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Multiplicity of Solutions For p-Laplacian Equations Which Need Not be Coercive

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Abstract

In this paper, we consider nonlinear Dirichlet problem driven by the p-Laplacian differential operator. Using variational methods based on the critical point theory and truncation techniques, we prove the existence of at least three nontrivial smooth solutions. The hypotheses on the nonlinearity incorporate in our framework of analysis both coercive and noncoercive problems. For the semilinear problem ($p = 2$), using Morse theory, we show the existence of four nontrivial smooth solutions.

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1 Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 boundary ∂Z. In this paper we study the existence of multiple nontrivial solutions for the following nonlinear Dirichlet problem:

$$\left\{ \begin{array}{c} -\triangle_p x(z) = f(z, x(z)) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0. \end{array} \right\} \tag{1.1}$$

Here $\triangle_p x = \text{div}(\|Dx\|^{p-2} Dx)$ $(1 < p < \infty)$ is the p-Laplacian differential operator and $f(z, x)$ is a Caratheodory nonlinearity. Note that if $p = 2$, then $\triangle_p = \triangle$ is the usual

Laplacian differential operator. We are interested in the existence of multiple nontrivial solutions for problem (1.1). More precisely, we want to prove the existence of at least three nontrivial solutions for problem (1.1).

Recently, three solutions theorems for Dirichlet problems, driven by the p-Laplacian, were obtained by Bartsch-Liu [5], Carl-Perera [6], Dancer-Perera [13], Cingolani-Degiovanni [9], Degiovanni-Lancelotti [12], Garcia Azorero-Peral Alonso [14], Liu [21], Liu-Liu [22], Papageorgiou-Papageorgiou [25] and Zhang-Chen-Li [28]. With the exception of Bartsch-Liu [5] and Degiovanni-Lancelotti [12], in all the other works the Euler functional of the problem is coercive. Moreover, in the works of Carl-Perera [6], Dancer-Perera [13] and Zhang-Chen-Li [13], the asymptotic limits

$$a_{\pm} = \lim_{x \to 0^{\pm}} \frac{f(z,x)}{|x|^{p-2}x}$$

play an essential role in their arguments. In the work of Bartsch-Liu [5] the nonlinearity $f(z,x)$ is continuous on $\overline{Z} \times \mathbb{R}$ and satisfies the well-known Ambrosetti-Rabinowitz condition. This implies that the function $x \to f(z,x)$ is p-superlinear near infinity. Moreover, in their work near the origin the quotients $\frac{f(z,x)}{|x|^{p-2}x}$ remain below $\lambda_1 > 0$, the principal eigenvalue of $(-\triangle_p, W_0^{1,p}(Z))$. A superlinear nonlinearity is also considered in the recent interesting work of Degiovanni-Lancelotti [12]. We should mention that the results of Bartsch-Liu [5], Carl-Perera [6] and Zhang-Chen-Li [28] provide information about the sign of the solutions they obtain.

Here, our hypotheses on the nonlinearity are such that they do not necessarily imply that the Euler functional of the problem is coercive. In fact, we do not impose any asymptotic conditions at infinity on the nonlinearity and we do not employ the Ambrosetti-Rabinowitz condition. We only assume that the function $x \to f(z,x)$ has subcritical growth. In the last section (Section 4), we consider the semilinear problem (i.e. $p = 2$). By strengthening the hypothesis on the nonlinearity near the origin and using Morse theory (in particular critical groups), we are able to show that the problem has at least four nontrivial solutions. Our approach is variational based on the critical point theory for C^1-functionals.

2 Mathematical background

Let X be a Banach space, and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Suppose that $\varphi \in C^1(X)$. We say that φ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (the PS_c-condition for short), if every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ and $\varphi'(x_n) \to 0$ in X^*, has a strongly convergent subsequence. We say that φ satisfies the PS-condition, if it satisfies the PS_c-condition for every $c \in \mathbb{R}$.

The topological notion of linking sets is important in the minimax characterization of the critical values of a functional $\varphi \in C^1(X)$.

Definition 2.1 Let Y be a Hausdorff topological space and E_0, E and D nonempty closed subsets of Y such that $E_0 \subseteq E$. We say that the pair $\{E_0, E\}$ *is linking with* D in Y, if and only if

(a) $E_0 \cap D = \emptyset$;

(b) for all $\gamma \in C(E,Y)$ such that $\gamma|_{E_0} = id|_{E_0}$, we have $\gamma(E) \cap D \neq \emptyset$.

Theorem 2.1 *If X is a Banach space, E_0, E and D are nonempty closed subsets of X such that the pair $\{E_0, E\}$ is linking with D in X, and $\varphi \in C^1(X)$ such that*

$$\sup_{E_0} \varphi < \inf_{D} \varphi,$$
$$\Gamma = \{\gamma \in C(E,X) : \gamma|_{E_0} = id|_{E_0}\},$$
$$c = \inf_{\gamma \in \Gamma} \sup_{v \in E} \varphi(\gamma(v)), \text{ and}$$
$$\varphi \text{ satisfies the } PS_c\text{-condition}$$

then $c \geq \inf_D \varphi$ and c is a critical value of φ.

Remark 2.1 By appropriately choosing the linking sets, from Theorem 2.1 we can produce as corollaries, the mountain pass theorem, the saddle point theorem and the generalized mountain pass theorem (see for example Gasinski-Papageorgiou [16], Section 5.2).

In our study of problem (1.1), we will also use the so-called "second deformation theorem". For the convenience of the reader, we state the result here. First, we need to introduce some notation.

