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Existence of Two Solutions for a Second-Order Discrete Boundary Value Problem

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Abstract

The existence of two nontrivial solutions for a class of nonlinear second-order discrete boundary value problems is established. The approach adopted is based on variational methods.

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1 Introduction

The theory of difference equations has been widely studied by many authors and a large number of applications has been developed in many fields of research such as science, economics, neural network, ecology and cybernetics; see [7].

In order to investigate nonlinear discrete boundary value problems, different arguments have been used. For instance, upper and lower solutions methods and topological degree are the main tools in [2], while, in [6], the existence of multiple solutions is established through fixed point theory. Moreover, some recent papers (see [1, 3, 5, 9] and the references given therein) show an approach based on the critical point theory.

In this connection, our aim is to apply a two-critical-point theorem, due to Brézis and Nirenberg in [4], to prove the existence of at least two nontrivial solutions for the following nonlinear second-order discrete boundary value problem:

$$(P_g) \qquad \begin{cases} -\Delta^2 u_{k-1} = g(k, u_k), & k \in [1, N], \\ u_0 = u_{N+1} = 0, \end{cases}$$

where N is a positive integer, $[1, N]$ is the discrete interval $\{1, ..., N\}$, $\Delta u_k := u_{k+1} - u_k$ is the forward difference operator, $\Delta^2 u_k := \Delta(\Delta u_k)$, and $g : [1, N] \times \mathbf{R} \to \mathbf{R}$ is a continuous function.

To be precise, to achieve our goal, we adopt the following strategy. First, we obtain the existence of at least one solution, let us say u_0, under a suitable sub-quadratic growth condition on the antiderivative G of the nonlinearity g (Theorem 2.2). Next, by using an appropriate splitting of the following N-dimensional Hilbert space

$$H := \{u : [0, N + 1] \to \mathbf{R} : u_0 = u_{N+1} = 0\},$$

and requiring a suitable behavior of G at zero, we guarantee that u_0 is nontrivial (Theorem 3.1). Finally, the existence of at least a second nontrivial solution is obtained when G satisfies a linking condition at zero (Theorem 3.2).

For completeness, we just observe that, in [5], a more general Dirichlet problem involving the discrete p–Laplacian is studied. Anyway, Remark 2.2 and Example 3.1 point out that our results are mutually independent compared to those achieved, for $p = 2$, in the above mentioned paper.

A particular case of our result is the following one.

Theorem 1.1 *Let* $h : [1, N] \times \mathbf{R} \to \mathbf{R}$ *be a continuous function. Assume that*

(i) $\lim_{|t|\to\infty} h(k, t) = 0$ *and* $h(k, 0) = 1$, *for every* $k \in [1, N]$.

Then, for every $\lambda \in \left]4 \sin^2\left(\frac{\pi}{2(N+1)}\right), 4 \sin^2\left(\frac{N\pi}{2(N+1)}\right)\right[$, *the problem*

$$(P^h_\lambda) \qquad \begin{cases} -\Delta^2 u_{k-1} = \lambda u_k h(k, u_k), & k \in [1, N], \\ u_0 = u_{N+1} = 0 \end{cases}$$

possesses at least two nontrivial solutions.

The plan of the paper is as follows. In the next Section we introduce our abstract framework. Successively, in Section 3, we show our multiplicity results. Concrete examples of application are also presented.

2 Auxiliary results and variational framework

Let $(X, \|\cdot\|)$ be a real reflexive Banach space, $E \subseteq X$ an open subset and $f : E \to \mathbf{R}$ a function. We say that $x_0 \in E$ is a critical point of f provided that f is Gâteaux differentiable on E and its Gâteaux derivative $f'(x_0) = 0$. Now, we recall a basic critical point result; see, for instance, the classical book [8].

