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A Minimum Problem with Free Boundary for the $p(x)$−Laplace Operator

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Abstract

In this paper we consider the problem of minimizing the functional

$$J(u) = \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + Q(x)\chi_{[u>0]} \right) dx.$$

We prove Lipschitz continuity for each minimizer u and establish the nondegeneracy at the free boundary $(\partial[u > 0]) \cap \Omega$ and the locally uniform positive density of the sets $[u > 0]$ and $[u = 0]$. In particular we obtain that the Lebesgue measure of the free boundary is zero.

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1 Introduction

Let Ω be a lipschitz bounded domain of $\mathbb{R}^n$ and let p be a measurable real valued function defined in Ω and satisfying for some positive numbers p_- and p_+

$$1 < p_- = \inf_\Omega p(x) \leq p(x) \leq p_+ = \sup_\Omega p(x) \quad \text{a.e. } x \in \Omega. \tag{1.1}$$

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We recall some definitions of Lebesgue and Sobolev spaces with variable exponents (see for example [5], [12], [13] and [15])

$$L^{p(x)}(\Omega)=\Big\{u:\Omega\to\mathbb{R}\ \text{ measurable }/\ \rho(u)=\int_\Omega|u(x)|^{p(x)}dx<\infty\Big\}.$$

Equipped with the Luxembourg norm

$$||u||_{p(x)}=\inf\Big\{\lambda>0\ \ /\ \ \rho\Big(\frac{|u|}{\lambda}\Big)dx\le 1\Big\},$$

$L^{p(x)}(\Omega)$ is a separable and reflexive Banach space. Moreover we have

Proposition 1.1 *i)* $||u||_{p(x)}\le 1\ \Leftrightarrow\ \rho(u)\le 1$.
ii) $\min\big(||u||^{p_-}_{p(x)},||u||^{p_+}_{p(x)}\big)\le\rho(u)\le\max\big(||u||^{p_-}_{p(x)},||u||^{p_+}_{p(x)}\big)$.
iii) If $p_1(x)$ and $p_2(x)$ satisfy (1.1) and $p_1(x)\le p_2(x)$ a.e. in Ω, then $L^{p_2(x)}(\Omega)\subset L^{p_1(x)}(\Omega)$ and the embedding is continuous.
iv) The dual space of $L^{p(x)}(\Omega)$ is the space $L^{q(x)}$ where $q(x)$ is defined by $\frac{1}{p(x)}+\frac{1}{q(x)}=1$.

The Sobolev space with variable exponent is defined by

$$W^{1,p(x)}(\Omega)=\Big\{u\in L^{p(x)}\Omega)\ /\ \nabla u\in\big(L^{p(x)}(\Omega)\big)^n\Big\}.$$

Equipped with the following norm

$$||u||_{1,p(x)}=||u||_{p(x)}+||\nabla u||_{p(x)},\quad ||\nabla u||_{p(x)}=\sum_{i=1}^n\Big\|\frac{\partial u}{\partial x_i}\Big\|_{p(x)}$$

$W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space. The space $W^{1,p(x)}_0(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Assume that p satisfies for some $L>0$

$$-|p(x)-p(y)|\log|x-y|\le L,\quad \forall x,y\,\epsilon\,\bar\Omega. \tag{1.2}$$

Then we have (see [12])

Proposition 1.2 *i)* $C^\infty(\overline{\Omega})$ *is dense in* $W^{1,p(x)}(\Omega)$.
ii) $W^{1,p(x)}_0(\Omega)=W^{1,p(x)}(\Omega)\cap W^{1,1}_0(\Omega)$.
iii) $\forall u\in W^{1,p(x)}_0(\Omega)\quad ||u||_{p(x)}\le C||\nabla u||_{p(x)}$ (*Poincaré's inequality*).