So, let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We define

$$\varphi^c = \{x \in X : \varphi(x) \leq c\} \text{ (the sublevel set of } \varphi \text{ at } c)$$
$$K = \{x \in X :, \varphi'(x) = 0\} \text{ (the critical set of } \varphi)$$
$$K_c = \{x \in K : \varphi(x) = c\} \text{ (the critical set of } \varphi \text{ at the level } c).$$

The next theorem, is known in the literature as the "second deformation theorem" (see Chang [7], p.23 and Gasinski-Papageorgiou [16], p.628). In the formulation that follows, we allow $b = +\infty$, in which case $\varphi^b \backslash K_b = X$.

Theorem 2.2 *If X is a Banach space, $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \leq +\infty$, φ satisfies the PS_c condition for every $c \in [a,b)$, φ has no critical values in (a,b) and $\varphi^{-1}(a)$ has at most a finite number of critical points of φ, then there exists a continuous deformation $h : [0,1] \times \varphi^b \backslash K_b \to \varphi^b$ such that*

(a) $h(1, \varphi^b \backslash K_b) \subseteq \varphi^a$;

(b) $h(t,x) = x$ for all $t \in [0,1]$ and all $x \in \varphi^a$;

(c) $\varphi(h(t,x)) \leq \varphi(h(s,x))$ for all $t,s \in [0,1]$, $s \leq t$ and all $x \in \varphi^b \backslash K_b$.

Remark 2.2 According to this theorem, φ^a is a strong deformation retract of $\varphi^b \backslash K_b$.

Next, we present some basic facts about the spectrum of the negative p-Laplacian with Dirichlet boundary conditions, which we will need in the sequel. So, we consider the following nonlinear eigenvalue problem:

$$\left\{ \begin{array}{c} -\triangle_p x(z) = \lambda |x(z)|^{p-2} x(z) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0, \ \ \lambda \in \mathbb{R}, \ \ 1 < p < \infty. \end{array} \right\} \tag{1.2}$$

Every number $\lambda \in \mathbb{R}$ for which problem (1.2) has a nontrivial solution is said to be an *eigenvalue* of $(-\triangle_p, W_0^{1,p}(Z))$ and the nontrivial solution is an *eigenfunction*. We know that $(-\triangle_p, W_0^{1,p}(Z))$ has a smallest eigenvalue λ_1 such that

- $\lambda_1 > 0$ and it is isolated;
- $\lambda_1 > 0$ is simple (i.e., the corresponding eigenspace is one-dimensional);
- λ_1 has the following variational characterization

$$\lambda_1 = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z),\ x \neq 0\right] \tag{1.3}$$

(see Anane [2]).

The infimum in (1.3) is attained on the corresponding one-dimensional eigen space. In the sequel, by u_1 we denote the L^p-normalized principal eigenfunction (i.e., $\|u_1\|_p = 1$). From (1.3), it is clear that u_1 does not change sign and so we may assume that $u_1 \geq 0$. In fact, nonlinear regularity theory (see for example Gasinski-Papagergiou [16], pp.738-739) implies that

$$u_1 \in C_0^1(\overline{Z}) = \{u \in C^1(\overline{Z}) : u|_{\partial Z} = 0\}.$$

The Banach space $C_0^1(\overline{Z})$ is an ordered Banach space, with order cone

$$C_+ = \{u \in C_0^1(\overline{Z}) : u(z) \geq 0 \ \text{ for all } \ z \in \overline{Z}\}.$$

This cone has a nonempty interior given by

$$\text{int}C_+ = \{u \in C_+ : u(z) > 0 \ \text{ for all } \ z \in Z \ \text{ and } \ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } \ z \in \partial Z\}.$$

By $n(\cdot)$ we denote the unit outward normal on ∂Z. ¿From the nonlinear strong maximum principle of Vazquez [27], we have that $u_1 \in \text{int}C_+$ (see also [11]).

Since $\triangle_p$ is a $(p-1)$-homogeneous operator, from the Lusternik-Schnirelmann theory, in addition to $\lambda_1 > 0$, we have a whole strictly increasing sequence $\{\lambda_k\}_{k\geq 1} \subseteq \mathbb{R}_+$ of eigenvalues of $(-\triangle_p, W_0^{1,p}(Z))$ such that $\lambda_k \to +\infty$. These are the so-called "LS-eigenvalues" of $(-\triangle_p, W_0^{1,p}(Z))$. If $p = 2$ (linear eigenvalue problem), then these are all the eigenvalues of $(-\triangle, H_0^1(Z))$. If $p \neq 2$ (nonlinear eigenvalue problem), then we do not know if this is true. However, because the set $\sigma(p)$ of eigenvalues is closed and $\lambda_1 > 0$ is isolated, we can define

$$\lambda_2^* = \inf\{\lambda : \lambda \in \sigma(p), \lambda > \lambda_1\} > 0$$

and we have $\lambda_2^* \in \sigma(p)$ (i.e., λ_2^* is the second eigenvalue of $(-\triangle_p, W_0^{1,p}(Z))$). In fact, we have

$$\lambda_2^* = \lambda_2,$$

namely, the second eigenvalue and the second LS-eigenvalue coincide (see Anane-Tsouli [3]).

