

**Periodic Bifurcation  
For Semilinear Differential Equations  
With Lipschitzian Perturbations in Banach Spaces**

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### Abstract.

Let  $A : D(A) \rightarrow E$ ,  $D(A) \subset E$ , be an infinitesimal generator either of an analytic compact semigroup or of a contractive  $C_0$ -semigroup of linear operators acting in a Banach space  $E$ . In this paper we give both necessary and sufficient conditions for bifurcation of  $T$ -periodic solutions for the equation  $\dot{x} = Ax + f(t, x) + \varepsilon g(t, x, \varepsilon)$  from a  $k$ -parameterized family of  $T$ -periodic solutions of the unperturbed equation corresponding to  $\varepsilon = 0$ . We show that by means of a suitable modification of the classical Mel'nikov approach we can construct a bifurcation function and to formulate the conditions for the existence of bifurcation in terms of the topological index of the bifurcation function. To do this, since the perturbation term  $g$  is only Lipschitzian we need to extend the classical Lyapunov-Schmidt reduction to the present nonsmooth case.

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## 1 Introduction

The aim of this paper is to give both necessary and sufficient conditions for the bifurcation of  $T$ -periodic solutions of the semi-linear differential equation

$$\dot{x} = Ax + f(t, x) + \varepsilon g(t, x, \varepsilon) \quad (1.1)$$

from a  $k$ -parameterized family of  $T$ -periodic solutions of the unperturbed system, obtained from (1.1) by letting  $\varepsilon = 0$ . Here  $A : D(A) \rightarrow E$ ,  $D(A) \subset E$ , is an infinitesimal generator either of an analytic compact semigroup or of a contractive  $C_0$ -semigroup of linear operators acting in the Banach space  $E$ , the nonlinear operators  $f \in C^1(\mathbb{R} \times E, E)$  and  $g \in C^0(\mathbb{R} \times E \times [0, 1], E)$  are  $T$ -periodic in the first variable.

In the case when the unperturbed system is autonomous, the problem was studied by Henry in ([7], Ch. 8), where it is assumed that  $g$  is differentiable in the second variable. The author provided sufficient conditions for bifurcation of  $T$ -periodic solutions from a  $T$ -periodic cycle  $x_0$ . The main tool employed in that paper is the classical Lyapunov-Schmidt reduction, see for instance Chow and Hale ([4], Ch. 2, § 4). These conditions are formulated in terms of the existence of nondegenerate zeros of an analogue of the Malkin's bifurcation function [12] for an infinite dimensional Banach space.

In the finite dimensional case, using topological degree arguments, Felmer and Manásevich [5] replaced the assumption of the existence of nondegenerate zeros of the bifurcation function by the requirement that the topological degree of the bifurcation function be different from zero with respect to a suitable set. Beginning with [5] there has been a great amount of work for developing bifurcation results by using the topological degree theory, see e.g. Henrard and Zanolin [6] for bifurcation from a cycle of a Hamiltonian system and Kamenskii, Makarenkov and Nistri [8] for bifurcation from a cycle of a self-oscillating system. In the present paper we avoid the requirement that the zeros of the bifurcation function are nondegenerate; instead we formulate suitable assumptions on the bifurcation function in terms of the topological degree to obtain for (1.1) results similar to those of ([7], Ch. 8). To this end, we prove an extension of the classical Lyapunov-Schmidt reduction

as presented in ([4], Ch. 2, § 4) to the case when the perturbation  $g$  is Lipschitzian. We mention in the sequel some problems involving partial differential equations which reduce to the situation considered in this paper. In Chow and Hale [4, Ch. 8, § 6] and Schaeffer and Golubitsky [14] the problem of the dependence of the steady states in chemical reaction models on the relative diffusion coefficients leads to the consideration of perturbed equations in Banach spaces with the property that the corresponding unperturbed equations have a family of solutions.

Another example of such a situation is presented in Berti and Bolle [2], where the problem of finding periodic solutions of a nonlinear wave equation by variational methods gives rise to an unperturbed equation with a family of periodic solutions.

The paper is organized as follows. A modified Lyapunov-Schmidt reduction for Lipschitzian perturbations of an operator of the form  $(P - I)$ , with  $P \in C^1(E, E)$ , is obtained in Section 2. In order to apply the results of Section 2, some relevant properties of the Poincaré map for system (1.1) are established in Section 3. Both necessary and sufficient conditions for bifurcation of periodic solutions to (1.1) are obtained in Section 4. Finally, in the appendix of Section 5 we give a proof of a technical result needed in Section 3.

## 2 Lyapunov-Schmidt reduction

Let  $E$  be a Banach space and consider the function  $F : E \times [0, 1] \rightarrow E$  given by

$$F(\xi, \varepsilon) = P(\xi) - \xi + \varepsilon Q(\xi, \varepsilon),$$

where  $P : E \rightarrow E$  and  $Q : E \times [0, 1] \rightarrow E$ . Assume that

(A<sub>1</sub>) there exist  $h_0 \in \mathbb{R}^k$ ,  $r_0 > 0$  and a function  $S \in C^1(B_{\mathbb{R}^k}(h_0, r_0), E)$  such that

$$P(\xi) = \xi \quad \text{for any } \xi \in \mathcal{Z} = \{S(h) : h \in B_{\mathbb{R}^k}(h_0, r_0)\}.$$

Here and in what follows,  $B_X(c, r)$  denotes the ball in the normed space  $X$  centered at  $c$  with radius  $r > 0$ . It is well known that, under the assumption (A<sub>1</sub>) with  $P \in C^1(E, E)$  and  $Q \in C^1(E \times [0, 1], E)$ , the Lyapunov-Schmidt reduction ([4], Ch. 2, § 4) allows to solve the equation

$$F(\xi, \varepsilon) = 0, \tag{2.1}$$

for  $\varepsilon > 0$  sufficiently small. The next theorem extends this result to the case when  $Q$  satisfies the following Lipschitz condition:

(L) For any  $R > 0$  there exists  $L(R) > 0$  such that

$$\|Q(\xi_1, \varepsilon) - Q(\xi_2, \varepsilon)\| \leq L(R) \|\xi_1 - \xi_2\|$$

whenever  $\xi_1, \xi_2 \in B_E(0, R)$  and  $\varepsilon \in [0, 1]$ .

**Theorem 2.1** *Let  $P \in C^1(E, E)$ ,  $Q \in C^0(E \times [0, 1], E)$ , where  $E$  is a Banach space. Assume that  $Q$  satisfies (L). Moreover, assume (A<sub>1</sub>) and*

$$(A_2) \dim S'(h_0)\mathbb{R}^k = k.$$

Let  $E_{1,h} = S'(h)\mathbb{R}^k$ . Let  $E_{2,h}$  be any subspace of  $E$  such that  $E = E_{1,h} \oplus E_{2,h}$  and assume that

(A<sub>3</sub>) there exists  $r_0 > 0$  such that both the projectors  $\pi_{1,h}$  of  $E$  onto  $E_{1,h}$  along  $E_{2,h}$  and  $\pi_{2,h}$  of  $E$  onto  $E_{2,h}$  along  $E_{1,h}$  are continuous in  $h \in B_{\mathbb{R}^k}(h_0, r_0)$ ,

(A<sub>4</sub>) for  $\xi_0 = S(h_0)$  we have

$$\pi_{2,h_0}(P'(\xi_0) - I)\pi_{2,h_0} \text{ is invertible on } E_{2,h_0}. \tag{2.2}$$

Then there exist  $0 < r_2 < r_1 < r_0$  and functions  $H : B_E(\xi_0, r_1) \rightarrow \mathbb{R}^k$ , with  $H(\xi) \rightarrow h_0$  as  $\xi \rightarrow \xi_0$  and  $\beta : B_{\mathbb{R}^k}(h_0, r_1) \times [0, r_1] \rightarrow E$ ,  $\beta(\cdot, \varepsilon) \in C^0(B_{\mathbb{R}^k}(h_0, r_1), E)$ ,  $\|\beta(h, \varepsilon)\| \leq M\varepsilon$  for some  $M > 0$ , any  $h \in B_{\mathbb{R}^k}(h_0, r_1)$  and any  $\varepsilon \in [0, r_1]$  with

$$\beta(h, \varepsilon) \in E_{2,h}. \tag{2.3}$$

Furthermore;

$$\beta(h, \varepsilon)/\varepsilon \rightarrow -(\pi_{2,h}(P'(S(h)) - I)\pi_{2,h})^{-1} \pi_{2,h}Q(S(h), 0) \text{ as } \varepsilon \rightarrow 0, \tag{2.4}$$

uniformly in  $h \in B_{\mathbb{R}^k}(h_0, r_1)$

and the following properties hold:

1) if  $(\xi, \varepsilon) \in B_E(\xi_0, r_2) \times [0, r_2]$  is a solution to equation (2.1) then  $(h, \varepsilon)$ , where  $h = H(\xi)$ , is a solution to

$$(S'(h))^{-1} \pi_{1,h} [P(\beta(h, \varepsilon) + S(h)) - (\beta(h, \varepsilon) + S(h)) + \varepsilon Q(\beta(h, \varepsilon) + S(h), \varepsilon)] = 0. \tag{2.5}$$

2) if  $(h, \varepsilon) \in B_{\mathbb{R}^k}(h_0, r_1) \times [0, r_1]$  solves (2.5) then  $(\xi, \varepsilon)$  solves (2.1), with

$$\xi = \beta(h, \varepsilon) + S(h). \tag{2.6}$$

Note, that the existence of  $(S'(h))^{-1}$  on  $E_{1,h}$  for  $h \in \mathbb{R}^k$  sufficiently close to  $h_0$  is guaranteed by (A<sub>2</sub>) and (A<sub>3</sub>). To prove Theorem 2.1, we need the following version of the implicit function theorem.

**Lemma 2.1** Let  $E$  be a Banach space and  $V \subset \mathbb{R}^k$  be an open bounded set. Consider a family of projectors  $\{\pi_h\}_{h \in V}$  on  $E$  continuous in  $h$  and let  $E_h = \pi_h E$  for any  $h \in V$ . Let  $\Phi_{h,\varepsilon} : E_h \rightarrow E_h$  be defined by

$$\Phi_{h,\varepsilon}(\beta) = \tilde{P}(h, \beta) + \varepsilon \tilde{Q}(h, \beta, \varepsilon), \tag{2.7}$$

where  $\tilde{P} \in C^0(\mathbb{R}^k \times E, E)$ ,  $\tilde{Q} \in C^0(\mathbb{R}^k \times E \times [0, 1], E)$ ,  $\tilde{P}(h, \cdot), \tilde{Q}(h, \cdot, \varepsilon) : E_h \rightarrow E_h$  for any  $h \in V, \varepsilon \in [0, 1]$ . Assume that

1. the continuity of  $\tilde{P}$  in the first variable is uniform on any bounded subset of  $V \times E$ ,
2.  $\tilde{P}$  is differentiable with respect to the second variable and the derivative is continuous in  $\bar{V} \times E$ ,
3.  $\tilde{Q}$  is Lipschitzian in the second variable uniformly on any bounded subset of  $V \times E \times [0, 1]$ .
4.  $\tilde{P}(h, 0) = 0$  for any  $h \in V$ ,
5.  $\pi_h \tilde{P}'_{\beta}(h, 0) : E_h \rightarrow E_h$  is invertible for any  $h \in V$  and  $(\pi_h \tilde{P}'_{\beta}(h, 0))^{-1} \pi_h$  is continuous in  $h \in V$ .

Then there exist  $r > 0, M > 0$  and a function  $\beta : V \times [0, r] \rightarrow E, \beta(\cdot, \varepsilon) \in C^0(V, E)$  such that

- a)  $\beta(h, \varepsilon) \in E_h$  for any  $h \in V, \varepsilon \in [0, r]$ ,
- b)  $\Phi_{h,\varepsilon}(\beta(h, \varepsilon)) = 0$  for any  $h \in V, \varepsilon \in [0, r]$ ,
- c)  $\beta(h, \varepsilon)$  is the only zero of  $\Phi_{h,\varepsilon}$  in  $B_{E_h}(0, r)$  for any  $h \in V, \varepsilon \in [0, r]$ ,
- d)  $\|\beta(h, \varepsilon)\| \leq M\varepsilon$  for any  $h \in V, \varepsilon \in [0, r]$ .

Although Lemma 2.1 looks well-known, the authors were unable to find a proof of it in the literature, thus for the reader convenience we provide a proof of Lemma 2.1 in the Appendix of Section 5.

*Proof of Theorem 2.1.* In order to define the function  $\beta$  we consider the following auxiliary function  $\Phi_{h,\varepsilon} \in C^0(E_{2,h}, E_{2,h})$  given by

$$\Phi_{h,\varepsilon}(\beta) = \pi_{2,h} [P(\pi_{2,h}\beta + S(h)) - (\pi_{2,h}\beta + S(h)) + \varepsilon Q(\beta + S(h), \varepsilon)].$$

Since  $P \in C^1(E, E)$  and  $S \in C^1(B_{\mathbb{R}^k}(h_0, r_0), E)$  then assumptions 1 and 2 of Lemma 2.1 are satisfied.

By our assumptions we have that the application  $(h, \beta, \varepsilon) \rightarrow \Phi_{h,\varepsilon}(\beta)$  is Lipschitzian in  $\beta$  uniformly on any bounded subset of  $B_{\mathbb{R}^k}(h_0, r_0) \times E \times [0, 1]$  and taking into account  $(A_1)$  we have

- 1)  $\Phi_{h,0}(0) = 0$  for any  $h \in B_{\mathbb{R}^k}(h_0, r_0)$ .

By assumptions  $(A_3)$ - $(A_4)$   $r_0 > 0$  can be diminished in such a way that

- 2)  $(\Phi_{h,0})'(0) = \pi_{2,h}(P'(S(h)) - I)\pi_{2,h}$  is an invertible operator from  $E_{2,h}$  to  $E_{2,h}$  for  $h \in B_{\mathbb{R}^k}(h_0, r_0)$ .

Therefore, Lemma 2.1 applies with

$$\tilde{P}(h, \beta) = \pi_{2,h}[P(\pi_{2,h}\beta + S(h)) - (\pi_{2,h}\beta + S(h))],$$

$$\tilde{Q}(h, \beta, \varepsilon) = \pi_{2,h}Q(\beta + S(h), \varepsilon) \quad \text{and} \quad V = B_{\mathbb{R}^k}(h_0, r_0).$$

Thus there exist  $r_1 \in [0, r_0]$ ,  $M > 0$  and a function  $\beta(\cdot, \varepsilon) \in C^0(B_{\mathbb{R}^k}(h_0, r_1), E)$  satisfying Properties a), b), c) and d) of Lemma 2.1. In particular, from  $(A_1)$  and Property b) we have

$$\begin{aligned} &\pi_{2,h}[P(\beta(h, \varepsilon) + S(h)) - (\beta(h, \varepsilon) + S(h)) - \\ &- (P(S(h)) - S(h)) + \varepsilon Q(\beta(h, \varepsilon) + S(h), \varepsilon)] = 0 \end{aligned}$$

or equivalently

$$\pi_{2,h} [(P'(S(h)) - I)\pi_{2,h}\beta(h, \varepsilon) + o(\beta(h, \varepsilon)) + \varepsilon Q(\beta(h, \varepsilon) + S(h), \varepsilon)] = 0,$$

for any  $h \in B_{\mathbb{R}^k}(h_0, r_1)$ . Therefore

$$\beta(h, \varepsilon) = -(\pi_{2,h}(P'(S(h)) - I)\pi_{2,h})^{-1}(\pi_{2,h}o(\beta(h, \varepsilon)) + \pi_{2,h}\varepsilon Q(\beta(h, \varepsilon) + S(h), \varepsilon)).$$

Due to Property d), the last equation implies (2.4).