Theorem 2.1 *Let $f : X \to \mathbf{R}$ be a weakly lower semicontinuous and coercive function. Then, there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$. Moreover, if f has bounded Gâteaux derivative on X, then x_0 is also a critical point of f.*

In order to give the variational formulation of the problem (P_g), on the N-dimensional Hilbert space

$$H := \{u : [0, N+1] \to \mathbf{R} : u_0 = u_{N+1} = 0\},$$

with inner product

$$(u, z) := \sum_{k=1}^{N} u_k z_k, \quad \forall\, u, z \in H,$$

and endowed norm

$$\|u\|_2 := \Big(\sum_{k=1}^{N} u_k^2\Big)^{1/2},$$

we define the functional $f : H \to \mathbf{R}$ given by

$$f(u) := \frac{1}{2}\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2 - \sum_{k=1}^{N}\int_0^{u_k} g(k,t)dt,$$

for every $u \in H$.

Proposition 2.1 *The solutions of the problem (P_g) are exactly the critical points of the function f.*

Proof. Clearly, a function $u \in H$ is solution of (P_g) if and only if

$$-\sum_{k=1}^{N} \Delta^2 u_{k-1} v_k = \sum_{k=1}^{N} g(k, u_k) v_k,$$

for every $v \in H$. Moreover, one has (sum by parts),

$$-\sum_{k=1}^{N} \Delta^2 u_{k-1} v_k = \sum_{k=1}^{N+1} \Delta u_{k-1} \Delta v_{k-1}, \quad \forall\, v \in H.$$

Hence, $u \in H$ is solution of (P_g) if and only if

$$\sum_{k=1}^{N+1} \Delta u_{k-1} \Delta v_{k-1} - \sum_{k=1}^{N} g(k, u_k) v_k = 0, \quad \forall\, v \in H.$$

On the other hand, an easy computation ensures that $f \in C^1$ with Gâteaux derivative

$$f'(u)(v) = \sum_{k=1}^{N+1} \Delta u_{k-1} \Delta v_{k-1} - \sum_{k=1}^{N} g(k, u_k) v_k, \quad \forall\, v \in H.$$

Therefore, it is clear that $u \in H$ is a solution of (P_g) if and only if u is a critical point for f. This is the end of the proof.

Now, let us consider the functional $\Phi : H \to \mathbf{R}$ given by

$$\Phi(u) := \frac{1}{2}\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2, \quad \forall\, u \in H.$$

One can be see that

$$\Phi(u) = \frac{1}{2}u^t A u, \quad \forall\, u \in H,$$

where $u := (u_1, ..., u_N)^t$, and

$$A := \begin{pmatrix} 2 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & ... & 0 \\ & ... & ... & ... & \\ 0 & ... & -1 & 2 & -1 \\ 0 & ... & 0 & -1 & 2 \end{pmatrix}_{N\times N}.$$

The matrix A is symmetric and positive-definite. Indeed, it is easy to show (see [9]) that A admits N distinct positive eigenvalues given by

$$\lambda_k := 4\sin^2\left(\frac{k\pi}{2(N+1)}\right), \quad \forall\, k \in [1, N], \tag{2.1}$$

and by direct computations, see [2, p. 150], λ_k is the k-eigenvalue, for every $k \in [1, N]$, of the linear problem

$$\begin{cases} -\Delta^2 u_{k-1} = \lambda u_k, & k \in [1, N], \\ u_0 = u_{N+1} = 0. \end{cases}$$

In the sequel φ_k, for every $k \in [1, N]$, denotes the correspondent orthonormal eigenfunction of λ_k. Further, the symbol $\overline{\text{Span}}\{\varphi_{j_1}, ..., \varphi_{j_s}\}$ stands for the $\mathbf{R}$-linear subspace of H generated by $\varphi_{j_1}, ..., \varphi_{j_s}$.

