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Existence of Solutions to Quasilinear Elliptic Equations With Singular Weights

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Abstract

In this paper we show the existence of multiple solutions to a class of quasilinear elliptic equations with singular weights when the continuous nonlinearity satisfies a superlinear condition only at zero. In particular, our approach allows us to consider superlinear, critical and supercritical nonlinearities.

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1 Introduction

We consider the following quasilinear elliptic problem with singular weights

$$(P)_\lambda \qquad \begin{cases} -div(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda |x|^{-(a+1)p+c} f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^1 boundary, $0 \in \Omega$, $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $c > 0$, λ is a positive parameter and the function f satisfies the following conditions

(F_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous .

(F_2) There exists $q \in (p, r)$, where

$$r = \min\{Np/(N-p), p(N-(a+1)p+c)/(N-p(a+1))\},$$

such that

$$\lim_{s\to 0} \frac{f(s)}{|s|^{q-2}s} = 1.$$

The weighted function space we seek for solutions is $W_0^{1,p}(\Omega, |x|^{-ap})$ which is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|$ defined by

$$\|u\| = \Big(\int_\Omega |x|^{-ap} |\nabla u|^p \, dx \Big)^{1/p}.$$

Our main result reads as follows.

Theorem 1.1 *Suppose that the function f satisfies (F_1) and (F_2). Let $N(\lambda)$ be the number of solutions of $(P)_\lambda$. Then*

$$\lim_{\lambda\to+\infty} N(\lambda) = +\infty.$$

Degenerate elliptic problems with weights have been intensively studied, starting with the pioneering work of Murthy and Stampacchia [17]. Several references on the subject can be found, for instance in the monograph [12]. In the radial case, i.e., when $u = u(|x|)$, ordinary differential equations methods apply, and many results about existence, non–existence and asymptotic behavior of solutions are available (see for example [6],[7]). Regarding problems with other weights than powers of $|x|$, see [2],[9],[18].

In the non–radial case, progress has also been made. The equation $(P)_\lambda$ for $p = 2$, that is, the weighted Laplacian case where the nonlinearity f is a power of u on $\mathbb{R}^N$, is studied by Catrina and Wang in [4]. They obtained the existence of solutions within a prescribed symmetry group. Subsequently, it was proved by Felli and Schneider in [11] that solutions of Problem $(P)_\lambda$ are Hölder continuous in bounded domains Ω, provided that the nonlinearity has subcritical growth. Abdellaoui and Peral proved in [1] the existence of solutions in the sense of entropy.

They considered equation $(P)_\lambda$ with $1 < p < N$ and f satisfying various structural assumptions. Blow–up phenomenon of the solutions is discussed as well. In Xuan [22] the Mountain Pass Theorem and linking arguments are used to prove existence of a solution of Problem $(P)_\lambda$ for a special type of nonlinearity f. Existence and multiplicity of solutions for an asymptotically linear f is shown in [23]. The eigenvalue problem associated to $(P)_\lambda$, with $1 < p < N$ and $a \geq 0$, is studied in [21] and it is shown that the first eigenvalue is simple and that the first eigenfunctions do not change sign. Iturriaga used variational methods in [13] to study problem $(P)_\lambda$ in a more general form, obtaining regularity, existence and multiplicity results of solutions. However, most of the works mentioned above impose critical or subcritical conditions on the growth of the function f at infinity.

For problems with Laplacian, that is, $p = 2$, $a = 0$ and $c = p$, and where the nonlinearity f satisfy certain conditions only at zero, the ideas of [15] apply and it is possible to show that problem $(P)_\lambda$ has at least one nontrivial, nonnegative C^1 solution for λ large enough. Moreover, Chen and Li [5] show that the number of solutions of the problem $(P)_\lambda$ increase to infinity as λ tends to infinity. Our aim is to extend these above ideas for our singular problem $(P)_\lambda$.

Theorem 1.1 is proved in the next section, where we begin by establishing a space function setting. Then we prove an L^∞ estimate (Lemma 2.2) that allows us to truncate the nonlinearity f and show that a solution of the truncated problem is a genuine solution of $(P)_\lambda$, see Lemma 2.3. In Lemmas 2.4 to 2.8 we develop a minimax argument to generate multiple critical points. Notice that equivariant theory of critical points [19] is not directly applicable to our energy functional corresponding to problem $(P)_\lambda$, but since it is bounded by even functionals, according to assumption (F_2), we are able to overcome this difficulty.