We now proceed to define the function  $H$ . By  $(A_2)$  we have that  $r_1 > 0$  can be taken sufficiently small such that  $S'(h) : \mathbb{R}^k \rightarrow E_{1,h}$  is invertible. Thus we can define the function  $\Phi_\xi : \mathbb{R}^k \rightarrow \mathbb{R}^k$ ,  $\xi \in E$ , as follows

$$\Phi_\xi(h) = (S'(h))^{-1} \pi_{1,h}(\xi - S(h)), \quad h \in B_{\mathbb{R}^k}(h_0, r_1).$$

We have the following properties for  $\Phi_\xi$ .

- 1)  $\Phi_{\xi_0}$  is differentiable at  $h_0$ .
- 2)  $(\Phi_{\xi_0})'(h_0) = (S'(h_0))^{-1} \pi_{1,h_0}(-S'(h_0)) = -I$ , namely  $(\Phi_{\xi_0})'(h_0)$  is an invertible  $k \times k$ -matrix.

Observe that property 1) is a direct consequence of the fact that  $\xi_0 - S(h_0) = 0$  and the continuity of the function  $h \rightarrow S^{-1}(h)\pi_{1,h}$ . Therefore the differentiability of  $\pi_{1,h}$  at  $h = h_0$  is not necessary for the validity of 1).

Let  $\delta > 0$  be such that  $h_0$  is the only zero of  $\Phi_{\xi_0}$  in  $B_{\mathbb{R}^k}(h_0, \delta)$ . By ([10], Theorem 6.3) we can take  $\delta > 0$  sufficiently small in such a way that  $d(\Phi_{\xi_0}, B_{\mathbb{R}^k}(h_0, \delta)) = (-1)^k$ . By the continuity property of the topological degree  $r_1 > 0$  can be diminished, if necessary, in such a way that  $d(\Phi_\xi, B_{\mathbb{R}^k}(h_0, \delta)) = (-1)^k$  for any  $\xi \in B_E(\xi_0, r_1)$ . Therefore, for any  $\xi \in B_E(\xi_0, r_1)$  there exists  $H(\xi) \in B_{\mathbb{R}^k}(h_0, \delta)$  such that  $\Phi_\xi(H(\xi)) = 0$ . Let us show that  $H(\xi) \rightarrow h_0$  as  $\xi \rightarrow \xi_0$ . Indeed, arguing by contradiction we would have a sequence  $\{\xi_n\}_{n \in \mathbb{N}} \subset B_E(\xi_0, r_1)$ ,  $h_* \in B_{\mathbb{R}^k}(h_0, \delta)$  such that  $H(\xi_n) \rightarrow h_* \neq h_0$  as  $n \rightarrow \infty$  and thus  $\Phi_{\xi_0}(h_*) = 0$ , contradicting the choice of  $\delta > 0$ . Therefore

$$\pi_{1,H(\xi)}(\xi - S(H(\xi))) = 0, \quad \xi \in B_E(\xi_0, r_1). \tag{2.8}$$

Moreover, we consider  $r_2 \in (0, r_1]$  sufficiently small to have

$$\|\xi - S(H(\xi))\| \leq r_1, \quad \xi \in B_E(\xi_0, r_2). \tag{2.9}$$

We are now in a position to complete the proof. For this let  $(\xi, \varepsilon) \in B_E(\xi_0, r_2) \times [0, r_2]$  satisfying (2.1). Then  $(\xi, \varepsilon)$  also satisfies

$$\begin{cases} \pi_{1,H(\xi)}[P(\xi - S(H(\xi)) + S(H(\xi))) - \\ - (\xi - S(H(\xi)) + S(H(\xi))) + \varepsilon Q(\xi - S(H(\xi)) + S(H(\xi)), \varepsilon)] = 0, \\ \pi_{2,H(\xi)}[P(\xi - S(H(\xi)) + S(H(\xi))) - \\ - (\xi - S(H(\xi)) + S(H(\xi))) + \varepsilon Q(\xi - S(H(\xi)) + S(H(\xi)), \varepsilon)] = 0. \end{cases}$$

From (2.8), (2.9) and Property c) of Lemma 2.1 we have

$$\begin{cases} \pi_{1,H(\xi)}[P(\xi - S(H(\xi)) + S(H(\xi))) - \\ - (\xi - S(H(\xi)) + S(H(\xi))) + \varepsilon Q(\xi - S(H(\xi)) + S(H(\xi)), \varepsilon)] = 0, \\ \beta(H(\xi), \varepsilon) = \xi - S(H(\xi)). \end{cases} \tag{2.10}$$

Therefore,

$$\pi_{1,h}[P(\beta(h, \varepsilon) + S(h)) - (\beta(h, \varepsilon) + S(h)) + \varepsilon Q(\beta(h, \varepsilon) + S(h), \varepsilon)] = 0 \tag{2.11}$$

has a solution  $h = H(\xi)$ . Since  $r_1 > 0$  has been chosen in such a way that  $S'(h)$  is invertible on  $E_{1,h}$  for  $h \in B_{\mathbb{R}^k}(h_0, r_1)$ , then (2.11) can be rewritten as (2.5). Assume now that (2.5) is satisfied with some  $(h_*, \varepsilon_*) \in B_{\mathbb{R}^k}(h_0, r_1) \times [0, r_1]$ . Define  $\tilde{\xi} \in E$  as

$$\tilde{\xi} = \beta(h_*, \varepsilon_*) + S(h_*). \tag{2.12}$$

Since  $(S'(h_*))^{-1}$  is invertible then  $\pi_{1,h_*}[P(\tilde{\xi}) - \tilde{\xi} + \varepsilon_* Q(\tilde{\xi}, \varepsilon_*)] = 0$ . On the other hand from (2.12) we have

$$\begin{aligned} \pi_{2,h_*}[P(\pi_{2,h_*}\beta(h_*, \varepsilon_*) + S(h_*)) - (\pi_{2,h_*}\beta(h_*, \varepsilon_*) + S(h_*)) + \\ + \varepsilon_* Q(\beta(h_*, \varepsilon_*) + S(h_*), \varepsilon_*)] = \pi_{2,h_*}[P(\tilde{\xi}) - \tilde{\xi} + \varepsilon_* Q(\tilde{\xi}, \varepsilon_*)]. \end{aligned}$$

Thus  $(\xi_*, \varepsilon_*)$  solves (2.1) and so the proof is complete. □

The following two results are consequences of Theorem 2.1 and they provide, respectively, a necessary and a sufficient condition for the existence of solutions to (2.1) near  $\xi_0$  when  $\varepsilon > 0$  is sufficiently small. These conditions are expressed in terms of the following bifurcation function

$$\begin{aligned} M(h) = & (S'(h))^{-1} \pi_{1,h}[Q(S(h_0), 0) - \\ & - (P'(S(h)) - I) (\pi_{2,h}(P'(S(h)) - I)\pi_{2,h})^{-1} \pi_{2,h}Q(S(h), 0)], \end{aligned}$$

where  $h$  varies in a sufficiently small neighborhood of  $h_0 \in \mathbb{R}^k$ .