Lemma 2.1 *For every $u \in H$, one has*

$$\|u\|_2^2 \le \frac{1}{4\sin^2\left(\frac{\pi}{2(N+1)}\right)} u^t A u. \tag{2.2}$$

Proof. If $u \in H = \overline{\text{Span}}\{\varphi_1, \ldots, \varphi_N\}$, $u = \sum_{k=1}^N b_k\varphi_k$, for suitable $b_k \in \mathbf{R}$, with $k \in [1, N]$. Moreover, A is a positive definite matrix. Hence, standard calculations give

$$u^t A u = \sum_{k=1}^{N} \lambda_k b_k^2 \ge \lambda_1 \sum_{k=1}^{N} b_k^2 = \lambda_1 \|u\|_2^2,$$

that is (2.2) holds.

With the above notation, the energy functional $f : H \to \mathbf{R}$ can be expressed as

$$f(u) = \Phi(u) - \Psi(u),$$

where

$$\Psi(u) := \sum_{k=1}^{N} \int_0^{u_k} g(k,t)dt, \qquad \forall\, u \in H.$$

We will denote

$$\|u\|_\infty := \max_{k\in[1,N]} |u_k|,$$

for every $u \in H$.

The following result guarantees the coercivity of the functional f associated to problem (P_g).

Theorem 2.2 *Let $g : [1,N] \times \mathbf{R} \to \mathbf{R}$ be a continuous function and suppose that*

$$(\mathrm{g}_1)\ \ \limsup_{|t|\to\infty} \frac{\int_0^{\xi} g(k,t)dt}{\xi^2} < 2\sin^2\left(\frac{\pi}{2(N+1)}\right), \qquad \forall\, k \in [1,N].$$

Then, the problem (P_g) possesses at least one solution in H.

Proof. We shall verify that the C^1-functional f has a critical point in H. From (g_1) there are constants $\epsilon \in \left]0, 2\sin^2\left(\frac{\pi}{2(N+1)}\right)\right[$ and $\sigma > 0$ such that

$$\frac{\int_0^{\xi} g(k,t)dt}{\xi^2} < \left(2\sin^2\left(\frac{\pi}{2(N+1)}\right) - \epsilon\right), \tag{2.3}$$

for every $|t| \geq \sigma$ and $k \in [1,N]$. Let us put

$$M_1 := \max_{(k,\xi)\in[1,N]\times[-\sigma,\sigma]} \int_0^{\xi} g(k,t)dt. \tag{2.4}$$

Therefore, for every $\xi \in \mathbf{R}$ and $k \in [1,N]$, one has

$$\int_0^{\xi} g(k,t)dt \leq M_1 + M_2\xi^2,$$

where

$$M_2 := \left(2\sin^2\left(\frac{\pi}{2(N+1)}\right) - \epsilon\right).$$

Moreover, the following inequality holds

$$f(u) \geq \frac{1}{2}u^tAu - \sum_{k=1}^{N}\left[M_1 + M_2u_k^2\right], \quad \forall\, u \in H.$$

Hence,

$$f(u) \geq \frac{1}{2}u^tAu - M_2\|u\|_2^2 - NM_1, \quad \forall\, u \in H,$$

and, by (2.2), one has

$$f(u) \geq \epsilon\|u\|_2^2 - NM_1, \quad \forall\, u \in H, \tag{2.5}$$

which clearly shows that

$$\lim_{\|u\|_2 \to +\infty} f(u) = +\infty. \tag{2.6}$$

From this, and taking into account that H is a finite dimensional space, it follows that f satisfies the $(\mathrm{PS})_f$ condition. In view of Theorem 2.1, f has a critical point $u_0 \in H$ such that $f(u_0) = \inf_{u\in H} f(u)$.

Remark 2.1 If in addition, in Theorem 2.2, $g(k,0) \neq 0$ for some $k \in [1,N]$, clearly, our result guarantees the existence of at least one nontrivial solution for the problem (P_g).