2 Proof of our main result

The following integral inequality due to Caffarelli, Kohn and Nirenberg [3] plays a central role in our variational approach to equation $(P)_\lambda$.

$$\Big(\int_{\mathbb{R}^N} |x|^{-bq}|u|^q\,dx\Big)^{p/q} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx \tag{2.1}$$

where

$$\begin{aligned} &-\infty < a < \frac{N-p}{p}, \qquad \text{for } a \leq b \leq a+1,\\ &q = p^*(a,b) = \frac{Np}{N-dp}, \qquad \text{for } d = 1+a-b. \end{aligned} \tag{2.2}$$

It follows from the boundedness of Ω and a standard approximation argument that, for any $u \in W_0^{1,p}(\Omega, |x|^{-ap})$, inequality (2.1) holds, in the sense that, for $1 \leq r \leq \dfrac{Np}{N-p}$ and $\alpha \leq (1+a)r + N(1-\frac{r}{p})$, we have

$$\Big(\int_\Omega |x|^{-\alpha}|u|^r\,dx\Big)^{p/r} \leq C\int_\Omega |x|^{-ap}|\nabla u|^p\,dx \tag{2.3}$$

in other words, the embedding $W_0^{1,p}(\Omega,|x|^{-ap}) \hookrightarrow L^r(\Omega,|x|^{-\alpha})$ is continuous, where $L^r(\Omega,|x|^{-\alpha})$ is endowed with the norm

$$\|u\|_{r,\alpha} := \|u\|_{L^r(\Omega,|x|^{-\alpha})} = \Big(\int_\Omega |x|^{-\alpha}|u|^r\,dx\Big)^{1/r}.$$

We will need the following compactness imbedding from [22], [23].

Lemma 2.1 *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and that $0 \in \Omega$, where $1 < p < N$, $-\infty < a < (N-p)/p$, $1 \le l < Np/(N-p)$ and $\alpha < (1+a)l + N(1-(l/p))$. Then the embedding $W_0^{1,p}(\Omega,|x|^{-ap}) \hookrightarrow L^l(\Omega,|x|^{-\alpha})$ is compact.*

We need an a priori estimate of our solutions, for that matter we use a Moser iterative scheme see for example [10, 16].

Lemma 2.2 *Consider $\varphi \in C(\mathbb{R})$ with $|\varphi(t)| \le L|t|^{q-1}$ for any $t \in \mathbb{R}$ and some positive constant L. Let $u \in W_0^{1,p}(\Omega,|x|^{-ap})$ be a solution of*

$$(P)\qquad \begin{cases} -div(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-(a+1)p+c}\varphi(u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, there exists a positive constant $C_1 = C_1(\Omega,a,p,c,q)$ such that

$$\|u\|_{L^\infty(\Omega)} \le C_1 L^{\frac{1}{r-q}}\|u\|_{r,(a+1)p-c}^{\frac{r-p}{r-q}}.$$

Proof. Under our condition on the function φ, we know that the solutions of the problem belong to $C^{0,\alpha}(\overline{\Omega})$, see for instance [13]. Then, we can take $\operatorname{sign}(u)|u|^{kp+1}$ as a test function. From equation (P), we obtain

$$\begin{aligned} \frac{kp+1}{(k+1)^p}\int_\Omega |x|^{-ap}|\nabla(u^{k+1})|^p &= (kp+1)\int_\Omega |x|^{-ap}|\nabla u|^p|u|^{kp}\\ &= \int_\Omega |x|^{-(a+1)p+c}\operatorname{sign}(u)\varphi(u)|u|^{kp+1} \qquad (2.4)\\ &\le L\int_\Omega |x|^{-(a+1)p+c}|u|^{q+kp} \end{aligned}$$

Using the inequality (2.3), we have

$$C\int_\Omega |x|^{-ap}|\nabla(u^{k+1})|^p \ge \Big(\int_\Omega |x|^{-(a+1)p+c}|u^{k+1}|^r\Big)^{\frac{p}{r}}.$$

The inequality (2.4) reads as,

$$\Big(\int_\Omega |x|^{-(a+1)p+c}|u^{k+1}|^r\Big)^{\frac{p}{r}} \le LC\left(\frac{(k+1)^p}{kp+1}\right)\int_\Omega |x|^{-(a+1)p+c}|u|^{q+kp}.$$

