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Finite Speed of Propagation in Monostable Degenerate Reaction-Diffusion-Convection Equations

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Abstract

We study the existence and properties of travelling wave solutions of the Fisher-KPP reaction-diffusion-convection equation $u_t + h(u)u_x = \Big[D(u)u_x\Big]_x + g(u)$, where the diffusivity $D(u)$ is simply or doubly degenerate. Both the cases when $\dot{D}(0)$ and $\dot{D}(1)$ are possibly zero real values or infinity, are treated. We discuss the effects, due to the presence of a convective term, concerning the property of finite speed of propagation. Moreover, in the doubly degenerate case we show the appearance of new types of profiles and provide their classification according to sharp relations between the nonlinear terms of the model. An application is also presented, concerning the evolution of a bacterial colony.

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1 Introduction

This paper deals with the reaction-diffusion-convection equation

$$u_t + h(u)u_x = \Big[D(u)u_x\Big]_x + g(u) \qquad t \geq 0,\, x \in \mathbb{R} \tag{1.1}$$

where $g \in C[0,1]$ is a Fisher-KPP reaction term, that is it satisfies

$$g(u) > 0 \text{ for } u \in (0,1) \quad \text{and} \quad g(0) = g(1) = 0, \tag{1.2}$$

$h \in C[0,1]$ is a nonlinear convective term and $D \in C^1[0,1]$ is a degenerate diffusivity, satisfying

$$D(0) = 0,\;\; D(1) \geq 0 \;\text{ and }\; D(u) > 0 \;\text{ for }\; 0 < u < 1. \tag{1.3}$$

When $D(1) > 0$ Eq. (1.1) has a unique degeneracy at 0 (simply degenerate case), instead when $D(1) = 0$ it has a pair of degeneracies at 0 and 1 (doubly degenerate case).

Reaction-diffusion equations with a degenerate diffusion coefficient $D(u)$ were mainly introduced in the investigation of various biological phenomena. In particular, in population migration and dispersal models (see, i.e., [2], [15], [17]). Recently, also in the study of the pattering behavior of bacterial colonies (see [18]).

This paper deals with the existence and properties of travelling wave solutions (t.w.s.) of Eq. (1.1) connecting the stationary states 0 and 1. As it is known, they play a relevant role in describing the asymptotic behavior of general classes of solutions of (1.1) (see e.g. [8] and [9] for recent contributions).

Recall that a t.w.s. is a solution $u(t,x)$ having a constant profile, that is such that $u(t,x) = \varphi(x - ct) = \varphi(\xi)$ for some function $\varphi(\xi)$ (the wave shape) and constant c (the wave speed). Observe that φ satisfies the ordinary differential equation

$$(D(\varphi)\varphi')' + (c - h(\varphi))\varphi' + g(\varphi) = 0 \tag{1.4}$$

and the boundary conditions

$$\varphi(-\infty) = 1\,, \quad \varphi(+\infty) = 0. \tag{1.5}$$

Front propagation for reaction-diffusion equations (without convective effects) was extensively investigated (see e.g. [6], [7], [15], [16], [19] and references there contained). In particular, it is well-known that Fisher-KPP equations always support a half-line $[c^*, +\infty)$ of admissible wave speeds and estimates of the threshold speed c^* are available (see [3], [7], [14], [17]). Also variational characterizations of c^* exist and we refer to [1] for a recent contribution in this direction. Moreover, when the diffusion term presents a unique degeneracy at 0 and $\dot{D}(0) \neq 0$, it was showed that the t.w.s. having speed c^* is of a special type (sharp-type), since it is right compactly supported, that is it reaches the equilibrium 0 at a finite value, with the precise non-zero slope $-c^*/\dot{D}(0)$.

In order to justify this behavior let us consider the singular associated system

$$\begin{cases} \varphi' = v \\ D(\varphi)v' = -cv - \dot{D}(\varphi)v^2 - g(\varphi) \end{cases} \tag{1.6}$$

(where $\dot{}$ stands for differentiation with respect to φ) whose singularity can be removed by introducing a parameter $\tau = \tau(\xi)$ satisfying $\frac{d\tau}{d\xi} = \frac{1}{D(\varphi(\xi))}$ (see [2]; or [15], page 290). The new system

$$\begin{cases} \varphi' = D(\varphi)v \\ v' = -cv - \dot{D}(\varphi)v^2 - g(\varphi) \end{cases} \tag{1.7}$$

is obtained where now $'$ denotes differentiation with respect to τ. System (1.7) is equivalent to (1.6) in the strip $\{(\varphi, v) : 0 < \varphi < 1, v \in \mathbb{R}\}$ and when $\dot{D}(0) > 0$ it has the equilibria $P_0 = (0, 0)$, $P_1 = (1, 0)$, $P_c = (0, -c/\dot{D}(0))$. It is then natural to expect that, for suitable c, (1.7) admits orbits both connecting P_0 with P_1 and P_0 with P_c: while the former ones correspond to classical t.w.s., the latter originate sharp-type t.w.s.

In [14] we investigated such a problem under mild assumptions and we also treated the doubly degenerate case showing that no other sharpness phenomenon arises due to the degeneracy at 1.

When the equilibrium 0 is attained at a finite value, the dynamics is said to have the property of *finite speed of propagation*. If g is not locally continuously differentiable at 0, this phenomenon can occur also for $c > c^*$ (see [7, Theorem 38]) but in this case the profile reaches the equilibrium with zero slope, so it can be continued up to infinity (see e.g. [17]).

The introduction of a convective term $h(u)$ inside the model changes its dynamics and, according to our knowledge, up to now the study of front propagation for reaction-diffusion-convection equations with a degenerate diffusivity was an open problem. The aim of the present paper is to investigate such a problem, both in the simply and in the doubly degenerate case. In particular, we will show that here the double degeneracy can determine the appearance of new types of sharp t.w.s, which can be right compactly supported, left compactly supported, or both of them. In other words, they can leave the equilibrium 1 and/or reach the equilibrium 0, at finite values and with non-zero slope. This means that, in this framework, the density $u(t, x)$ not only spreads with finite speed but also reaches its maximum value 1 in finite times.

All these different situations can be handled by using the following unifying definition of t.w.s.

Definition 1.1 A function $u = u(t, x)$ is said to be a t.w.s. (with wave speed c) if there exist a (maximal) interval (α, β), with $-\infty \leq \alpha < \beta \leq +\infty$, and a function $\varphi \in C(cl(\alpha, \beta)) \cap C^2(\alpha, \beta)$, satisfying equation (1.4) in (α, β), such that $u(t, x) = \varphi(x - ct) = \varphi(\xi)$ and

$$\varphi(\alpha) = 1 \ , \quad \varphi(\beta) = 0 \tag{1.8}$$

$$\lim_{\xi \to \alpha^+} D(\varphi(\xi))\varphi'(\xi) = \lim_{\xi \to \beta^-} D(\varphi(\xi))\varphi'(\xi) = 0. \tag{1.9}$$

A t.w.s. φ is said to be of *front-type* if $(\alpha, \beta) = \mathbb{R}$, *sharp of type (I)* if $\alpha = -\infty$ and $\beta \in \mathbb{R}$, *sharp of type (II)* if $\alpha \in \mathbb{R}$ and $\beta = +\infty$; *sharp of type (III)* if both $\alpha, \beta \in \mathbb{R}$.

In Remark 2.1 we discuss the relations between this definition and other well-known ones. In particular, when $(\alpha,\beta)=\mathbb{R}$ condition (1.9) is automatically satisfied and then the previous definition reduces to the classical one for the non-degenerate case. Moreover, in the present regularity setting this condition is equivalent to that given in weak form in [7]. Finally, when $\dot{D}(0)\neq 0$ and $\beta\in\mathbb{R}$ the t.w.s. reaches the equilibrium 0 with the slope $(h(0)-c)/\dot{D}(0)$ and also in this case the previous definition coincides with that used in [17]. Similarly, when $D(1)=0$, $\dot{D}(1)\neq 0$ and $\alpha\in\mathbb{R}$, the t.w.s. reaches the equilibrium 1 with the slope $(h(1)-c)/\dot{D}(1)$.

Again one can use a dynamical systems approach in order to understand the appearance of these solutions. Indeed, with the additional convective term $h(\varphi)$, system (1.7) maintains the equilibria P_0 and P_1, while P_c becomes $(0,(h(0)-c)/\dot{D}(0))$ (when $\dot{D}(0)\neq 0$) and the new equilibrium $P_q:=(1,(h(1)-c)/\dot{D}(1))$ arises when $D(1)=0$ and $\dot{D}(1)\neq 0$. Both connections from P_0 to P_q and from P_c to P_q are allowed and they originate sharp t.w.s. of type (II) and (III) respectively.

Observe that, when the reactivity g is not continuously differentiable at 0, precisely when $\int_0^\delta \frac{1}{g(u)}\,du<\infty$, all the t.w.s. having wave speed $c>c^*$ attain the equilibrium 0 at a finite value (see Theorem 8.1); however, their slope is zero so they can be smoothly continued up to infinity. According to Definition 1.1, such t.w.s. are of front-type; instead the sharp t.w.s. of type (I) reach the equilibrium 0 with a non-zero slope (eventually infinite), hence they are not continuable up to infinity. In this way, concerning the property of finite speed of propagation, we are able to distinguish between a smooth and a non-smooth attaining of the equilibrium. A similar situation holds for the equilibrium 1 too. We discuss this topic in Section 8.

The present investigation starts with the proof of the appearance of a threshold speed c^* satisfying

$$h(0)\le c^*\le 2\sqrt{\sup_{s\in(0,1]}\frac{D(s)g(s)}{s}}+\max_{u\in[0,1]}h(u) \tag{1.10}$$

such that Eq. (1.1) has t.w.s. (of front or sharp-type) if and only if $c\ge c^*$ (see Theorem 2.2). We remark that, in this general framework, at our knowledge this existence was never previously stated. However, the main purpose of this paper is to classify these t.w.s. In particular, in the simply degenerate case we will show that the t.w.s. having speed c^* can be sharp of type (I) or of front-type, according to the value of $c^*-h(0)$. This phenomenon is clearly in contrast to what happens for the reaction-diffusion equation without convection where the threshold speed t.w.s. is always sharp of type (I). The following result provides a classification of t.w.s. in the simply degenerate case, under the assumption $\dot{D}(0)\neq 0$. The case when $\dot{D}(0)=0$ is more complicated and will be treated in Section 6.

Theorem 1.1 *Let g, D satisfy (1.2), (1.3) and $\dot{D}(0)\neq 0$. Assume that $D(1)>0$. Then, there exists a threshold value c^* satisfying (1.10) such that Eq. (1.1) has*

(i) no t.w.s. for $c<c^*$;

(ii) a unique (up to shifts) t.w.s. for $c=c^*$, which is of front-type if $c^*=h(0)$, whereas it is sharp of type (I) if $c^*>h(0)$;

(iii) a unique (up to shifts) front-type t.w.s. for every $c > c^*$.

A natural question arises, from the previous classification result, concerning the study of the sign of $c^* - h(0)$, since the exact value of c^* is unknown in general, being available only the estimate (1.10). The following results deal with the comparison between c^* and $h(0)$.

Theorem 1.2 *Let g, D satisfy (1.2) and (1.3). Assume that $h(\varphi) < h(0)$ for $\varphi \in (0,1)$ and*

$$D(\varphi)g(\varphi) \le \frac{1}{4}\,[h(\varphi) - h(0)] \int_0^{\varphi} [h(s) - h(0)]\mathrm{d}s \tag{1.11}$$

then $c^* = h(0)$.

Theorem 1.3 *Let g, D satisfy (1.2) and (1.3). If $h(\varphi) \ge h(0)$ in a right neighborhood of 0, then $c^* > h(0)$.*
Moreover, if for some $\delta > 0$ and $\ell > \frac{1}{4}$ we have $h(\varphi) \le h(0)$ in $[0,\delta]$, but

$$D(\varphi)g(\varphi) \ge \ell[h(\varphi) - h(0)] \int_0^{\varphi} [h(s) - h(0)]\mathrm{d}s \quad \text{for } 0 \le \varphi \le \delta\ , \tag{1.12}$$

then again $c^* > h(0)$.

Note that conditions (1.11) and (1.12) are very close each other and they respectively imply that the t.w.s having speed c^* is of front-type or sharp of type (I). The previous results show that both the cases $c^* = h(0)$ and $c^* > h(0)$ can really occur (see also Example 2.1); in particular, when $h(\varphi)$ is monotone, the former case can occur only when h is decreasing and the validity of $c^* = h(0)$ necessarily implies $h(\varphi)$ not identically zero.

In the doubly degenerate case, the situation is more complicated since also the value $h(1) - c$ is involved in the classification of t.w.s. The results related to this case are the following ones.

