,

$$\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx + \int_{\mathbb{R}^N} |u_n|^\alpha \, dx = o(1) + \mu(\varepsilon),$$

which is a contradiction with  $\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx + \int_{\mathbb{R}^N} |u_n|^\alpha \, dx \rightarrow \bar{\lambda} > 0$ . Hence,  $\{y_n\}$  is bounded. So for  $R$  sufficiently large, we have

$$\int_{\{|x| \geq R\}} (|\nabla u_n|^p + |u_n|^\alpha) \, dx < \varepsilon. \tag{32}$$

Assume  $u_n \rightharpoonup u$  in  $E$ , by (32),

$$\begin{aligned} & \int_{\mathbb{R}^N} |u_n - u|^\alpha \, dx \\ & = \int_{\{|x| \leq R\}} |u_n - u|^\alpha \, dx + \int_{\{|x| \geq R\}} |u_n - u|^\alpha \, dx \\ & = o(1) + C_1 \int_{\{|x| \geq R\}} |u_n|^\alpha \, dx + C_2 \int_{\{|x| \geq R\}} |u|^\alpha \, dx. \\ & \leq o(1) + \mu(\varepsilon). \end{aligned}$$

It implies  $u_n \rightarrow u$  in  $L^\alpha(\mathbb{R}^N)$ . Lemma 4 then gives that  $\|u_n - u\| \rightarrow 0$ .

### 3 Existence of positive solutions

To show problem  $(P_\lambda)$  possesses a solution, we shall verify that the functional  $I_\lambda$  has the mountain pass geometry.

We recall an element inequality in [5].

**Lemma 5** *Assume  $1 \leq q \leq 3$ , then there is a constant  $C(q)$ , such that for all  $\alpha, \beta \in R$ , we have*

$$||\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2})| \leq \begin{cases} C|\alpha||\beta|^{q-1}, & \text{if } |\alpha| \geq |\beta|, \\ C|\alpha|^{q-1}|\beta|, & \text{if } |\alpha| \leq |\beta|. \end{cases} \quad (33)$$

On the other hand, if  $q \geq 3$ , we have

$$||\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2})| \leq C(|\alpha|^{q-2}\beta^2 + \alpha^2|\beta|^{q-2}).$$

**Proposition 2** *Assume  $(H1)$ ,  $(A1)$ ,  $(A2)$ ,  $(G1) - (G3)$ ,  $\lambda \notin \sigma(-\Delta_p, \Omega^0, h(x))$  and  $\bar{\Omega}^+ \cap \bar{\Omega}^- = \emptyset$ , then the functional  $I_\lambda$  satisfies*

(a) *If  $\lambda < \lambda_1(h)$ , then  $u = 0$  is a local minimum of  $I_\lambda$ .*

(b) *If  $\int_{R^N} a(x)\varphi_1^\alpha dx < 0$ , then there exists  $\bar{\alpha} > 0$  such that for  $\lambda_1(h) \leq \lambda < \lambda_1(h) + \bar{\alpha}$ , there exists  $e \in E$  such that  $I_\lambda(e) < 0$ .*

*Proof.* (a). Since  $\lambda < \lambda_1(h)$  and

$$\int_{R^N} |\nabla u|^p dx \geq \lambda_1 \int_{R^N} h(x)|u|^p dx,$$

for  $R$  large so that  $a(x) < 0$  if  $|x| \geq R$ , we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p} \left[ 1 - \frac{\lambda}{\lambda_1} \right] \int_{R^N} |\nabla u|^p dx - \int_{R^N} a(x)G(u) dx \\ &\geq C \int_{R^N} |\nabla u|^p dx + \int_{\{|x| \geq R\}} G(u) dx - \int_{\{|x| \leq R\}} a(x)G(u) dx \\ &\geq C \int_{R^N} |\nabla u|^p dx - C \int_{\{|x| \leq R\}} (|u|^\alpha + |u|^\beta) dx \\ &\geq C|\nabla u|_p^p - C|\nabla u|_p^\alpha - C|\nabla u|_p^\beta. \end{aligned}$$

It implies that 0 is a local minimum of  $I_\lambda$ .