Applying Hölder inequality at the right hand side of last expression, we get

$$\int_\Omega |x|^{-(a+1)p+c}|u|^{q+kp} \leq \left(\int_\Omega |x|^{-(a+1)p+c}|u|^{(k+1)lp}\right)^{\frac{1}{l}} \left(\int_\Omega |x|^{-(a+1)p+c}|u|^{r}\right)^{\frac{q-p}{r}},$$

where $l = r/(r-q+p)$. Combining the last inequalities, we obtain

$$\|u\|_{r(k+1),(a+1)p-c} \leq \left[LC\left(\frac{(k+1)^p}{kp+1}\right)\right]^{\frac{1}{p(k+1)}} \|u\|_{(k+1)pl,(a+1)p-c} \cdot \|u\|_{r,(a+1)p-c}^{\frac{q-p}{p(k+1)}}.$$

We define k_1 in such way that $r(k_1+1)pl = r$. Note that $k_1 + 1 = r/pl = (r-q)/p + 1 > 1$. Then

$$\begin{aligned}\|u\|_{r(k_1+1),(a+1)p-c} &\leq \left[LC\left(\frac{(k_1+1)^p}{k_1p+1}\right)\right]^{\frac{1}{p(k_1+1)}} \|u\|_{r,(a+1)p-c} \cdot \|u\|_{r,(a+1)p-c}^{\frac{q-p}{p(k_1+1)}} \\ &= \left[LC\left(\frac{(k_1+1)^p}{k_1p+1}\right)\|u\|_{r,(a+1)p-c}^{q-p}\right]^{\frac{1}{p(k_1+1)}} \|u\|_{r,(a+1)p-c}.\end{aligned}$$

Define by induction $(k_n+1)pl = r(k_{n-1}+1)$. Then $k_n + 1 = (r/(pl))^n$ and

$$\begin{aligned}\|u\|_{r(k_n+1),(a+1)p-c} &\leq \left[LC\left(\frac{(k_n+1)^p}{k_np+1}\right)\right]^{\frac{1}{p(k_n+1)}} \|u\|_{(k_{n-1}+1)r,(a+1)p-c} \cdot \|u\|_{r,(a+1)p-c}^{\frac{q-p}{p(k_n+1)}} \\ &\leq \left[\prod_{i=1}^{n}\left(LC\frac{(k_i+1)^p}{k_ip+1}\right)^{\frac{1}{p(k_i+1)}}\right] \|u\|_{r,(a+1)p-c}^{1+\frac{q-p}{p}\sum_{i=1}^n \frac{1}{k_i+1}}.\end{aligned} \tag{2.5}$$

Setting

$$C_1 = C^{\frac{1}{r-q}} \lim_{n\to\infty} \prod_{i=1}^{n}\left(\frac{(k_i+1)^p}{k_ip+1}\right)^{\frac{1}{p(k_i+1)}},$$

and letting $n \to \infty$ in (2.5), we obtain

$$\|u\|_{L^\infty(\Omega)} \leq C_1 L^{\frac{1}{r-q}} \|u\|_{r,(a+1)p-c}^{\frac{r-p}{r-q}}.$$

□

The rest of the section is devoted to the construction of a variational setting and a minimax procedure to generate multiple critical points.

We define $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = f(t) - |t|^{q-2}t$. Then by (F_1) and (F_2), we have that g is continuous and

$$\lim_{t\to 0} \frac{g(t)}{|t|^{q-2}t} = 0.$$

For a given $0 < \varepsilon < \frac{q-p}{2(q+p)}$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$|g(t)| \leq \varepsilon |t|^{q-1}, \quad t \in [-2\delta, 2\delta].$$

Let ϕ_ε be a $C^\infty-$function, such that $\phi_\varepsilon(t) = 1$ if $|t| \leq \delta$, $\phi_\varepsilon(t) = 0$ if $|t| \geq 2\delta$, and $0 \leq \phi_\varepsilon(t) \leq 1$ for any $t \in \mathbb{R}$. Let $g_\varepsilon(t) = \phi_\varepsilon(t)g(t)$ and consider the following problem

$$(P)_{\lambda,\varepsilon} \quad \begin{cases} -div(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-(a+1)p+c}(|u|^{q-2}u + \lambda^{\frac{q-1}{q-p}} g_\varepsilon(\lambda^{\frac{-1}{q-p}} u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.3 *Suppose that v is a solution of $(P)_{\lambda,\varepsilon}$ and $\|v\|_{L^\infty(\Omega)} \leq \delta\lambda^{\frac{1}{q-p}}$. Then, $u = \lambda^{\frac{-1}{q-p}} v$ is a solution of $(P)_\lambda$.*

