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A Note on The Schrödinger-Poisson-Slater Equation on Bounded Domains

David Ruiz*

SISSA, Via Beirut 2-4, Trieste 34014, Italy
Permanent address: Dpto. Análisis Matemático, Univ. Granada
18071 Granada, Spain
e-mail: daruiz@ugr.es

Gaetano Siciliano†

Dpto di Matematica, Univ. degli Studi di Bari
Via E. Orabona 4, Bari 70125, Italy
e-mail: siciliano@dm.uniba.it

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Abstract

In this note we consider the problem

$$\begin{cases} -\Delta u + \lambda\phi u = |u|^{p-1}u, \\ -\Delta\phi = u^2, \end{cases}$$

where $u, \phi \in H_0^1(\Omega)$ and $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain. We are interested in existence and multiplicity of solutions depending on the parameters p and λ. Our results extend previous work made in $\mathbb{R}^3$.

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1 Introduction

In this paper we are interested in the following system of elliptic equations:

$$\begin{cases} -\Delta u + \omega u + \phi(x)u = |u|^{p-1}u & x \in \Omega, \\ -\Delta \phi = u(x)^2 & x \in \Omega, \\ u(x) = \phi(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here $1 < p < 5$ is a real parameter and Ω is a smooth bounded domain.

This equation appears in the mean field approach for the Hartree-Fock model. Specifically, in 1951 Slater [24] proposed a local approximation of the so-called exchange term of the Hartree-Fock equations. With this approach, the HF system becomes much more tractable. Moreover, if we pass to a singleton equation via a mean field approach, we get (1.1) with $p = 5/3$, which is called the Schrödinger-Poisson-Slater equation. For more information about this topic, see [6, 18, 23].

Equation (1.1) has also appeared in a different framework, as a nonlinear Schrödinger equation that takes into account the electrostatic field generated by the wave, see [5].

Even if most of our results hold for problem (1.1), for the sake of simplicity we prefer to restrict ourselves to $\omega = 0$. Moreover, we have introduced the parameter $\lambda > 0$ motivated by some perturbation results ([11, 21]). Hence, we will study:

$$\begin{cases} -\Delta u + \lambda\phi(x)u = |u|^{p-1}u & x \in \Omega, \\ -\Delta \phi = u(x)^2 & x \in \Omega, \\ u(x) = \phi(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Similar problems have been studied quite a bit in recent years, see [3, 4, 8, 9, 10, 11, 12, 13, 17, 21, 22, 26]. All those papers consider the case $\Omega = \mathbb{R}^3$, giving different results about existence, multiplicity and concentration phenomena of the solutions. In particular, in [3, 22] it is pointed out that the value $p = 2$ is somehow critical.

However, the bounded domain case has been investigated less, see [19, 20]. In this paper we continue the study made in [20] about existence and multiplicity of solutions for (1.2). Regarding existence of positive solutions, the following table sums up the main results of this paper:

	λ small	$\lambda >> 1$
$1 < p < 2$	Two solutions	No solutions
$p = 2$	One solution	No solutions
$2 < p < 5$	One solution	One solution for a.e. λ

In the above table, "Two solutions" (respectively, "One solution") means that there exist at least two (respectively, one) positive solutions. On the other hand, "No solution" means that there are no nontrivial solutions.

We begin the paper with a result of existence of a positive solution for λ small. In a sense, this is accomplished by considering (1.2) as a small perturbation of the problem:

$$\begin{cases} -\Delta u(x) = u(x)^p & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.3}$$

However, we do not know whether the positive solutions of (1.3) are degenerate or not. So, we cannot use the same arguments as in the euclidean space case (see [21], Proposition 2.1). Here we use a topological argument, see Section 2.

In section 3 we deal with our problem through a variational argument; that is, we look for solutions of (1.2) as critical points of the energy functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$,

$$I_\lambda(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{\lambda}{4}\phi_u u^2 - \frac{1}{p+1}|u|^{p+1} \right) dx, \tag{1.4}$$

where ϕ_u is the solution of the linear problem $-\Delta\phi_u = u^2$ with homogeneous Dirichlet boundary conditions.