(b). Write  $u \in E$  as  $u = t\varphi_1 + v$ , where  $v$  and  $\varphi_1$  are linear independent. Let us take  $R_0 > 0$  large enough so that  $a(x) < -1$  if  $|x| \geq R_0$  and  $\int_{|x| \leq R_0} a(x)\varphi_1^\alpha dx < 0$ . We consider the case  $p \geq 3$ , the case  $1 \leq p \leq 3$  can be proved in the same way. We first assume  $\lambda = \lambda_1(h)$  and estimate  $I_\lambda(u)$  for  $\|u\|$  small. By Lemma 5,

$$I_{\lambda_1}(u)$$

$$\begin{aligned}
 &= \frac{1}{p} \int_{\mathbb{R}^N} |t \nabla \varphi_1 + \nabla v|^p dx - \frac{\lambda_1}{p} \int_{\mathbb{R}^N} h(x) |t \varphi_1 + v|^p dx - \int_{\mathbb{R}^N} a(x) G(u) dx \\
 &\geq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla v|^p dx - \frac{\lambda_1}{p} \int_{\mathbb{R}^N} h(x) |v|^p dx \\
 &+ \int_{\mathbb{R}^N} t \nabla \varphi_1 \nabla v (|t|^{p-2} |\nabla \varphi_1|^{p-2} + |\nabla v|^{p-2}) dx \\
 &- \int_{\mathbb{R}^N} \lambda_1 h(x) t \varphi_1 v (|t|^{p-2} |\varphi_1|^{p-2} + |v|^{p-2}) dx \\
 &- \frac{C}{p} \int_{\mathbb{R}^N} (|t|^{p-2} |\nabla \varphi_1|^{p-2} |\nabla v|^2 + |t|^2 |\nabla \varphi_1|^2 |\nabla v|^{p-2}) dx \\
 &- \frac{C}{p} \int_{\mathbb{R}^N} \lambda_1 (|t|^{p-2} |\varphi_1|^{p-2} |v|^2 + \lambda_1 |t|^2 |\varphi_1|^2 |v|^{p-2}) dx \\
 &- \int_{\mathbb{R}^N} a(x) G(u) dx \\
 &\geq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla v|^p dx - \frac{\lambda_1}{p} \frac{1}{\lambda_2} \int_{\mathbb{R}^N} |\nabla v|^p dx \\
 &- \int_{\mathbb{R}^N} [|t|^{p-1} |\nabla \varphi_1|^{p-1} |\nabla v| + |t| |\nabla \varphi_1| |\nabla v|^{p-1}] dx \\
 &- \int_{\mathbb{R}^N} [\lambda_1 h(x) |t|^{p-1} \varphi_1^{p-1} |v| + \lambda_1 h(x) |t| |\varphi_1| |v|^{p-1}] dx \\
 &- \frac{C}{p} \int_{\mathbb{R}^N} [|t|^{p-2} |\nabla \varphi_1|^{p-2} |\nabla v|^2 + |t|^2 |\nabla \varphi_1|^2 |\nabla v|^{p-2}] dx \\
 &- \frac{C}{p} \int_{\mathbb{R}^N} [\lambda_1 |t|^{p-2} \varphi_1^{p-2} v^2 + \lambda_1 |t|^2 \varphi_1^2 |v|^{p-2}] dx \\
 &- \int_{\mathbb{R}^N} a(x) G(u) dx,
 \end{aligned}$$

where

$$\lambda_1 = \inf_{0 \neq v \in E} \frac{\int_{\mathbb{R}^N} |\nabla v|^p}{\int_{\mathbb{R}^N} h(x) |v|^p}, \quad \lambda_2 = \inf_{0 \neq v \in E \setminus \{t \varphi_1\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^p}{\int_{\mathbb{R}^N} h(x) |v|^p}.$$

We note  $\lambda_1 < \lambda_2$ . Indeed, if  $\lambda_1 = \lambda_2$ , there exists  $\tilde{v} \in E \setminus \{t \varphi_1\}$  such that

$$-\Delta_p \tilde{v} = \lambda_2 h(x) |\tilde{v}|^{p-2} \tilde{v} = \lambda_1 h(x) |\tilde{v}|^{p-2} \tilde{v}.$$