Define $g_{\varepsilon,\lambda}(u) = \lambda^{\frac{q-1}{q-p}} g_\varepsilon(\lambda^{\frac{-1}{q-p}} u)$, $G_{\varepsilon,\lambda}(t) = \int_0^t g_{\varepsilon,\lambda}(s)ds$,

$$J_1(u) = \frac{1}{p}\int |x|^{-ap}|\nabla u|^p - \frac{3}{2q}\int_\Omega |x|^{-(a+1)p+c}|u|^q, \quad u \in W_0^{1,p}(\Omega, |x|^{-ap}),$$
$$J_2(u) = \frac{1}{p}\int |x|^{-ap}|\nabla u|^p - \frac{1}{2q}\int_\Omega |x|^{-(a+1)p+c}|u|^q, \quad u \in W_0^{1,p}(\Omega, |x|^{-ap})$$

and

$$I_{\varepsilon,\lambda}(u) = \frac{1}{p}\int |x|^{-ap}|\nabla u|^p - \frac{1}{q}\int_\Omega |x|^{-(a+1)p+c}|u|^q - \int_\Omega |x|^{-(a+1)p+c} G_{\varepsilon,\lambda}(u), \quad u \in W_0^{1,p}(\Omega, |x|^{-ap}).$$

It follows that, $J_1(u) \leq I_{\varepsilon,\lambda}(u) \leq J_2(u)$ for any $u \in W_0^{1,p}(\Omega, |x|^{-ap})$. Under the hypotheses (F_1) and (F_2) it is easy to see that J_1, J_2 and $I_{\varepsilon,\lambda}$ are C^1-functionals and they satisfy the $(PS)-$conditions, see for instance [13]. In addition, there exists $r > 0$ such that $J_1(u) > 0$ if $0 < \|u\|_{W_0^{1,p}(\Omega,|x|^{-ap})} < r$ and $I(u) \geq c > 0$ if $\|u\|_{W_0^{1,p}(\Omega,|x|^{-ap})} = r$ for some c.

Now we follow an idea from [8] applied to weighted spaces. Let $\{E_m\}_m$ be a sequence of subspaces of $W_0^{1,p}(\Omega, |x|^{-ap})$ such that

(i) $\dim(E_m) = m$,

(ii) $E_m \subset E_{m+1}$,

(iii) $\mathcal{L}(\bigcup E_m)$, linear manifold generated by $\bigcup_{m\in\mathbb{N}} E_m$, is dense in $W_0^{1,p}(\Omega, |x|^{-ap})$.

By Z_m we denote the topological and algebraic complement of E_{m-1} for $m \geq 2$.

Lemma 2.4 *Let*

$$\beta_k = \sup_{\substack{u\in Z_k \\ \|u\|=1}} \left(\int_\Omega |x|^{-(a+1)p+c}|u|^q\right)^{\frac{1}{q}}.$$

Then $\beta_k \to 0$ as $k \to \infty$.

Proof. It is clear that $0 < \beta_{k+1} \leq \beta_k$, so that $\beta_k \to \beta \geq 0$. For every $k \geq 0$ there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u\|_{L^q(\Omega,|x|^{-(a+1)p+c})} > \frac{\beta_k}{2}$. Since $u_k \rightharpoonup 0$ in $W_0^{1,p}(\Omega, |x|^{-ap})$, the inequality (2.3) together with Theorem 2.1 imply that $u_k \to 0$ in $L^p(\Omega, |x|^{-(a+1)p+c})$. Thus, we have proved that $\beta = 0$. □

Lemma 2.5 *There are $r_k, \rho_k > 0$ such that $r_{k+1} > r_k$, $r_k \to \infty$, $\rho_k > r_k$, $\rho_{k+1} > \rho_k$,*

$$\max_{\substack{u\in E_k\\ \|u\|\geq\rho_k}} J_2(u) < 0$$

and

$$\inf_{\substack{u\in Z_k\\ \|u\|=r_k}} J_1(u) \to \infty \text{ as } k \to \infty.$$