Corollary 2.1 *Let $g : [1,N] \times \mathbf{R} \to \mathbf{R}$ be a continuous function and suppose that*

(g_1') $\displaystyle \limsup_{|t|\to\infty} \frac{g(k,t)}{t} < 4\sin^2\left(\frac{\pi}{2(N+1)}\right), \quad \forall\, k \in [1,N].$

Then, the same conclusion of Theorem 2.2 holds.

Proof. Our aim is to prove that (g_1') implies (g_1). From (g_1') there are constants

$$\epsilon' \in \left]0, 4\sin^2\left(\frac{\pi}{2(N+1)}\right)\right[,$$

and $\sigma > 0$ such that

$$\frac{g(k,t)}{t} < \left(4\sin^2\left(\frac{\pi}{2(N+1)}\right) - \epsilon'\right), \tag{2.7}$$

for every $|t| \geq \sigma$ and $k \in [1,N]$. Since g is a continuous function, we also have

$$M := \sup_{(k,t)\in[1,N]\times[-\sigma,\sigma]} |g(k,t)| < +\infty. \tag{2.8}$$

Therefore, if $\xi \geq \sigma$, one has

$$\int_0^\xi g(k,t)dt = \int_0^\sigma g(k,t)dt + \int_\sigma^\xi g(k,t)dt \leq M\sigma + \frac{1}{2}\left(4\sin^2\left(\frac{\pi}{2(N+1)}\right) - \epsilon'\right)\xi^2,$$

while, for $\xi \leq -\sigma$, it follows that

$$\int_0^\xi g(k,t)dt = -\left[\int_\xi^{-\sigma} g(k,t)dt + \int_{-\sigma}^0 g(k,t)dt\right] \leq M\sigma + \frac{1}{2}\left(4\sin^2\left(\frac{\pi}{2(N+1)}\right) - \epsilon'\right)\xi^2.$$

Consequently,

$$\int_0^\xi g(k,t)dt \leq M\sigma + \frac{1}{2}\left(4\sin^2\left(\frac{\pi}{2(N+1)}\right) - \epsilon'\right)\xi^2, \quad \forall\, (k,\xi) \in [1,N] \times \mathbf{R}. \tag{2.9}$$

Thus, our claim holds and the proof is completed.

Remark 2.2 Obviously, there are functions that satisfy (g_1) but for which (g_1') is not verified, see, for instance, example 3.1 below. As you can see, conditions like (g_1') have been used to ensure the coercivity also for functionals related to boundary value problems involving the discrete p–Laplacian. However here, for $p = 2$, we use an explicit representation for λ_1. Moreover, in [5], the coercivity of the energy functional (see Remarks 3.1 and 3.3) is obtained under the growth condition of the potential

$$\limsup_{|t|\to\infty} \frac{\int_0^\xi g(k,t)dt}{\xi^p} < \frac{1}{N(N+1)}, \quad \forall\, k \in [1,N], \tag{2.10}$$

or by using

$$\limsup_{|t|\to\infty} \frac{\int_0^\xi g(k,t)dt}{\xi^s} < +\infty, \quad \forall\, k \in [1,N], \tag{2.11}$$

where $s < p$. Taking into account that the following inequality holds

$$\frac{1}{N(N+1)} < \sin^2\left(\frac{\pi}{2(N+1)}\right), \ \ \forall\, N > 1,$$

we just observe that assumption (g_1) improves, for $p = 2$, conditions (2.10) and (2.11).

Now, we recall the statement of the main tool used here to investigate (P_g). Let $\delta > 0$ and $x \in X$. We denote $B(x,\delta) := \{z \in X : \|z - x\| < \delta\}$ and $\overline{B}(x,\delta) := \{z \in X : \|z - x\| \leq \delta\}$ the open (respectively closed) ball of center x and radius δ. A C^1-function $f : X \to \mathbf{R}$ is said to fulfill the Palais-Smale condition if:

$(\mathrm{PS})_f$ *Every sequence* $\{x_n\} \subseteq X$ *such that* $f(x_n)$ *is bounded and* $f'(u_n) \to 0$ *as* $n \to \infty$ *possesses a convergent subsequence.*