The case $p > 3$ has already been treated in [20], so we will focus on the remaining cases. As it is well-known, for the use of variational methods the so-called Palais-Smale condition is very important. In this paper we prove the (PS) condition for $p < 2$, which allows us to use classical variational techniques. However, the Palais-Smale property is not known to hold for $p \in [2, 3)$. In order to avoid this problem, we use a monotonicity argument, originally due to Struwe. By using this argument, we are able to prove the existence of a solution for almost all values of λ. We conjecture that the existence result actually holds for every $\lambda > 0$.

As we see, the results are similar to those of [22], where the case $\Omega = \mathbb{R}^3$ is considered. We point out that the arguments in [22] use in a strong way the fact that the domain is invariant under dilatations. Moreover, the Pohozaev equality is used. In [3] a different argument is introduced, by using the monotonicity trick combined with the Pohozaev equality. In our problem, though, the domain is not invariant under dilatations and the Pohozaev equality includes some boundary terms which do not allow us to conclude as in [3].

Finally, in Section 4 we give results about multiplicity of solutions, in the spirit of [3].

2 The case λ small

In this section we will give a first result concerning the existence of solutions for λ small enough. Actually, we have the following:

Theorem 2.1 *Let $p \in (1, 5)$. Then, there exists $\lambda_0 > 0$ such that problem (1.2) has at least a positive solution for $\lambda \in (0, \lambda_0)$.*

Proof. Let $\alpha \in (0, 1)$, and consider $X = C^{0,\alpha}(\Omega)$, $X_0 = \{u \in X : u(x) = 0 \; \forall x \in \partial\Omega\}$, both equipped with the norm $\|\cdot\|_{C^{0,\alpha}}$. Define $K = \Delta^{-1} : X \to X_0$, that is, $K(u) = v$ if $v \in C^2(\Omega) \cap X_0$ and $-\Delta v = u$. It is well-known that K is a well-defined compact operator.

Define also the continuous and bounded operators:

$$N_\lambda : X \to X, \;\; N_\lambda(u) = (u^+)^p - \lambda K(u^2)u,$$

$$Q_\lambda : X \to X, \;\; Q_\lambda(u) = u - K \circ N_\lambda(u).$$

Clearly Q_λ is a compact perturbation of the identity map, and the zeroes of Q_λ correspond to the solutions of (1.2).

It is easy to check that Q_λ is C^1 and its derivative is:

$$Q'_\lambda(u)[v] = v - K[p(u^+)^{p-1}v - \lambda K(u^2)v - 2\lambda K(uv)u].$$

Note that 0 is a zero of Q_λ, and $Q'_\lambda(0) = Id$. Then, the Inverse Function Theorem implies that 0 is an isolated zero of Q_λ. Moreover $deg(Q_\lambda, B(0,\varepsilon), 0) = deg(Q'_\lambda(0), B(0,\varepsilon), 0) = 1$, where deg refers to the Leray-Schauder degree (see, for instance, Lemma 3.18 in [1]).

For $\lambda = 0$, there exist a priori estimates of the zeroes of Q_0, that is, there exists $R > 0$ such that any zero of Q_0 is contained in $B(0,R) \subset X_0$. Moreover, it is known that $deg(Q_0, B(0,R), 0) = 0$, see [14]. Since Q_0 is a compact perturbation of the identity map, then $Q_0(\partial B(0,R))$ is closed (see [15], Chapter 2, Section 8). Then, there exists $\delta > 0$ such that

$$\|Q_0(u)\|_{C^{0,\alpha}} > \delta \; \forall \; u \in \partial B(0,R).$$

Choose $\lambda_0 > 0$ such that $\|Q_\lambda(v) - Q_0(v)\|_{C^{0,\alpha}} < \delta$ for any $v \in \overline{B}(0,R)$ and $\lambda \in [0,\lambda_0]$. Then, the homotopy:

$$H : X \times [0,\lambda_0] \to X, \;\; H(u,\lambda) = Q_\lambda(u)$$

is valid for applying the homotopy property of the Leray-Schauder degree. Therefore, $deg(Q_\lambda, B(0,R), 0) = 0$. Since we also have that $deg(Q_\lambda, B(0,\varepsilon), 0) = 1$, the excision property of the degree implies the existence of a nonzero solution. By the maximum principle, this solution must be positive.