**Theorem 2.2** *Let all the assumptions of Theorem 2.1 be satisfied. Assume that there exist sequences  $\varepsilon_n \rightarrow 0$  and  $\xi_n \rightarrow \xi_0$  as  $n \rightarrow \infty$  such that  $(\xi_n, \varepsilon_n)$  solves (2.1). Then*

$$M(h_0) = 0. \tag{2.13}$$

*Proof.* By Theorem 2.1, for  $n \geq n_0$ , with  $n_0 \in \mathbb{N}$  sufficiently large, we have that

$$\begin{aligned} & (S'(h_n))^{-1} \pi_{1,h_n} [P(\beta(h_n, \varepsilon_n) + S(h_n)) - \\ & - (\beta(h_n, \varepsilon_n) + S(h_n)) + \varepsilon_n Q(\beta(h_n, \varepsilon_n) + S(h_n), \varepsilon_n)] = 0 \end{aligned} \quad (2.14)$$

where  $h_n = H(\xi_n)$ . On the other hand, by  $(A_1)$ ,  $n_0$  can be chosen sufficiently large in such a way that

$$P(S(h_n)) - S(h_n) = 0 \quad \text{for } n \geq n_0.$$

Thus, for  $n \geq n_0$ , (2.14) can be rewritten as

$$\begin{aligned} & (S'(h_n))^{-1} \pi_{1,h_n} [(P'(S(h_n)) - I) \frac{\beta(h_n, \varepsilon_n)}{\varepsilon_n} + \\ & + \frac{o(\beta(h_n, \varepsilon_n))}{\varepsilon_n} + Q(\beta(h_n, \varepsilon_n) + S(h_n), \varepsilon_n)] = 0. \end{aligned} \quad (2.15)$$

By means of property (2.4) we can pass to the limit as  $n \rightarrow \infty$  in (2.15) to obtain (2.13).  $\square$

**Theorem 2.3** *Let all the assumptions of Theorem 2.1 be satisfied. Assume that*

$$h_0 \text{ is an isolated zero of } M \quad (2.16)$$

and

$$\text{ind}(h_0, M) \neq 0. \quad (2.17)$$

Then, for any  $\varepsilon > 0$  sufficiently small there exists  $\xi_\varepsilon \in E$  such that

$$F(\xi_\varepsilon, \varepsilon) = 0$$

and

$$\xi_\varepsilon \rightarrow \xi_0 \quad \text{as } \varepsilon \rightarrow 0. \quad (2.18)$$

*Proof.* Let  $r_1 > 0$  be as given by Theorem 2.1. Since

$$P(S(h)) = S(h) \quad \text{for any } h \in B_{\mathbb{R}^k}(h_0, r_1) \quad (2.19)$$

then the zeros of the function

$$\begin{aligned} \Phi(h, \varepsilon) &= (S'(h))^{-1} \pi_{1,h} [P(\beta(h, \varepsilon) + S(h)) - \\ & - (\beta(h, \varepsilon) + S(h)) + \varepsilon Q(\beta(h, \varepsilon) + S(h), \varepsilon)] \end{aligned}$$

coincide with the zeros of the function

$$\begin{aligned} M_\varepsilon(h) &= (S'(h))^{-1} \pi_{1,h} [(P'(S(h)) - I) \frac{\beta(h, \varepsilon)}{\varepsilon} + \\ & + \frac{o(\beta(h, \varepsilon))}{\varepsilon} + Q(\beta(h, \varepsilon) + S(h), \varepsilon)]. \end{aligned}$$

In order to apply Theorem 2.1, we show now that  $r \in (0, r_1]$  can be chosen in such a way that the function  $M_\varepsilon$  has zeros in  $B_{\mathbb{R}^k}(h_0, r)$  for any  $\varepsilon > 0$  sufficiently small.

By condition (2.16),  $r > 0$  can be chosen sufficiently small in such a way that

$$\text{the only zero of } M \text{ in } B_{\mathbb{R}^k}(h_0, r) \text{ is } h_0. \tag{2.20}$$

Therefore, by condition (2.17) we have

$$d(M, B_{\mathbb{R}^k}(h_0, r)) = \text{ind}(h_0, M) \neq 0.$$

On the other hand from property (2.4) we have that

$$M_\varepsilon(h) \rightarrow M(h) \quad \text{as } \varepsilon \rightarrow 0 \tag{2.21}$$

uniformly with respect to  $h \in B_{\mathbb{R}^k}(h_0, r)$ . Thus we conclude that

$$d(M_\varepsilon, B_r(h_0)) \neq 0$$

for  $\varepsilon \in (0, \varepsilon_0]$ , where  $\varepsilon_0 > 0$  is sufficiently small. Thus for any  $\varepsilon \in (0, \varepsilon_0]$  there exists  $h_\varepsilon$  such that  $M_\varepsilon(h_\varepsilon) = 0$ . Moreover, we have that

$$h_\varepsilon \rightarrow h_0 \quad \text{as } \varepsilon \rightarrow 0$$

otherwise  $M$  would have zeros in  $B_{\mathbb{R}^k}(h_0, r)$  different from  $h_0$ , contradicting (2.20). Finally, (2.18) follows from (2.6). □

In finite dimensional spaces, results similar to previous Theorems 2.2 and 2.3 have been recently obtained by Buica, Llibre and Makarenkov [3], where the uniqueness of the bifurcating periodic solutions is also proved.

### 3 The Poincaré map

Since the definition of the Poincaré map for system (1.1) on the time interval  $[0, T]$  depends on the assumptions on the linear unbounded operator  $A$ , we specify in (C1) and (C2) below the two cases that we consider for  $A$  in the paper.

(C1) The operator  $A$  is a generator of an analytic compact semigroup  $e^{At}$  in  $E$ . The operators  $f, g$  are subordinated to some  $A^{-\alpha}$ ,  $0 < \alpha < 1$  (see e.g. [11]), the operator  $f(\cdot, A^{-\alpha}\cdot)$  is differentiable in the second variable and the operators  $f'_{(2)}(\cdot, A^{-\alpha}\cdot)$ ,  $g(\cdot, A^{-\alpha}\cdot, \cdot)$  are continuous in  $\mathbb{R} \times E$  and  $\mathbb{R} \times E \times [0, 1]$  respectively and they satisfy a Lipschitz condition in the second variable uniformly with respect to the others.

(C2) The operator  $A$  is a generator of a  $C_0$ -semigroup  $e^{At}$ . The semigroup  $e^{At}$  is contractive, namely

$$\|e^{At}\| \leq e^{-\gamma t},$$

where  $\gamma > 0$ . The operators  $f$  and  $g$  are continuous from  $\mathbb{R} \times E \rightarrow E$  and verify the inequality

$$\chi(f(t, \Omega)) \leq k\chi(\Omega), \quad \chi(g(t, \Omega, \varepsilon)) \leq k\chi(\Omega),$$

where  $\chi$  is the Hausdorff measure of noncompactness<sup>1</sup> in the space  $E$ ,  $k \geq 0$  and  $q = k/\gamma < 1$ . The operator  $f$  is differentiable in the second variable and the operators  $f'_{(2)}$  and  $g$  are continuous in  $\mathbb{R} \times E$  and  $\mathbb{R} \times E \times [0, 1]$  respectively and they satisfy a Lipschitz condition in the second variable uniformly with respect to the others.

It is a classical result (see e.g. [11]) that (C1) and (C2) ensures respectively that the integral equations

$$x(t) = e^{At}\xi + \int_0^t A^\alpha e^{A(t-s)} [f(s, A^{-\alpha}x(s)) + \varepsilon g(s, A^{-\alpha}x(s), \varepsilon)] ds, \tag{3.1}$$

$$x(t) = e^{At}\xi + \int_0^t e^{A(t-s)} [f(s, x(s)) + \varepsilon g(s, x(s), \varepsilon)] ds \tag{3.2}$$

have a unique solution  $x(\cdot)$  defined on some interval  $[0, d]$ ,  $d > 0$ . By means of this function  $x$  we can define the shift operator as follows.

**Definition 3.1** Let  $x : [0, d] \times E \times [0, 1] \rightarrow E$  be defined at  $(t, \xi, \varepsilon)$  as  $x(t, \xi, \varepsilon) = x(t)$  for all  $t \in [0, d]$ . If for some  $\xi \in E$  and  $\varepsilon \in [0, 1]$  we have that  $x(\cdot, \xi, \varepsilon)$  is defined on the whole time interval  $[0, T]$ , then for these values  $\xi$  and  $\varepsilon$ , we define the *Poincaré map* for system (1.1) as

$$\mathcal{P}_\varepsilon(\xi) = x(T, \xi, \varepsilon).$$

A crucial role in what follows is played by the following technical lemma.