The Lusternik-Schnirelmann theory provides a variational characterization of $\lambda_2 > 0$. However, for our purposes that characterization is not convenient. Instead, we will use an alternative one, produced by Cuesta-de Figueiredo-Gossez [10]. So let

$$\begin{aligned}
&\partial B_1^{L^p} = \{x \in L^p(Z) : \|x\|_p = 1\},\\
&S = W_0^{1,p}(Z) \cap \partial B_1^{L^p}(Z),\ \textit{and}\\
&\Gamma_0 = \{\gamma_0 \in C([-1,1], S) : \gamma_0(-1) = -u_1, \gamma_0(1) = u_1\}.
\end{aligned}$$

Then, from [7] we have

$$\lambda_2 = \inf_{\gamma_0 \in \Gamma_0} \max_{x \in \gamma_0([-1,1])} \|Dx\|_p^p. \tag{1.4}$$

For further details about the spectrum of the p-Laplacian, we refer the reader to Lê [18] and Gasinski-Papageorgiou [16].

Next, let us recall some basic definitions and facts from Morse theory, that we will need in Section 4. So, let H be a Hilbert space and $\varphi \in C^1(H)$ a functional satisfying the PS-condition. Let $x_0 \in H$ be an isolated critical point of φ and $c_0 = \varphi(x_0)$. The critical groups (over $\mathbb{Z}$) of φ at x_0 are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^{c_0} \cap U, \varphi^{c_0} \cap U \backslash \{x_0\}) \ \text{ for all } \ k \geq 0,$$

where U is a neighborhood of x_0 and $H_k(\cdot,\cdot)$ is the k^{th}-singular relative homology group with integer coefficients. From the excision property of singular homology theory, we see that this definition is independent of the neighborhood of U of x_0. Suppose that $-\infty < \inf \varphi(K)$ and let $c < \inf \varphi(K)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(H, \varphi^c) \ \text{ for all } \ k \geq 0$$

(see Bartsch-Li [4]). From the deformation lemma (see Gasinski-Papageorgiou [16], p.626), which is valid since by hypothesis φ satisfies the PS-condition, we see that this definition is independent of c. If K is finite, then the Morse-type numbers of φ are defined by

$$M_k = \sum_{x \in K} \text{rank} C_k(\varphi, x)$$

and the Betti-type numbers of φ are defined by

$$\beta_k = \text{rank} C_k(\varphi, \infty).$$

From Morse theory (see Bartsch-Li [4], Chang [7] and Mawhin-Willem [24]), we have

$$\sum_{k=0}^{m} (-1)^{m-k} M_k \geq \sum_{k=0}^{m} (-1)^{m-k} \beta_k, \ \textit{and} \tag{1.5}$$

$$\sum_{k \geq 0} (-1)^k M_k = \sum_{k \geq 0} (-1)^k \beta_k. \tag{1.6}$$

From (1.5), we infer that $\beta_k \leq M_k$ for all $k \geq 0$. Therefore, if $\beta_k \neq 0$ for some $k \geq 0$, then φ must have a critical point $x \in H$ and the critical group $C_k(\varphi, x)$ is non-trivial. Equality (1.6) is also known as the "Poincare-Hopf formula". If $K = \{x_0\}$, then $C_k(\varphi, \infty) = C_k(\varphi, x_0)$.

Next, recall that, if X is a Banach space and $A : X \to X^*$, we say that A is of type $(S)_+$ if $x_n \xrightarrow{w} x$ in X and $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$ imply $x_n \to x$ in X. Hereafter, by $\xrightarrow{w}$ we denote the weak convergence and by $\to$ the strong convergence in a Banach space.

If $X = W_0^{1,p}(Z)$, $X^* = W^{-1,p'}(Z)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) and $A : X \to X^*$ is defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N} dz \ \text{ for all } \ x, y \in W_0^{1,p}(Z), \tag{1.7}$$

then from Gasinski-Papageorgiou [16], we have:

Proposition 2.1 *The map $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by (1.7) is of type $(S)_+$.*

In this paper, as usual, by p^* we denote the critical Sobolev exponent, i.e.,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } \ p < N \\ +\infty & \text{if } \ p \geq N \end{cases}.$$

Also, $r^{\pm} = \max\{\pm r, 0\}$ for all $r \in \mathbb{R}$ and $\|x\| = \|Dx\|_p$ for all $x \in W_0^{1,p}(Z)$.

Finally we mention that, if X is an ordered Banach space with order cone K, $\text{int}K \neq \emptyset$ and $\varrho \in \text{int}K$, then for every $y \in X$, we can find $t = t(y) > 0$ such that $t\varrho - y \in \text{int}K_+$ (see Papageorgiou-Papageorgiou [16], Lemma 3.5).

3 Three nontrivial solutions

In this section, we prove a "three-nontrivial-solutions theorem" for problem (1.1). The hypotheses on the nonlinearity $f(z, x)$, are the following: $\underline{H(f)_1}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z, 0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to f(z, x)$ is continuous;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$,

$$|f(z, x)| \leq a(z) + c|x|^{r-1}$$

with $a \in L^\infty(Z)_+$, $c > 0$ and $1 < r < p^*$;

(iv) there exist $a_- < 0 < a_+$ such that

$$f(z, a_-) = f(z, a_+) = 0 \ \text{ a.e. on } \ Z$$

and

$$\begin{cases} 0 \leq f(z, x) & \text{for a.a. } z \in Z, \ \text{ all } x \in [0, a_+] \\ f(z, x) \leq 0 & \text{for a.a. } z \in Z, \ \text{ all } x \in [a_-, 0] \end{cases},$$

with some $\xi > 0$ and a.a. $z \in Z$, $x \to f(z, x) + \xi|x|^{p-2}x$ is nondecreasing on $[a_-, a_+]$ and there exist $0 < \delta < \min\{a_-, a_+\}$ and $\eta > \lambda_2$ such that

$$\frac{\eta}{p}|x|^p \leq F(z, x) \ \text{ for a.a. } \ z \in Z \ \text{ and all } \ |x| \leq \delta,$$

where $F(z, x) = \int_0^x f(z, s)ds$.