Theorem 1.4 *Let g, D satisfy (1.2), (1.3) and $\dot{D}(0) \neq 0$. Assume that $D(1) = 0$, $\dot{D}(1) \neq 0$ and $h(0) < h(1)$. Then, there exists a threshold value c^*, satisfying (1.10) such that Eq. (1.1) has*

(i) no t.w.s. for $c < c^*$;

(ii) a unique (up to shifts) t.w.s. for $c = c^*$, which is:

- *sharp of type (II) if $c^* = h(0)$;*
- *sharp of type (III) if $h(0) < c^* < h(1)$;*
- *sharp of type (I) if $c^* \ge h(1)$;*

(iii) a unique (up to shifts) t.w.s. for every $c > c^*$ which is:

- *sharp of type (II) if* $c < h(1)$*;*
- *of front-type if* $c \geq h(1)$*.*

Theorem 1.5 *Let* g*,* D*, satisfy (1.2), (1.3) and* $\dot{D}(0) \neq 0$*. Assume that* $D(1) = 0$*,* $\dot{D}(1) \neq 0$ *and* $h(0) \geq h(1)$*. Then, there exists a threshold value* c^**, satisfying (1.10) such that Eq. (1.1) has*

(i) no t.w.s. for $c < c^*$*;*

(ii) a unique (up to shifts) t.w.s. for $c = c^*$*, which is of front-type if* $c^* = h(0)$*, whereas it is sharp of type (I) if* $c^* > h(0)$*;*

(iii) a unique (up to shifts) front-type t.w.s. for every $c > c^*$*.*

Also in this case it is important to know the sign of $c^* - h(1)$ and the following results account for this study.

Theorem 1.6 *Let* g*,* D *satisfy (1.2) and (1.3). Let* $D(1) = 0$*. Assume that* $h(\varphi) < h(1)$ *for* $\varphi \in [0, 1)$ *and*

$$D(\varphi)g(\varphi) < \frac{1}{4}\,[h(\varphi) - h(1)] \int_0^{\varphi} [h(s) - h(1)]\mathrm{d}s \quad \textit{for every} \;\; \varphi \in (0, 1) \tag{1.13}$$

$$\lim_{\varphi \to 0^+} \frac{D(\varphi)g(\varphi)}{\int_0^{\varphi}(h(s) - h(1))\mathrm{d}s} = 0 \tag{1.14}$$

then $c^* < h(1)$*.*

Theorem 1.7 *Let* g*,* D *satisfy (1.2) and (1.3). Let* $D(1) = 0$*. If* $h(\varphi) \geq h(1)$ *in a right neighborhood of* 0*, then* $c^* > h(1)$*. Moreover, if for some* $\delta > 0$ *and* $\ell > \frac{1}{4}$ *we have* $h(\varphi) \leq h(1)$ *in* $[0, \delta]$*, but*

$$D(\varphi)g(\varphi) \geq \ell[h(\varphi) - h(1)] \int_0^{\varphi} [h(s) - h(1)]\mathrm{d}s \quad \textit{for} \;\; 0 \leq \varphi \leq \delta \tag{1.15}$$

then again $c^* > h(1)$*.*

As an example, it is easy to check that for $g(u) := u(1 - u)$, $D(u) := u(1 - u)$ and $h(u) := 2u$, all the conditions of Theorems 1.3 and 1.6 are satisfied and then $h(0) < c^* < h(1)$. Theorem 1.4 then implies that the t.w.s. having speed c^* is sharp of type (III), those with speed $c^* < c < h(1)$ are sharp of type (II) and the remaining ones are front-type t.w.s.

In Section 5 we present some applications of Theorem 1.3 (see Corollaries 5.1, 5.2 and 5.3). We also provide several examples of dynamics with polynomial convective terms $h(u)$ for which all the possibilities can occur concerning the t.w.s. having speed c^*: front-type or sharp of type (I) in the simply degenerate case; front-type or sharp of type (I), (II),

(III) in the doubly degenerate one. Finally we give a generalization of Theorem 1.3 and Theorem 1.7 (see Proposition 5.1) suitable to treat also the case when $h(u)$ is oscillatory.

As already mentioned, a different situation occurs when $\dot{D}(0) = 0$ (or $D(1) = 0$ and $\dot{D}(1) = 0$). Indeed, in this case, when $c^* = h(0)$ the threshold speed t.w.s. can be of front-type or sharp of type (I), according to the various relations between $g(u)$, $h(u)$ and $D(u)$ (see Theorem 6.2). This study enables, in particular, to analytically solve a recently proposed model on the spatiotemporal evolution of a bacterial colony (see Eq. (6.12) in Section 6). The obtained conclusions confirm previous numerical computations discussed in [18].

Our approach allows us to treat also dynamics where the diffusivity has infinite slope at the equilibria 0 and 1. This occurs, for instance, in porous media equations $u_t = (u^m)_{xx}$ for $1 < m < 2$ (see e.g. [5]) and also in the model studied by Kamin-Rosenau [8]. When $\dot{D}(0)$ or $\dot{D}(1)$ is infinite, Eq. (1.1) does not support sharp type t.w.s. Roughly speaking, this can be justified with the coincidence of the equilibrium point P_c of system (1.7) with P_0. We discuss this matter in Section 7.

As regards the methodology, our approach is based on a refined upper and lower solutions method, combined with phase-plane techniques, applied to an associated first order singular problem (see Section 2). The classification of the t.w.s. having speed c^* is carried out by an approximation technique based on a sequence of reaction-diffusion equations whose reaction terms vanish in a right neighborhood of 0 (see Section 3). Section 4 contains the proofs of classification Theorems 1.1, 1.4, 1.5, while the proofs of the other ones are presented in Section 5, together with examples and applications.

2 First order singular analysis

The aim of this section is to develop our method in searching for t.w.s. which is based on a first order singular analysis. We start with some preliminary properties satisfied by all types of t.w.s.

Proposition 2.1 *Let φ be a t.w.s. of Eq. (1.1) satisfying boundary conditions (1.8). Then $\varphi'(\xi) < 0$ whenever $0 < \varphi(\xi) < 1$ and its wave speed c satisfies $c \geq h(0)$.*

Proof. Let $\varphi(\xi)$ be a t.w.s. of Eq. (1.1), with wave speed c, as stated in Definition 1. Assume, by contradiction, the existence of a value ξ_0 such that $\varphi'(\xi_0) = 0$ and $0 < \varphi(\xi_0) < 1$. From (1.2) and Eq. (1.4) it follows that ξ_0 is a local maximum point for φ. Therefore, by boundary conditions (1.8), there exists another point $x_0 < \xi_0$ such that $\varphi(x_0) = \varphi'(x_0) = 0$. Hence, integrating (1.4) in $[x_0, \beta)$, by (1.8) and (1.9) we obtain $\int_{x_0}^{\beta} g(\varphi(\sigma))d\sigma = 0$, contradicting the positivity of g. Similarly we can exclude that $\varphi'(\xi)$ takes positive values. Moreover, since $\varphi'(\xi) < 0$ whenever $0 < \varphi(\xi) < 1$, for every $c < h(0)$ we have

$$(D(\varphi)\varphi')' = (h(\varphi) - c)\varphi' - g(\varphi) < 0$$

in a left neighborhood of β, contradicting (1.9), being $D(\varphi(\xi))\varphi'(\xi) < 0$. □

Notice that, when $h(\varphi) \equiv 0$, that is when there is no convective effect, the admissible wave speeds satisfy the sharper constraint $c > 0$ as already showed in [14]. As a consequence of the monotonicity of each t.w.s. showed above, there exists the inverse function $\xi = \xi(\varphi)$ in $(0,1)$. Put

$$z(\varphi) := D(\varphi)\varphi'(\xi(\varphi)) \quad 0 < \varphi < 1, \tag{2.1}$$

by (1.9) it satisfies the singular first order boundary value problem

$$\begin{cases} \dot{z} = h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{z} \\ z(\varphi) < 0, \quad \varphi \in (0,1) \\ z(0^+) = z(1^-) = 0 \end{cases} \tag{2.2}$$

This means that each t.w.s. of Eq. (1.1) satisfying (1.8) gives rise to a solution of (2.2). As we will show in Theorem 2.1, also the converse is valid, i.e. there is a one-to-one correspondence between t.w.s. of (1.1) satisfying conditions (1.8) and solutions of (2.2). Our analysis is based on this correspondence.

In order to prove it we need to show the differentiability of z at the extreme points $\varphi = 0$ and $\varphi = 1$. In Theorem 2.1 the value of $\dot{z}(0)$, as well as of $\dot{z}(1)$ when $D(1) = 0$, will be also used as a criterion for distinguishing between front and sharp-type t.w.s.

Lemma 2.1 *Let $z(\varphi)$ be a solution of (2.2) for some given real $c \geq h(0)$. Then, $\dot{z}(0)$ exists and it holds $\dot{z}(0) = 0$ or $\dot{z}(0) = h(0) - c$. Moreover, if $D(1) = 0$, also $\dot{z}(1)$ exists and it holds $\dot{z}(1) = 0$ or $\dot{z}(1) = h(1) - c$.*

Proof. Let $z = z(\varphi)$ be a solution of (2.2) for some real $c \geq h(0)$. Of course we have $D^+z(0) := \limsup_{\varphi \to 0^+} \dfrac{z(\varphi)}{\varphi} \leq 0$. Since

$$\Big(h(\varphi) - c - \dot{z}(\varphi)\Big)\frac{z(\varphi)}{\varphi} = \frac{D(\varphi)g(\varphi)}{\varphi}, \qquad 0 < \varphi < 1,$$

and $\dot{D}(0) \in \mathbb{R}$, we have

$$\lim_{\varphi \to 0^+} \Big(h(\varphi) - c - \dot{z}(\varphi)\Big)\frac{z(\varphi)}{\varphi} = 0. \tag{2.3}$$

Therefore, if $D^+z(0) < 0$ then $h(0) - c = \lim_{\varphi \to 0^+} \dot{z}(\varphi) = \lim_{\varphi \to 0^+} \dfrac{z(\varphi)}{\varphi} < 0.$

Assume now, by contradiction, $D_+z(0) < D^+z(0) = 0$. Given a real number $\eta \in (D_+z(0), 0)$, we can find two sequences $(\varphi_n)_n$, $(\omega_n)_n$ in $(0,1)$ convergent to 0 such that

$$\frac{z(\varphi_n)}{\varphi_n} = \eta \quad \text{and} \quad \frac{1}{\varphi_n}\left(\dot{z}(\varphi_n) - \frac{z(\varphi_n)}{\varphi_n}\right) = \frac{d}{d\varphi}\left(\frac{z(\varphi)}{\varphi}\right)_{|\varphi = \varphi_n} \geq 0$$

$$\frac{z(\omega_n)}{\omega_n} = \eta \quad \text{and} \quad \frac{1}{\omega_n}\left(\dot{z}(\omega_n) - \frac{z(\omega_n)}{\omega_n}\right) = \frac{d}{d\varphi}\left(\frac{z(\varphi)}{\varphi}\right)_{|\varphi = \omega_n} \leq 0.$$

Hence, if $c = h(0)$, being $\dot z(\omega_n) \le \frac{z(\omega_n)}{\omega_n} = \eta$, we get $\dot z(\omega_n) \cdot \frac{z(\omega_n)}{\omega_n} \ge \eta^2 > 0$, in contradiction with (2.3). Instead, if $c > h(0)$ assuming with no loss of generality $h(0)-c < \eta$ we obtain

$$\dot z(\varphi_n) = h(\varphi_n) - c - \frac{\frac{D(\varphi_n)}{\varphi_n} g(\varphi_n)}{\eta} \to h(0) - c$$

when $n \to +\infty$, in contradiction with $\dot z(\varphi_n) \ge \eta$. So, we conclude that either $D^+ z(0) < 0$ and there exists $\dot z(0) = h(0) - c$, or $D^+ z(0) = D_+ z(0) = 0 = \dot z(0)$.