In this paper, we would like to consider the following minimization problem

$$(P)\begin{cases}\text{Find } u\in\mathcal{K}_g=g+W^{1,p(x)}_0(\Omega)\text{ such that :}\\ J(u)\le J(v)\quad\forall v\in\mathcal{K}_g,\end{cases}$$

where $J(u)=\displaystyle\int_\Omega\Big(\frac{1}{p(x)}|\nabla u|^{p(x)}+Q(x)\chi_{[u>0]}\Big)dx$, $\chi_{[u>0]}$ is the characteristic function of the set $[u>0]$, g is a function in $W^{1,p(x)}(\Omega)$ and $Q(x)$ is a function satisfying for some nonnegative constants Q_- and Q_+

$$Q_-=\inf_\Omega\ Q(x)\le Q(x)\le Q_+=\sup_\Omega\ Q(x)\quad\text{for a.e. } x\in\Omega. \tag{1.3}$$

This problem has been studied first in [1] in the case $p(x) \equiv 2$ in Ω. The authors proved Lipschitz continuity of the minimizers and $C^{1,\alpha}$ regularity of the free boundary. In [2] the authors extended the results in [1] to a nonlinear uniformly elliptic operator. The same results were also generalized in [9] when $p(x)$ is identically equal to a constant $p > 1$. Recently the problem was addressed in [17] in the framework of Orlicz-Sobolev spaces.

The main result of this paper is the Lipschitz continuity of the minimizers, which plays an important role in studying the regularity of the free boundary for this category of problems in the spirit of [1]. In Section 1, we prove the existence of a solution to the problem (P). In Section 2, we give some properties of the solutions. In Section 3, we prove Hölder continuity of the minimizers. In Section 4, we prove Lipschitz continuity of the minimizers. In Section 5, we establish the nondegeneracy of a minimizer at the free boundary $(\partial[u > 0]) \cap \Omega$ and the locally uniform positive density of the sets $[u > 0]$ and $[u = 0]$. As a consequence we obtain that the Lebesgue measure of the free boundary is zero.

2 Existence of a minimizer

Proposition 2.1 *There exists a minimizer for the functional $J(u)$.*

Proof. Note that $\mathcal{K}_g \neq \emptyset$ since $g \in \mathcal{K}_g$. Moreover $J(u) \geq 0$. So there exists a minimizing sequence $u_k \in \mathcal{K}_g$ such that $\lim_{k\to\infty} J(u_k) = \inf_{u\in\mathcal{K}_g} J(u) = \alpha$.

If $J(g) = 0$, then g is a minimizer and the proposition is trivial.

If $J(g) > 0$, then using a subsequence if necessary, we have for all k

$$J(u_k) \leq \alpha + J(g) \leq J(g) + J(g) = 2J(g).$$

Using the convexity of $t \to t^{p(x)}$ and (1.1), we get

$$\begin{aligned}
\int_\Omega |\nabla(u_k - g)|^{p(x)} dx &\leq \int_\Omega 2^{p(x)-1}(|\nabla u_k|^{p(x)} + |\nabla g|^{p(x)}) dx \\
&\leq \int_\Omega \frac{p_+ 2^{p(x)-1}}{p(x)} \Big(|\nabla u_k|^{p(x)} + |\nabla g|^{p(x)}\Big) dx \\
&\leq p_+ 2^{p_+-1} \int_\Omega \frac{1}{p(x)} \Big(|\nabla u_k|^{p(x)} + |\nabla g|^{p(x)}\Big) dx \\
&\leq p_+ 2^{p_+-1}(J(u_k) + J(g)) \\
&\leq p_+ 2^{p_+-1}(2J(g) + J(g)) \\
&= 3p_+ 2^{p_+-1} J(g) = C. \qquad (2.1)
\end{aligned}$$

Using Proposition 1.2 *iii*) (the Poincaré inequality) and Proposition 1.1 *ii*), we obtain from (2.1), for some positive constant C independent of k

$$\|u_k - g\|_{1,p(x)} \leq C. \qquad (2.2)$$

Since $W^{1,p(x)}(\Omega)$ is reflexive, there exists a subsequence, still denoted by (u_k) and a function u in $W^{1,p(x)}(\Omega)$ such that

$$u_k \rightharpoonup u \quad \text{in} \quad W^{1,p(x)}(\Omega). \tag{2.3}$$

Since $p(x) \geq p_- > 1$, we have $W^{1,p(x)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega)$ which leads by (2.3), up to a subsequence, to

$$u_k \rightharpoonup u \quad \text{in} \quad W^{1,p_-}(\Omega) \tag{2.4}$$

$$u_k \to u \quad \text{in} \quad L^{p_-}(\Omega) \quad \text{and a.e. in } \Omega. \tag{2.5}$$

Recall that $W_0^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \bigcap W_0^{1,1}(\Omega)$. Since $u_k - g \in W_0^{1,p(x)}(\Omega)$, we have $u_k - g \in W_0^{1,p_-}(\Omega)$. From (2.4) we get $u - g \in W_0^{1,p(x)}(\Omega)$.