**Lemma 3.1** Assume that either (C1) or (C2) is satisfied. Assume that for some  $\xi_0 \in E$  the shift operator  $(t, \xi, \varepsilon) \rightarrow x(t, \xi, \varepsilon)$  is well defined for  $t = T$ ,  $\xi = \xi_0$  and  $\varepsilon = 0$ . Then there exists  $r > 0$  such that this operator is well defined for  $t = T$ , any  $\xi \in B_E(\xi_0, r)$ , any  $\varepsilon \in [0, r]$  and the function

$$u(t, \xi, \varepsilon) = \frac{x(t, \xi, \varepsilon) - x(t, \xi, 0)}{\varepsilon}$$

is Lipschitz in the second variable uniformly in  $[0, T] \times B_E(\xi_0, r) \times (0, r]$ , namely there exists  $L > 0$  such that

$$\|u(t, \xi_1, \varepsilon) - u(t, \xi_2, \varepsilon)\| \leq L \|\xi_1 - \xi_2\|$$

for any  $t \in [0, T]$ ,  $\xi_1, \xi_2 \in B_E(\xi_0, r)$  and  $\varepsilon \in (0, r]$ .

<sup>1</sup>We recall (see [1]) that for a bounded set  $\Omega \subset E$  the Hausdorff measure of noncompactness is defined by the formula

$$\chi(\Omega) = \inf \{r > 0 : \text{there exists } (y_1, \dots, y_m) \text{ such that } \Omega \subset \cup_{i=1}^m B_E(y_i, r)\},$$

where  $m \in \mathbb{N}$ .

The continuous operator  $F : E \rightarrow E$  is called  $(q, \chi)$ -condensing if

$$\chi(F(t, \Omega)) \leq q\chi(\Omega)$$

for any bounded  $\Omega \in E$ .

*Proof.* The fact that the assumptions of the Lemma imply the existence of  $r > 0$  such that the operator  $(t, \xi, \varepsilon) \mapsto x(t, \xi, \varepsilon)$  is well defined, bounded and continuous on  $[0, T] \times B_E(\xi_0, r) \times [0, r]$  is well known, see, for instance, ([9], Theorem 5.2.5). In the sequel we have  $\alpha \neq 0$  if (C1) holds, while  $\alpha = 0$  if we assume (C2). Since  $A^{-\alpha}$  is either a compact operator or the identity then the operator  $(t, \xi, \varepsilon) \mapsto A^{-\alpha}x(t, \xi, \varepsilon)$  is well defined, bounded and continuous on  $[0, T] \times B_E(\xi_0, r) \times [0, r]$ . Therefore, taking into account that  $f'_x$  satisfies Lipschitz condition, there exists  $\widehat{M} > 0$  such that

$$\|f'_x(s, A^{-\alpha}\{\theta x(s, \xi_2, \varepsilon) + (1 - \theta)x(s, \xi_1, \varepsilon)\})\| \leq \widehat{M}$$

for any  $s \in [0, T]$ ,  $\theta \in [0, 1]$ ,  $\xi_1, \xi_2 \in B_E(\xi_0, r)$ .

From the continuous differentiability of  $f$  and the Lipschitz condition on  $g$  assumed in (C1) and (C2) we deduce the existence of  $\widetilde{M} > 0$  such that

$$\|f(t, A^{-\alpha}\xi)\| + \|g(t, A^{-\alpha}\xi, \varepsilon)\| \leq \widetilde{M}$$

for any  $t \in [0, T]$ ,  $\xi \in x([0, T], B_E(\xi_0, r), [0, r])$  and  $\varepsilon \in [0, r]$ .

Since  $A^{-\alpha}x([0, T], B_E(\xi_0, r), [0, r])$  is bounded then by using the Lipschitz condition on  $g$  we obtain the existence of  $\widetilde{L} > 0$  such that

$$\|g(s, A^{-\alpha}\xi_1, \varepsilon) - g(s, A^{-\alpha}\xi_2, \varepsilon)\| \leq \widetilde{L}\|\xi_1 - \xi_2\|$$

for any  $s \in [0, T]$ ,  $\xi_1, \xi_2 \in x([0, T], B_E(\xi_0, r), [0, r])$  and  $\varepsilon \in [0, r]$ . Furthermore, by [13, Theorem 6.13] there exists  $c > 0$  such that  $\sup_{t \in [0, T]} \|e^{At}\| < c$  and  $\|A^\alpha e^{At}\| < c/t^\alpha$ , where

either  $\alpha = 0$  or  $\alpha > 0$ .

Now, given an arbitrary  $\phi \in B_{E^*}(0, 1)$ , where  $E^*$  denotes the dual space of  $E$ , we evaluate  $\langle \phi, x(t, \xi_1, \varepsilon) - x(t, \xi_2, \varepsilon) \rangle$  as follows

$$\begin{aligned} & \langle \phi, x(t, \xi_1, \varepsilon) - x(t, \xi_2, \varepsilon) \rangle = \\ & = \langle \phi, e^{At}(\xi_1 - \xi_2) \rangle + \int_0^t \langle \phi, A^\alpha e^{A(t-s)} f'_x\left(s, A^{-\alpha}\{\theta(s, \xi_1, \xi_2, \varepsilon)x(s, \xi_1, \varepsilon) + \right. \\ & \quad \left. + (1 - \theta(s, \xi_1, \xi_2, \varepsilon))x(s, \xi_2, \varepsilon)\} A^{-\alpha}(x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon))\right) \rangle ds + \\ & \quad + \varepsilon \int_0^t \left\langle \phi, A^\alpha e^{A(t-s)} (g(s, A^{-\alpha}x(s, \xi_2, \varepsilon), \varepsilon) - g(s, A^{-\alpha}x(s, \xi_1, \varepsilon), \varepsilon)) \right\rangle ds \leq \\ & \leq c\|\xi_1 - \xi_2\| + \int_0^t \frac{c\widehat{M}}{(t-s)^\alpha} \|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\| ds + \\ & \quad + \varepsilon \int_0^t \frac{c\widetilde{L}}{(t-s)^\alpha} \|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\| ds. \end{aligned} \tag{3.3}$$

Since  $\phi$  is arbitrary we have

$$\begin{aligned} \|x(t, \xi_1, \varepsilon) - x(t, \xi_2, \varepsilon)\| &\leq c\|\xi_1 - \xi_2\| + \int_0^t \frac{c\widehat{M}}{(t-s)^\alpha} \|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\| ds + \\ &+ \varepsilon \int_0^t \frac{c\widetilde{L}}{(t-s)^\alpha} \|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\| ds. \end{aligned} \tag{3.4}$$

Dividing the last inequality by  $\|\xi_1 - \xi_2\|$ , one obtains that

$$\frac{\|x(t, \xi_1, \varepsilon) - x(t, \xi_2, \varepsilon)\|}{\|\xi_1 - \xi_2\|} \leq c + \int_0^t \frac{c\widehat{M} + \varepsilon c\widetilde{L}}{(t-s)^\alpha} \cdot \frac{\|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\|}{\|\xi_1 - \xi_2\|} ds.$$

Using the generalized Gronwall–Bellman lemma, see ([7], Lemma 7.1.1), from the last inequality we obtain that there exists  $M_x > 0$  such that

$$\frac{\|x(t, \xi_1, \varepsilon) - x(t, \xi_2, \varepsilon)\|}{\|\xi_1 - \xi_2\|} \leq M_x \tag{3.5}$$

for any  $(t, \xi_1, \xi_2, \varepsilon) \in [0, T] \times B_E(\xi_0, r) \times B_E(\xi_0, r) \times [0, r]$ .