Now let us consider the extreme point $\varphi = 1$ in the case $D(1) = 0$. Similarly to the previous case we obtain

$$\lim_{\varphi \to 1^-} \frac{z(\varphi)}{\varphi - 1} \left(h(\varphi) - c - \dot z(\varphi)\right) = \lim_{\varphi \to 1^-} g(\varphi) \frac{D(\varphi)}{\varphi - 1} = 0. \tag{2.4}$$

Hence, if $D_- z(1) > 0$ then

$$h(1) - c = \lim_{\varphi \to 1^-} \dot z(\varphi) = \lim_{\varphi \to 1^-} \frac{z(\varphi)}{\varphi - 1} > 0.$$

Assume now, by contradiction, $0 = D_- z(1) < D^- z(1)$. As in the study at $\varphi = 0$, given a positive real number $\lambda \in (0, D^- z(1))$, we can find two sequences $(\phi_n)_n$, $(\psi_n)_n$ in $(0,1)$ convergent to 1 such that

$$\frac{z(\phi_n)}{\phi_n - 1} = \lambda \quad \text{and} \quad \frac{1}{\phi_n - 1}\left(\dot z(\phi_n) - \frac{z(\phi_n)}{\phi_n - 1}\right) = \frac{d}{d\varphi}\left(\frac{z(\varphi)}{\varphi - 1}\right)_{|\varphi = \phi_n} \ge 0;$$

$$\frac{z(\psi_n)}{\psi_n - 1} = \lambda \quad \text{and} \quad \frac{1}{\psi_n - 1}\left(\dot z(\psi_n) - \frac{z(\psi_n)}{\psi_n - 1}\right) = \frac{d}{d\varphi}\left(\frac{z(\varphi)}{\varphi - 1}\right)_{|\varphi = \psi_n} \le 0.$$

Hence, if $c \ge h(1)$, being $\dot z(\psi_n) \ge \frac{z(\psi_n)}{\psi_n - 1} = \lambda$, we obtain $\dot z(\psi_n) \cdot \frac{z(\psi_n)}{\psi_n - 1} \ge \lambda^2 > 0$ in contradiction with (2.4). Otherwise, when $c < h(1)$, assuming $\lambda < h(1) - c$, we get

$$\dot z(\phi_n) = h(\phi_n) - c - \frac{\frac{D(\phi_n)}{\phi_n - 1} g(\phi_n)}{\lambda} \to h(1) - c > \lambda$$

when $n \to +\infty$, in contradiction with $\dot z(\phi_n) \le \lambda$. Therefore either $\dot z(1) = h(1) - c$ or $\dot z(1) = 0$. □

The following result provides a classification of the appearing t.w.s. under the assumption $\dot D(0) \ne 0$ (and $\dot D(1) \ne 0$ for the doubly degenerate case). The case when $\dot D(0) = 0$ and/or $\dot D(1) = 0$ will be treated in Section 6.

Theorem 2.1 *Let g, D satisfy (1.2), (1.3) and $\dot D(0) \ne 0$. There is a one-to-one correspondence between the t.w.s. (of front or sharp-type) of Eq. (1.1) satisfying (1.8) (modulo shifts of the origin) and the solutions of problem (2.2) with $c \ge h(0)$. Moreover, if $D(1) > 0$ (simply degenerate case), to every solution $z(\varphi)$ corresponds:*

- *a front-type t.w.s. iff $\dot z(0) = 0$;*

- *a sharp t.w.s. of type (I) iff* $\dot{z}(0) = h(0) - c < 0$.

Instead, if $D(1) = 0$ *and* $\dot{D}(1) \neq 0$ *(doubly degenerate case), to every solution* $z(\varphi)$ *corresponds:*

- *a front-type t.w.s. iff* $\dot{z}(0) = \dot{z}(1) = 0$*;*
- *a sharp t.w.s. of type (I) iff* $\dot{z}(1) = 0$ *and* $\dot{z}(0) = h(0) - c < 0$*;*
- *a sharp t.w.s. of type (II) iff* $\dot{z}(1) = h(1) - c > 0$ *and* $\dot{z}(0) = 0$*;*
- *a sharp t.w.s. of type (III) iff* $\dot{z}(1) = h(1) - c > 0$ *and* $\dot{z}(0) = h(0) - c < 0$.

Proof. We already showed that each t.w.s. φ of equation (1.1) satisfying (1.8) gives rise to a solution of problem (2.2), defined by (2.1). Now let z be a solution of (2.2) for some $c \geq h(0)$. Let $\varphi = \varphi(\xi)$ be the (unique) solution of the Cauchy problem

$$\varphi' = \frac{z(\varphi)}{D(\varphi)}, \qquad \varphi(0) = \frac{1}{2} \tag{2.5}$$

defined on its maximal existence interval $(\alpha, \beta) \subseteq \mathbb{R}$. Of course, φ is a decreasing solution of Eq. (1.4) in (α, β) and $\varphi(\alpha^+) = 1$, $\varphi(\beta^-) = 0$.

We have to verify the validity of conditions (1.9). To this aim, if $\alpha = -\infty$, for every $\xi < 0$ integrating (1.4) in $[\xi, 0]$ we get

$$D(\varphi(\xi))\varphi'(\xi) - D(\varphi(0))\varphi'(0) = c(\varphi(0) - \varphi(\xi)) - \int_{\varphi(\xi)}^{\varphi(0)} h(\sigma)d\sigma + \int_{\xi}^{0} g(\varphi(\sigma))d\sigma. \tag{2.6}$$

So, according to conditions (1.2) and (1.8), $D(\varphi(\xi))\varphi'(\xi)$ has limit as $\xi \to -\infty$ and the boundedness of φ implies that it is equal to zero. A similar reasoning holds when $\beta = +\infty$. Then, if $\alpha = -\infty$ and $\beta = +\infty$, φ is a front-type t.w.s.

Instead, if $\alpha \in \mathbb{R}$, we have $\lim_{\xi \to \alpha^+} \varphi'(\xi) = \lim_{\varphi \to 1^-} \frac{z(\varphi)}{D(\varphi)}$. So, if $D(1) > 0$ we have $\lim_{\xi \to \alpha^+} \varphi'(\xi) = 0$; whereas, if $D(1) = 0$ then by virtue of Lemma 2.1 we have $\varphi'(\alpha^+) = 0$ or $\varphi'(\alpha^+) = (h(1) - c)/\dot{D}(1)$. When $\varphi'(\alpha^+) = 0$, the function φ can be continued in all $(-\infty, \beta)$, remaining a solution of (1.4). In both cases the first condition in (1.9) holds.

Similarly, when $\beta \in \mathbb{R}$, it is possible to show that $\varphi'(\beta^-) = 0$ or $\varphi'(\beta^-) = (h(0) - c)/\dot{D}(0)$. When $\varphi'(\beta^-) = 0$, the function φ can be continued in all $(\alpha, +\infty)$, remaining a solution of (1.4). In both cases the second condition in (1.9) holds.

Summarizing, if $\varphi'(\alpha^+) = 0$ and $\varphi'(\beta^-) = 0$, φ is a front-type t.w.s., instead in the other cases φ is a sharp t.w.s. of type (I), (II), (III) according to the various eventualities.

Finally, observe that by the definition of φ we have $D(\varphi)\varphi'(\xi(\varphi)) = z(\varphi)$, and this shows that the correspondence is one-to-one. □

Remark 2.1 In view of the proof of the above Theorem 2.1, it is clear that when $(\alpha, \beta) = \mathbb{R}$ then Definition 1.1 reduces to the classical one, since condition (1.9) is automatically satisfied. Instead, when $\alpha = -\infty$, $\beta \in \mathbb{R}$ and $\dot{D}(0) \neq 0$, then each t.w.s. satisfying (1.9) reaches the equilibrium 0 with the slope $c - h(0)/\dot{D}(0)$. Vice versa, each solution of (1.4) satisfying boundary conditions (1.8), satisfies conditions (1.9) too. So, also in this case Definition 1.1 reduces to that used in [17].

Finally, in [7] a more general definition of t.w.s. was proposed in the setting of the theory of distributions. We point out that, in this regularity framework, we obtain the same solutions as in the quoted monograph. In fact, according to property (1.9), all our t.w.s. satisfy the definition proposed in [7]. Vice versa, since for each solution in the sense of [7] the corresponding $z(\varphi)$ defined as in (2.1) satisfies (2.2), then previous theorem states that it is a front-type t.w.s. or sharp of type (I), (II) or (III).

The following result deals with the solvability of (2.2) depending on its parameter c. It can be proved with the same techniques developed in [12] for the non-degenerate case (i.e. for $D(0)D(1) > 0$). In fact these methods only depend on the product function $D(u)g(u)$ which vanishes at $u = 0$ and $u = 1$ in every case. A detailed investigation of the simply and doubly degenerate cases in absence of the convectivity ($h \equiv 0$) can be found in [14, Theorems 9 and 14]

Theorem 2.2 *Let g, D satisfy (1.2) and (1.3). Then a real value c^* exists such that (2.2) is solvable if and only if $c \geq c^*$. Moreover the solution is unique and c^* satisfies inequalities (1.10).*

Proof. According to [12, Theorem 4.1], there exists a real value c^* such that the boundary value problem

$$\begin{cases} u'' - (c - h(u))u' + D(u)g(u) = 0 \\ u(-\infty) = 0, \;\; u(+\infty) = 1 \end{cases}$$

has a unique solution (up to space shifts) for all $c \geq c^*$ and inequalities (1.10) hold. As a consequence of [12, Lemma 2.2], this implies that also

$$\begin{cases} \dot{\zeta} = c - h(\varphi) - \frac{D(\varphi)g(\varphi)}{\zeta} \\ \zeta(\varphi) > 0, \;\; 0 < \varphi < 1 \\ \zeta(0^+) = 0, \, \zeta(1^-) = 0 \end{cases}$$

is uniquely solvable, for all $c \geq c^*$. The change of variable $z(\varphi) = -\zeta(\varphi)$ then leads to the conclusion. $\square$

Example 2.1 Take $g(u) := u^2(1-u)$, $D(u) := u(1-u)$ and $h(u) := u(4u-3)$ for $0 \leq u \leq 1$. It is easy to check that problem (2.2) is solvable with $c = 0$ and $z(u) = u^2(u-1)$. According to (1.10), $c^* = 0$ and since $\dot{z}(0) = 0$ and $\dot{z}(1) = h(1)$, Theorem 2.1 implies that the corresponding t.w.s. is sharp of type (II).

3 Existence of constant profile solutions for a related combustion model

This section deals with front propagation for a class of equations obtained by (1.1) when assuming that the reaction term vanishes near 0. Such a kind of equations frequently occurs in combustion models, where an ignition temperature is present, before which no reaction is active (see [4]). This analysis will be used in the proof of Theorems 1.1, 1.4, 1.5 in order to prove the appearance of sharp profiles of type (I) or (III) for Eq. (1.1) corresponding to the threshold speed c^*, when $c^* > h(0)$.

Given $\vartheta \in (0, \frac{1}{2})$ we define

$$g_\vartheta(\varphi) := \begin{cases} 0 & 0 \le \varphi \le \vartheta \\ \min\{g(\varphi),\ \frac{g(2\vartheta)}{\vartheta}(\varphi - \vartheta)\} & \vartheta < \varphi < 2\vartheta \\ g(\varphi) & 2\vartheta \le \varphi \le 1, \end{cases} \tag{3.1}$$

and consider the family of boundary value problems

$$\begin{cases} \dot{z} = h(\varphi) - c - \frac{D(\varphi) g_\vartheta(\varphi)}{z}, \\ z(\varphi) < 0, \quad \text{for } 0 < \varphi < 1 \\ z(0^+) = z(1^-) = 0 \end{cases} \tag{3.2}$$

depending on the parameter ϑ.

Also in this case it is possible to show that the solvability of (3.2) is equivalent to the existence of decreasing t.w.s. for the reaction-diffusion equation

$$u_t + h(u)u_x = [(D(u)u_x]_x + g_\vartheta(u), \tag{3.3}$$

which satisfy the boundary conditions $\varphi(-\infty) = 1$ and $\varphi(+\infty) = 0$. Indeed, when $h \equiv 0$ and $D \equiv 1$ the proof of this statement can be found in [6]; while for h and D in $C^1[0,1]$ and D positive, it is in [13, Theorem 1]. The same argument as in [13] holds also when $D(0) = 0$, since $g_\vartheta(\varphi) \equiv 0$ in $(0, \vartheta)$. However we do not make use of this equivalence in the present paper, but we are only interested in the solvability of the first order problem (3.2).

When $D(\varphi)$ is positive on $[0,1]$ and $h \equiv 0$, Berestycki, Nicolaenko and Scheurer [4] proved the existence of a unique admissible $c > 0$ corresponding to a unique (up to space shifts) wave profile of (3.3) between 0 and 1, for each $\vartheta \in (0,1)$. On the other hand, in [13] we showed that when the additional convective term $h(\varphi)$ prevails over D and g_ϑ, then it can prevent the appearance of front propagation for (3.3) between its stationary states 0 and 1 for any speed $c \in \mathbb{R}$.

Concerning the special problem (3.2), in this Section we prove that if $c^* > h(0)$, whatever $h(\varphi)$ may be, problem (3.2) is solvable, for sufficiently small ϑ. This is enough for our aims.