Remark 3.1 We emphasize that no asymptotic conditions at $\pm\infty$ are assumed. This is in sharp contrast with all the previous works. Hypothesis $H(f)_1(iv)$ and the truncation techniques allow us to avoid asymptotic conditions at $\pm\infty$. We only assume a subcritical growth (see hypothesis $H(f)_1(iii)$). So, our framework of analysis incorporates coercive but also noncoercive problems, such as the p-superlinear ones, studied by Bartsch-Liu [5].

We introduce the following truncations of the nonlinearity $f(z,x)$:

$$\widehat{f}_+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ f(z,x) & \text{if } x \in [0,a_+] \\ 0 & \text{if } x > a_+ \end{cases}, \quad \widehat{f}_-(z,x) = \begin{cases} 0 & \text{if } x < a_- \\ f(z,x) & \text{if } x \in [a_-,0] \\ 0 & \text{if } x > 0 \end{cases}$$

and

$$\widehat{f}(z,x) = \begin{cases} 0 & \text{if } x < a_- \\ f(z,x) & \text{if } x \in [a_-,a_+] \\ 0 & \text{if } x > a_+ \end{cases}.$$

Evidently, $\widehat{f}_\pm$ and $\widehat{f}$ are Caratheodory functions, i.e., they are measurable in $z \in Z$ and continuous in $x \in \mathbb{R}$. We set

$$\widehat{F}_\pm(z,x) = \int_0^x \widehat{f}_\pm(z,s)ds \text{ and } \widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)ds, \quad x \in \mathbb{R},$$

and then consider the functionals $\widehat{\varphi}_\pm, \widehat{\varphi} : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\pm(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{F}_\pm(z,x(z))dz, \text{ and}$$
$$\widehat{\varphi}(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{F}(z,x(z))dz \text{ for all } x \in W_0^{1,p}(Z).$$

Clearly $\widehat{\varphi}_\pm, \widehat{\varphi} \in C^1(W_0^{1,p}(Z))$.

In the next proposition , we produce two solutions of constant sign.

Proposition 3.1 *If hypotheses $H(f)_1$ hold, then problem (1.1) has two solutions x_0 and v_0 of constant sign,*

$$x_0 \in intC_+ \text{ and } v_0 \in -intC_+,$$

which are local minimizers of the functional $\widehat{\varphi}$.

Proof. First we establish the existence of a positive solution $x_0 \in \text{int}C_+$. Clearly the functional $\widehat{\varphi}_+$ is coercive. Moreover, exploiting the compact embedding of $W_0^{1,p}(Z)$ into $L^r(Z)$ (Sobolev embedding theorem), we can easily verify that $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, by the theorem of Weierstrass, we can find $x_0 \in W_0^{1,p}(Z)$ such that

$$\widehat{\varphi}_+(x_0) = \inf[\widehat{\varphi}_+(x) : x \in W_0^{1,p}(Z)] = \widehat{m}_+. \tag{1.8}$$

Let $u_1 \in \text{int}C_+$ be the L^p-normalized principal eigenfunction of $(-\triangle_p, W_0^{1,p}(Z))$. We can find $t > 0$ small such that

$$tu_1(z) \in [0,\delta] \text{ for all } z \in \overline{Z},$$

with $\delta > 0$ being as in hypothesis $H(f)_1(v)$. Then according to this hypothesis, we have

$$\widehat{F}_+(z, tu_1(z)) \geq \frac{\eta}{p} t^p u_1(z)^p \ \text{ a.e. on } \ Z. \tag{1.9}$$

Therefore,

$$\begin{aligned}
\widehat{\varphi}_+(tu_1) &= \frac{t^p}{p}\|Du_1\|_p^p - \int_Z \widehat{F}_+(z, tu_1)dz \\
&\leq \frac{t^p}{p}\|Du_1\|_p^p - \frac{\eta}{p} t^p \|u_1\|_p^p \ \text{ (see (1.9))} \\
&= \frac{t^p}{p}\int_Z (\lambda_1 - \eta) u_1(z)^p dz \ \text{ (since } \|Du_1\|_p^p = \lambda_1 \|u_1\|_p^p) \\
&< 0 \ \text{ (since } \eta > \lambda_2 > \lambda_1).
\end{aligned}$$

So, from (1.8) we have

$$\begin{aligned}
& \widehat{m}_+ = \widehat{\varphi}_+(x_0) \leq \widehat{\varphi}_+(tu_1) < 0 = \widehat{\varphi}_+(0), \\
\Rightarrow \quad & x_0 \neq 0.
\end{aligned}$$

Also, we have

$$\begin{aligned}
& \widehat{\varphi}'_+(x_0) = 0, \\
\Rightarrow \quad & A(x_0) = \widehat{N}_+(x_0),
\end{aligned} \tag{1.10}$$

where $\widehat{N}_+(x)(\cdot) = \widehat{f}_+(\cdot, x(\cdot))$ for all $x \in W_0^{1,p}(Z)$. On (1.10), we act with the test function $-x_0^- \in W_0^{1,p}(Z)$ and we obtain

$$\begin{aligned}
& \|Dx_0^-\|_p^p = 0 \\
\Rightarrow \quad & x_0^- = 0, \ \text{i.e.,} \ \ x_0 \geq 0, \ \ x_0 \neq 0.
\end{aligned}$$