This is a contradiction with  $\lambda_1$  is simple. Hence

$$\begin{aligned}
 &I_{\lambda_1}(u) \\
 &\geq C \int_{\mathbb{R}^N} |\nabla v|^p dx - \int_{\mathbb{R}^N} [|t|^{p-1} |\nabla \varphi_1|^{p-1} |\nabla v| + |t| |\nabla \varphi_1| |\nabla v|^{p-1} \\
 &+ \lambda_1 h(x) |t|^{p-1} \varphi_1^{p-1} |v| + \lambda_1 h(x) |t| |\varphi_1| |v|^{p-1}] dx
 \end{aligned}$$

$$\begin{aligned}
& -\frac{C}{p} \int_{\mathbb{R}^N} [|t|^{p-2} |\nabla \varphi_1|^{p-2} |\nabla v|^2 + |t|^2 |\nabla \varphi_1|^2 |\nabla v|^{p-2} \\
& + \lambda_1 |t|^{p-2} \varphi_1^{p-2} v^2] dx + \lambda_1 |t|^2 \varphi_1^2 |v|^{p-2} - \int_{\mathbb{R}^N} a(x) G(u) dx \\
& = \text{I} + \text{II} + \text{III} + \text{IV}
\end{aligned}$$

Now, we estimate the II and III. Let  $\tau = |\nabla v|_p$ .

$$\begin{aligned}
& \int_{\mathbb{R}^N} |t|^{p-1} |\nabla \varphi_1|^{p-1} |\nabla v| dx \\
& \leq \left( \int_{\mathbb{R}^N} |\nabla v|^p \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} [|t|^{p-1} |\nabla \varphi_1|^{p-1}]^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \\
& = \left( \int_{\mathbb{R}^N} |\nabla v|^p \right)^{\frac{1}{p}} |t|^{p-1} \left( \int_{\mathbb{R}^N} |\nabla \varphi_1|^p \right)^{\frac{p-1}{p}} \\
& = C\tau |t|^{p-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_{\mathbb{R}^N} |t| |\nabla \varphi_1| |\nabla v|^{p-1} dx, \quad \int_{\mathbb{R}^N} \lambda_1 h(x) |t| |\varphi_1| |v|^{p-1} dx \leq C\tau^{p-1} |t|, \\
& \int_{\mathbb{R}^N} \lambda_1 h(x) |t|^{p-1} |\varphi_1|^{p-1} |v| \leq C\tau |t|^{p-1}.
\end{aligned}$$

IV can be estimated similarly. As a result,

$$\begin{aligned}
& I_{\lambda_1}(u) \\
& \geq C\tau^p - (C_1\tau |t|^{p-1} + C_2\tau^{p-1} |t| + C_3\tau^2 |t|^{p-2} + C_4\tau^{p-2} t^2) - \int_{\mathbb{R}^N} a(x) G(u) dx.
\end{aligned}$$

Since

$$\begin{aligned}
& - \int_{\mathbb{R}^N} a(x) G(u) dx \\
& = -\frac{1}{\alpha} \int_{\{|x| \leq R_0\}} a(x) |t\varphi_1|^\alpha dx - \int_{\{|x| \leq R_0\}} a(x) \left[ G(t\varphi_1) - \frac{|t\varphi_1|^\alpha}{\alpha} \right] dx \\
& - \int_{\{|x| \leq R_0\}} a(x) [G(t\varphi_1 + v) - G(t\varphi_1)] dx - \int_{\{|x| \geq R_0\}} a(x) G(u) dx \\
& \geq - \int_{\{|x| \leq R_0\}} a(x) \left[ G(t\varphi_1) - \frac{|t\varphi_1|^\alpha}{\alpha} \right] dx - \\
& \int_{\{|x| \leq R_0\}} a(x) [G(t\varphi_1 + v) - G(t\varphi_1)] dx - \int_{\{|x| \geq R_0\}} a(x) G(u) dx.
\end{aligned}$$

and

$$\begin{aligned} & \int_{\{|x| \leq R_0\}} a(x)[G(t\varphi_1 + v) - G(t\varphi_1)] dx \\ & \leq C \int_{\{|x| \leq R_0\}} a(x)|g(t\varphi_1 + \theta v)|v dx \\ & \leq C \left[ \int_{\{|x| \leq R_0\}} |t\varphi_1 + \theta v|^{\alpha-1} v dx + \int_{\{|x| \leq R_0\}} |t\varphi_1 + \theta v|^{\beta-1} v dx \right] \\ & \leq C [|t|^{\alpha-1} \left( \int_{\{|x| \leq R_0\}} |v|^\alpha dx \right)^{\frac{1}{\alpha}} + |t|^{\beta-1} \left( \int_{\{|x| \leq R_0\}} |v|^\beta dx \right)^{\frac{1}{\beta}}] \\ & \quad + \int_{\{|x| \leq R_0\}} |v|^\alpha dx + \int_{\{|x| \leq R_0\}} |v|^\beta dx \\ & \leq C [|t|^{\alpha-1} |\nabla v|_p + |t|^{\beta-1} |\nabla v|_p + |\nabla v|_p^\alpha + |\nabla v|_p^\beta]. \end{aligned}$$