Theorem 3.1 *Let g, D satisfy (1.2), (1.3) and $c^* > h(0)$. Then there exists $\bar{\vartheta} \in (0, \frac{1}{2})$ such that, for every $0 < \vartheta < \bar{\vartheta}$ problem (3.2) is solvable for a unique value $\tilde{c} = \tilde{c}_\vartheta$. Moreover, $h(0) < \tilde{c}_\vartheta < c^*$ and the solution is unique.*

The proof of this theorem requires the following result.

Lemma 3.1 *If $c^* > h(0)$, for all $h(0) < c < c^*$ there exists $\bar{w}_0 = \bar{w}_0(c) < 0$ such that for every $\bar{w}_0 < w_0 < 0$ the solution $z(\varphi)$ of*

$$\dot{z} = h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{z}, \tag{3.4}$$

with initial condition $z(0) = w_0$, satisfies $z(\varphi_0^+) = 0$ for some $0 < \varphi_0 < 1$.

Proof. Let $z^*(\varphi)$ denote the unique solution of (2.2) with $c = c^*$. Given $c < c^*$, first we show that whenever $\zeta(\varphi)$ is a negative solution of (3.4) defined on all $(0, 1)$, then it holds $\zeta(\varphi) < z^*(\varphi)$ for every $\varphi \in (0, 1)$. Assuming in fact $\zeta(\bar{\varphi}) \geq z^*(\bar{\varphi})$ for some $\bar{\varphi} \in (0, 1)$, we obtain

$$\dot{\zeta}(\bar{\varphi}) = h(\bar{\varphi}) - c - \frac{D(\bar{\varphi})g(\bar{\varphi})}{\zeta(\bar{\varphi})} > h(\bar{\varphi}) - c^* - \frac{D(\bar{\varphi})g(\bar{\varphi})}{z^*(\bar{\varphi})} = \dot{z}^*(\bar{\varphi})$$

and this leads to the contradictory conclusion $0 \geq \zeta(1^-) > z^*(1^-) = 0$.

Now, for some $c < c^*$ assume by contradiction the existence of a negative increasing sequence $(w_n)_n$ convergent to 0 such that for all $n \in \mathbb{N}$ the solution $z_n(\varphi)$ of (3.4) with $z_n(0) = w_n$ is defined on all $(0, 1)$. By what is proved above, $z_n(\varphi) < z^*(\varphi)$ for $n \in \mathbb{N}$ and $\varphi \in (0, 1)$, and $(z_n(\varphi))_n$ is an increasing sequence on all $[0, 1]$. So we can define $z(\varphi) := \sup_{n \in \mathbb{N}} z_n(\varphi)$, $0 \leq \varphi < 1$. The monotone convergence theorem implies that $z(\varphi)$ is a solution of (3.4) on all $(0, 1)$ satisfying $z(0^+) = 0$. By the definition of c^* (see Theorem 2.2), we get $z(1^-) < 0$.

Let us consider the monotone sequence $(\zeta_n(\varphi))_n$ of solutions of (3.4) satisfying $\zeta_n(1) = \frac{z(1^-)}{n}$. Notice that $\zeta_n(\bar{\varphi}) \geq z^*(\bar{\varphi})$ for some $n \in \mathbb{N}$ and $\bar{\varphi} \in (0, 1)$ implies $\dot{\zeta}_n(\bar{\varphi}) > \dot{z}^*(\bar{\varphi})$ in contradiction with $\zeta_n(1^-) < 0$. Hence $\zeta_n(\varphi) < z^*(\varphi)$ for all $n \in \mathbb{N}$ and $\varphi \in (0, 1)$. In particular ζ_n is defined on all $[0, 1]$ and since $z(\varphi) < \zeta_n(\varphi)$ for all φ, $\zeta_n(0^+) = 0$. The monotone convergence theorem guarantees that the limit function of $(\zeta_n)_n$ is a solution of (2.2), in contradiction with Theorem 2.2, since $c < c^*$. □

Proof of Theorem 3.1. For every given $c > h(0)$ and $\vartheta \in (0, \frac{1}{2})$, let $\zeta_{c,\vartheta}(\varphi)$ be the unique solution of the Cauchy problem

$$\begin{cases} \dot{z} = h(\varphi) - c - \frac{D(\varphi)g_\vartheta(\varphi)}{z}, \\ z(0) = 0 \end{cases} \tag{3.5}$$

defined on its maximal existence interval $[0, \bar{\varphi}_\vartheta(c)) \subset (0, 1)$. Note that if $c_1 < c_2$ then $\zeta_{c_1,\vartheta}(\varphi) > \zeta_{c_2,\vartheta}(\varphi)$ for every $0 < \varphi < \min\{\bar{\varphi}_\vartheta(c_1), \bar{\varphi}_\vartheta(c_2)\}$. In fact, we have $\zeta_{c_1,\vartheta}(\varphi) > \zeta_{c_2,\vartheta}(\varphi)$ in $(0, \vartheta)$; moreover in each interval $(0, \alpha)$ where $\zeta_{c_1,\vartheta}(\varphi) > \zeta_{c_2,\vartheta}(\varphi)$ we have

$$\zeta_{c_1,\vartheta}(\alpha) = \int_0^\alpha \left(h(\varphi) - c_1 - \frac{D(\varphi)g_\vartheta(\varphi)}{\zeta_{c_1,\vartheta}(\varphi)} \right) d\varphi >$$
$$> \int_0^\alpha \left(h(\varphi) - c_2 - \frac{D(\varphi)g_\vartheta(\varphi)}{\zeta_{c_2,\vartheta}(\varphi)} \right) d\varphi = \zeta_{c_2,\vartheta}(\alpha).$$

This implies, in particular,

$$\bar{\varphi}_\vartheta(c_1) \le \bar{\varphi}_\vartheta(c_2) \quad \text{and} \quad \zeta_{c_1,\vartheta}(\bar{\varphi}_\vartheta(c_1)^-) > \zeta_{c_2,\vartheta}(\bar{\varphi}_\vartheta(c_1)^-). \tag{3.6}$$

Put $\hat{c} := \frac{1}{2}(c^* + h(0))$, let $\hat{z}(\varphi)$ be a solution of (3.4) for $c = \hat{c}$ with $\hat{z}(0) = w_0 < 0$ such that $\hat{z}(\varphi_0) = 0$ for some $\varphi_0 \in (0,1)$ as stated in Lemma 3.1. It is easy to show the existence of $\hat{\varphi} \in (0, \varphi_0)$ satisfying

$$\hat{z}(\varphi) < \int_0^\varphi h(\sigma)\, d\sigma - \hat{c}\varphi < 0 \quad \text{for all } 0 < \varphi < \hat{\varphi}. \tag{3.7}$$

Put $\bar{\vartheta} := \frac{1}{2}\hat{\varphi}$ and fix $0 < \vartheta < \bar{\vartheta}$. According to (3.7) and the definition of g_ϑ we obtain that $\hat{z}(\varphi) < \zeta_{\hat{c},\vartheta}(\varphi)$ for every $0 \le \varphi \le 2\vartheta$. When $2\vartheta \le \varphi \le 1$ both $\zeta_{\hat{c},\vartheta}(\varphi)$ and $\hat{z}(\varphi)$ are solutions of the equation in (2.2). The uniqueness of the solution of the associated Cauchy problems then implies $\bar{\varphi}_\vartheta(\hat{c}) \le \varphi_0 < 1$.

Consider now $\zeta_{c^*,\vartheta}$ and again let $z^*(\varphi)$ be the solution of (2.2) for $c = c^*$. It is easy to see that $\zeta_{c^*,\vartheta}(\varphi) < z^*(\varphi)$ for $0 < \varphi \le \vartheta$. Moreover, when assuming $\zeta_{c^*,\vartheta}(\tilde{\varphi}) < z^*(\tilde{\varphi})$ for some $\tilde{\varphi} \in (\vartheta, 1)$, it holds $\dot{\zeta}_{c^*,\vartheta}(\tilde{\varphi}) < \dot{z}_{c^*}(\tilde{\varphi})$. This implies $\bar{\varphi}_\vartheta(c^*) = 1$ and $\zeta_{c^*,\vartheta}(1^-) < 0$.

Put $A := \{c > h(0) : \bar{\varphi}_\vartheta(c) < 1\}$, $B := \{c > h(0) : \bar{\varphi}_\vartheta(c) = 1 \text{ and } \zeta_{c,\vartheta}(1^-) < 0\}$, the previous discussion shows that $\hat{c} \in A$ and $c^* \in B$. Moreover, by the continuous dependence on initial data of the Cauchy problem

$$\dot{z} = h(\varphi) - c - \frac{D(\varphi)g_\vartheta(\varphi)}{z}\,, \quad z(\vartheta) = \int_0^\vartheta (h(\varphi) - c)d\varphi, \tag{3.8}$$

it is easy to check that B is open. Hence, put $\tilde{c}_\vartheta := \inf B$, we get $\tilde{c}_\vartheta \notin B$ and $h(0) < \tilde{c}_\vartheta < c^*$, so by (3.6) it follows $\tilde{c}_\vartheta = \sup A$. The proof of the solvability of (3.2) for $c = \tilde{c}_\vartheta$ will be complete if we show that $\tilde{c}_\vartheta \notin A$.

To this purpose, let us assume by contradiction that $\bar{\varphi}_\vartheta(\tilde{c}_\vartheta) < 1$. Consider a decreasing sequence $(c_n)_n$ convergent to $\tilde{c}_\vartheta$ and define $\zeta_n(\varphi) := \zeta_{c_n,\vartheta}(\varphi)$. Of course, $(\zeta_n(\varphi))_n$ is an increasing sequence of negative functions, convergent in $[0,1]$ to a function $\tilde{\zeta}(\varphi)$. By applying the monotone convergence theorem, we get

$$\tilde{\zeta}(1) - \tilde{\zeta}(\varphi) = \lim_{n\to+\infty}(\zeta_n(1) - \zeta_n(\varphi)) = \lim_{n\to+\infty} \int_\varphi^1 \left(h(s) - c_n - \frac{D(s)g_\vartheta(s)}{\zeta_n(s)} \right) \mathrm{d}s =$$

$$= \int_\varphi^1 \lim_{n\to+\infty} \left(h(s) - c_n - \frac{D(s)g_\vartheta(s)}{\zeta_n(s)} \right) \mathrm{d}s.$$

This implies that $\tilde{\zeta}(\varphi) < 0$ for a.e. $\varphi \in (0,1)$ and is continuous in $[0,1]$. Moreover, in every interval where it is negative, it is a solution of the equation in (3.2) for $c = \tilde{c}_\vartheta$. The unique solvability of (3.8), considered when $c = \tilde{c}_\vartheta$ then implies $\zeta_{\tilde{c}_\vartheta,\vartheta} = \tilde{\zeta}$ on all $[0, \bar{\varphi}_\vartheta(\tilde{c}_\vartheta))$. Therefore, there exists an interval $(a,b) \subset (\bar{\varphi}_\vartheta(\tilde{c}_\vartheta), 1)$ such that $\tilde{\zeta}(\varphi) < 0$ in (a,b) and $\tilde{\zeta}(a) = 0$, but this implies $\dot{\tilde{\zeta}}(a^+) = \lim_{\varphi\to\alpha^+} h(\varphi) - \tilde{c}_\vartheta - \dfrac{D(\varphi)g(\varphi)}{\tilde{\zeta}(\varphi)} = +\infty$, a contradiction.

Thus, we conclude that $\bar{\varphi}_\vartheta(\tilde{c}_\vartheta) = 1$ and $\zeta_{\tilde{c}_\vartheta,\vartheta}(1^-) = 0$, that is $\zeta_{\tilde{c}_\vartheta,\vartheta}(\varphi)$ is the solution of (3.2) we were looking for.

The uniqueness of $\tilde{c}_\vartheta$ is a consequence of the strict monotonicity in (3.6) and the uniqueness of the solution follows from the uniqueness property of problem (3.8).

4 Proofs of classification theorems 1.1, 1.4, 1.5

Note that the analysis of the admissible wave speeds as well as the estimate of the threshold value c^* are contained in Theorem 2.2. So here we have to prove the classification of the various types of t.w.s. according to the value of the speed c.

Proof of Theorem 1.1. Given $c > c^*$ consider the t.w.s. $\varphi(\xi)$ with speed c and let $z(\varphi)$ be defined by (2.1), satisfying the first order dynamic (2.2) (see Theorems 2.1 and 2.2). Let z^* denote the solution of (2.2) with $c = c^*$. Reasoning as at the beginning of the proof of Lemma 3.1, we obtain $z^*(\varphi) < z(\varphi)$ for all $\varphi \in (0,1)$. From Proposition 2.1, we get $h(0) - c < h(0) - c^* \leq 0$, hence if $\dot{z}(0) = h(0) - c$ then $z(\varphi) < z^*(\varphi)$ for sufficiently small positive φ, a contradiction. Therefore, by applying Lemma 2.1 we have $\dot{z}(0) = 0$. Theorem 2.1 then implies that $\varphi(\xi)$ has a front-type profile.