Remark 2.1 The above result states the existence of at least one positive solution of (1.2). Actually, in next Section we will find another positive solution for $p \in (1,2)$ and λ small, see Theorem 3.2.

3 Main results

In this section we use a variational approach to problem (1.2). So, we look for solutions of (1.2) as critical points of a suitable functional defined on the Sobolev space $H_0^1(\Omega)$. In this way we obtain weak solutions but standard regularity results show that the solutions we find are indeed regular (see for instance [22], subsection 2.1). We consider on $H_0^1(\Omega)$ the equivalent norm $\|u\| = \|\nabla u\|_2$ (hereafter $\|\cdot\|_p$ stands for the usual L^p-norm).

The functional I_λ given by (1.4) is well defined and C^1 on $H_0^1(\Omega)$ and its critical points turn out to be solutions of (1.2); for a proof, see for instance [5, 9]. In this section we will analyze some properties of I_λ depending on the values of the parameters p and λ.

First of all, let us recall some properties of the operator $\Phi : H_0^1(\Omega) \to \mathbb{R}$, $\Phi(u) = \int_\Omega \phi_u u^2 \, dx$, where $\phi_u \in H_0^1(\Omega)$ satisfies $-\Delta\phi_u = u^2$. It is quite easy to show that:

$$\|\phi_u\| \leq C\|u\|^2 \tag{3.5}$$

for some $C > 0$. Moreover, we have the following compactness property:

$$u_n \rightharpoonup u \;\Rightarrow \Phi(u_n) \to \Phi(u). \tag{3.6}$$

Both convergences above are understood in the space $H_0^1(\Omega)$. For these and related properties, the reader is referred to [5, 9] (see also [22], pages 658-659).

Finally, we notice that $\phi_u = \phi$ in $H_0^1(\Omega)$ satisfies the relation

$$\int_\Omega |\nabla\phi|^2 dx = \int_\Omega \phi u^2 dx.$$

3.1 The case $p \in (1, 2]$

We begin the study of this case by giving a nonexistence result:

Theorem 3.1 *There exists $\bar\lambda > 0$ such that for every $\lambda \geq \bar\lambda$ the system (1.2) has no nontrivial solutions.*

Proof. Take a sequence $\lambda_n \to +\infty$ and (u_n, ϕ_n) solutions of (1.2) for $\lambda = \lambda_n$. Then

$$\int_\Omega |\nabla u_n|^2 dx + \lambda_n \int_\Omega \phi_n u_n^2 dx = \int_\Omega |u_n|^{p+1} dx.$$

On the other hand, multiplying the second equation in (1.2) by $|u_n| \in H_0^1(\Omega)$ and integrating we get

$$\int_\Omega |u_n|^3 dx = \int_\Omega \nabla\phi_n \nabla|u_n|\, dx \leq \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{1}{2}\int_\Omega |\nabla\phi_n|^2 dx. \tag{3.7}$$

We combine both previous expressions:

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \left(\lambda_n - \frac{1}{2}\right)\int_\Omega \phi_n u_n^2 dx + \int_\Omega |u_n|^3 dx \leq \int_\Omega |u_n|^{p+1} dx.$$

In particular, for $\lambda_n \geq \frac{1}{2}$, we have

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx \leq \int_\Omega |u_n|^{p+1} dx - \int_\Omega |u_n|^3 dx. \tag{3.8}$$

By (3.8) we conclude, since $p + 1 \leq 3$, that u_n must be bounded as $\lambda_n \to +\infty$. So we can assume that $u_n \rightharpoonup u$. Standard regularity results imply that ϕ_n converges strongly to a limit ϕ, that satisfies $-\Delta\phi = u^2$. Now we distinguish two cases:

Case a) $u \neq 0$.

Clearly, $I'_{\lambda_n}(u_n)[u] = 0$, that is:

$$\int_\Omega \nabla u_n \nabla u\, dx + \lambda_n \int_\Omega \phi_n u_n u\, dx = \int_\Omega |u_n|^{p-1} u_n u\, dx.$$

As $\lambda_n \to +\infty$, we get a contradiction since

$$\int_\Omega \nabla u_n \nabla u\, dx \to \int_\Omega |\nabla u|^2 dx, \qquad \int_\Omega \phi_n u_n u\, dx \to \int_\Omega \phi u^2 dx > 0,$$

$$\int_\Omega |u_n|^{p-1}u_n u\,dx, \to \int_\Omega |u|^{p+1}dx.$$

Case b) $u = 0$.