Proof. Let $\beta_k = \sup\limits_{\substack{u\in Z_k\\ \|u\|=1}} \left(\int_\Omega |x|^{-(a+1)p+c}|u|^q\right)^{\frac{1}{q}}$. Then

$$\left(\int_\Omega |x|^{-(a+1)p+c}|u|^q\right)^{\frac{1}{q}} \leq \beta_k \left(\int_\Omega |x|^{-ap}|\nabla u|^p\right)^{\frac{1}{p}}, \text{ for all } u \in Z_k$$

which implies

$$\begin{aligned} J_1(u) &= \frac{1}{p}\int_\Omega |x|^{-ap}|\nabla u|^p - \frac{3}{2q}\int_\Omega |x|^{-(a+1)p+c}|x|^q \\ &\geq \frac{1}{p}\|u\|^p - \frac{3}{2q}\beta_k^q\|u\|^q, \text{ for all } u \in Z_k. \end{aligned}$$

Choose $r_k = (\frac{3}{2}\beta_k^q)^{\frac{1}{p-q}}$. If $u \in Z_k$ and $\|u\| = r_k$, we have

$$J_1(u) \geq r_k^p\left(\frac{q-p}{qp}\right).$$

On the other hand, since the norms are equivalent in E_k, we obtain

$$J_2(u) \leq \frac{1}{p}\|u\|^p - \frac{c_k}{2q}\|u\|^q, \qquad \text{for all } u \in E_k.$$

We may take ρ_k such that $J_2(u) < 0$ if $\|u\| > \rho_k$. Without loss of generality we choose ρ_k such that $\rho_k > r_k$ and $\rho_{k+1} > \rho_k$. □

Observe that

$$\max_{\substack{u\in E_k\\ \|u\|\geq\rho_k}} J_1(u) \leq \max_{\substack{u\in E_k\\ \|u\|\geq\rho_k}} I_{\varepsilon,\lambda}(u) \leq \max_{\substack{u\in E_k\\ \|u\|\geq\rho_k}} J_2(u) < 0$$

$$\inf_{\substack{u\in Z_k\\ \|u\|=r_k}} J_2(u) \geq \inf_{\substack{u\in Z_k\\ \|u\|=r_k}} I_{\varepsilon,\lambda}(u) \geq \inf_{\substack{u\in Z_k\\ \|u\|=r_k}} J_1(u) \to \infty \text{ as } k \to \infty.$$

Define

$$B_k = \{u \in E_k : \|u\| \le \rho_k\},$$

$$\Lambda_1 = \{\psi \in C(B_1, X) : \psi|_{\partial B_1} = id\},$$

$$\Lambda_k = \{\psi \in C(B_k, X) : \psi \text{ is odd}, \psi(u) \in N_k \text{ for some } u \in B_k,\ \psi|_{B_{k-1}} \in \Lambda_{k-1} \text{ and } \psi|_{\partial B_k} = id\},$$

$$N_k = \{u \in Z_k : \|u\| = r_k\}$$

and

$$c_k = \inf_{\psi \in \Lambda_k} \max_{u \in B_k} I_{\varepsilon,\lambda}(\psi(u)).$$

Taking $\psi \in \Lambda_{k+1}$, we have $\psi|_{B_k} \in \Lambda_k$ and

$$\max_{u \in B_{k+1}} I_{\varepsilon,\lambda}(\psi(u)) \ge \max_{u \in B_k} I_{\varepsilon,\lambda}(\psi(u)),$$

implying

$$c_{k+1} \ge c_k.$$

On the other hand, if $\psi \in \Lambda_k$, there is $u_k \in B_k$ such that $\psi(u_k) \in N_k$ and

$$\max_{u \in B_k} I_{\varepsilon,\lambda}(\psi(u)) \ge I_{\varepsilon,\lambda}(\psi(u_\psi)) \ge J_1(\psi(u_\psi)) \ge \inf_{v \in N_k} J_1(v) \to \infty$$

implying

$$c_{k+1} \ge c_k \ge \inf_{v \in N_k} J_1(v) \to \infty.$$

There are subsequences c_{k_n} and $\epsilon_{k_n} > 0$ such that $c_{k_n+1} > c_{k_n} + \epsilon_{k_n}$.