For the function  $u(t, \xi, \varepsilon)$  we have the following inequality

$$\begin{aligned} &\langle \phi, u(t, \xi, \varepsilon) \rangle = \\ &= \left\langle \phi, \frac{1}{\varepsilon} \int_0^t A^\alpha e^{A(t-s)} [f(s, A^{-\alpha}x(s, \xi, \varepsilon)) - f(s, A^{-\alpha}x(s, \xi, 0))] ds + \right. \\ &\quad \left. + \int_0^t A^\alpha e^{A(t-s)} g(s, A^{-\alpha}x(s, \xi, \varepsilon), \varepsilon) ds \right\rangle \leq \\ &\leq \int_0^t \frac{c\widehat{M}\|u(s, \xi, \varepsilon)\|}{(t-s)^\alpha} ds + \int_0^t \frac{c\widetilde{M}}{(t-s)^\alpha} ds. \end{aligned}$$

Using again the generalized Gronwall–Bellman lemma from the last inequality we obtain that there exists  $M_u > 0$  such that

$$\|u(t, \xi, \varepsilon)\| \leq M_u \quad \text{for any } (t, \xi, \varepsilon) \in [0, T] \times B_E(\xi_0, r) \times [0, r]. \tag{3.6}$$

Observe that if a function  $\Psi : E \rightarrow E$  is differentiable and there exists  $L > 0$  such that  $\|\Psi'(\xi) - \Psi'(\zeta)\| \leq L\|\xi - \zeta\|$  for any  $\xi, \zeta \in E$  then

$$\begin{aligned} &\|\Psi(\xi_2) - \Psi(\xi_1) - \Psi(\zeta_2) + \Psi(\zeta_1)\| \leq \\ &\leq \sup_{0 \leq \theta \leq 1} \|\Psi'(\zeta_2 + \theta(\xi_2 - \zeta_2))\| \|\xi_2 - \xi_1 - \zeta_2 + \zeta_1\| + \\ &\quad + L \max\{\|\xi_2 - \xi_1\|, \|\zeta_2 - \zeta_1\|\} \|\xi_1 - \zeta_1\|. \end{aligned} \tag{3.7}$$

To prove this, it is sufficient to consider the real function  $\gamma : [0, 1] \rightarrow \mathbb{R}$  given by

$$\gamma(\tau) = \langle \phi, \Psi(\zeta_2 + \tau(\xi_2 - \zeta_2)) - \Psi(\zeta_1 + \tau(\xi_1 - \zeta_1)) \rangle, \quad \tau \in [0, 1].$$

By Lagrange theorem there exists  $\theta \in [0, 1]$  such that

$$\gamma(1) - \gamma(0) = \gamma'(\theta)$$

and then

$$\begin{aligned} & |\langle \phi, \Psi(\xi_2) - \Psi(\xi_1) - \Psi(\zeta_2) + \Psi(\zeta_1) \rangle| = \gamma(1) - \gamma(0) = \gamma'(\theta) = \\ & = |\langle \phi, \Psi'(\zeta_2 + \theta(\xi_2 - \zeta_2))(\xi_2 - \zeta_2) - \Psi'(\zeta_1 + \theta(\xi_1 - \zeta_1))(\xi_1 - \zeta_1) \rangle| \leq \\ & \leq |\langle \phi, \Psi'(\zeta_2 + \theta(\xi_2 - \zeta_2))(\xi_2 - \xi_1 - \zeta_2 + \zeta_1) \rangle| + \\ & \quad + |\langle \phi, (\Psi'(\zeta_2 + \theta(\xi_2 - \zeta_2)) - \Psi'(\zeta_1 + \theta(\xi_1 - \zeta_1)))(\xi_1 - \zeta_1) \rangle| \leq \\ & \leq \|\Psi'(\zeta_2 + \theta(\xi_2 - \zeta_2))\| \|\xi_2 - \xi_1 - \zeta_2 + \zeta_1\| + \\ & \quad + \|\Psi'(\zeta_2 + \theta(\xi_2 - \zeta_2)) - \Psi'(\zeta_1 + \theta(\xi_1 - \zeta_1))\| \|\xi_1 - \zeta_1\| \leq \\ & \leq \sup_{\theta \in [0,1]} \|\Psi'(\zeta_2 + \theta(\xi_2 - \zeta_2))\| \|\xi_2 - \xi_1 - \zeta_2 + \zeta_1\| + \\ & \quad + L\|(1 - \theta)\zeta_2 + \theta\xi_2 - (1 - \theta)\zeta_1 - \theta\xi_1\| \cdot \|\xi_1 - \zeta_1\| = \\ & = \sup_{\theta \in [0,1]} \|\Psi'(\zeta_2 + \theta(\xi_2 - \zeta_2))\| \|\xi_2 - \xi_1 - \zeta_2 + \zeta_1\| + \\ & \quad + L\|(1 - \theta)(\zeta_2 - \zeta_1) + \theta(\xi_2 - \xi_1)\| \|\xi_1 - \zeta_1\| \leq \\ & \leq \sup_{\theta \in [0,1]} \|\Psi'(\zeta_2 + \theta(\xi_2 - \zeta_2))\| \|\xi_2 - \xi_1 - \zeta_2 + \zeta_1\| + \\ & \quad + L \max\{\|\xi_2 - \xi_1\|, \|\zeta_2 - \zeta_1\|\} \|\xi_1 - \zeta_1\|. \end{aligned}$$

By the Lipschitz assumption on  $f'_x$  there exists  $\widehat{L} > 0$  such that

$$\|f'_x(s, A^{-1}\xi_1) - f'_x(s, A^{-1}\xi_2)\| \leq \widehat{L} \|\xi_1 - \xi_2\|$$

for any  $s \in [0, T]$ ,  $\xi_1, \xi_2 \in x([0, T], B_E(\xi_0, r), [0, r])$ .

Now, consider

$$\begin{aligned} & \frac{u(t, \xi_1, \varepsilon) - u(t, \xi_2, \varepsilon)}{\|\xi_1 - \xi_2\|} = \frac{x(t, \xi_1, \varepsilon) - x(t, \xi_1, 0) - x(t, \xi_2, \varepsilon) + x(t, \xi_2, 0)}{\varepsilon \|\xi_1 - \xi_2\|} = \\ & = \frac{1}{\varepsilon \|\xi_1 - \xi_2\|} \int_0^t A^\alpha e^{A(t-s)} [f(s, A^{-\alpha}x(s, \xi_1, \varepsilon)) - f(s, A^{-\alpha}x(s, \xi_1, 0)) - \\ & \quad - f(s, A^{-\alpha}x(s, \xi_2, \varepsilon)) + f(s, A^{-\alpha}x(s, \xi_2, 0))] ds + \\ & \quad + \frac{1}{\|\xi_1 - \xi_2\|} \int_0^t A^\alpha e^{A(t-s)} [g(s, A^{-\alpha}x(s, \xi_1, \varepsilon), \varepsilon) - g(s, A^{-\alpha}x(s, \xi_2, \varepsilon), \varepsilon)] ds \leq \\ & \leq \sup_{s \in [0, T], \theta \in [0, 1]} \|f'_x(s, A^{-\alpha}(x(s, \xi_1, 0) + \theta(x(s, \xi_2, 0) - x(s, \xi_1, 0))))\| \cdot \\ & \quad \cdot \int_0^t \frac{c \|A^{-\alpha}\|}{(t-s)^\alpha} \frac{\|x(s, \xi_1, 0) - x(s, \xi_1, \varepsilon) - x(s, \xi_2, 0) + x(s, \xi_2, \varepsilon)\|}{\varepsilon \|\xi_1 - \xi_2\|} ds + \\ & \quad + \sup_{s \in [0, T]} \max \left\{ \frac{\|x(s, \xi_1, 0) - x(s, \xi_1, \varepsilon)\|}{\varepsilon}, \frac{\|x(s, \xi_2, 0) - x(s, \xi_2, \varepsilon)\|}{\varepsilon} \right\}. \end{aligned}$$

$$\begin{aligned} & \int_0^t \frac{c\widehat{L}}{(t-s)^\alpha} \frac{\|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\|}{\|\xi_1 - \xi_2\|} ds + \\ & + \int_0^t \frac{c\widetilde{L}}{(t-s)^\alpha} \frac{\|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\|}{\|\xi_1 - \xi_2\|} ds. \end{aligned}$$

By (3.6) and (3.5) there exists  $M > 0$  such that the last inequality can be rewritten as

$$\frac{\|u(t, \xi_1, \varepsilon) - u(t, \xi_2, \varepsilon)\|}{\|\xi_1 - \xi_2\|} \leq \int_0^t \frac{c\|A^{-\alpha}\|\widehat{M}}{(t-s)^\alpha} \frac{\|u(s, \xi_1, \varepsilon) - u(s, \xi_2, \varepsilon)\|}{\|\xi_1 - \xi_2\|} ds + M$$

and the assertion follows from the generalized Gronwall–Bellman lemma . □

### 4 Existence of periodic solutions

In this section we assume that either (C1) or (C2) is satisfied. Moreover, we assume the following condition:

( $\widetilde{A}_0$ ) the solution  $x$  of (1.1) with  $\varepsilon = 0$  satisfying  $x(0) = \xi_0$  is defined on  $[0, T]$ , namely the Poincaré map  $\mathcal{P}_0$  is defined at  $\xi_0$ .