By using the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, we obtain

$$\int_\Omega |\nabla u_n|^2 dx \leq \int_\Omega |\nabla u_n|^2 dx + \lambda_n \int_\Omega \phi_n u_n^2 dx = \int_\Omega |u_n|^{p+1} dx \to 0.$$

Therefore, $u_n \to 0$ strongly. Then

$$0 = \int_\Omega |\nabla u_n|^2 dx + \lambda_n \int_\Omega \phi_n u_n^2 dx - \int_\Omega |u_n|^{p+1} dx \geq \int_\Omega |\nabla u_n|^2 dx - \int_\Omega |u_n|^{p+1} dx.$$

We now use Sobolev embedding to conclude: $0 \geq \|u_n\|^2 - C\|u_n\|^{p+1}$. Since $\|u_n\| \to 0$, this readily implies that $\|u_n\| = 0$ for all $n \geq n_0$. This finishes the proof.

We are now interested in existence results for $p \in (1, 2)$. For that, we will need the following lemma:

Lemma 3.1 *Assume $p \in (1, 2)$ and $\lambda > 0$. Then the functional I_λ given in (1.4) satisfies the following properties:*

1. *I_λ is weakly lower semi-continuous and coercive.*
2. *I_λ satisfies the Palais-Smale property.*

Proof. Take $u_n \rightharpoonup u$; the compactness of the Sobolev embedding and (3.6) show that

$$\int_\Omega |u_n|^{p+1}\,dx \to \int_\Omega |u|^{p+1}\,dx, \quad \int_\Omega \phi_{u_n} u_n^2\,dx \to \int_\Omega \phi_u u^2\,dx.$$

Then, it follows that I_λ is w.l.s.c.

Let us now show that I_λ is coercive. By multiplying the second equation of (1.2) by $|u|$ and integrating by parts, as in (3.7), we get:

$$\frac{\sqrt{\lambda}}{2}\int_\Omega |u|^3\,dx \leq \frac{1}{4}\int_\Omega |\nabla u|^2 dx + \frac{\lambda}{4}\int_\Omega |\nabla\phi|^2 dx,$$

Hence, we get

$$I_\lambda(u) \geq \frac{1}{4}\int_\Omega |\nabla u|^2 dx + \int_\Omega \left(\frac{\sqrt{\lambda}}{2}|u|^3 - \frac{1}{p+1}|u|^{p+1}\right) dx \geq \frac{1}{4}\|u\|^2 - C.$$

To prove 2, take a Palais-Smale sequence $u_n \in H_0^1(\Omega)$, that is:

$$I_\lambda(u_n) \text{ is bounded and } I'_\lambda(u_n) \to 0.$$

Let us write $\phi_n = \phi_{u_n}$. We show that u_n admits a converging subsequence. First, since $I_\lambda(u_n)$ is bounded, this implies that u_n is also bounded (by coercivity). Hence, up to subsequence, $u_n \rightharpoonup u$; by the compactness of the Sobolev embedding, we have that $u_n \to u$ strongly in L^q for any $q \in (2, 6)$. We evaluate:

$$I'_\lambda(u_n)[u_n - u] = \int_\Omega \nabla u_n \nabla(u_n - u)\, dx + \lambda \int_\Omega \phi_n u_n (u_n - u)\, dx - \int_\Omega |u_n|^{p-1} u_n (u_n - u)\, dx.$$

By using Holder inequality, we have:

$$\int_\Omega \phi_n u_n (u_n - u)\, dx \le \|\phi_n\|_3 \|u_n\|_3 \|u_n - u\|_3.$$

By (3.5), the above expression tends to zero as $n \to +\infty$. Moreover,

$$\int_\Omega |u_n|^{p-1} u_n (u_n - u)\, dx \to 0.$$

Then we conclude that $u_n \to u$ strongly in $H^1_0(\Omega)$.

As a consequence, we have the following existence result.