From (1.10), we infer that

$$-\triangle_p x_0(z) = \widehat{f}_+(z, x_0(z)) \ \text{ a.e. on } \ Z, \ \ x_0|_{\partial Z} = 0. \tag{1.11}$$

From Theorem 7.1 of Ladyzhenskaya-Uraltseva [17], we have $x_0 \in L^\infty(Z)$ and then Theorem 1 of Lieberman [19] implies that $x_0 \in C_+$. Moreover, from (1.11), the definition of $\widehat{f}_+$ and hypothesis $H(f)_1(iv)$, we have

$$\triangle_p x_0(z) \leq 0 \ \text{ a.e. on } \ Z. \tag{1.12}$$

Invoking the nonlinear strong maximum principle of Vazquez [27], from (1.12) we infer that $x_0 \in \text{int} C_+$.

Next, note that

$$\begin{aligned}
& A(a_+ - x_0) = -A(x_0) \\
\Rightarrow \quad & -A(a_+ - x_0) = \widehat{N}_+(x_0) \ \text{ (see (1.10)).}
\end{aligned} \tag{1.13}$$

On (1.13), we act with the test function $(x_0 - a_+)^+ \in W_0^{1,p}(Z)$. Note that

$$\widehat{f}_+(z, x_0(z)) = 0 \ \text{ a.e. on } \ \{x_0 > a_+\}.$$

Therefore, we obtain

$$\begin{aligned} & \|D(x_0 - a_+)^+\|_p^p = 0, \\ \Rightarrow \qquad & (x_0 - a_+)^+ = 0, \ \text{ i.e., } \ x_o \leq a_+. \end{aligned}$$

Then, by virtue of hypothesis $H(f)_1(iv)$ and the strong comparison principle (tangency principle), we have

$$x_0(z) < a_+ \ \text{ for all } \ z \in \overline{Z}.$$

Hence, we can find $r > 0$ small such that

$$\widehat{\varphi}_+|_{\overline{B}_r^{C_0^1(\overline{Z})}(x_0)} = \widehat{\varphi}|_{\overline{B}_r^{C_0^1(\overline{Z})}(x_0)}$$

where $\overline{B}_r^{C_0^1(\overline{Z})}(x_0) = \{x \in C_0^1(\overline{Z}) : \|x - x_0\|_{C_0^1(\overline{Z})} \leq r\}$. This means that $x_0 \in \text{int}C_+$ is a local $C_0^1(\overline{Z})$-minimizer of $\widehat{\varphi}$. According to Theorem 1.1 of Garcia Azorero Manfredi-Peral Alonso [15], $x_0 \in \text{int}C_+$ is also a local $W_0^{1,p}(Z)$-minimizer of $\widehat{\varphi}$.

Similarly, working with the functional $\widehat{\varphi}_-$, we obtain a solution $v_0 \in -\text{int}C_+$ which too is a local $W_0^{1,p}(Z)$-minimizer of $\widehat{\varphi}$.

Next, we will produce one more nontrivial smooth solution of (1.1) and have the "three solutions theorem" for problem (1.1).

Theorem 3.1 *If hypotheses $H(f)_1$ hold, then problem (1.1) has at least three nontrivial solutions*

$$x_0 \in intC_+, \ v_0 \in -intC_+, \ y_0 \in C_0^1(\overline{Z}).$$

Proof. Note that the functional $\widehat{\varphi} : W_0^{1,p}(Z) \to \mathbb{R}$ is coercive. We claim that it satisfies the PS-condition. Indeed, let $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$ be such that

$$|\widehat{\varphi}(x_n)| \leq M_1 \ \text{ for some } \ M_1 > 0, \ \text{ all } \ n \geq 1, \ and \tag{1.14}$$

$$\widehat{\varphi}'(x_n) \to 0 \ \text{ in } \ W^{-1,p'}(Z) \ \text{ as } \ n \to \infty. \tag{1.15}$$

From (1.14) and since $\widehat{\varphi}$ is coercive, we infer that $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. So, by passing to a suitable subsequence, if necessary, we may assume that

$$x_n \xrightarrow{w} x \ \text{ in } \ W_0^{1,p}(Z) \ \text{ and } \ x_n \to x \ \text{ in } \ L^p(Z). \tag{1.16}$$

From (1.15), we have

$$|\langle A(x_n), x_n - x\rangle - \int_Z \widehat{f}(z, x_n)(x_n - x)dz| \leq \varepsilon_n \|x_n - x\| \ \text{ with } \ \varepsilon_n \downarrow 0. \tag{1.17}$$

Note that

$$\int_Z \widehat{f}(z, x_n)(x_n - x)dz \to 0 \ \text{ (see (1.16)).}$$

Therefore, from (1.17) it follows that

$$\begin{aligned} & \lim\langle A(x_n), x_n - x\rangle = 0, \\ \Rightarrow \quad & x_n \to x \ \text{ in } \ W_0^{1,p}(Z) \ \text{ (see Proposition 2.1)}, \\ \Rightarrow \quad & \widehat{\varphi} \ \text{ satisfies the PS-condition.} \end{aligned}$$

We may assume, without any loss of generality, that

$$\widehat{\varphi}(v_0) \leq \widehat{\varphi}(x_0) < 0 = \widehat{\varphi}(0).$$

Arguing as in Aizicovici-Papageorgiou-Staicu [1] (see Proposition 29), we can find $r > 0$ small such that

$$\widehat{\varphi}(x_0) < \inf[\widehat{\varphi}(x) : \|x - x_0\| = r] = \theta_r \leq 0.$$