Theorem 2.3 *[4, Theorem 4] Suppose that* $X := X_1 \oplus X_2$ *with* $\dim(X_1) > 0$ *and* $0 < \dim(X_2) < \infty$. *Let* f *be a* C^1*-function on* X *with* $f(0) = 0$, *satisfying* $(\mathrm{PS})_f$ *and, for some* $\rho > 0$,

$$f(u) \leq 0 \quad \forall\, u \in \overline{B}(0,\rho) \cap X_2,$$

$$f(u) \geq 0 \quad \forall\, u \in \overline{B}(0,\rho) \cap X_1.$$

Assume also that f *is bounded from below and* $\inf_{x\in X} f(x) < 0$. *Then* f *has at least two nonzero critical points.*

3 Main Results

In the following result we establish the existence of at least one nontrivial solution of our problem.

Theorem 3.1 *Let* $g : [1,N] \times \mathbf{R} \to \mathbf{R}$ *be a continuous function and suppose that* (g_1) *holds. Assume that there exists an integer* $\ell \in [1,N]$ *such that* (g$_2$) $\liminf_{\xi\to 0} \dfrac{\int_0^{\xi} g(k,t)dt}{\xi^2} \geq 2\sin^2\left(\dfrac{\ell\pi}{2(N+1)}\right), \quad \forall\, k \in [1,N].$

Then, the problem (P_g) possesses at least one nontrivial solution in H.

Proof. Clearly, from (g$_1$), problem (P_g) admits at least one solution $u_0 \in H$. To be precise, from Theorem 2.2, u_0 is a global minimum for the energy functional f. Hence, our conclusion will follows if we prove that u_0 is nontrivial. Obviously, this is true if $g(k,0) \neq 0$ for some $k \in [1,N]$. On the other hand, we will prove that, for some $\rho > 0$, $f(u) \leq 0$ for every $u \in X_2 \cap \overline{B}(0,\rho)$, where $X_2 = \overline{\mathrm{Span}}\{\varphi_1,\ldots,\varphi_\ell\}$. Thus, by (g$_2$), there exists $\delta > 0$ such that

$$\int_0^{\xi} g(k,t)dt \geq 2\sin^2\left(\frac{\ell\pi}{2(N+1)}\right)\xi^2, \quad \forall\, k \in [1,N],$$

provided $0 < |\xi| \leq \delta$.

Now, taking into account the discrete Cauchy-Schwarz inequality, one has

$$|u_k| \leq \sum_{k=1}^{N} |u_k| \leq N^{1/2}\Big(\sum_{k=1}^{N} |u_k|^2\Big)^{1/2},$$

for every $k \in [1,N]$.

Then,

$$\|u\|_\infty \leq N^{1/2}\|u\|_2, \quad \forall\, u \in H.$$

Hence, for every $u \in \overline{B}(0,\rho) \cap X_2$, it follows that

$$\|u\|_\infty \leq N^{1/2}\rho. \tag{3.1}$$

Consequently, if we take $\rho \leq \delta/N^{1/2}$, we get

$$\int_0^{u_k} g(k,t)dt \geq 2\sin^2\left(\frac{\ell\pi}{2(N+1)}\right)u_k^2, \tag{3.2}$$

for every $k \in [1,N]$. Moreover, if $u \in X_2$, there exists $a_k \in \mathbf{R}$ for every $k \in [1,N]$, such that $u = \sum_{k=1}^{\ell} a_k\varphi_k$ and $u^tAu = \sum_{k=1}^{\ell} \lambda_k a_k^2 \leq \lambda_\ell \sum_{k=1}^{\ell} a_k^2$. Hence,

$$u^tAu \leq 4\sin^2\left(\frac{\ell\pi}{2(N+1)}\right)\|u\|_2^2, \ \ \forall\, u \in X_2. \tag{3.3}$$