Theorem 3.2 *Assume $p \in (1, 2)$ and λ small enough. Then system (1.2) has (at least) two different positive nontrivial solutions.*

Proof. By Lemma 3.1, I_λ is bounded from below and attains its minimum. Moreover, since I_0 is unbounded from below, we can choose λ small such that $\min I_\lambda < 0$; then, the minimizer is different from zero and corresponds to a nontrivial solution of (1.2). Observe also that 0 is a local minimum of the functional I_λ, and I_λ satisfies the (PS) property. Then the mountain pass theorem ([2]) applies and I_λ admits another critical point at a positive level.

In order to obtain the positivity of the solutions, it suffices to observe that we can apply the previous procedure to the functional

$$I^+_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{\lambda}{4} \int_\Omega \phi u^2 dx - \frac{1}{p+1} \int_\Omega (u^+)^{p+1} dx. \tag{3.9}$$

In so doing, we obtain two different nontrivial critical points of I^+_λ. Thanks to the maximum principle, those solutions are positive and hence they solve (1.2).

3.2 The case $2 < p < 5$

The main difficulty here is that we do not know whether the (PS) condition holds or not. Hence, we cannot simply apply classical variational arguments. Instead we will use an argument which dates back to Struwe [25] (see also Jeanjean [16]). Basically, we introduce a parameter in our problem and succeed to solve it for almost all values of this parameter.

Consider a positive real parameter μ and define the functional:

$$I_{\lambda,\mu}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{\lambda}{4} \int_\Omega \phi u^2 dx - \frac{\mu}{p+1} \int_\Omega |u|^{p+1} dx. \tag{3.10}$$

Let us begin with the following result.

Proposition 3.1 *For any λ and μ positive the functional $I_{\lambda,\mu}$ is unbounded from below.*

Proof. As in [9], the proof consists in the evaluation of the functional $I_{\lambda,\mu}$ along the curve $u_t(x) = t^2 u(tx)$ where $t \to +\infty$ and u is a fixed function. The main difference now is that we are in a bounded domain which is not invariant by dilatations.

Without loss of generality, we can suppose that $0 \in \Omega$. Let $B \subset \Omega$ be a ball centered at zero and choose $u \in C_0^\infty(B)$. Observe that the functions u_t's are supported in B for $t \geq 1$. Easy computations show that

$$\int_\Omega |\nabla u_t|^2 dx = t^3 \int_\Omega |\nabla u|^2 dx \ , \ \int_\Omega |u_t|^{p+1} dx = t^{2p-1} \int_\Omega |u|^{p+1} dx.$$

Let us now deal with the remaining term. We denote $\phi_t = \phi_{u_t}$ and G the Green function of the Laplacian in the domain Ω with Dirichlet homogeneous boundary conditions. By the representation of ϕ_t by means of the Green function G, we have, after a change of variables,

$$\int_\Omega \phi_t(x) u_t(x)^2 dx = \int_{\Omega\times\Omega} G(x,y) t^4 u^2(ty) t^4 u^2(tx)\, dxdy$$
$$= t^2 \int_{\Omega\times\Omega} G\left(\frac{x}{t}, \frac{y}{t}\right) u^2(x) u^2(y)\, dxdy.$$

By the structure of the Green function G, we have

$$\frac{1}{t} G\left(\frac{x}{t}, \frac{y}{t}\right) = \frac{1}{|x-y|} + \frac{1}{t} H\left(\frac{x}{t}, \frac{y}{t}\right),$$

where H is a smooth function and then

$$\frac{1}{t} H\left(\frac{x}{t}, \frac{y}{t}\right) \to 0 \text{ in } L^\infty(\Omega\times\Omega) \text{ as } t \to +\infty.$$

Hence,

$$I_{\lambda,\mu}(u_t) = \frac{t^3}{2} \int_\Omega |\nabla u|^2 dx + \frac{\lambda t^3}{4} \int_{\Omega\times\Omega} \frac{u^2(x) u^2(y)}{|x-y|} dxdy$$
$$+\lambda t^3 \int_{\Omega\times\Omega} \frac{1}{t} H\left(\frac{x}{t}, \frac{y}{t}\right) u^2(x) u^2(y)\, dxdy - \frac{\mu t^{2p-1}}{p+1} \int_\Omega |u|^{p+1}\, dx \to -\infty$$

when $t \to +\infty$ since $2p - 1 > 3$.