Consider the sets

$$\begin{aligned} & E_0 = \{v_0, x_0\}, \\ & E = [v_0, x_0] = \{x \in W_0^{1,p}(Z) : v_0(z) \leq x(z) \leq x_0(z) \ \text{ a.e. on } \ Z\}, \ \textit{and} \\ & D = \partial B_r(x_0). \end{aligned}$$

It is easy to check (see also Gasinski-Papageorgiou [16], p.462) that the pair $\{E_0, E\}$ is linking with D in $W_0^{1,p}(Z)$. So, we can apply Theorem 2.1 and obtain $y_0 \in W_0^{1,p}(Z)$, a critical point of $\widehat{\varphi}$, such that

$$\widehat{\varphi}(v_0) \leq \widehat{\varphi}(x_0) < \theta_r \leq \widehat{\varphi}(y_0), \ \textit{and} \tag{1.18}$$

$$\widehat{\varphi}(y_0) = \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} \widehat{\varphi}(\gamma(t)) \tag{1.19}$$

where $\Gamma = \{\gamma \in C([-1,1], W_0^{1,p}(Z)) : \gamma(-1) = v_0, \ \gamma(0) = x_0\}$.

From (1.18), it follows that $y_0 \neq x_0, v_0$. We need to show that it is also nontrivial. To this end, we will use (1.19). According to (1.19), to prove the nontriviality of y_0, it suffices to produce a path $\gamma^* \in \Gamma$ such that $\widehat{\varphi}|_{\gamma^*} < 0$.

Let $\delta_0 > 0$ be such that

$$\lambda_2 + \delta_0 < \eta \ \text{ (recall that } \eta > \lambda_2, \text{ see hypothesis } H(f)_1(v)). \tag{1.20}$$

Also, we set

$$\begin{aligned} & S = W_0^{1,p}(Z) \cap \partial B_1^{L^p(z)} \ \text{ endowed with the relative } W_0^{1,p}(Z)\text{-topology, and} \\ & S_c = U \cap C_0^1(\overline{Z}) \ \text{ endowed with the relative } C_0^1(\overline{Z})\text{-topology.} \end{aligned}$$

Then S_c is dense in S. Therefore, if

$$\begin{aligned} & \widehat{\Gamma}_S = \{\widehat{\gamma} \in C([-1,1], U) : \widehat{\gamma}(-1) = -u_1, \widehat{\gamma}(1) = u_1\}, \ \textit{and} \\ & \widehat{\Gamma}_{S_c} = \{\widehat{\gamma} \in C([-1,1], U_c) : \widehat{\gamma}(-1) = u_1, \widehat{\gamma}(1) = u_1\}, \end{aligned}$$

then $\widehat{\Gamma}_{S_c}$ is dense in $\widehat{\Gamma}_S$ for the $C([-1,1],S)$-topology. So, because of (1.4), we can find $\widehat{\gamma}_0 \in \Gamma_{S_c}$ such that

$$\max[\|Dx\|_p^p : x \in \widehat{\gamma}_0([-1,1])] \leq \lambda_2 + \delta_0. \tag{1.21}$$

Since $\widehat{\gamma}_0 \in C([-1,1],S_c)$ and $-v_0, x_0 \in \text{int}C_+$, we can choose $\varepsilon > 0$ small such that

$$\varepsilon|x(z)| \leq \delta \ \text{ for all } \ z \in \overline{Z} \ \text{ and } \ x \in \widehat{\gamma}_0([-1,1]), \ \textit{and} \tag{1.22}$$

$$\varepsilon x \in [v_0, x_0] \ \text{ for all } \ x \in \widehat{\gamma}_0([-1,1]). \tag{1.23}$$

Hence, if $x \in \widehat{\gamma}_0([-1,1])$, then

$$\begin{aligned}
\widehat{\varphi}(\varepsilon x) &= \frac{\varepsilon^p}{p}\|Dx\|_p^p - \int_Z F(z,\varepsilon x)dz \ \text{ (see (1.23))} \\
&\leq \frac{\varepsilon^p}{p}\|Dx\|_p^p - \frac{\eta}{p}\varepsilon^p\|x\|_p^p \ \text{ (see (1.22) and hypothesis } H(f)_1(v)) \\
&= \frac{\varepsilon^p}{p}\|Dx\|_p^p - \frac{\varepsilon^p\eta}{p} \ \text{ (since } \|x\|_p = 1) \\
&\leq \frac{\varepsilon^p}{p}(\lambda_2 + \delta_0 - \eta) \ \text{ (see (1.21))} \\
&< 0 \ \text{ (see (1.20)).}
\end{aligned}$$

So, if $\widehat{\gamma}_0^\varepsilon = \varepsilon\widehat{\gamma}_0$, then

$$\widehat{\varphi}|_{\widehat{\gamma}_0^\varepsilon} < 0. \tag{1.24}$$

Without any loss of generality, we may assume that $x_0 \in \text{int}C_+$ is the only nontrivial critical point of $\widehat{\varphi}_+$ and $v_0 \in -\text{int}C_+$ is the only nontrivial critical point of $\widehat{\varphi}_-$. Indeed, if $y \in W_0^{1,p}(Z)$ is a nontrivial critical point of $\widehat{\varphi}_+$, then as for x_0 (see the proof of Proposition 3.1), we can check that $y \in [0, a_+]$ and $y \in \text{int}C_+$. So, if $y \neq x_0$, then $y \in \text{int}C_+$ is a third nontrivial constant sign solution of (1.1). Similarly for φ_-.