For  $|t|$  and  $|\nabla v|_p$  small, we have

$$\begin{aligned} & \int_{\{|x| \leq R_0\}} a(x)[G(t\varphi_1 + v) - G(t\varphi_1)] dx \\ & \leq C [|t|^{\alpha-1} |\nabla v|_p + |\nabla v|_p^\alpha] = C [\tau |t|^{\alpha-1} + \tau^\alpha]. \end{aligned}$$

Now we estimate I. By  $(G_1)$ ,  $(G_2)$  and Lebesgue's Dominated Convergence Theorem, we have

$$\left| \int_{\{|x| \leq R_0\}} a(x) \left[ G(t\varphi_1) - \frac{|t\varphi_1|^\alpha}{\alpha} \right] dx \right| = o(1) |t|^\alpha \text{ as } t \rightarrow 0.$$

Hence for  $|t|$  sufficiently small,

$$I_{\lambda_1}(u) \geq J(t, |\nabla v|_p) + \int_{\{|x| \geq R_0\}} G(u) dx,$$

where

$$J(t, \tau) = a\tau^p - b\tau^\alpha + c|t|^\alpha - d|t|^{\alpha-1}\tau - e\tau|t|^{p-1} - f\tau^{p-1}|t| - g\tau^2|t|^{p-2} - h\tau^{p-2}|t|^2.$$

Consider  $K(t, \tau) = C|t|^\alpha - e\tau|t|^{p-1}$ , we may verify that

$$\min_{|t| \text{ small}} K(t, \tau) = -C|t|^\alpha < 0.$$

Similar results hold for  $K_1(t, \tau) = C|t|^\alpha - f\tau^{p-1}|t|$ ,  $K_2(t, \tau) = C|t|^\alpha - g\tau^2|t|^{p-2}$ ,  $K_3(t, \tau) = C|t|^\alpha - h\tau^{p-2}|t|^2$ . So  $J(t, \tau) \geq C_0\delta^p$  if  $|t|$  and  $|\nabla v|_p$  are sufficiently small. Consider on the space E the equivalent norm

$$\|u\|_e = (\max\{|t|, |\nabla v|_p\}^p + |u|_\alpha^p)^{\frac{1}{p}} = (\|u\|^p + |u|_\alpha^p)^{\frac{1}{p}}.$$

For some  $C_0 > 0$ , and  $\|u\| = \delta$ , we have  $I_{\lambda_1}(u) \geq C_0\delta^p + \int_{\{|x| \geq R_0\}} G(u) dx$ . We claim  $\lim_{\|u\|=\rho, \rho \rightarrow 0} \frac{\int_{\{|x| \geq R_0\}} G(u) dx}{\|u\|^\alpha} \geq d_0\rho^r$  for some  $d_0 > 0$  and  $r \geq 0$ . Indeed, let  $\|u\|_e = \rho$ . If  $\|u\| = \delta \rightarrow 0$ , then  $|u|_\alpha \rightarrow \rho$  and

$$\int_{\{|x| \leq R_0\}} |u|^\alpha dx \leq C(R_0) \left( \int_{\{|x| \leq R_0\}} |u|^{p^*} dx \right)^{\frac{\alpha}{p^*}} \leq C(R_0) |\nabla u|_p^\alpha \leq C(R_0) \|u\|^\alpha \rightarrow 0,$$

it implies

$$\int_{\{|x| \geq R_0\}} |u|^\alpha dx = \int_{\mathbb{R}^N} |u|^\alpha dx + o(1) \rightarrow \rho^\alpha.$$

By (4) we obtain

$$\int_{\{|x| \geq R_0\}} G(u) dx \geq C \int_{\{|x| \geq R_0\}} |u|^\alpha dx \geq C\rho^\alpha.$$

Thus,  $I_{\lambda_1}(u) \geq \alpha(\rho) > 0$  for  $\|u\| = \rho$  small and

$$|I_\lambda(u) - I_\mu(u)| = |\lambda - \mu| \int_{\mathbb{R}^N} h(x)|u|^p dx,$$

which implies that there exists  $\bar{\alpha} > 0$  such that

$$I_\lambda(u) \geq \alpha(\rho) > 0, \quad \lambda_1 \leq \lambda < \lambda_1 + \bar{\alpha}.$$