Define

$$\Pi_{k_n} = \{\phi \in \Lambda_{k_n+1} : \max_{u \in B_{k_n}} I(\phi(u)) < c_{k_n} + \frac{\epsilon_{k_n}}{2}\}.$$

Lemma 2.6 *$\Pi_{k_n} \neq \emptyset$.*

Proof. Let $\psi \in \Lambda_{k_n}$ such that

$$\max_{u \in B_{k_n}} I_{\varepsilon,\lambda}(\psi(u)) < c_{k_n} + \frac{\epsilon_{k_n}}{2}.$$

Extend ψ to a function $\overline{\psi} \in C(E_{k_n}, X)$ such that $\overline{\psi} = Id$ in $E_{k_n} \setminus B_{k_n}$. Since E_{k_n+1} is isomorphic to $\mathbb{R}^{k_n+1}$ we may work with the "euclidean" norm $\|\cdot\|_2$ which is equivalent to the norm in E_{k_n+1}. Let

$$\widetilde{\rho_{k_n}} = \max_{u \in B_{k_n}} \|u\|_2,$$

in this way, we may extend the function $\overline{\psi}$ as an odd function Ψ over the ball $\{u \in E_{k_n+1} : \|u\|_2 \leq \widetilde{\rho_{k_n}}\}$ and, also as the identity outside this ball. Here, we may choose ρ_{k_n} and ρ_{k_n+1}, as in the proof of Lemma 2.5, in such a way that $B_{k_n} \subseteq \{u \in E_{k_n+1} : \|u\|_2 \leq \widetilde{\rho_{k_n}}\} \subsetneq B_{k_n+1}$. Calling this extension of Ψ by $\widetilde{\psi} \in C(B_{k_n+1}, X)$ we have that $\widetilde{\psi}|_{\partial B_{k_n+1}} = Id$. Thus, using the intersection lemma (see Lemma 3.4 of [20]), we have that $\widetilde{\psi} \in \Lambda_{k_n+1}$. Moreover

$$\max_{u \in B_{k_n}} I_{\varepsilon,\lambda}(\widetilde{\psi}(u)) < c_{k_n} + \frac{\epsilon_{k_n}}{2}.$$

Therefore, $\widetilde{\psi} \in \Pi_{k_n}$. □

Define

$$\widetilde{c_{k_n}} = \inf_{\phi \in \Pi_{k_n}} \max_{u \in B_{k_n+1}} I_{\varepsilon,\lambda}(\phi(u)).$$

Lemma 2.7 $\widetilde{c_{k_n}} \geq c_{k_n+1}$.

Proof. It follows from the fact that $\Pi_{k_n} \subseteq \Lambda_{k_n+1}$. □

Lemma 2.8 $\widetilde{c_{k_n}}$ *is a critical value.*

Proof. Suppose on the contrary, then by Deformation Lemma for every sufficiently small $\epsilon > 0$ there exists

$$\eta := \eta_\epsilon \in C([0,1] \times W_0^{1,p}(\Omega, |x|^{-ap}), W_0^{1,p}(\Omega, |x|^{-ap}))$$

such that

$$\eta(1, I_{\varepsilon,\lambda}^{\widetilde{c_{k_n}}+\epsilon}) \subset I_{\varepsilon,\lambda}^{\widetilde{c_{k_n}}-\epsilon}$$

(we have used the notation

$$I_{\varepsilon,\lambda}^d = \{u : I_{\varepsilon,\lambda}(u) \leq d)\}.$$

Moreover

$$\eta(t,u) = u, \quad \text{if } u \in W_0^{1,p}(\Omega, |x|^{-ap}) \setminus \{\widetilde{c_{k_n}} - 2\epsilon \leq I_{\varepsilon,\lambda}(u) \leq \widetilde{c_{k_n}} + 2\epsilon\},$$

for all $t \in [0,1]$, and $\eta(t,\cdot)$ is an odd function.