Putting together (3.2) and (3.3), we have

$$f(u) = \frac{1}{2}u^tAu - \sum_{k=1}^{N}\int_0^{u_k} g(k,t)dt \leq$$

$$\leq 2\sin^2\left(\frac{\ell\pi}{2(T+1)}\right)(\|u\|_2^2 - \|u\|_2^2) = 0, \quad \forall\, u \in \overline{B}(0,\rho) \cap X_2,$$

that is,

$$f(u) \leq 0 \quad \forall\, u \in \overline{B}(0,\rho) \cap X_2. \tag{3.4}$$

Now, if $f(u_0) = 0$, every $u \in X_2$, with $\|u\|_2 \leq \rho$, is a solution of the problem (P_g). On the other hand, if $f(u_0) < 0$, owing to $f(0) = 0$, it follows that $u_0 \neq 0$ and this complete the proof.

Example 3.1 Consider the following problem

$$\begin{cases} -\Delta^2 u_{k-1} = k\sin(2ku_k e^{u_k})(1+u_k)e^{u_k} + \dfrac{29}{50}u_k, & \forall\, k \in [1,3], \\ u_0 = u_4 = 0. \end{cases}$$

Since

$$\int_0^\xi g(k,t)dt = \sin^2(ku_k e^{u_k}) + \frac{29}{100}u_k^2, \quad \forall\,(k,\xi) \in [1,3]\times \mathbf{R},$$

owing to Theorem 3.1, such problem admits at least one nontrivial solution. In fact, by direct computations, one has

$$\limsup_{|t|\to\infty} \frac{\displaystyle\int_0^\xi g(k,t)dt}{\xi^2} = \frac{29}{100} < 2\sin^2\left(\frac{\pi}{8}\right),$$

and

$$\liminf_{\xi\to 0} \frac{\displaystyle\int_0^\xi g(k,t)dt}{\xi^2} = k^2 + \frac{29}{100} > 2\sin^2\left(\frac{\pi}{8}\right),$$

for every $k \in [1,3]$. Moreover, we stress that, in this example, condition (g_1') fails taking into account that

$$\limsup_{t\to+\infty} \frac{g(k,t)}{t} = +\infty,$$

for every $k \in [1,3]$.

Now, we give our main result which guarantees the existence of two nontrivial solutions to the problem (P_g).

Theorem 3.2 *Let $g : [1,N]\times\mathbf{R} \to \mathbf{R}$ be a continuous function. Assume that* (g_1) *holds and that there exists an integer $\ell \in [1,N-1]$ such that* (g_2) *is verified. Further, suppose that*

$$(\mathrm{g}_3)\ \limsup_{\xi\to 0} \frac{\displaystyle\int_0^\xi g(k,t)dt}{\xi^2} \le 2\sin^2\left(\frac{(\ell+1)\pi}{2(N+1)}\right), \quad \forall\, k \in [1,N].$$

Then, the problem (P_g) possesses at least two nontrivial solutions in H.

Proof. Our aim is to apply Theorem 2.3. From (g_1), since H is a finite dimensional space, it is easy to see that f satisfies condition $(\mathrm{PS})_f$. Moreover, arguing as before, taking into account (g_2), it follows that

$$f(u) \le 0, \quad \forall\, u \in \overline{B}(0,\rho_1)\cap X_2, \tag{3.5}$$

for some $0 < \rho_1 \le \delta/N^{1/2}$. Hence, it remains to show that there exists $\rho_2 > 0$ such that

$$f(u) \ge 0, \quad \forall\, u \in \overline{B}(0,\rho_2)\cap X_1,$$

where $X_1 := \overline{\mathrm{Span}}\{\varphi_{\ell+1},\ldots,\varphi_N\}$. Fix $u \in X_1$, for suitable $b_k \in \mathbf{R}$ and for every $k \in [\ell+1,N]$, one has that $u = \sum_{k=\ell+1}^N b_k\varphi_k$ and

$$f(u) \geq \sum_{k=\ell+1}^{N} \frac{\lambda_k}{2} b_k^2 - \sum_{k=1}^{N} \int_0^{u_k} g(k,t)dt.$$