It remains to prove that u is a minimizer of J. As in [1], the pointwise convergence implies that

$$\int_\Omega Q(x)\chi_{[u>0]} \leq \liminf_{k\to\infty} \int_\Omega Q(x)\chi_{[u_k>0]}. \tag{2.6}$$

Next we prove that

$$\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \leq \liminf_{k\to\infty} \int_\Omega \frac{1}{p(x)}|\nabla u_k|^{p(x)}dx. \tag{2.7}$$

Indeed, we have by Young's inequality

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)}dx &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla(u-u_k)dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla u_k dx \\ &\leq \int_\Omega \frac{1}{q(x)}|\nabla u|^{p(x)}dx + \frac{1}{p(x)}|\nabla u_k|^{p(x)}dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla(u-u_k)dx.\end{aligned}$$

Then

$$\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \leq \int_\Omega \frac{1}{p(x)}|\nabla u_k|^{p(x)}dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla(u-u_k)dx.$$

The assertion follows then from (2.3) by letting $k \to \infty$ in the last inequality.

Finally by combining (2.6) and (2.7), we get

$$\alpha \leq J(u) \leq \liminf_{k\to\infty} \int_\Omega \frac{1}{p(x)}|\nabla u_k|^{p(x)}dx + \liminf_{k\to\infty} \int_\Omega Q(x)\chi_{[u_k>0]}dx \leq \liminf_{k\to\infty} J(u_k) = \alpha$$

Hence

$$J(u) = \inf_{v\in\mathcal{K}_g} J(v).$$

□

3 Some properties

We will denote by $\mathcal{S}(g,\Omega)$ the set of all minimizers of the functional $J(u)$ subject to the boundary condition $u = g$.

Proposition 3.1 *Let $u \in \mathcal{S}(g,\Omega)$. Then u is $p(x)-$Subharmonic in Ω, i.e. we have*

$$\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u) \geq 0 \quad in \quad \mathcal{D}'(\Omega). \tag{3.1}$$

Proof. Let $\zeta \in W_0^{1,p(x)}(\Omega)$, $\zeta \geq 0$, and $\epsilon \in (0,1)$. Since $u - \epsilon\zeta$ is a test function for (P), we have

$$\begin{aligned} 0 \leq J(u-\epsilon\zeta) - J(u) &= \int_\Omega \frac{1}{p(x)}(|\nabla(u-\epsilon\zeta)|^{p(x)}dx - |\nabla u|^{p(x)})dx \\ &+ \int_\Omega Q(x)\left(\chi_{[u-\epsilon\zeta>0]} - \chi_{[u>0]}\right)dx. \end{aligned} \tag{3.2}$$

Moreover we have for a.e. $x \in \Omega$

$$\left(\chi_{[u-\epsilon\zeta>0]} - \chi_{[u>0]}\right)(x) = \begin{cases} \chi_{[u-\epsilon\zeta>0]}(x) - 1 \leq 0 & \text{if} \quad u(x) > 0 \\ \chi_{[u-\epsilon\zeta>0]}(x) = 0 & \text{if} \quad u(x) \leq 0. \end{cases}$$

We deduce from (3.2) that

$$0 \leq \int_\Omega \frac{1}{p(x)}(|\nabla(u-\epsilon\,\zeta)|^{p(x)} - |\nabla u|^{p(x)})dx.$$

Using the convexity inequality for $p \geq 1$: $|b|^p - |a|^p \geq p(|a|^{p-2}a, b-a)$, we obtain

$$\int_\Omega |\nabla(u-\epsilon\,\zeta)|^{p(x)-2}\nabla(u-\epsilon\,\zeta).\nabla\zeta\;dx \leq 0 \qquad \forall\zeta \in W_0^{1,p(x)}(\Omega),\;\; \zeta \geq 0.$$