Observe that I_λ has a local minimum at zero and is unbounded from below: hence, we are under the geometric conditions of the mountain pass theorem. As we said before, the problem here is the possible lack of the (PS) condition.

Theorem 3.3 *For almost every $\lambda > 0$ there exists a positive solution of (1.2).*

Proof. Let us fix $k \in \mathbb{N}$, $J_k = [1/k, k]$, and take $\mu \in J_k$ a real parameter. Since $I_{\lambda,\mu}$ is unbounded from below, we can choose $u_0 \in H_0^1(\Omega)$ such that $I_{1,1/k}(u_0) < 0$. Let us define the class

$$\Gamma = \left\{\gamma \in C\left([0,1], H_0^1(\Omega)\right) : \gamma(0) = 0, \gamma(1) = u_0\right\}$$

and the numbers

$$c_\mu := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{1,\mu}(\gamma(t)),\ \mu\in J_k.$$

Observe that under the above conditions, there exists a Palais-Smale sequence for any value of μ. However, Theorem 1.1 of [16] implies the existence of a bounded Palais-Smale sequence for almost every value of μ. Specifically, there exists $A_k \subset J_k$ such that:

1. The set $J_k \setminus A_k$ has Lebesgue measure equal to zero.
2. For any $\mu \in A_k$, there exists a bounded sequence u_n such that:

$$I_{1,\mu}(u_n) \to c_\mu\ ,\ \ I'_{1,\mu}(u_n) \to 0.$$

We point out that the min-max scheme defined is independent of μ, which is essential in order to apply Theorem 1.1 of [16]. This is the reason why we need to restrict ourselves to values $\mu \in J_k$.

Take $\mu \in A_k$, u_n a bounded Palais-Smale sequence at level c_μ. By the arguments at the end of the proof of Lemma 3.1, we conclude that a subsequence of u_n converges strongly to a certain u. Therefore, there exists a critical point of $I_{1,\mu}$ at level $c_\mu > 0$.

Define now $A = \cup_{k=2}^{\infty} A_k$. Then, for any $\mu \in A$, there exists a nonzero critical point of $I_{1,\mu}$. It is easy to see that $(0,+\infty)\setminus A$ has zero measure.

Finally, once we have a critical point u of $I_{1,\mu}$, we easily find out that the rescaled function $v = \mu^{\frac{1}{p-1}} u$ satisfies

$$\begin{cases} -\Delta v + \mu^{\frac{2}{1-p}}\phi(x)v = |v|^{p-1}v & x\in\Omega,\\ -\Delta\phi = v(x)^2 & x\in\Omega,\\ v(x)=\phi(x)=0 & x\in\partial\Omega. \end{cases}$$

Therefore, we have a solution of (1.2) for every $\lambda = \mu^{\frac{2}{1-p}}$ with $\mu \in A$.

4 Remarks on multiplicity of solutions

In this section we are interested in multiplicity of (possibly sign-changing) solutions. Our arguments here will be sketchy since they follow the ideas of [3].

We start with the following result:

Theorem 4.1 *For $p \in (2,5)$ and for almost every value of $\lambda > 0$ there exist infinitely many solutions of the problem (1.2).*

Proof. Again, let us fix $k \in \mathbb{N}$, $J_k = [1/k, k]$, take $\mu \in J_k$ a real parameter, and $I_{\lambda,\mu}$ as given in (3.10). Suppose again that $0 \in \Omega$, and take $B \subset \Omega$ a ball centered at zero. Take $E_1 \subset E_2 \subset \ldots \subset H_0^1(B)$ a nested sequence of subspaces, $dim\ E_j = j$, and consider $H_0^1(B) \subset H_0^1(\Omega)$ via the trivial extension. Define B_j the unit ball in E_j, and S_j its boundary. Unless otherwise stated, in what follows we consider j fixed and $\lambda = 1$.