Let

$$a = \widehat{\varphi}_+(x_0) < 0 \ \text{ and } \ b = 0.$$

Note that $\varphi_+^a = \{x_0\}$. By virtue of Theorem 2.2, we can find a continuous deformation $h : [0,1] \times (\widehat{\varphi}_+^0 \backslash K_0) \to \widehat{\varphi}_+^0$ such that

$$h(1, \widehat{\varphi}_+^0 \backslash K_0) \subseteq \widehat{\varphi}_+^a = \{x_0\}, \tag{1.25}$$

$$\begin{aligned}
&h(t,x_0) = x_0 \ \text{ for all } \ t \in [0,1], \\
&\widehat{\varphi}_+(h(t,x)) \leq \widehat{\varphi}_+(h(s,x)) \qquad\qquad (1.26) \\
&\text{for all } \ s,t \in [0,1], \ \ s \leq t \ \text{ and all } \ x \in \widehat{\varphi}_+^0 \backslash K_0.
\end{aligned}$$

We set

$$\gamma_+(t) = h(t, \varepsilon u_1), \ \ t \in [0,1].$$

Evidently, $\gamma_+ : [0,1] \to \widehat{\varphi}_+^0$ is a continuous path. Also,

$$\begin{aligned}
&\gamma_+(0) = h(0, \varepsilon u_1) = \varepsilon u_1 \ \text{ (since } h \text{ is a deformation),} \\
&\gamma_+(1) = h(1, \varepsilon u_1) = x_0 \ \text{ (see (1.25)).}
\end{aligned}$$

Therefore, γ_+ is a continuous path, which joins εu_1 with x_0. Moreover, from (1.26) and (1.24), we have

$$\widehat{\varphi}_+(\gamma(t)) \leq \widehat{\varphi}_+(\varepsilon u_1) < 0 \ \text{ for all } \ t \in [0,1]. \tag{1.27}$$

But note that

$$\begin{aligned} & \widehat{\varphi} \leq \widehat{\varphi}_+ \ \text{ (see hypothesis } H(f)_1(iv)) \\ \Rightarrow \quad & \widehat{\varphi}|_{\gamma_+} < 0 \ \text{ (see (1.27)).} \end{aligned} \tag{1.28}$$

In a similar fashion, we produce a continuous path γ_-, which connects $-\varepsilon u_1$ with v_0, such that

$$\widehat{\varphi}|_{\gamma_-} < 0. \tag{1.29}$$

Concatenating paths $\gamma_-, \widehat{\gamma}_0^{\varepsilon}$ and γ_+, we produce a $\gamma^* \in \Gamma$ such that

$$\widehat{\varphi}|_{\gamma^*} < 0 \ \text{ (see (1.24), (1.28) and (1.29)).}$$

This implies that $y_0 \neq 0$. Moreover,

$$\begin{aligned} & \widehat{\varphi}'(y_0) = 0 \\ \Rightarrow \quad & A(y_0) = \widehat{N}(y_0), \ \text{where } \widehat{N}(x)(\cdot) = \widehat{f}(\cdot, x(\cdot)) \text{ for all } x \in W_0^{1,p}(Z) \\ \Rightarrow \quad & -\triangle_p y_0(z) = \widehat{f}(z, y_0(z)) \ \text{ a.e. on } \ Z, \ y_0|_{\partial Z} = 0. \end{aligned}$$

But as for $\widehat{\varphi}_+$ and $\widehat{\varphi}_-$ (see the proof of Proposition 3.1), we can easily show that $y_0 \in [a_-, a_+]$, and hence

$$-\triangle_p y_0(z) = f(z, y_0(z)) \ \text{ a.e. on } \ Z, \ y_0|_{\partial Z} = 0.$$

Therefore, y_0 is a nontrivial solution of problem (1.1) and by nonlinear regularity theory, we have $y_0 \in C_0^1(\overline{Z})$.

4 Semilinear problem

In this section, we consider the semilinear case (i.e., $p = 2$) and by strengthening the condition on $f(z,\cdot)$ near the origin and using Morse theory, we produce a fourth nontrivial smooth solution. So, the problem under consideration is the following:

$$\left\{ \begin{array}{l} -\triangle x(z) = f(z, x(z)) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0. \end{array} \right\} \tag{1.30}$$

The hypotheses on the nonlinearity $f(z,x)$ are the following: $\underline{H(f)_2}$:] $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable;

(ii) for almost all $z \in Z$, the function

$$x \to f(z,x) \ \text{ is } \ C^1;$$

(iii) for almost all $z \in Z$, and all $x \in \mathbb{R}$,

$$|f'_x(z,x)| \leq a(z) + c|x|^{r-2}$$

with $a \in L^\infty(Z)_+$, $c > 0$ and $1 < r < 2^*$;

(iv) there exist $a_- < 0 < a_+$ such that

$$\begin{array}{l} f(z,a_-) = f(z,a_+) = 0 \ \text{ a.e. on } \ Z, \ \textit{and} \\ \left\{ \begin{array}{ll} 0 \leq f(z,x) & \text{for a.a. } z \in Z \text{ all } x \in [0,a_+] \\ f(z,x) \leq 0 & \text{for a.a. } z \in Z \text{ all } x \in [a_-,0] \end{array} \right. ; \end{array}$$

(v) there exist $\delta > 0$ and an integer $m \geq 3$ such that

$$\lambda_m \leq \frac{f(z,x)}{x} \leq \lambda_{m+1} \ \text{ for a.a. } \ z \in Z \ \text{ and all } \ x \in [-\delta, \delta],$$

with $\{\lambda_k\}_{k \geq 1}$ being the distinct eigenvalues of $(-\triangle, H^1_0(Z))$.