Letting $\epsilon \to 0$ and taking into account the Lebesgue theorem, we get

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla\zeta\;dx \leq 0 \qquad \forall\zeta \in W_0^{1,p(x)}(\Omega),\;\; \zeta \geq 0.$$

□

Proposition 3.2 *Assume that there exists $M > 0$ such that $0 \leq g \leq M$ on $\partial\Omega$. Then we have*

$$\forall u \in \mathcal{S}(g,\Omega) \quad 0 \leq u \leq M \quad a.e.\ in \quad \Omega. \tag{3.3}$$

Proof. Let $u \in \mathcal{S}(g,\Omega)$.

i) $\underline{u \leq M}$:

Since $\zeta = (u-M)^+ \in W_0^{1,p(x)}(\Omega)$, $\zeta \geq 0$, we obtain from (3.1)

$$\int_\Omega |\nabla u|^{p(x)-2}\;\nabla u.\nabla(u-M)^+dx \leq 0 \quad \text{or} \quad \int_\Omega |\nabla(u-M)^+|^{p(x)}dx \leq 0.$$

Since $u = g \leq M$ on $\partial\Omega$, we obtain $(u-M)^+ = 0$ a.e. in Ω, which means that $u \leq M$ a.e. in Ω.

ii) $\underline{u \geq 0}$:

Note that for each $\epsilon \in (0,1)$, $u+\epsilon u^-$ is a test function for (P). Moreover we have $\chi_{[u+\epsilon u^->0]}-\chi_{[u>0]} \leq 0$ for a.e. $x \in \Omega$. Indeed

$$\left(\chi_{[u+\epsilon u^->0]}-\chi_{[u>0]}\right)(x) = \begin{cases} \chi_{[u+\epsilon u^->0]}(x) - 1 \leq 0 & \text{if} \quad u(x) > 0 \\ \chi_{[(\epsilon-1)u^->0]}(x) = 0 & \text{if} \quad u(x) \leq 0. \end{cases}$$

It follows that

$$0 \leq J(u+\epsilon u^-) - J(u) \leq \int_\Omega \frac{1}{p(x)}(|\nabla(u+\epsilon\, u^-)|^{p(x)} - |\nabla u|^{p(x)})dx. \tag{3.4}$$

Arguing as in the proof of Proposition 3.1, we get

$$\int_\Omega |\nabla(u+\epsilon u^-)|^{p(x)-2}\nabla(u+\epsilon u^-).\nabla u^- dx \geq 0.$$

Letting $\epsilon \to 0$, we obtain

$$\int_\Omega |\nabla u|^{p(x)-2}\, \nabla u \cdot \nabla u^- dx \geq 0 \quad \text{or} \quad \int_\Omega |\nabla u^-|^{p(x)} dx \leq 0.$$

Since $u^- = g^- = 0$ on $\partial\Omega$, we obtain $u^- = 0$ a.e. in Ω. Hence $u \geq 0$ a.e. in Ω. □

Proposition 3.3 *Let $u \in \mathcal{S}(g,\Omega)$. Then u is $p(x)-$Harmonic in $[u>0]$, i.e. we have*

$$\Delta_{p(x)} u = 0 \quad in \quad \mathcal{D}'([u>0]). \tag{3.5}$$

Proof. Let $\zeta \in \mathcal{D}([u>0])$ and $\epsilon > 0$. Then $u \pm \epsilon\zeta$ is a test function for (P). Moreover we have $\chi_{[u\pm\epsilon\zeta>0]} - \chi_{[u>0]} \leq 0$ a.e. in Ω. Indeed

$$\left(\chi_{[u\pm\epsilon\zeta>0]}-\chi_{[u>0]}\right)(x) = \begin{cases} \chi_{[u\pm\epsilon\zeta>0]}(x) - 1 \leq 0 & \text{if} \quad u(x) > 0 \\ \chi_{[\pm\epsilon\zeta>0]}(x) = 0 & \text{if} \quad u(x) = 0. \end{cases}$$

It follows that

$$0 \leq J(u\pm\epsilon\zeta) - J(u) \leq \int_\Omega \frac{1}{p(x)}(|\nabla(u\pm\epsilon\zeta)|^{p(x)} - |\nabla u|^{p(x)})dx.$$