Therefore, from Lemma 3.1 we have that there exists  $r > 0$  such that the Poincaré map  $\mathcal{P}_\varepsilon$  for system (1.1) is defined on  $B_E(\xi_0, r)$  for any  $\varepsilon \in [0, r]$  and it has the form

$$\mathcal{P}_\varepsilon(\xi) = \mathcal{P}_0(\xi) + \varepsilon Q(\xi, \varepsilon),$$

where  $\mathcal{P}_0$  is differentiable and  $Q$  satisfies a Lipschitz condition in the first variable  $\xi$  uniformly on  $B_E(\xi_0, r) \times [0, r]$ .

Let  $F(\xi, \varepsilon) = \mathcal{P}_\varepsilon(\xi)$ . Then assumptions (A<sub>1</sub>)-(A<sub>4</sub>) of Theorem 2.1 can be rewritten as

( $\widetilde{A}_1$ ) there exists a function  $S \in C^1(V, E)$  defined on some open neighborhood  $V \subset \mathbb{R}^k$  of  $h_0$  such that  $S(h_0) = \xi_0$  and  $\mathcal{P}_0(\xi) = \xi$  for any  $\xi \in \mathcal{Z} = \bigcup_{h \in V} S(h)$ ,

( $\widetilde{A}_2$ )  $\dim S'(h_0)\mathbb{R}^k = k$ .

Let  $E_{1,h} = S'(h)\mathbb{R}^k$  and let  $E_{2,h}$  be any subspace of  $E$  such that  $E = E_{1,h} \oplus E_{2,h}$  and

( $\widetilde{A}_3$ ) both the projectors  $\pi_{1,h}$  of  $E$  onto  $E_{1,h}$  along  $E_{2,h}$  and  $\pi_{2,h}$  of  $E$  onto  $E_{2,h}$  along  $E_{1,h}$  are continuous in  $h \in V$ ,

( $\widetilde{A}_4$ ) for  $\xi_0 = S(h_0)$  we have

$$\pi_{2,h_0}(\mathcal{P}'_0(\xi_0) - I)\pi_{2,h_0} \text{ is invertible on } E_{2,h_0}. \tag{4.1}$$

Furthermore, it can be observed that  $Q(\xi, 0)$  is the value of the solution of the problem

$$\begin{aligned} \dot{y} &= Ay + f'_x(t, x(t, \xi, 0))y + g(t, x(t, \xi, 0), 0), \\ y(0) &= 0 \end{aligned} \tag{4.2}$$

at time  $t = T$ . To see this, observe that the function  $u$  of Lemma 3.1 satisfies the following integral equation:

$$u(t, \xi, \varepsilon) = \int_0^t A^\alpha e^{\Lambda(t-s)} f'_x(s, A^{-\alpha}x(s, \xi, 0))u(s, \xi, \varepsilon)ds + \int_0^t A^\alpha e^{\Lambda(t-s)} \frac{o(x(s, \xi, \varepsilon) - x(s, \xi, 0))}{\varepsilon} ds + \int_0^t A^\alpha e^{\Lambda(t-s)} g(s, A^{-\alpha}x(s, \xi, \varepsilon), \varepsilon)ds$$

and so  $u(T, \xi, 0) = Q(\xi, 0)$ . Therefore, we can give an equivalent definition of the bifurcation function  $M$  introduced in Section 2, that is,  $M \in C^0(\mathbb{R}^k, \mathbb{R}^k)$  can be defined as follows

$$M(h) = (S'(h))^{-1} \pi_{1,h}[\eta(S(h_0)) - (\mathcal{P}'_0(S(h)) - I) (\pi_{2,h}(\mathcal{P}'_0(S(h)) - I)\pi_{2,h})^{-1} \pi_{2,h}\eta(S(h))],$$

where  $h \in B_{\mathbb{R}^k}(h_0, r)$ , and  $\eta$  is the value of the solution of (4.2) at time  $t = T$ .

From Theorem 2.2 we have the following necessary condition for the existence of  $T$ -periodic solutions to (1.1).

**Theorem 4.1** *Assume that (C1) or (C2) is satisfied. Assume  $(\tilde{A}_0)$ - $(\tilde{A}_4)$ . Assume that there exists a sequence  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and a sequence of  $T$ -periodic functions  $x_n \in C^0([0, T], E)$ ,  $x_n \rightarrow x(\cdot, \xi_0, 0)$  as  $n \rightarrow \infty$  such that  $(x_n, \varepsilon_n)$  solves (1.1). Then*

$$M(h_0) = 0.$$

Analogously from Theorem 2.3 we derive the following sufficient condition for the existence of  $T$ -periodic solutions to (1.1).

**Theorem 4.2** *Assume that (C1) or (C2) is satisfied. Assume  $(\tilde{A}_0)$ - $(\tilde{A}_4)$  and that*

$$h_0 \text{ is an isolated zero of } M$$

with

$$\text{ind}(h_0, M) \neq 0.$$

Then, for  $\varepsilon > 0$  sufficiently small, (1.1) has a  $T$ -periodic solution  $x_\varepsilon \in C^0([0, T], E)$  and

$$x_\varepsilon(0) \rightarrow \xi_0 \quad \text{as } \varepsilon \rightarrow 0.$$

## 5 Appendix

*Proof of Lemma 2.1.* Let  $\bar{\Phi}_{h,\varepsilon} : E \rightarrow E$  be defined by

$$\bar{\Phi}_{h,\varepsilon}(\xi) = \Phi_{h,\varepsilon}(\pi_h \xi) + (I - \pi_h)\xi. \tag{5.1}$$

Observe, that if there exists  $r > 0$ ,  $M > 0$  and  $\xi : \mathbb{R}^k \times [0, r] \rightarrow E$  satisfying

$$\xi(\cdot, \varepsilon) \in C^0(V, E), \xi(h, \varepsilon) \rightarrow \xi(h, 0) \text{ as } \varepsilon \rightarrow 0 \text{ uniformly in } h \in V \tag{5.2}$$

such that

- b')  $\bar{\Phi}_{h,\varepsilon}(\xi(h, \varepsilon)) = 0$  for any  $h \in V, \varepsilon \in [0, r]$ ,
- c')  $\xi(h, \varepsilon)$  is the only zero of  $\bar{\Phi}_{h,\varepsilon}$  in  $B_E(0, r)$ ,
- d')  $\|\xi(h, \varepsilon)\| \leq M\varepsilon$  for any  $h \in V, \varepsilon \in [0, r]$ ,

then  $\beta(h, \varepsilon) = \pi_h \xi(h, \varepsilon)$  satisfies a), b), c) and d).

To prove this assertion, from assumption 4 we have

$$\bar{\Phi}_{h,0}(0) = \Phi_{h,0}(0) = \tilde{P}(h, 0) = 0.$$

For the derivative  $(\bar{\Phi}_{h,0})'(\cdot)$ , taking into account that  $\tilde{P}(h, \cdot)$  acts on  $E_h$ , we have

$$(\bar{\Phi}_{h,0})'(0) = \pi_h \tilde{P}'_\beta(h, 0) \pi_h + (I - \pi_h).$$

Let us show that  $(\bar{\Phi}_{h,0})'(0)$  is invertible on  $E$  for  $h \in V$ . To do this, we show that given  $b \in E$  there exists a unique  $a_b \in E$  such that

$$(\bar{\Phi}_{h,0})'(0) a_b = b. \tag{5.3}$$

Indeed, applying  $I - \pi_h$  to (5.3), we have  $(I - \pi_h) a_b = (I - \pi_h) b$ . On the other hand, by assumption 5  $\pi_h \tilde{P}'_\beta(h, 0)$  is invertible and thus applying  $(\pi_h \tilde{P}'_\beta(h, 0))^{-1} \pi_h$  to (5.3) we obtain  $\pi_h a_b = (\pi_h \tilde{P}'_\beta(h, 0))^{-1} \pi_h b$ . Therefore the unique solution  $a_b$  of (5.3) is given by  $a_b = (\pi_h \tilde{P}'_\beta(h, 0))^{-1} \pi_h b + (1 - \pi_h) b$ . This means that  $((\bar{\Phi}_{h,0})'(0))^{-1} \pi_h$  is continuous in  $h$ . Now, introducing  $\bar{P}(h, \xi) = \tilde{P}(h, \pi_h \xi) + (I - \pi_h) \xi$ , we have that

- 1')  $\bar{\Phi}_{h,\varepsilon}(\xi) = \bar{P}(h, \xi) + \varepsilon \tilde{Q}(h, \xi, \varepsilon)$ ,
- 2')  $\bar{P}(h, 0) = 0$ ,
- 3')  $\bar{P}'_\xi(h, 0)$  is invertible and  $(\bar{P}'_\xi(h, 0))^{-1}$  is continuous in  $h$ .