Let us now consider the t.w.s having speed $c = c^*$. First note that if $c^* = h(0)$, then according to Lemma 2.1, we have $\dot{z}(0) = 0$ and this implies that $\varphi(\xi)$ has a front-type profile.

Now, let $c^* > h(0)$. By virtue of Lemma 2.1 and Theorem 2.1, we have to show that $\dot{z}^*(0) = h(0) - c^*$. In order to do this, let $\bar{\vartheta}$ be the positive value given by Theorem 3.1 and consider the decreasing sequence $\vartheta_n := 2^{-n}\bar{\vartheta}$, $n \geq 1$. By virtue of Theorem 3.1, problem (3.2) with $\vartheta = \vartheta_n$ is solvable for a unique $c = c_n$ satisfying $h(0) < c_n < c^*$. Let us denote by $z_n(\varphi)$ its unique solution.

Following the same proof of [14, Lemma 12], with the only introduction of the additional convective term $h(\varphi)$, one can show that $(c_n)_n$ is a strictly increasing sequence converging towards c^*. As a consequence, by using the same argument of [14, Lemma 13], that works also with the additional function $h(\varphi)$, it is possible to prove the existence of a positive decreasing sequence $(b_n)_n$ convergent to 0 and a sequence $(\psi_n(\varphi))_n$ of continuous functions defined on $[0, u_n)$ with $u_n \leq u_{n+1}$ for all n, satisfying the following properties

$$
\begin{array}{ll}
(i)\ \ \psi_n(\varphi) = \int_0^\varphi (h(s) - c_n)\mathrm{d}s & \text{for all } \varphi \in [0, b_n]; \\
(ii)\ \ \psi_n(\varphi) \geq \int_0^\varphi (h(s) - c^*)\mathrm{d}s & \text{for every } \varphi \in [0, u_n); \\
(iii)\ \ \dot{\psi}_n(\varphi) = h(\varphi) - c_n - \frac{D(\varphi)g(\varphi)}{\psi_n(\varphi)} & \text{for every } \varphi \in [b_n, u_n); \\
(iv)\ \ \psi_n(u_n^-) = 0; & \\
(v)\ \ \psi_n(\varphi) \geq \psi_{n+1}(\varphi) & \text{for all } \varphi \in [0, u_n).
\end{array}
$$

Now put $\bar{u} := \sup_{n\in\mathbb{N}} \bar{u}_n \leq 1$ and $\psi(\varphi) := \inf_{n\in\mathbb{N}} \psi_n(\varphi)$. By (ii) it follows that $\psi(\varphi) \geq \int_0^\varphi (h(s) - c^*)\mathrm{d}s$ for $0 < \varphi < \bar{u}$. Moreover, by (iii) and the dominated convergence theorem we get that $\psi(\varphi)$ is a solution in $(0, \bar{u})$ of the equation in (2.2) and by (i)

we have

$$\frac{1}{\varphi}\int_0^{\varphi} h(s)\mathrm{d}s - c^* \le \frac{\psi(\varphi)}{\varphi} \le \frac{\psi_n(\varphi)}{\varphi} = \frac{1}{\varphi}\int_0^{\varphi} h(s)\mathrm{d}s - c_n\,, \quad 0<\varphi<b_n$$

which implies the existence of $\dot{\psi}(0) = h(0) - c^*$.

Moreover, following the same argument used in the proof of [14, Theorem 2], it is possible to prove that $\psi(\bar{u}^-) = 0$. Therefore, if $\bar{u} = 1$ then by the uniqueness of the solution of (2.2) for $c \ge c^*$ (see Theorem 2.2) it follows $\psi(\varphi) = z^*(\varphi)$ and then $\dot{z}^*(0) = h(0) - c^*$. Instead, when $\bar{u} < 1$, since $z^*(\bar{u}) < 1$ the uniqueness of the Cauchy problem associated to (3.4) implies $z^*(\varphi) < \psi(\varphi)$ for all $0<\varphi<\bar{u}$ and consequently

$$\dot{z}^*(0) = \lim_{\varphi\to 0^+} \frac{z^*(\varphi)}{\varphi} \le \lim_{\varphi\to 0^+} \frac{\psi(\varphi)}{\varphi} = h(0) - c^*. \qquad \square$$

Proof of Theorems 1.4 and 1.5. Taking account of Theorem 2.1 and the proof of Theorem 1.1, it only remains to discuss the value of $\dot{z}(1^-)$ depending on the wave speed c. Since

$$\dot{z}(\varphi) = h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{z(\varphi)} \ge h(\varphi) - c, \quad 0<\varphi<1,$$

we get $\dot{z}(1) \ge h(1) - c$. Hence, if $c < h(1)$ then necessarily $\dot{z}(1^-) = h(1) - c$ by Lemma 2.1, and the t.w.s. is sharp of type (II) or (III) according to the value of $h(0) - c$. Instead, when $h(1) - c \le 0$ we have $\dot{z}(1) = 0$, since $z(\varphi)$ is a negative function with $z(1) = 0$. $\square$

5 Comparison between c^* and $h(0)$, $h(1)$

In this Section we give the proofs of Theorems 1.2, 1.3, 1.6 and 1.7, together with some corollaries and examples. To this aim we need the following lemma whose proof is omitted since it is quite similar to the one given in [14, Lemma 8] for the same result but in the special case when $h(\varphi) \equiv 0$.

Lemma 5.1 *Assume that there exists a negative function* $\eta \in C^1(0,1)$ *such that* $\eta(0^+) = 0$ *and*

$$\dot{\eta}(\varphi) \ge h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\eta(\varphi)} \quad \textit{for every } \varphi \in (0,1)$$

(that is η is an upper solution for the equation in (2.2)). Then problem (2.2) admits a solution $z(\varphi) \ge \eta(\varphi)$.

Proof of Theorem 1.2. Let us consider the function $H(\varphi) := \frac{1}{2}\int_0^{\varphi}(h(s) - h(0))\mathrm{d}s$. Since $h(\varphi) < h(0)$ for every $\varphi \in (0,1)$, $H(\varphi) < 0$ for every $\varphi \in (0,1]$. Moreover, from (1.11) it follows that

$$\dot{H}(\varphi) = \frac{1}{2}(h(\varphi) - h(0)) \ge h(\varphi) - h(0) - \frac{D(\varphi)g(\varphi)}{H(\varphi)} \quad \text{for } 0<\varphi<1$$

that is $H(\varphi)$ is an upper solution for the equation in (2.2) for $c = h(0)$. Hence, by Lemma 5.1 we infer that problem (2.2) admits a solution for $c = h(0)$. Being $c^* \geq h(0)$, we deduce $c^* = h(0)$. $\square$

Proof of Theorem 1.6. Let us consider the function $H(\varphi) := \frac{1}{2}\int_0^{\varphi}(h(s) - h(1))\mathrm{d}s$. Since $H(\varphi) < 0$ for every $\varphi \in (0, 1]$, by (1.13) it follows that $H(\varphi)$ is an upper solution for the equation in (2.2). Therefore, by applying Lemma 5.1 we get that problem (2.2) admits a solution $\tilde{z}(\varphi)$ for $c = h(1)$.

From assumption (1.14) we infer the existence of a positive real $\delta_1 < 1$ such that

$$\frac{D(\varphi)g(\varphi)}{H(\varphi)} - \frac{1}{2}(h(\varphi) - h(1)) > \frac{1}{4}(h(1) - h(0)) \quad \text{for } 0 < \varphi \leq \delta_1. \tag{5.1}$$

Now, for every $\epsilon > 0$, let us consider the solution $z_\epsilon(\varphi)$ of the following terminal value problem

$$(\mathrm{TP}_\epsilon) \qquad \begin{cases} \dot{z}(\varphi) = h(\varphi) - h(1) + \epsilon - \frac{D(\varphi)g(\varphi)}{z(\varphi)}, & 0 < \varphi < 1 \\ z(1) = \frac{1}{2}H(1). \end{cases}$$

We have $z_\epsilon(\varphi) < \tilde{z}(\varphi)$ for all $\varphi \in (0, 1]$, since $z_\epsilon(1) < \tilde{z}(1)$ and $z_\epsilon(\bar{\varphi}) = \tilde{z}(\bar{\varphi})$ for some $\bar{\varphi} \in (0, 1)$ implies $\dot{z}_\epsilon(\bar{\varphi}) > \dot{\tilde{z}}(\bar{\varphi})$. Let $\delta_2 \in (0, 1 - \delta_1)$ be such that putting $\epsilon^* = \frac{1}{4}(h(1) - h(0))$ we have $z_{\epsilon^*}(\varphi) > H(\varphi)$ for every $\varphi \in [1 - \delta_2, 1]$. Reasoning as before, when we compared z_ϵ and $\tilde{z}$, we get

$$z_\epsilon(\varphi) > H(\varphi) \quad \text{for every } \epsilon \leq \epsilon^* \quad \text{and } \varphi \in [1 - \delta_2, 1]. \tag{5.2}$$

Put $m := \min\left(\frac{D(\varphi)g(\varphi)}{H(\varphi)} - \frac{1}{2}\left(h(\varphi) - h(1)\right)\right)$ for $\varphi \in [\delta_1, 1 - \delta_2]$. By assumption (1.13) we deduce that $m > 0$. Hence, consider now $\rho := \min\{m, \epsilon^*\} > 0$ and the solution $z_\rho(\varphi)$ of problem (TP_ρ). From (5.1) and the definitions of ρ and m, it follows that

$$\dot{H}(\varphi) > h(\varphi) - h(1) + \rho - \frac{D(\varphi)g(\varphi)}{H(\varphi)} \quad \text{for every } 0 < \varphi < 1 - \delta_2,$$

that is $H(\varphi)$ is an upper-solution for the equation in (TP_ρ) in $[0, 1 - \delta_2]$. Hence, by (5.2), it follows that $z_\rho(\varphi) > H(\varphi)$ in its maximal existence interval $(0, 1]$. Since also $z_\rho(\varphi) < \tilde{z}(\varphi)$ in $(0, 1]$, we infer that $z_\rho(0^+) = 0$. According to Lemma 5.1, this implies that problem (2.2) admits a solution for $c = h(1) - \rho$, that is $c^* < h(1)$. $\square$

Theorems 1.3 and 1.7 are consequence of the following result, which extends and unifies them.

Proposition 5.1 *Let g, D satisfy (1.2) and (1.3). For $i = 0, 1$, put*

$$k_i(\varphi) = \min\{h(\varphi) - h(i),\, 0\}. \tag{5.3}$$

If for some $\delta > 0$ and $\ell > \frac{1}{4}$ it holds

$$D(\varphi)g(\varphi) \geq \ell k_i(\varphi) \int_0^{\varphi} k_i(s)\mathrm{d}s \quad \textit{for } 0 \leq \varphi \leq \delta \tag{5.4}$$

then $c^* > h(i)$.

Proof. Fix $i \in \{0,1\}$ and assume, by contradiction, the existence of a solution $z(\varphi)$ of problem (2.2) for $c = h(i)$.

We can take, without restrictions, $\ell \in (\frac{1}{4}, 1)$ in assumption (5.4). Observe that, in this case $\dot{z}(\varphi) \geq \alpha k_i(\varphi)$ for $\varphi \in (0,\delta)$ and some $\alpha > 0$ implies

$$\dot{z}(\varphi) \geq (1 - \frac{\ell}{\alpha})k_i(\varphi) \tag{5.5}$$

in the same interval. In fact, since $\alpha \int_0^\varphi k_i(\varphi)\mathrm{d}s \leq z(\varphi) < 0$ in $(0,\delta)$, we get by (5.4)

$$\dot{z}(\varphi) \geq k_i(\varphi) - \frac{\ell k_i(\varphi)\int_0^\varphi k_i(s)\mathrm{d}s}{\alpha \int_0^\varphi k_i(s)\mathrm{d}s} = (1 - \frac{\ell}{\alpha})k_i(\varphi).$$

Put $a_1 := 1$ and $a_{n+1} := 1 - \frac{\ell}{a_n}$ until $a_n > 0$. Since $\ell > \frac{1}{4}$, we obtain $a_{n+1} < a_n$, so if $a_n > 0$ for every $n \in \mathbb{N}$, then $a_n \to \lambda$ for some $\lambda \geq 0$, a contradiction. Therefore, there exists $n^* \in \mathbb{N}$ such that $a_{n^*+1} \leq 0$.