For any $u \in S_j$ and $t \geq 1$, we take $u_t(x) = t^2 u(tx)$. Define $T > 1$ such that

$$I_{1,1/k}(u_t) < 0\ \ \forall\, t \geq T,\ \forall\, u \in S_j. \tag{4.11}$$

The choice of such T is always possible, see the proof of Proposition 3.1. Define $h : B_j \to H_0^1(\Omega)$, $h(u) = u_{s(\|u\|)}$, where:

$$s(r) = \begin{cases} 1 & \text{if } r \leq 1/T, \\ Tr & \text{if } r \in (1/T, 1]. \end{cases}$$

We now define the min-max scheme by following [2], Theorem 2.11. Let

$$\tilde{\Gamma}_j = \{g \in C(B_j, H_0^1(\Omega)) : \ g \text{ is odd, injective and } I_{1,1/k}(g(S_j)) \subset (-\infty, 0]\}.$$

Clearly, $\tilde{\Gamma}_j$ is not empty since h belongs to it. We now define the values:

$$\tilde{b}_{j,\mu} = \inf_{g \in \tilde{\Gamma}_j} \max_{u \in B_j} I_{1,\mu}(g(u)), \ \mu \in J_k.$$

Observe that $\tilde{b}_{j,\mu} \geq \delta > 0$ since $g(B_j)$ always intersects a small sphere around zero, and 0 is a local minimum of $I_{1,\mu}$.

We now compare our situation with that of Theorem 2.11 in [2]. In our case, condition (I_5) does not hold, but such condition is used in [2] only to show that $\tilde{\Gamma}_j$ is not empty. Moreover, condition (I_3) (the Palais-Smale condition) is not known in our case: we will overcome this difficulty again by using the monotonicity trick of Struwe. Specifically, we are under the conditions of Proposition 2.3 of [3]. Therefore, there exists $A_{k,j} \subset J_k$ such that:

1. The set $J_k \setminus A_{k,j}$ has Lebesgue measure equal to zero.
2. For any $\mu \in A_{k,j}$, there exists a bounded sequence u_n such that:

$$I_{1,\mu}(u_n) \to \tilde{b}_{j,\mu} \ , \ \ I'_{1,\mu}(u_n) \to 0.$$

As in the proof of Theorem 3.3, we can pass to the limit and find a critical point at level $\tilde{b}_{j,\mu}$ for any $\mu \in A_{k,j}$.

We now claim that $\tilde{b}_{j,\mu} \to +\infty$ as $j \to +\infty$. Indeed, $I_{0,k} \leq I_{1,\mu}$, and if we define:

$$\hat{\Gamma}_j = \{g \in C(B_j, H_0^1(\Omega)) : \ g \text{ is odd, injective and } I_{0,k}(g(S_j)) \subset (-\infty, 0]\},$$

we have that $\tilde{\Gamma}_j \subset \hat{\Gamma}_j$. Therefore,

$$\hat{b}_j = \inf_{g \in \hat{\Gamma}_j} \max_{u \in B_j} I_{0,k}(g(u)) \leq \tilde{b}_{j,\mu}.$$

Since $\hat{b}_j \to +\infty$ (see [2], in particular Remark 3.19), we conclude the claim.

Therefore, for any $\mu \in A_k = \cap_{j=1}^{\infty} A_{k,j}$, there exist infinitely many different critical points for $I_{1,\mu}$. If we now take $A = \cup_{k=2}^{\infty} A_k$, we have that $(0, \infty) \setminus A$ has zero Lebesgue measure. Finally, given any $\mu \in A$ and any critical point u of $I_{1,\mu}$, the function $v = \mu^{\frac{1}{p-1}} u$ solves

$$\begin{cases} -\Delta v + \mu^{\frac{2}{1-p}} \phi(x) v = |v|^{p-1} v & x \in \Omega, \\ -\Delta \phi = v(x)^2 & x \in \Omega, \\ v(x) = \phi(x) = 0 & x \in \partial\Omega. \end{cases}$$

This concludes the proof.

The case $p \in (1,2)$ is easier since the (PS) condition holds: we have the following result.

Theorem 4.2 *Let $p \in (1,2)$, $k \in \mathbb{N}$. Then there exists $\Lambda_k > 0$ such that for any $\lambda \in (0, \Lambda_k)$ the problem (1.2) has at least k different solutions.*

The proof can be made by following the proof of Theorem 3.1 in [3].

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