Hence

$$f(u) \geq 2\sin^2\left(\frac{(\ell+1)\pi}{2(N+1)}\right)\|u\|_2^2 - \sum_{k=1}^{N} \int_0^{u_k} g(k,t)dt.$$

Further, from (g_3), there exists $\sigma > 0$ such that

$$\int_0^{\xi} g(k,t)dt \leq 2\sin^2\left(\frac{(\ell+1)\pi}{2(N+1)}\right)\xi^2, \quad \forall\, k \in [1,N],$$

provided $0 < |\xi| \leq \sigma$. Thus, taking $\rho_2 \leq \sigma/N^{1/2}$, for every $u \in \overline{B}(0,\rho_2) \cap X_1$, we have

$$f(u) \geq 2\sin^2\left(\frac{(\ell+1)\pi}{2(N+1)}\right)(\|u\|_2^2 - \|u\|_2^2) = 0.$$

Therefore, choosing $\rho \leq \min\{\rho_1, \rho_2\}$, if $\inf_{u\in H} f(u) < 0 = f(0)$ our claim is proved. On the other hand, if $\inf_{u\in H} f(u) = 0$, we argue as above that is, every $u \in X_2$ with $\|u\|_2 \leq \rho$ is solution of the problem (P_g). So, our goal is achieved.

Remark 3.1 Observe that, in order to guarantee condition (g_3), it is enough to verify

(g_3') $\displaystyle\limsup_{|t|\to 0} \frac{g(k,t)}{t} < 4\sin^2\left(\frac{(\ell+1)\pi}{2(N+1)}\right)$, for every $k \in [1,N]$.

Indeed, by (g_3'), it follows that there exist $\beta > 0$ such that

$$\frac{g(k,t)}{t} < 4\sin^2\left(\frac{(\ell+1)\pi}{2(N+1)}\right),$$

for every $|t| \in]0,\beta[$ and $k \in [1,N]$. Hence, for every $|\xi| \in]0,\beta[$, one has

$$\frac{\int_0^{\xi} g(k,t)dt}{\xi^2} \leq 2\sin^2\left(\frac{(\ell+1)\pi}{2(N+1)}\right), \quad \forall\, k \in [1,N],$$

which, clearly, implies (g_3).

The next result is achieved by using Theorem 3.2.

Corollary 3.1 *Let* $g : [1,N] \times \mathbf{R} \to \mathbf{R}$ *be a continuous function. Assume that* (g_1) *holds and, in addition, suppose that*

(g_4) $\displaystyle 2\sin^2\left(\frac{\pi}{2(N+1)}\right) \leq \lim_{\xi\to 0} \frac{\int_0^{\xi} g(k,t)dt}{\xi^2} \leq 2\sin^2\left(\frac{N\pi}{2(N+1)}\right), \quad \forall\, k \in [1,N].$

Then, the same conclusion of Theorem 3.2 holds.

Finally, a simple and direct consequence of Corollary 3.1 is Theorem 1.1 in Introduction.

Proof of Theorem 1.1: Fix λ as in our conclusion and let $g : [1, N] \times \mathbf{R} \to \mathbf{R}$ given by

$$g(k,t) := \lambda t h(k,t), \quad \forall\, (k,t) \in [1,N] \times \mathbf{R}.$$

For every $k \in [1, N]$, we get

$$\lim_{|t|\to\infty} \frac{g(k,t)}{t} = 0,$$

and

$$\lim_{\xi\to 0} \frac{\int_0^{\xi} g(k,t)dt}{\xi^2} = \frac{\lambda}{2}.$$

Hence, condition (g_4) holds and the proof is complete.

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