Observe now that $\dot{z}(\varphi) = h(\varphi) - h(i) - \frac{D(\varphi)g(\varphi)}{z(\varphi)} \geq a_1 k_i(\varphi)$, so from (5.5) we deduce that

$$z(\varphi) \geq a_{n^*+1}\int_0^\varphi k_i(s)\mathrm{d}s \geq 0 \quad \text{in } (0,\delta),$$

a contradiction. Hence $c^* > h(i)$. □

We remark that Theorems 1.2, 1.3, 1.6 and 1.7 are especially suitable for treating monotone convectivities $h(\varphi)$. In particular, Theorem 1.3 states that, whenever h is increasing, $c^* > h(0)$. In the simply degenerate case this implies that the threshold t.w.s. is right compactly supported, i.e. it is sharp of type (I). The same is true also in the doubly degenerate case, when in addition (1.15) holds (see Theorem 1.7). Instead, when assuming (1.13) and (1.14) and the strict monotonicity of h, the threshold profile is sharp of type (III) and both equilibria 0 and 1 are reached at finite values (see Theorem 1.6).

Instead, Proposition 5.1 only concerns the behavior of the negative part respectively of $h(\varphi) - h(0)$ and $h(\varphi) - h(1)$ with respect to the product $D(\varphi)g(\varphi)$ and can be applied to oscillatory convective terms too, as the following example shows.

Example 5.1 Let $h(u) := u \sin\frac{1}{u}$ for $u \in (0,1]$ and $h(0) = 0$. Define $k_0(u)$ as in (5.3). Since $k_0(u) \geq -u$, we get $k_0(u)\int_0^u k_0(s)\mathrm{d}s \leq \frac{u^3}{2}$ for all $0 \leq u \leq 1$. Therefore, whenever $D(u)g(u) > ku^3$ in a right neighborhood of 0 for some $k > \frac{1}{8}$, by applying the previous proposition one concludes that $c^* > h(0)$.

Now we state some applications of Theorem 1.3, useful when the calculation of the integral function of h is not easy. The following results take into consideration the case when $h(\varphi) \leq h(0)$ in a right neighborhood of 0. In the class of functions $h(\varphi)$ which are locally monotone in some interval $[0,\delta]$, this is the only remaining case; in fact whenever $h(\varphi) \geq h(0)$ in $[0,\delta]$ then $c^* > h(0)$ (see Theorem 1.3).

Corollary 5.1 *Let there exist* $\alpha' \geq \alpha > 0$, $\beta' \geq \beta \geq 0$ *and* $\delta > 0$ *such that*

$$\alpha\varphi^{\beta'} \leq h(0) - h(\varphi) \leq \alpha'\varphi^{\beta} \quad \text{for every } 0 \leq \varphi \leq \delta \tag{5.6}$$

$$\limsup_{\varphi \to 0^+} \frac{\varphi^{1+\beta-\beta'}[h(0) - h(\varphi)]^2}{D(\varphi)g(\varphi)} < \frac{\alpha}{\alpha'}(\beta + 1). \tag{5.7}$$

Then $c^* > h(0)$.

Proof. By (5.7) there exists a positive $\rho \leq \delta$ such that

$$D(\varphi)g(\varphi) > \frac{\alpha'}{\alpha(\beta+1)} \frac{\varphi[h(0) - h(\varphi)]^2}{\varphi^{\beta'-\beta}} \quad \text{for every } \varphi \in (0, \rho]. \tag{5.8}$$

Moreover, by (5.6) it follows that

$$\frac{\int_0^\varphi [h(0) - h(s)]\mathrm{d}s}{\varphi[h(0) - h(\varphi)]} \leq \frac{\alpha'}{\alpha(\beta+1)}\varphi^{\beta-\beta'}.$$

Hence, taking (5.8) into account, we get

$$[h(0) - h(\varphi)] \int_0^\varphi [h(0) - h(s)]\mathrm{d}s \leq \frac{\alpha'}{\alpha(\beta+1)} \frac{\varphi[h(0) - h(\varphi)]^2}{\varphi^{\beta'-\beta}} < D(\varphi)g(\varphi),$$

that is (1.12) holds for $\ell = 1$ and the assertion follows as an application of Theorem 1.3. □

Example 5.2 Take $h(u) := -ku^p$ for some $k, p > 0$. By applying Corollary 5.1 with $\alpha = \alpha' = k$, $\beta = \beta' = p$, we deduce that if $\limsup_{u \to 0^+} \frac{u^{2p+1}}{D(u)g(u)} < \frac{p+1}{k^2}$, then $c^* > h(0) = 0$.

Moreover, take $h(u) := -ku^p |\log u|^q$ for some $k, p, q > 0$. In this case, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$ku^p \leq ku^p |\log u|^q \leq ku^{p-\epsilon} \quad \text{for every } u \in (0, \delta].$$

So, if $\lim_{u \to 0^+} \frac{u^{2p+1-\epsilon}}{D(u)g(u)} = 0$ for some $\epsilon > 0$, then the conditions of Corollary 5.1 are satisfied for $\alpha = \alpha' = k$ and $\beta = p - \epsilon/2$, $\beta' = p$. Hence $c^* > h(0) = 0$.

Corollary 5.2 *Let* $h(\varphi) \in C^n[0, \delta]$ *for some* $\delta > 0$, *with*

$$h'(0) = h''(0) = \ldots = h^{(n-1)}(0) = 0 \quad \text{and} \quad h^{(n)}(0) < 0. \tag{5.9}$$

If

$$\limsup_{\varphi \to 0^+} \frac{\varphi[h(0) - h(\varphi)]^2}{D(\varphi)g(\varphi)} < n + 1 \tag{5.10}$$

then $c^* > h(0)$.

Proof. Put $L := \limsup_{\varphi\to 0^+} \dfrac{\varphi[h(0)-h(\varphi)]^2}{D(\varphi)g(\varphi)} < n+1$ and $\lambda := -h^{(n)}(0)/n! > 0$. Since $h(0)-h(\varphi) = \lambda\varphi^n + o(\varphi^n)$, taken $\epsilon > 0$ in such a way that $\frac{\lambda-\epsilon}{\lambda+\epsilon}(n+1) > L$, there is $\delta > 0$ such that

$$(\lambda-\epsilon)\varphi^n \le h(0)-h(\varphi) \le (\lambda+\epsilon)\varphi^n \quad \text{for every } \varphi \in [0,\delta].$$

Therefore, the assertion follows as a consequence of Corollary 5.1 with $\beta = \beta' = n$, $\alpha = \lambda-\epsilon$, $\alpha' = \lambda+\epsilon$. □

Corollary 5.3 *Assume that $h(\varphi) \le h(0)$ in a right neighborhood of 0 and*

$$\limsup_{\varphi\to 0^+} \frac{\varphi[h(0)-h(\varphi)]}{D(\varphi)g(\varphi)} < +\infty. \tag{5.11}$$

Then $c^* > h(0)$.

Proof. Let $k > \limsup_{\varphi\to 0^+} \dfrac{\varphi[h(0)-h(\varphi)]}{D(\varphi)g(\varphi)}$. Let $\delta > 0$ be such that $h(\varphi) \le h(0)$,

$$D(\varphi)g(\varphi) > \frac{1}{k}\varphi[h(0)-h(\varphi)] \quad \text{and} \quad \frac{1}{\varphi}\int_0^\varphi [h(0)-h(s)]\mathrm{d}s < \frac{1}{k}$$

for every $\varphi \in (0,\delta]$. We have

$$D(\varphi)g(\varphi) > [h(0)-h(\varphi)]\int_0^\varphi [h(0)-h(s)]\mathrm{d}s \quad \text{for every } \varphi \in (0,\delta]$$

and the assertion follows from Theorem 1.3. □

Of course, analogous conditions to those in Corollaries 5.1, 5.2 and 5.3 could be given in order to guarantee that assumption (1.15) holds and then no sharp t.w.s. of type (II) and (III) exists.

6 Analysis of $\dot{D}(0) = 0$ and $D(1) = \dot{D}(1) = 0$

The present section is devoted to the study of dynamics such that $\dot{D}(0) = 0$ and/or $D(1) = \dot{D}(1) = 0$, for which the classification of the t.w.s. is not as simple as when the derivatives do not vanish. Indeed, we show that in the simply degenerate case also when $c^* = h(0)$ the corresponding t.w.s. can be of front-type or sharp of type I, according to the relations between the nonlinear terms $D(u)$, $g(u)$, $h(u)$. Note, moreover, that in this case the associated system (1.7) has no more the additional equilibrium $P_c = h(0) - c^*/\dot{D}(0)$ and the sharp-type t.w.s. can reach the equilibrium 0 with infinite slope.

We begin analyzing the situation which arises when $c > h(0)$.

Theorem 6.1 *Let g satisfy (1.2) and let $D \in C^1[0,1]$ be such that*

$$D(0) = \dot{D}(0) = 0 \ , \qquad D(\varphi) > 0 \ \ \textit{for every} \ \ 0 < \varphi \leq 1.$$

Assume that

$$0 \leq \dot{D}(\varphi)\varphi \leq kD(\varphi) \quad \textit{for every } 0 \leq \varphi \leq \delta \tag{6.1}$$

for some $\delta, k > 0$. Then, for every $c > h(0)$ there is a one-to-one correspondence between the t.w.s. (of front or sharp-type) of Eq. (1.1) satisfying (1.8) (modulo shifts of the origin) and the solutions of problem (2.2).
Moreover, to every solution $z(\varphi)$ corresponds:

- *a front-type t.w.s. iff $\dot{z}(0) = 0$;*
- *a sharp t.w.s. of type (I) iff $\dot{z}(0) = h(0) - c < 0$.*

Proof. First observe that Proposition 2.1 and Lemma 2.1 hold even when $\dot{D}(0) = 0$ and then each t.w.s. gives rise to a solution of problem (2.2) defined by (2.1). Vice versa, let $\varphi(\xi)$ be the unique solution of (2.5) defined on its maximal existence interval $(\alpha, \beta) \subset \mathbb{R}$. Also in this case $\varphi(\xi)$ is a decreasing solution of Eq. (1.4) in (α, β) satisfying $\varphi(\alpha^+) = 1$ and $\varphi(\beta^-) = 0$. Then, if $\alpha = -\infty$ and $\beta = +\infty$, by repeating the same argument used in the proof of Theorem 2.1, it is easy to show that conditions (1.9) hold and φ is a front-type t.w.s. Moreover, being $D(1) > 0$ it is easy to show that if $\alpha \in \mathbb{R}$ then $\varphi'(\alpha^+) = 0$ and again φ can be continued in $(-\infty, \beta)$ as a solution of Eq. (1.4).

Let us now consider the case $\beta \in \mathbb{R}$. If $\dot{z}(0) = h(0) - c < 0$ then $\varphi'(\beta^-) = -\infty$ and $\lim_{\xi \to \beta^-} D(\varphi(\xi))\varphi'(\xi) = z(0) = 0$. So, φ is a sharp t.w.s. of type (I). Let us take into consideration the case $\dot{z}(0) = 0$. By (2.5) we get $\lim_{\xi \to \beta^-} \varphi'(\xi) \dfrac{D(\varphi(\xi))}{\varphi(\xi)} = \lim_{\xi \to \beta^-} \dfrac{z(\varphi(\xi))}{\varphi(\xi)} = 0$, so by (6.1) we obtain

$$\lim_{\xi \to \beta^-} \varphi'(\xi)\dot{D}(\varphi(\xi)) = 0. \tag{6.2}$$

Let us consider system (1.7) with the additional convective term $h(\varphi)$, for $\tau = \tau(\xi) := \int_{\bar{\xi}}^{\xi} 1/D(\varphi(s))\, ds$, with $\bar{\xi} \in (\alpha, \beta)$. Of course, $\tau(\xi)$ is a strictly increasing function defined in (α, β). Moreover, since $\varphi'(\xi) = \dfrac{z(\varphi)}{D(\varphi)}$ and $z(\varphi) = o(\varphi)$, we get

$$\lim_{\xi \to \beta^-} \tau(\xi) = \int_{\bar{\xi}}^{\beta} \frac{1}{D(\varphi(\xi))} d\xi = \int_{\varphi(\bar{\xi})}^{0} \frac{1}{z(\varphi)} d\varphi = +\infty.$$

Let $\xi(\tau)$ be the inverse function of $\tau(\xi)$. Let us define $v(\tau) := \varphi'(\xi(\tau))$ and $\psi(\tau) := v(\tau)\dot{D}(\varphi(\xi(\tau)))$. By (6.2) we have

$$\lim_{\tau \to +\infty} \psi(\tau) = \lim_{\xi \to \beta^-} v(\xi)\dot{D}(\varphi(\xi)) = 0. \tag{6.3}$$

Moreover, the second equation of system (1.7) becomes

$$\dot{v}(\tau) = -(c - h(\varphi(\xi(\tau))) + \psi(\tau))v(\tau) - g(\varphi(\xi(\tau))). \tag{6.4}$$