Take ϵ even smaller in order to satisfy $\epsilon < \frac{\epsilon_{k_n}}{4}$ and $\widetilde{c_{k_n}} - \epsilon > 0$. Let $\phi \in \Pi_{k_n+1}$ such that

$$\max_{u \in B_{k_n+1}} I_{\varepsilon,\lambda}(\phi(u)) \leq \widetilde{c_{k_n}} + \epsilon.$$

Observe that

$$\max_{u \in B_{k_n}} I_{\varepsilon,\lambda}(\phi(u)) < c_{k_n} + \frac{\epsilon_{k_n}}{2} < c_{k_n+1} - \frac{\epsilon_{k_n}}{2} \leq \widetilde{c_{k_n}} - \frac{\epsilon_{k_n}}{2} \leq \widetilde{c_{k_n}} - 2\epsilon,$$

and $I_{\varepsilon,\lambda}(\phi(u)) = I_{\varepsilon,\lambda}(u) < 0$ for all $u \in \partial B_{k_n+1}$. Thus, we have $\eta(1, \phi(u)) = \phi(u)$ for every $u \in \partial B_{k_n+1} \cup B_{k_n}$ and for all $t \in [0,1]$. Moreover, let $u \in B_{k_n+1}$ such that $\phi(u) \in N_{k_n+1}$. Then we have $I_{\varepsilon,\lambda}(\phi(u)) < 0$ and therefore $\eta(1,\phi(u)) = \phi(u)$. Hence, $\eta(1,\cdot)\circ\phi \in \Pi_{k_n+1}$ and

$$\max_{u\in B_{k_n+1}} I_{\varepsilon,\lambda}(\eta(1,\phi(u))) \le \widetilde{c_{k_n}} - \epsilon,$$

a contradiction. Therefore, $\widetilde{c_{k_n}}$ is a critical value of $I_{\varepsilon,\lambda}$. □

We conclude the Section with the proof of our Theorem.

Proof of Theorem 1.1. If u_{k_j} is a critical point of $I_{\varepsilon,\lambda}$, then

$$\int |x|^{-ap}|\nabla u|^p = \int_\Omega |x|^{-(a+1)p+c}|u_j|^q - \int_\Omega |x|^{-(a+1)p+c}u_j g_{\varepsilon,\lambda}(u_j)$$

and

$$\widetilde{c_{k_j}} = \frac{1}{p}\int |x|^{-ap}|\nabla u_j|^p - \frac{1}{q}\int_\Omega |x|^{-(a+1)p+c}|u_j|^q - \int_\Omega |x|^{-(a+1)p+c}G_{\varepsilon,\lambda}(u_j).$$

Then,

$$\begin{aligned}\widetilde{c_{k_j}} &= \frac{1}{p}\int_\Omega |x|^{-(a+1)p+c}|u_j|^q + \frac{1}{p}\int_\Omega |x|^{-(a+1)p+c}g_{\varepsilon,\lambda}(u_j)u_j \\ &\quad - \frac{1}{q}\int_\Omega |x|^{-(a+1)p+c}|u_j|^q - \int_\Omega |x|^{-(a+1)p+c}G_{\varepsilon,\lambda}(u_j) \\ &\ge \left(\frac{q-p}{qp}\right)\|u_j\|^q_{q,(a+1)p-c} - \varepsilon\left(\frac{q+p}{qp}\right)\|u_j\|^q_{q,(a+1)p-c} \\ &> \left(\frac{q-p}{2qp}\right)\|u_j\|^q_{q,(a+1)p-c},\end{aligned}$$

that is,

$$\|u_j\|^q_{q,(a+1)p-c} < \frac{2qp\widetilde{c_{k_j}}}{q-p} \le \frac{2qpd_{k_j}}{q-p},$$

where

$$d_{k_j} = \inf_{\phi\in\Pi_{k_j}} \max_{u\in B_{k_j+1}} J_2(\phi(u)).$$

Since,

$$||t|^{q-2}t + g_{\varepsilon,\lambda}(t)| \le (1+\varepsilon)|t|^{p-1} \le \left(1+\frac{q-p}{2(q+p)}\right)|t|^{p-1}$$

and d_{k_j} is independent of ε and λ, we known that for any $j\in\mathbb{N}$, there is $\lambda_j > 0$ such that for any $\lambda \ge \lambda_j$

$$\|u_i\|_{L^\infty(\Omega)} \le \delta\lambda^{\frac{1}{q-p}}, \quad 1 \le i \le j.$$

Then by Lemma 2.3, $\lambda^{\frac{-1}{q-p}}u_i$ is a solution of $(P)_\lambda$. Thus

$$\lim_{\lambda\to+\infty} N(\lambda) = +\infty.$$

□

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