## 4 Proof of Theorem 1

We know from Proposition 2 that  $u = 0$  is a local minimum of  $I_\lambda$  if  $\lambda \leq \lambda_1(h)$ , and there exists  $\rho_0 > 0$  and  $\bar{\alpha} > 0$  such that

$$I_\lambda(u) \geq \alpha(\rho_0) > 0 \quad \text{if } \|u\| = \rho_0, \quad \lambda < \lambda_1(h) + \bar{\alpha}.$$

Now, choose  $\varphi \in C_0^\infty(\Omega^+)$ ,  $0 \leq \varphi \leq 1$   $\varphi(x) = 1$  if  $x \in \bar{B}_r(x_0) \subset \Omega^+$  for some  $r > 0$  and  $x_0 \in \Omega^+$ . Assumption (G2) implies that

$$\int_{\mathbb{R}^N} a(x)G(t\varphi) dx \geq \int_{B_r(x_0)} a(x)G(t\varphi) dx \geq Ct^\beta,$$

it follows  $I_\lambda(t\varphi) \leq C_1t^p - C_2t^\beta < 0$  for all  $t$  sufficiently large. By the Mountain-Pass Theorem, there exists a critical point  $u_\lambda$  of  $I_\lambda$ . Since  $I_\lambda(u) = I_\lambda(|u|)$ , we may assume the  $u_\lambda$  is positive critical point of  $I_\lambda$ .

Next we prove the existence of a second positive solution for the case  $\lambda_1(h) < \lambda < \lambda_1(h) + \bar{\alpha}$  for some  $0 < \bar{\alpha} \leq \bar{\alpha}$ . From the proof of Proposition 2, we see that there exists

an  $\tilde{\alpha} > 0$  so that for  $\lambda_1(h) < \lambda < \lambda_1(h) + \tilde{\alpha}$ , we may find  $\rho(\lambda), C(\lambda) > 0$  with  $\rho(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \lambda_1(h)$  and  $I_\lambda(u) \geq C(\lambda) > 0$  on  $\|u\| = \rho(\lambda)$ . Let

$$m_\lambda := \inf_{\|u\| \leq \rho(\lambda)} I_\lambda(u).$$

$m_\lambda$  is attained in the open ball  $B_\rho(\lambda) = \{u \mid \|u\| < \rho(\lambda)\}$ . Actually,  $m_\lambda < 0$ . Indeed, let  $\xi$  be a function such that  $0 \leq \xi \in C_0^\infty(R)$  such that  $\xi(x) = 1$  if  $|x| \leq 1$ ;  $\xi(x) = 0$  if  $|x| \geq 2$ . Let  $\xi_R(x) = \xi(\frac{x}{R})$ . Multiplying (1) and integrating by parts, we have

$$\int_{\mathbb{R}^N} |\nabla \varphi_1|^p \xi_R^p dx + p \int_{\mathbb{R}^N} |\nabla \varphi_1|^{p-2} \nabla \varphi_1 \varphi_1 \xi_R^{p-1} \nabla \xi_R dx = \int_{\mathbb{R}^N} \lambda_1 h(x) \varphi_1^p \xi_R^p dx, \quad (34)$$