Let us fix $\epsilon > 0$. Since $\lim_{\tau\to+\infty} g(\varphi(\xi(\tau))) = 0$, there exists $\tau^* \in \mathbb{R}$ such that

$$g(\varphi(\xi(\tau))) < \epsilon \quad \text{for every } \tau \geq \tau^*. \tag{6.5}$$

Let us now consider the function

$$w_\epsilon(\tau) := e^{-\int_{\tau^*}^{\tau}[c-h(\varphi(\xi(s)))+\psi(s)]\,ds}\left\{v(\tau^*) - \epsilon\int_{\tau^*}^{\tau} e^{\int_{\tau^*}^{s}[c-h(\varphi(\xi(\sigma)))+\psi(\sigma)]d\sigma}\,ds\right\}$$

which is the unique solution of the linear Cauchy problem

$$\dot{w} + (c - h(\varphi(\xi(\tau))) + \psi(\tau))w = -\epsilon, \quad w(\tau^*) = v(\tau^*).$$

By (6.3) we have

$$\lim_{\tau\to+\infty} c - h(\varphi(\xi(\tau))) + \psi(\tau) = c - h(0) > 0,$$

then $\int_{\tau^*}^{+\infty}[c - h(\varphi(\xi(s))) + \psi(s)]\,ds = +\infty$, hence

$$\lim_{\tau\to+\infty} w_\epsilon(\tau) = \frac{-\epsilon}{c - h(0)}. \tag{6.6}$$

On the other hand, by (6.5) $w_\epsilon(\tau)$ is a lower-solution for Eq. (6.4) in $[\tau^*, +\infty)$, so $w_\epsilon(\tau) \leq v(\tau) < 0$ for every $\tau \geq \tau^*$. Hence, by (6.6) we get $\lim_{\xi\to\beta^-}\varphi'(\xi) = \lim_{\tau\to+\infty} v(\tau) = 0$. Theorefore, $\varphi(\xi)$ can be continued on all $(\alpha, +\infty)$ as a solution of (1.4). This means that $\varphi(\xi)$ is a front-type t.w.s. □

Remark 6.1 Observe that when $\dot{D}(0) \neq 0$ then condition (6.1) is trivially satisfied. Moreover, note that if $D \in C^n[0,\delta]$ with $\dot{D}(0) = 0, \ldots, D^{(n-1)}(0) = 0$ and $D^{(n)}(0) \neq 0$ for some $\delta > 0$ and $n \in \mathbb{N}$, then again condition (6.1) holds. So, in particular, it is satisfied whenever $D(\varphi)$ is analytic in some right neighborhood of 0. In fact, we have

$$D(\varphi) = \frac{1}{n!}\,D^{(n)}(0)\varphi^n + o(\varphi^n)\,, \quad \varphi\dot{D}(\varphi) = \frac{1}{(n-1)!}\,D^{(n)}(0)\varphi^n + o(\varphi^n)$$

hence, $\dfrac{\varphi\dot{D}(\varphi)}{D(\varphi)} \to n$ as $\varphi \to 0^+$ and (6.1) holds.

Theorem 6.1 showed that if $c^* > h(0)$ the same analysis carried out in the case $\dot{D}(0) > 0$ holds also in the present situation. In particular this is true when the assumptions of Theorem 1.3 hold. However, this result can not be applied when $c^* = h(0)$. Recall that if $\dot{D}(0) > 0$ then the t.w.s. $\varphi^*(\xi)$ having speed $h(0)$ is indeed of front-type. But now, when $\dot{D}(0) = 0$ this conclusion is not always valid. We will show that both the situations can in fact occur, that is $\varphi^*(\xi)$ can be of front-type or sharp of type (I).

Theorem 6.2 *Assume that $\dot{D}(0) = 0$ and $c^* = h(0)$. Let $\varphi^*(\xi)$ denote the t.w.s. having speed $h(0)$. If*

$$\lim_{\varphi\to 0^+} \frac{\int_0^\varphi [h(s) - h(0)]\mathrm{d}s}{D(\varphi)} = 0, \tag{6.7}$$

$\varphi^*(\xi)$ is of front-type. Whereas if

$$-\infty < \liminf_{\varphi\to 0^+} \frac{h(\varphi) - h(0)}{\dot{D}(\varphi)} < 0 \tag{6.8}$$

and

$$\liminf_{\varphi\to 0^+} \frac{g(\varphi)}{\dot{D}(\varphi)} > 0, \tag{6.9}$$

$\varphi^*(\xi)$ is sharp of type (I).

Proof. Let $z^*(\varphi)$ denote the solution of problem (2.2) for $c = c^* = h(0)$. Since

$$\frac{\int_0^\varphi [h(s) - h(0)]\mathrm{d}s}{D(\varphi)} \le \frac{z^*(\varphi)}{D(\varphi)} \le 0 \quad \text{in } (0,1)$$

then assumption (6.7) implies $\frac{z^*(\varphi)}{D(\varphi)} \to 0$ as $\varphi \to 0^+$. Consequently $(\varphi^*)'(\beta^-) = 0$, where β is the right endpoint of the maximal existence interval of φ^*, satisfying $\varphi^*(\beta^-) = 0$. Therefore φ^* is of front-type.

Let (6.8), (6.9) hold and introduce positive constants M, λ and δ with $0 < \delta < 1$, such that

$$\frac{h(\varphi) - h(0)}{\dot{D}(\varphi)} > -M \quad \text{and} \quad \frac{g(\varphi)}{\dot{D}(\varphi)} > \lambda \quad \text{for all } \varphi \in (0, \delta]. \tag{6.10}$$

First assume that $\lim\limits_{\varphi\to 0^+} \dfrac{z^*(\varphi)}{D(\varphi)} = 0$. According to (6.9), when integrating the equation in (2.2) we get

$$\lim_{\varphi\to 0^+} \frac{\int_0^\varphi [h(s) - h(0)]ds}{D(\varphi)} = \lim_{\varphi\to 0^+} \frac{\int_0^\varphi \frac{D(s)g(s)}{z^*(s)}\mathrm{d}s + z^*(\varphi)}{D(\varphi)} = \lim_{\varphi\to 0^+} \frac{D(\varphi)g(\varphi)}{z^*(\varphi)\dot{D}(\varphi)} = -\infty$$

in contradiction with (6.8).

Consider now the case when $\liminf\limits_{\varphi\to 0^+} \dfrac{z^*(\varphi)}{D(\varphi)} < 0$ and $\limsup\limits_{\varphi\to 0^+} \dfrac{z^*(\varphi)}{D(\varphi)} = 0$ and let $\varphi_n \to 0^+$ be such that $\frac{d}{d\varphi}\left(\frac{z^*(\varphi)}{D(\varphi)}\right)_{|\varphi=\varphi_n} = 0$ and $\frac{z^*(\varphi_n)}{D(\varphi_n)} \to 0$ when $n \to +\infty$. Notice that it is not restrictive to assume $\varphi_n < \delta$ and $\frac{z^*(\varphi_n)}{D(\varphi_n)} > -\frac{\lambda}{2M}$ for all n. From the choice of φ_n we get $\dot{z}^*(\varphi_n) = \frac{z^*(\varphi_n)\dot{D}(\varphi_n)}{D(\varphi_n)}$. By the equation in (2.2) and (6.10) we deduce

$$-M < \frac{h(\varphi_n) - h(0)}{\dot{D}(\varphi_n)} = \frac{z^*(\varphi_n)}{D(\varphi_n)} + \frac{D(\varphi_n)g(\varphi_n)}{z^*(\varphi_n)\dot{D}(\varphi_n)} < -2M$$

again a contradiction. Hence $\limsup_{\varphi\to 0^+} \dfrac{z^*(\varphi)}{D(\varphi)} = \limsup_{\xi\to\beta^-} (\varphi^*)'(\xi) < 0$, and $\varphi^*(\xi)$ is sharp of type (I) since $\beta \in \mathbb{R}$. □

Remark 6.2 Assuming $\dot{D}(\varphi) > 0$ in some interval $[0, \delta]$, if we replace (6.8) and (6.9) with the stronger conditions

$$-\lambda := \lim_{\varphi\to 0^+} \frac{h(\varphi) - h(0)}{\dot{D}(\varphi)}, \quad L := \lim_{\varphi\to 0^+} \frac{g(\varphi)}{\dot{D}(\varphi)}$$

with $\lambda \in \mathbb{R}$, $L \in \mathbb{R} \cup \{+\infty\}$, observe that whenever $4L > \lambda^2$ then condition (1.12) of Theorem 1.3 holds and $c^* > h(0)$. So, in the framework of Theorem 6.2 we always have $4L \le \lambda^2$.

Moreover, in order to compare condition (6.7) with the pair of conditions (6.8), (6.9), note that if (1.11) holds, which implies $c^* = h(0)$, then $\lambda^2 \ge 4L$. Hence, whenever $L > 0$ then necessarily $\lambda > 0$ and consequently the validity of assumption (6.7) implies that $L = 0$.

Example 6.1 Put $h(u) := -u^2$ and $D(u)g(u) \le u^5/12$. It is immediate to verify that the assumptions of Theorem 1.2 are satisfied, hence $c^* = 0$. In particular, when $D(u) = u^2$, implying $g(u) \le u^3/12$, then (6.7) holds and the t.w.s. φ^* is of front-type. Instead, taking $D(u) = u^3$ and $g(u) = \frac{1}{12}u^2$, then (6.8) and (6.9) are satisfied and φ^* is sharp of type (I).

As regards the doubly degenerate case, the following result, analogous to Theorem 6.1, holds.

Theorem 6.3 *Let g satisfy (1.2) and let $D \in C^1[0,1]$ be such that*

$$D(0) = 0,\ \dot{D}(0) > 0,\ D(1) = \dot{D}(1) = 0, \quad D(\varphi) > 0 \text{ for every } 0 < \varphi < 1.$$

Assume that

$$0 \le \dot{D}(\varphi)(\varphi - 1) \le LD(\varphi) \quad \text{for every } 1 - \sigma \le \varphi \le 1 \tag{6.11}$$

and some $\sigma, L > 0$.

Then if $c \ne h(1)$ to every solution $z(\varphi)$ of problem (2.2) with $c \ne h(1)$ corresponds:

- *a front-type t.w.s. iff $\dot{z}(0) = \dot{z}(1) = 0$;*
- *a sharp t.w.s. of type (I) iff $\dot{z}(1) = 0$ and $\dot{z}(0) = h(0) - c < 0$;*
- *a sharp t.w.s. of type (II) iff $\dot{z}(1) = h(1) - c > 0$ and $\dot{z}(0) = 0$;*
- *a sharp t.w.s. of type (III) iff $\dot{z}(1) = h(1) - c > 0$ and $\dot{z}(0) = h(0) - c < 0$.*

Sketch of the proof. The proof is quite analogous to that of Theorem 6.1. In fact, also in this case, when $\dot{z}(1) = 0$, assumption (6.11) implies that $\lim_{\xi \to \alpha^+} \varphi'(\xi)\dot{D}(\varphi(\xi)) = 0$. Moreover, $\tau(\xi) \to -\infty$ as $\xi \to \alpha^+$. The only difference is that now we take the solution $w(\tau)$ of the linear Cauchy problem

$$\dot{w} + (c - h(\tau) + \psi(\tau))w = 0, \quad w(0) = v(0).$$

which is an upper-solution of equation (6.4) in $(-\infty, 0]$. So, again we deduce $w(\tau) \leq v(\tau) \leq 0$ for every $\tau \leq 0$, and $w(\tau) \to 0$ as $\tau \to -\infty$ leads to conclusion. □

Of course, when both $\dot{D}(0) = \dot{D}(1) = 0$, then the pair of conditions (6.1), (6.11) implies the validity of the analysis carried out in Theorem 2.1.

In [18] the following equation

$$b_t = \big(D(b)b_x\big)_x + (1-b)b \tag{6.12}$$

was proposed, modelling the evolution of a bacterial colony and its nutrient in a plane surface, when assuming the conservation of total mass. They take $D(b) > 0$ for $0 < b < 1$ and $D(0) = D(1) = 0$. By means of a numerical simulation the special case when $D(b) = (1-b)b$ is solved, showing the presence of a sharp-type t.w.s. corresponding to the threshold wave speed. The investigation developed in this paper enables to analytically solve the general situation when $D \in C^1[0,1]$ and it satisfies conditions (6.1) and (6.11); in particular for an arbitrary analytic D. Indeed, according to Theorems 6.1 and 6.3 we get a $c^* > 0$ satisfying (1.10) such that all conclusions *(i), (ii)* and *(iii)* of Theorem 1.1 hold also in this case.