Theorem 4.1 *If hypotheses $H(f)_2$ hold, then problem (1.30) has at least four nontrivial solutions*

$$\begin{array}{l} x_0 \in \textit{int}C_+, \ \ v_0 \in -\textit{int}C_+ \\ y_0, w_0 \in C^1_0(\overline{Z}). \end{array}$$

Proof. From Theorem 3.1, we already have three nontrivial solutions

$$x_0 \in \text{int}C_+, \ v_0 \in -\text{int}C_+ \ \text{ and } \ y_0 \in C^1_0(\overline{Z}).$$

We need to produce the fourth nontrivial solution.

We know that $x_0 \in \text{int}C_+$ and $v_0 \in -\text{int}C_+$ are both local minimizers of $\widehat{\varphi}$ (see Proposition 3.1). Hence

$$C_k(\widehat{\varphi}, x_0) = C_k(\widehat{\varphi}, v_0) = \delta_{k,0}\mathbb{Z} \tag{1.31}$$

(see Chang [7] and Mawhin-Willem [24]). Also, from the proof of Theorem 3.1 we know that y_0 is a critical point of $\widehat{\varphi}$ of mountain pass-type. Moreover, as we did for x_0 and v_0 (see the proof of Proposition 3.1), we can check that

$$a_- < y_0(z) < a_+ \ \text{ for all } \ z \in \overline{Z}.$$

So, we can find an $r > 0$ small such that $\widehat{\varphi}|_{\overline{B}^{C^1_0(\overline{Z})}_{r}(y_0)}$ is C^2. Hence, from Theorem 1.6, p. 91 of Chang [7] (see also Corollary 8.5, p.195, of Mawhin-Willem [24]), we have

$$C_k(\widehat{\varphi}|_{C^1_0(\overline{Z})}, y_0) = \delta_{k,1}\mathbb{Z}. \tag{1.32}$$

But from Chang [8] (see also Liu-Wu [23]), we know that

$$C_k(\widehat{\varphi}|_{C^1_0(\overline{Z})}, y_0) = C_k(\widehat{\varphi}, y_0) \ \text{ for all } \ k \geq 0. \tag{1.33}$$

From (1.32) and (1.33), it follows that

$$C_k(\widehat{\varphi}, y_0) = \delta_{k,1}\mathbb{Z}. \tag{1.34}$$

Also, from Proposition 1.1 of Li-Perera-Su [20], we know that

$$C_k(\widehat{\varphi}, 0) = \delta_{k,d_m}\mathbb{Z}, \tag{1.35}$$

where $d_m = \dim \overset{m}{\underset{i=1}{\oplus}} E(\lambda_i)$ and $E(\lambda_i)$ is the eigenspace corresponding to the eigenvalue λ_i.

Finally, consider the one-parameter family of functions $\widehat{\varphi}_t : H_0^1(Z) \to \mathbb{R}$, $t \in [0,1]$, defined by

$$\widehat{\varphi}_t(x) = \frac{1}{2}\|Dx\|_2^2 - t\int_Z \widehat{F}(z, x(z))dz \ \text{ for all } \ x \in H_0^1(Z).$$

Clearly for every $t \in [0,1]$, $\widehat{\varphi}_t \in C^1(H_0^1(Z))$ and $x \to \widehat{\varphi}_t'(x)$, $x \to \partial_t\widehat{\varphi}_t(x)$ are Lipschitz continuous. Moreover, for every $t \in [0,1]$, $x \to \widehat{\varphi}_t(x)$ is coercive. Therefore, we can apply Lemma 2.4 of Perera-Schecter [26] and obtain

$$C_k(\widehat{\varphi}_0, \infty) = C_k(\widehat{\varphi}_1, \infty) \ \text{ for all } \ k \geq 0. \tag{1.36}$$

But note that

$$\widehat{\varphi}_0(x_0) = \frac{1}{2}\|Dx\|_p^p \ \text{ for all } \ x \in W_0^{1,p}(Z), \ \widehat{\varphi}_1 = \widehat{\varphi}.$$

The functional $\widehat{\varphi}_0$ has only one critical point, the origin, which clearly is a global minimizer of $\widehat{\varphi}_0$. Therefore

$$C_k(\widehat{\varphi}_0, \infty) = C_k(\widehat{\varphi}_0, 0) = \delta_{k,0}\mathbb{Z}. \tag{1.37}$$

From (1.36) and (1.37), it follows that

$$C_k(\widehat{\varphi}, \infty) = \delta_{k,0}\mathbb{Z}. \tag{1.38}$$

Suppose that $\{0, x_0, v_0, y_0\}$ were the only critical points of $\widehat{\varphi}$. Then, from (1.31), (1.34), (1.35), (1.38) and the Poincare-Hopf formula (see (1.5)), we have

$$\begin{aligned} & (-1)^0 + (-1)^0 + (-1)^1 + (-1)^{d_m} = (-1)^0 \\ \Rightarrow \quad & (-1)^{d_m} = 0, \ \text{ a contradiction.} \end{aligned}$$

So, there exists a fourth nontrivial critical point w_0 of $\widehat{\varphi}$, distinct from x_0, v_0, y_0. Again, we have $w_0 \in [a_-, a_+]$; so it solves (1.1) and by regularity theory $w_0 \in C_0^1(\overline{Z})$.

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