this gives

$$\begin{aligned} & I_\lambda(t\xi_R\varphi_1) \\ & \leq \frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla \varphi_1|^p \xi_R^p dx + \frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla \xi_R|^p \varphi_1^p dx \\ & + t^p \int_{\mathbb{R}^N} \nabla \xi_R \nabla \varphi_1 \xi_R |\nabla \xi_R|^{p-2} \varphi_1^{p-1} dx + t^p \int_{\mathbb{R}^N} \nabla \xi_R \nabla \varphi_1 \varphi_1 \xi_R^{p-1} |\nabla \varphi_1|^{p-2} dx \\ & + \frac{t^p}{p} C \int_{\mathbb{R}^N} [|\nabla \xi_R|^{p-2} \varphi_1^{p-2} \xi_R^2 |\nabla \varphi_1|^2 + |\nabla \xi_R|^2 \varphi_1^2 \xi_R^{p-2} |\nabla \varphi_1|^{p-2}] dx \\ & - \lambda \frac{t^p}{p} \int_{\mathbb{R}^N} h(x) \xi_R^p \varphi_1^p dx - \int_{\mathbb{R}^N} a(x) G(t\xi_R\varphi_1) dx \\ & \leq \frac{t^p}{p} \left[ \int_{\mathbb{R}^N} \lambda_1 h \varphi_1^p \xi_R^p dx - p \int_{\mathbb{R}^N} |\nabla \varphi_1|^{p-2} \nabla \varphi_1 \varphi_1 \xi_R^{p-1} \nabla \xi_R dx \right] \\ & + \frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla \xi_R|^p \varphi_1^p dx + t^p \int_{\mathbb{R}^N} \nabla \xi_R \nabla \varphi_1 \xi_R |\nabla \xi_R|^{p-2} \varphi_1^{p-1} dx \\ & + t^p \int_{\mathbb{R}^N} \nabla \xi_R \nabla \varphi_1 \varphi_1 \xi_R^{p-1} |\nabla \varphi_1|^{p-2} dx \\ & + \frac{t^p}{p} C \int_{\mathbb{R}^N} (|\nabla \xi_R|^{p-2} \varphi_1^{p-2} \xi_R^2 |\nabla \varphi_1|^2 + |\nabla \xi_R|^2 \varphi_1^2 \xi_R^{p-2} |\nabla \varphi_1|^{p-2}) dx \\ & - \lambda \frac{t^p}{p} \int_{\mathbb{R}^N} h(x) \xi_R \varphi_1^p dx - \int_{\mathbb{R}^N} a(x) G(t\xi_R\varphi_1) dx \\ & = \frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla \xi_R|^p \varphi_1^p dx + \frac{t^p}{p} (\lambda_1 - \lambda) \int_{\mathbb{R}^N} h(x) \varphi_1^p \xi_R^p dx \\ & + t^p \int_{\mathbb{R}^N} \nabla \xi_R \nabla \varphi_1 \xi_R |\nabla \xi_R|^{p-2} \varphi_1^{p-1} dx \\ & + \frac{t^p}{p} C \int_{\mathbb{R}^N} (|\nabla \xi_R|^{p-2} \varphi_1^{p-2} \xi_R^2 |\nabla \varphi_1|^2 + |\nabla \xi_R|^2 \varphi_1^2 \xi_R^{p-2} |\nabla \varphi_1|^{p-2}) dx \\ & - \int_{\mathbb{R}^N} a(x) G(t\xi_R\varphi_1) dx \\ & = A_1 + A_2 + A_3 + A_4 + A_5 \end{aligned}$$

By Hölder's inequality,

$$A_1 = \frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla \xi_R|^p \varphi_1^p \leq C \left( \int_{\{R \leq |x| \leq 2R\}} \varphi_1^{p^*} \right)^{\frac{p}{p^*}},$$

where  $C$  is independent of  $R$ . So  $A_1 \rightarrow 0$  if  $R \rightarrow \infty$ . Similarly,  $A_3, A_4 \rightarrow 0$ . Therefore,

$$I_\lambda(t\xi_{R_1}\varphi_1) \leq \left[ \frac{\lambda_1 - \lambda}{2p} \int_{\mathbb{R}^N} h(x)\xi_R^p \varphi_1^p dx \right] t^p - \int_{\mathbb{R}^N} a(x)G(t\xi_R\varphi_1) dx + o(1).$$

Using (G1)-(G3), we obtain

$$I_\lambda(t\xi_{R_1}\varphi_1) \leq -at^p + bt^\alpha + ct^\beta,$$

it implies  $I_\lambda(t\xi_R\varphi_1) < 0$  for  $t > 0$  small. Thus  $m_\lambda = \inf_{\|u\| \leq \rho(\lambda)} I_\lambda(u) < 0$ .

Finally, if  $\tilde{v}_n$  is a minimizing sequence for  $m_\lambda$ , by Ekeland's Variational Principle, there is a sequence  $\{u_n\}$  such that  $I_\lambda(u_n) \rightarrow m_\lambda, I'_\lambda(u_n) \rightarrow 0$ . Proposition 1 implies that there exists a minimizer  $v_\lambda \in B_{\rho(\lambda)}$  of  $I_\lambda$  which is a critical point of  $I_\lambda$ , and  $I_\lambda(v_\lambda) = m_\lambda < 0$ . By Proposition 2 (b) and Proposition 1, we see that  $I_\lambda$  has the mountain pass structure and satisfies (PS) condition, so there is a mountain pass solution  $u_\lambda$  which is different from  $v_\lambda$ .

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