Arguing as in the proof of Proposition 3.1, we get

$$\int_\Omega |\nabla(u\pm\epsilon\zeta)|^{p(x)-2}\nabla(u\pm\epsilon\zeta)\cdot\nabla(\pm\zeta)dx \leq 0. \tag{3.6}$$

Letting $\epsilon \to 0$ in (3.6), we obtain

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\pm\zeta)dx \leq 0 \quad \text{or} \quad \int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\zeta dx = 0.$$

□

Remark 3.1 Since $\Delta_{p(x)}u = 0$ in $[u>0]$, we deduce (see [7], [10]) that $u \in C^{1,\alpha}_{loc}([u>0])$.

Proposition 3.4 *Assume that $p, Q \in C^1(\Omega)$. Then we have for each $u \in \mathcal{S}(g,\Omega)$ and $\eta \in \mathcal{D}(\Omega,\mathbb{R}^n)$*

$$\lim_{\epsilon\to 0}\int_{\partial[u>\epsilon]}\Big(\frac{p(x)-1}{p(x)}|\nabla u|^{p(x)} - Q\Big)\eta\cdot\nu\, d\sigma(x) = 0.$$

In particular if $(\partial[u>0])\cap\Omega$ is smooth enough, we obtain

$$|\nabla u(x)| = \left(\frac{p(x)}{p(x)-1}Q(x)\right)^{\frac{1}{p(x)}} \quad \forall x\in\ (\partial[u>0])\cap\Omega.$$

Proof. We adapt an idea from [1]. Let $\eta\in\mathcal{D}(\Omega,\mathbb{R}^n)$ and $\epsilon>0$. We consider $T_\epsilon(x) = x+\epsilon\eta(x)$. Since $DT_0(x) = I_n$, where I_n is the $n\times n$ identity matrix, it is clear by the inverse function theorem that for ϵ small enough T_ϵ is a C^1-diffeomorphism from Ω into $T_\epsilon(\Omega)\subset\Omega$. Let us now define u_ϵ by $u_\epsilon(x) = u(T_\epsilon(x))$. Then u_ϵ is a test function for (P) and we have

$$J(u)\le J(u_\epsilon). \tag{3.7}$$

Using the change of variables $y = T_\epsilon(x)$, we get for ϵ small enough

$$\begin{aligned}
J(u_\epsilon) &= \int_\Omega \frac{1}{p(x)}|\nabla u_\epsilon|^{p(x)} + Q(x)\chi_{[u_\epsilon>0]}\\
&= \int_\Omega \frac{1}{p(x)}|DT_\epsilon(x)\nabla u(T_\epsilon(x))|^{p(x)} + Q(x)\chi_{[uoT_\epsilon>0]}\\
&= \int_{T_\epsilon(\Omega)} \frac{1}{p(T_\epsilon^{-1}(y))}|DT_\epsilon(T_\epsilon^{-1}(y))\nabla u(y)|^{p(T_\epsilon^{-1}(y))}\Big(det\big(DT_\epsilon(T_\epsilon^{-1}(y))\big)\Big)^{-1}\\
&\quad + \int_{T_\epsilon(\Omega)} Q(T_\epsilon^{-1}(y))\chi_{[u>0]}\Big(det\big(DT_\epsilon(T_\epsilon^{-1}(y))\big)\Big)^{-1}.
\end{aligned} \tag{3.8}$$

Note that we have $T_\epsilon^{-1}(y) = y-\epsilon\eta(T_\epsilon^{-1}(y))$ and $DT_\epsilon(T_\epsilon^{-1}(y)) = I_n+\epsilon D\eta(T_\epsilon^{-1}(y))$.
To simplify things, we set $x_\epsilon = T_\epsilon^{-1}(y)$. Then we obtain easily

$$\Big(det\big(DT_\epsilon(T_\epsilon^{-1}(y))\big)\Big)^{-1} = 1-\epsilon\nabla\cdot\eta + o(\epsilon) \tag{3.9}$$

$$Q(x_\epsilon) = Q(y) - \epsilon\eta(x_\epsilon).\nabla Q(y) + o(\epsilon) \tag{3.10}$$

$$p(x_\epsilon) = p(y) - \epsilon\eta(x_\epsilon).\nabla p(y) + o(\epsilon) \tag{3.11}$$

$$\frac{1}{p(x_\epsilon)} = \frac{1}{p(y)} + \frac{\epsilon}{p^2(y)}\eta(x_\epsilon).\nabla p(y) + o(\epsilon). \tag{3.12}$$