Let  $\hat{\Phi}_{h,\varepsilon}(\xi) = (\bar{P}'_\xi(h, 0))^{-1} \bar{\Phi}_{h,\varepsilon}(\xi)$ . Since  $\bar{\Phi}_{h,\varepsilon}(\xi) = 0$  if and only if  $\hat{\Phi}_{h,\varepsilon}(\xi) = 0$ , we aim now at finding a solution  $\xi(h, \varepsilon)$  to  $\hat{\Phi}_{h,\varepsilon}(\xi) = 0$  satisfying properties b'), c') and d'). By assumption 2 for any  $h \in \bar{V}$  there exists  $r(h) > 0$  such that

$$\|I - (\hat{\Phi}_{\hat{h},0})'(\xi)\| \leq 1/4$$

for any  $\|\xi\| \leq r(h)$  and any  $\hat{h} \in B_{\mathbb{R}^k}(h, r(h)) \cap \bar{V}$ .

Since the family  $\bigcup_{h \in V} B_{\mathbb{R}^k}(h, r(h))$  covers the set  $\bar{V}$ , we can extract from it a finite subfamily covering  $V$ . This implies the existence of  $r > 0$  such that

$$\|I - (\widehat{\Phi}_{h,0})'(\xi)\| \leq 1/4$$

for any  $\|\xi\| \leq r$  and any  $h \in \bar{V}$ .

By assumption 3 there is  $L > 0$  such that

$$\|(\bar{P}'_{\xi}(h, 0))^{-1}(\varepsilon\tilde{Q}(h, \xi_1, \varepsilon) - \varepsilon\tilde{Q}(h, \xi_2, \varepsilon))\| \leq \varepsilon L$$

for any  $h \in V, \xi_1, \xi_2 \in B_E(0, 1), \varepsilon \in [0, 1]$ .

Therefore,  $r > 0$  can be considered sufficiently small to have

$$\|\xi_1 - \widehat{\Phi}_{h,\varepsilon}(\xi_1) - \xi_2 + \widehat{\Phi}_{h,\varepsilon}(\xi_2)\| \leq (1/2)\|\xi_1 - \xi_2\| \tag{5.4}$$

for any  $h \in V, \varepsilon \in [0, r], \|\xi_1\| \leq r, \|\xi_2\| \leq r$ .

Thus, there exists  $\xi : V \times [0, r] \rightarrow E$  satisfying b') and c'). It remains to show that  $\xi$  satisfies also (5.2) and d'). Indeed, by using b') and (5.4) for any  $h_1, h_2 \in V$  and  $\varepsilon \in [0, r]$  we have

$$\begin{aligned} & \|\xi(h_1, \varepsilon_1) - \xi(h_2, \varepsilon_2)\| \leq \\ & \leq \|\xi(h_1, \varepsilon_1) - \widehat{\Phi}_{h_2,\varepsilon_2}(\xi(h_1, \varepsilon_1)) - \xi(h_2, \varepsilon_2) + \widehat{\Phi}_{h_2,\varepsilon_2}(\xi(h_2, \varepsilon_2))\| + \\ & \quad + \|\widehat{\Phi}_{h_2,\varepsilon_2}(\xi(h_1, \varepsilon_1)) - \widehat{\Phi}_{h_1,\varepsilon_2}(\xi(h_1, \varepsilon_1))\| + \\ & \quad + \|\widehat{\Phi}_{h_1,\varepsilon_2}(\xi(h_1, \varepsilon_1)) - \widehat{\Phi}_{h_1,\varepsilon_1}(\xi(h_1, \varepsilon_1))\| \leq \\ & \leq (1/2)\|\xi(h_1, \varepsilon_1) - \xi(h_2, \varepsilon_2)\| + \|\widehat{\Phi}_{h_2,\varepsilon_2}(\xi(h_1, \varepsilon_1)) - \widehat{\Phi}_{h_1,\varepsilon_2}(\xi(h_1, \varepsilon_1))\| + \\ & \quad + |\varepsilon_1 - \varepsilon_2| \left\| \left(\bar{P}'_{\xi}(h_1, 0)\right)^{-1} \left(\tilde{Q}(h_1, \xi(h_1, \varepsilon_1), \varepsilon_2) - \tilde{Q}(h_1, \xi(h_1, \varepsilon_1), \varepsilon_1)\right) \right\|. \end{aligned}$$

Finally the continuity assumptions 1, 2 and 3 imply that  $\xi$  satisfies (5.2) and d'). □

## References

- [1] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, Measures of noncompactness and condensing operators. Translated from the 1986 Russian original by A. Iacob. Operator Theory: Advances and Applications, 55. Birkhauser Verlag, Basel, 1992.
- [2] M. Berti and P. Bolle, Multiplicity of periodic solutions of nonlinear wave equations, Nonlinear Anal. **56** (2004), 1011–1046.
- [3] A. Buica, J. Llibre and O. Makarenkov, Lyapunov–Schmidt reduction for nonsmooth functions with application to Malkin’s problem on the existence of periodic solutions, in progress.
- [4] S. N. Chow and J. K. Hale, Methods of bifurcation theory. Grundlehren der Mathematischen Wissenschaften, **251**, Springer-Verlag, New York-Berlin, 1982.

- [5] P. Felmer and R. Manásevich, *A global approach for bifurcation from a nondegenerate periodic solution*, *Nonlinear Anal.* **22** (1994), 353-361.
- [6] M. Henrard and F. Zanolin, *Bifurcation from a periodic orbit in perturbed planar Hamiltonian systems*, *J. Math. Anal. Appl.* **277** (2003), 79-103.
- [7] D. Henry, *Geometric theory of semilinear parabolic equations*. Lecture Notes in Mathematics, **840**, Springer-Verlag, Berlin-New York, 1981.
- [8] M. Kamenskii, O. Makarenkov and P. Nistri, *A continuation principle for a class of periodically perturbed autonomous systems*, *Math. Nachr.*, **281** (2008), 42-61.
- [9] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing multivalued maps and semilinear differential inclusions in Banach spaces*. De Gruyter Series in Nonlinear Analysis and Applications, **7**, Walter de Gruyter & Co., Berlin, 2001.
- [10] M. A. Krasnosel'skii and P. P. Zabreiko, *Geometrical methods of nonlinear analysis*. Fundamental Principles of Mathematical Sciences **263**, Springer-Verlag, Berlin, 1984.
- [11] M. A. Krasnoselskii, P. P. Zabreiko, E. I. Pustyl'nik and P. E. Sobolevski, *Integral operators in spaces of summable functions*. Translated from the Russian by T. Ando. Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis. Noordhoff International Publishing, Leiden, 1976.
- [12] I. G. Malkin, *On Poincaré's theory of periodic solutions*, *Akad. Nauk SSSR. Prikl. Mat. Meh.* **13** (1949), 633-646 (In Russian).
- [13] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*. Applied Mathematical Sciences, **44**, Springer-Verlag, New York, 1983.
- [14] D. G. Schaeffer and M. A. Golubitsky, *Bifurcation analysis near a double eigenvalue of a model chemical reaction*, *Arch. Rational Mech. Anal.* **75** (1980/81), 315-347.