7 Diffusion with infinite slope at the equilibria

In this section we briefly treat the case of diffusion with infinite slope at one or both the equilibria. Such a situation leads to a regularization of the appearing t.w.s., since it prevents sharpness phenomena, i.e. the attaining of the equilibria with non-zero slope.

In the simply degenerate case the following result holds.

Theorem 7.1 *Let g satisfy (1.2) and $D \in C[0,1] \cap C^1(0,1)$ satisfy (1.3) with $D(1) > 0$ and $\dot{D}(0) = +\infty$. Assume that*

$$D(\varphi)g(\varphi) = o(\varphi) \quad \text{as } \varphi \to 0^+. \tag{7.1}$$

Then, there exists a threshold value c^* satisfying (1.10) such that Eq. (1.1) has:

(i) no t.w.s. for $c < c^*$;

(ii) a unique (up to shifts) front-type t.w.s. for $c \geq c^*$.

Sketch of the proof. First note that Proposition 2.1 holds also in this case. As regards Lemma 2.1, its proof works in this setting thanks to condition (7.1). Finally, the correspondence proved in Theorem 2.1 between the t.w.s. and the solutions of problem (2.2) is still valid, but all the appearing t.w.s. are of front-type since the solution $\varphi(\xi)$ of the Cauchy problem (2.5) satisfies $\lim_{\xi\to\alpha^+}\varphi'(\xi) = \lim_{\xi\to\beta^-}\varphi'(\xi) = 0$. So, in this case we do not need to know the value of $\dot z(0)$. $\square$

Similarly, in the doubly degenerate case, if $\dot D(1) = -\infty$, we need to assume that

$$D(\varphi)g(\varphi) = o(1-\varphi) \quad \text{as } \varphi \to 1^-. \tag{7.2}$$

In this case, the degeneracy at 1 does not causes the attaining of the equilibrium 1 with a negative slope, and the t.w.s. are of front-type or sharp of type I, according to the value of $\dot D(0)$. In particular, when both $\dot D(0) = +\infty$ and $\dot D(1) = -\infty$ are satisfied, if (7.1) and (7.2) hold, then all the t.w.s. are of front-type.

8 Smooth attaining of the equilibria

According to Definition 1.1, in the class of front-type t.w.s. we did not distinguish between strictly decreasing solutions and solutions which smoothly attain one or both the equilibria at finite values. On the other hand, this is an interesting question. When a t.w.s reaches the value 0, it has the property of finite speed of propagation, whereas when equilibrium 1 is attained, we could say that the t.w.s. has the property of finite speed of saturation.

We discuss the attaining of 0 in detail, while we briefly treat the issue concerning 1, being quite similar.

Let (ξ_1, ξ_0) be the interval such that $0 < \varphi(\xi) < 1$ for every $\xi_1 < \xi < \xi_0$, with $\varphi(\xi_1^+) = 1$, $\varphi(\xi_0^-) = 0$. The existence and uniqueness of such an interval follows from Proposition 2.1. Of course, $\varphi(\xi)$ attains the value 0 if and only if $\xi_0 \in \mathbb{R}$.

As already observed in [7], note that

$$\xi_0 = +\infty \quad \text{if and only if} \int_0^{\frac12} \frac{D(s)}{-z(s)}ds = +\infty.$$

In fact, by (2.1) $\xi_0 = +\infty$ if and only if

$$\int_0^{\frac12} \frac{D(s)}{-z(s)}ds = \int_0^{\frac12} -\frac{1}{\varphi'(\varphi^{-1}(s))}ds = \int_{\frac12}^{0} (\varphi^{-1})'(s)ds = +\infty.$$

According to Lemma 2.1 and Theorem 2.1, when $\dot z(0) < 0$ and $D \in C^1[0,1]$ then necessarily $\xi_0 \in \mathbb{R}$. Therefore, we focus our investigation on the case $\dot z(0) = 0$. This situation occurs whenever $c > c^*$ or $c = c^* = h(0)$.

The first result deals with the t.w.s. having speed $c > c^*$. The proof of the necessary part is an adaptation of [7, Theorem 38] due to the presence of convective effects.

Theorem 8.1 *Let g, D satisfy (1.2) and (1.3) respectively. Moreover, assume that for some $\delta > 0$, $D(\varphi)g(\varphi)$ is differentiable in $(0, \delta)$ with $\frac{d(Dg)}{d\varphi}(\varphi) \to 0$ as $\varphi \to 0^+$ and*

$$\frac{h(\varphi) - c}{D(\varphi)g(\varphi)} \quad \textit{is monotone (increasing or decreasing) in } (0, \delta). \tag{8.1}$$

If $c > c^*$, then the front t.w.s. attains the value 0 if and only if $\int_0^{\frac{1}{2}} \frac{1}{g(s)} ds < +\infty$

Proof. First assume that $\int_0^{\frac{1}{2}} \frac{1}{g(s)}\, ds = +\infty$. Put $\tilde{c} := \frac{1}{3}(c - h(0)) > 0$. We can assume, without restriction, $c - h(\varphi) > 2\tilde{c}$ in $(0, \delta)$ and

$$\frac{d(Dg)}{d\varphi}(\varphi) < \frac{1}{3}\, \tilde{c}\, (c - h(0)) \quad \text{ in } (0, \delta). \tag{8.2}$$

Put $\tilde{z}(\varphi) := -D(\varphi)g(\varphi)/\tilde{c}$ and observe that, by (8.2), in $(0, \delta)$ we have

$$\dot{\tilde{z}}(\varphi) = -\frac{1}{\tilde{c}} \frac{d(Dg)}{d\varphi}(\varphi) - \tilde{c} - \frac{D(\varphi)g(\varphi)}{\tilde{z}(\varphi)} >$$

$$> -\frac{2}{3}(c - h(0)) - \frac{D(\varphi)g(\varphi)}{\tilde{z}(\varphi)} > h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\tilde{z}(\varphi)}.$$

Therefore, if $\tilde{z}(s) = z(s)$ for some $s \in (0, \delta)$, then $\dot{\tilde{z}}(s) > \dot{z}(s)$. Hence, two cases can occur: $\tilde{z}(\varphi) > z(\varphi)$ in $(0, \delta)$ or $\tilde{z}(\varphi) < z(\varphi)$ in $(0, \rho)$ for some $0 < \rho < \delta$.

Let us show that the former case does not hold. Indeed, otherwise we have

$$\dot{z}(\varphi) = h(\varphi) - c - \frac{g(\varphi)D(\varphi)}{z(\varphi)} < h(\varphi) - c - \frac{g(\varphi)D(\varphi)}{\tilde{z}(\varphi)} = h(\varphi) - c + \tilde{c}$$

implying $\frac{z(\varphi)}{\varphi} < \frac{1}{\varphi} \int_0^{\varphi} h(s) ds + \tilde{c} - c$. Thus, $\limsup_{\varphi \to 0^+} \frac{z(\varphi)}{\varphi} \leq h(0) - c + \tilde{c} < 0$, a contradiction since $\dot{z}(0) = 0$. Therefore, $-\frac{1}{\tilde{c}} D(\varphi)g(\varphi) = \tilde{z}(\varphi) < z(\varphi)$ in $(0, \rho)$ for some $0 < \rho < \delta$ and then

$$\int_0^{\frac{1}{2}} \frac{D(s)}{-z(s)}\, ds > \tilde{c} \int_0^{\frac{1}{2}} \frac{1}{g(s)}\, ds = +\infty.$$

Let us now consider the case when $\int_0^{\frac{1}{2}} \frac{1}{g(s)}\, ds < +\infty$. First observe that assumption (8.1) implies that $\dot{z}(\varphi) \leq 0$ in $(0, \epsilon)$ for some $\epsilon > 0$. Indeed, otherwise there exist points $0 < \varphi_1 < \varphi_2 < \varphi_3 < \delta$ such that $\dot{z}(\varphi_i) = 0$, $i = 1, 2, 3$, and $z(\varphi_1) < z(\varphi_2)$, $z(\varphi_3) < z(\varphi_2)$. But being $\dot{z}(\varphi_i) = 0$, from Eq. (2.2) we get $z(\varphi_i) = \frac{D(\varphi_i)g(\varphi_i)}{h(\varphi_i) - c}$, a contradiction with (8.1).

Then, we have

$$0 \leq \frac{D(\varphi)}{-z(\varphi)} = \frac{\dot{z}(\varphi) + c - h(\varphi)}{g(\varphi)} \leq \frac{K}{g(\varphi)} \quad \text{ in } (0, \epsilon)$$

for some $K > 0$, and then $\int_0^\epsilon \frac{D(s)}{-z(s)}\,ds \le K \int_0^\epsilon \frac{1}{g(s)}\,ds < +\infty$. □

Let us now consider the case $c = c^* = h(0)$. In this case, even if $\int_0^{\frac{1}{2}} \frac{1}{g(s)}\,ds = +\infty$, both the situations occur: 0 can be reached at a finite value or there is a positive t.w.s.

Proposition 8.1 *Let g, D satisfy (1.2) and (1.3) respectively. If all the conditions of Theorem 1.2 hold (implying $c^* = h(0)$) and*

$$\int_0^{\frac{1}{2}} \frac{D(u)}{\int_0^u (h(0)-h(s))\,ds}\,du = +\infty \tag{8.3}$$

then the t.w.s. having speed c^* is positive everywhere.

Instead, if (8.1) holds and

$$\int_0^{\frac{1}{2}} \frac{h(0)-h(s)}{g(s)}\,ds < +\infty \tag{8.4}$$

then the t.w.s. with speed c^* attains the equilibrium 0 at a finite value.

Proof. Assume all the conditions of Theorem 1.2 and let $H(\varphi) = \frac{1}{2}\int_0^\varphi ((h(s)-h(0))\,ds$. As showed in the proof of Theorem 1.2, $H(\varphi)$ is an upper-solution for the equation in (2.2) with $c = h(0)$ and by Lemma 5.1 we get $H(\varphi) \le z(\varphi)$. So, by (8.3)

$$\int_0^{\frac{1}{2}} \frac{D(s)}{-z(s)}\,ds \ge \int_0^{\frac{1}{2}} \frac{D(s)}{-H(s)}\,ds = +\infty.$$

Instead, if (8.1) and (8.4) hold, then $\dot z(\varphi) \le 0$ in a right neighborhood of 0 (see the proof of Theorem 8.1) and then

$$\frac{D(\varphi)}{-z(\varphi)} = \frac{\dot z(\varphi) + h(0) - h(\varphi)}{g(\varphi)} \le \frac{h(0)-h(\varphi)}{g(\varphi)}$$

and the assertion follows by (8.4). □

Remark 8.1 In order to compare assumptions (8.3) and (8.4), note that under condition (1.11) we have

$$\int_0^{\frac{1}{2}} \frac{D(u)}{\int_0^u (h(0)-h(s))\,ds}\,du \le \frac{1}{4}\int_0^{\frac{1}{2}} \frac{h(0)-h(s)}{g(s)}\,ds$$

and the above Proposition covers all the situations with the exception of the case when $\displaystyle\int_0^{\frac{1}{2}} \frac{h(0)-h(u)}{g(u)}\,du = +\infty$ and $\displaystyle\int_0^{\frac{1}{2}} \frac{D(u)}{\int_0^u (h(0)-h(s))\,ds}\,du < +\infty$.

Moreover, the previous result holds also when $\dot D(0) = +\infty$, provided that (7.1) is satisfied.

Example 8.1 Let $g(u) := u^2(1-u)^2$, $D(u) = u$, $h(u) = u(4u-3)$. Then, as already showed in Example 2.1, $c^* = 0 = h(0)$ with $z(u) = u^2(u-1)$. In this case $\int_0^{\frac{1}{2}} \frac{D(u)}{-z(u)}\, du = +\infty$ and the corresponding front-type t.w.s. is strictly decreasing. Note that condition (8.3) is satisfied. Instead, taking $g(u) := u(1-u)^2$ and $D(u) := u^2$, with the same $h(u)$, we have $\int_0^{\frac{1}{2}} \frac{D(s)}{-z(s)}\, ds < +\infty$ and the corresponding front-type t.w.s. reaches the equilibrium 0 at a finite value.

As regards the attaining of the equilibrium 1, we briefly note that in the simply degenerate case, i.e. $D(1) > 0$, when $\dot{g}(1) > -\infty$ then $\varphi(\xi) < 1$ for all $\xi \in \mathbb{R}$ (see [11, Proposition 2]). Instead, in the doubly degenerate case, a result analogous to Theorem 8.1 holds whenever $c > h(1)$, and the corresponding t.w.s. satisfies $\varphi(\xi) < 1$ for every $\xi \in (-\infty, \alpha)$ (see Definition 1.1). While, when $c < h(1)$ the t.w.s. is sharp of type (II) or (III); in both cases it attains the value 1 at a finite value.

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