Using (3.9) and (3.10), we get

$$\begin{aligned}
\int_{T_\epsilon(\Omega)} Q(T_\epsilon^{-1}(y))\chi_{[u>0]}\Big(det\big(DT_\epsilon(T_\epsilon^{-1}(y))\big)\Big)^{-1} &= \int_{T_\epsilon(\Omega)} Q(y)\chi_{[u>0]}\\
-\epsilon\int_{T_\epsilon(\Omega)} Q(y)\chi_{[u>0]}\nabla\cdot\eta - \epsilon\int_{T_\epsilon(\Omega)}\eta(x_\epsilon).\nabla Q(y)\chi_{[u>0]} &+ o(\epsilon).
\end{aligned} \tag{3.13}$$

Using (3.11), we compute

$$\begin{aligned}
&|DT_\epsilon(x_\epsilon)\nabla u(y)|^{p(x_\epsilon)} = |\nabla u(y) + \epsilon\, {}^tD\eta(x_\epsilon).\nabla u(y)|^{(p(y)-\epsilon\eta(x_\epsilon).\nabla p(y)+o(\epsilon))}\\
&= \exp\Big(\Big(\frac{p(y)-\epsilon\eta(x_\epsilon).\nabla p(y)+o(\epsilon)}{2}\Big)\ln\big(|\nabla u(y)|^2 + 2\epsilon(D\eta(x_\epsilon).\nabla u(y)).\nabla u(y)+o(\epsilon)\big)\Big)\\
&= \exp\Big(\Big(\frac{p(y)-\epsilon\eta(x_\epsilon).\nabla p(y)}{2}+o(\epsilon)\Big)\Big(2\ln(|\nabla u(y)|) + \frac{2\epsilon({}^tD\eta(x_\epsilon).\nabla u(y))}{|\nabla u(y)|^2}.\nabla u(y)+o(\epsilon)\Big)\Big)\\
&= |\nabla u(y)|^{p(y)}\exp\Big(-\epsilon(\eta(x_\epsilon).\nabla p(y))\ln(|\nabla u(y)|) + \frac{\epsilon p(y)(D\eta(x_\epsilon).\nabla u(y))}{|\nabla u(y)|^2}.\nabla u(y)+o(\epsilon)\Big)\\
&= |\nabla u(y)|^{p(y)} - \epsilon(\eta(x_\epsilon).\nabla p(y))|\nabla u(y)|^{p(y)}\ln(|\nabla u(y)|)\\
&\quad + \epsilon p(y)({}^tD\eta(x_\epsilon).\nabla u(y))|\nabla u(y)|^{p(y)-2}\nabla u(y) + o(\epsilon).
\end{aligned} \tag{3.14}$$

Using (3.9), (3.12) and (3.14), we get

$$\begin{aligned}
&\int_{T_\epsilon(\Omega)}\frac{1}{p(T_\epsilon^{-1}(y))}|DT_\epsilon(T_\epsilon^{-1}(y))\nabla u(y)|^{p(T_\epsilon^{-1}(y))}\Big(det(DT_\epsilon(T_\epsilon^{-1}(y)))\Big)^{-1}\\
&= \int_{T_\epsilon(\Omega)}\frac{1}{p(y)}|\nabla u(y)|^{p(y)} - \epsilon\int_{T_\epsilon(\Omega)}\frac{1}{p(y)}(\eta(x_\epsilon).\nabla p(y))|\nabla u(y)|^{p(y)}\ln(|\nabla u(y)|)\\
&+\epsilon\int_{T_\epsilon(\Omega)}(D\eta(x_\epsilon).\nabla u(y))|\nabla u(y)|^{p(y)-2}\nabla u(y) - \epsilon\int_{T_\epsilon(\Omega)}\frac{1}{p(y)}\nabla.\eta(x_\epsilon)|\nabla u(y)|^{p(y)}\\
&+\epsilon\int_{T_\epsilon(\Omega)}\frac{1}{p^2(y)}(\eta(x_\epsilon).\nabla p(y))|\nabla u(y)|^{p(y)} + o(\epsilon).
\end{aligned} \tag{3.15}$$

Using (3.7), (3.13), (3.15) and the fact that $T_\epsilon(\Omega)\subset\Omega$ for ϵ small enough, we get

$$\begin{aligned}
0 \le &-\epsilon\int_{T_\epsilon(\Omega)}\frac{1}{p(y)}(\eta(x_\epsilon).\nabla p(y))|\nabla u(y)|^{p(y)}\ln(|\nabla u(y)|)\\
&+\epsilon\int_{T_\epsilon(\Omega)}(D\eta(x_\epsilon).\nabla u(y))|\nabla u(y)|^{p(y)-2}\nabla u(y)\\
&-\epsilon\int_{T_\epsilon(\Omega)}\frac{1}{p(y)}\nabla.\eta(x_\epsilon)|\nabla u(y)|^{p(y)} + \epsilon\int_{T_\epsilon(\Omega)}\frac{1}{p^2(y)}(\eta(x_\epsilon).\nabla p(y))|\nabla u(y)|^{p(y)}\\
&-\epsilon\int_{T_\epsilon(\Omega)}Q(y)\chi_{[u>0]}\nabla\cdot\eta - \epsilon\int_{T_\epsilon(\Omega)}\eta(x_\epsilon).\nabla Q(y)\chi_{[u>0]} + o(\epsilon).
\end{aligned}$$

Repeating the above calculations for $-\eta$, dividing by ϵ and letting $\epsilon\to 0$, we get

$$\begin{aligned}
&\int_\Omega\frac{1}{p(y)}(\eta.\nabla p(y))|\nabla u(y)|^{p(y)}\ln(|\nabla u(y)|) - \int_\Omega(D\eta.\nabla u(y))|\nabla u(y)|^{p(y)-2}\nabla u(y)\\
&+\int_\Omega\frac{1}{p(y)}\nabla.\eta|\nabla u(y)|^{p(y)} - \int_\Omega\frac{1}{p^2(y)}(\eta.\nabla p(y))|\nabla u(y)|^{p(y)}\\
&+\int_\Omega Q(y)\chi_{[u>0]}\nabla\cdot\eta + \int_\Omega\eta.\nabla Q(y)\chi_{[u>0]} = 0.
\end{aligned} \tag{3.16}$$

Taking into account that $\Delta_{p(y)}u = 0$ in $[u > 0]$, we deduce from (3.16) that

$$\int_{[u>0]} div\Big(\Big(\frac{|\nabla u|^{p(y)}}{p(y)} + Q\Big)\eta - (\eta.\nabla u)|\nabla u|^{p(y)-2}\nabla u\Big) = 0. \tag{3.17}$$

Integrating by parts in (3.17) and using the fact that the unit normal vector to $\partial[u > \epsilon]$ is given by $\nu = \frac{\nabla u(y)}{|\nabla u(y)|}$, we end up with

$$\begin{aligned} 0 &= \lim_{\epsilon\to 0}\int_{[u>\epsilon]} div\Big(\Big(\frac{|\nabla u|^{p(y)}}{p(y)} + Q\Big)\eta - (\eta.\nabla u)|\nabla u|^{p(y)-2}\nabla u\Big) \\ &= \lim_{\epsilon\to 0}\int_{\partial[u>\epsilon]} \Big(\frac{|\nabla u|^{p(y)}}{p(y)} + Q - |\nabla u|^{p(y)}\Big)\eta.\nu \\ &= -\lim_{\epsilon\to 0}\int_{\partial[u>\epsilon]} \Big(\frac{p(y)-1}{p(y)}|\nabla u|^{p(y)} - Q\Big)\eta\cdot\nu. \end{aligned}$$

□

4 Hölder continuity of the minimizers

From now on, we assume that $p \in C^{0,\beta}(\Omega)$ i.e.

$$\exists\beta \in (0,1),\ \exists L > 0 :\ \forall x, y \in \Omega \quad |p(x) - p(y)| \le L|x - y|^{\beta}. \tag{4.1}$$

We also assume that there exists $M > 0$ such that $0 \le g \le M$ on $\partial\Omega$. According to Proposition 3.2 every solution of the problem (P) is nonnegative and bounded by M.

The main result of this section is the Hölder Continuity of the minimizers.

Theorem 4.1 *We have $\mathcal{S}(g,\Omega) \subset C^{0,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$.*

To prove Theorem 4.1, we follow a similar approach as in [4]. First we need the following lemma which gives local higher integrability for the gradient of the minimizers in the spirit of [20].

Lemma 4.1 *There exist positive numbers $r_0 = r_0(\beta, \delta(\Omega))$, $C_0 = C_0(n, p_-, p_+, L, \beta, M, Q_+)$ and $\epsilon_0 = \epsilon_0(n, p_-, p_+, L, \beta, M, Q_+)$ such that for each $u \in \mathcal{S}(g,\Omega)$, any $\epsilon \in (0, \epsilon_0]$, any ball B_r with $r \le r_0$ and $B_{2r} \subset\subset \Omega$, we have*

$$⨍_{B_r} |\nabla u|^{p(x)(1+\epsilon)}dx \le C_0\Big(⨍_{B_{2r}} \Big(1 + |\nabla u|^{p(x)}\Big)dx\Big)^{1+\epsilon}, \tag{4.2}$$

where $\delta(\Omega)$ is the diameter of Ω and $⨍_{B_r} v = \frac{1}{|B_r|}\int_{B_r} v = v_r$.

Proof. Let $r_0 > 0$ such that $r_0 < \min\Big(\frac{\delta(\Omega)}{2}, \frac{1}{e^{1/\beta}}\Big)$ and let $r \le r_0$ such that $B_{2r} \subset\subset \Omega$. To prove (4.2), it is enough (see [14] Corollary 6.1, p. 204) to prove that there exists $m \in (0,1)$ and a positive constant C such that

$$⨍_{B_r} |\nabla u|^{p(x)}dx \le C\Big(\Big(⨍_{B_{2r}} |\nabla u|^{mp(x)}dx\Big)^{\frac{1}{m}} + ⨍_{B_{2r}} (1 + Q(x))dx\Big). \tag{4.3}$$

Indeed we deduce from (4.3) that there exists $\epsilon_0 > 1$ and a positive constant $C > 1$ (if $C \leq 1$, one can take $1 + C$) such that

$$\fint_{B_r} |\nabla u|^{p(x)(1+\epsilon_0)}dx \quad\leq\quad C\Big(\Big(\fint_{B_{2r}} |\nabla u|^{p(x)}dx\Big)^{1+\epsilon_0} + \fint_{B_{2r}} (1+Q(x))^{1+\epsilon_0}dx\Big).$$

Then we obtain

$$\begin{aligned}
\fint_{B_r} |\nabla u|^{p(x)(1+\epsilon_0)}dx &\leq C\Big(\Big(\fint_{B_{2r}} |\nabla u|^{p(x)}dx\Big)^{1+\epsilon_0} + (1+Q_+)^{1+\epsilon_0}\Big)\\
&\leq C'\Big(\Big(\fint_{B_{2r}} |\nabla u|^{p(x)}dx\Big)^{1+\epsilon_0} + 1\Big), \quad C' = C(1+Q_+)^{1+\epsilon_0}\\
&= C'\Big(\Big(\fint_{B_{2r}} |\nabla u|^{p(x)}dx\Big)^{1+\epsilon_0} + \Big(\fint_{B_{2r}} dx\Big)^{1+\epsilon_0}\Big)\\
&\leq C'\Big(\Big(\fint_{B_{2r}} (1+|\nabla u|^{p(x)})dx\Big)^{1+\epsilon_0} + \Big(\fint_{B_{2r}} (1+|\nabla u|^{p(x)})dx\Big)^{1+\epsilon_0}\Big)\\
&= C_0\Big(\fint_{B_{2r}} (1+|\nabla u|^{p(x)})dx\Big)^{1+\epsilon_0}, \quad C_0 = 2C'.
\end{aligned}$$

Let $\epsilon \in (0, \epsilon_0)$. Using the inequality $\Big(\fint_{B_r} |v|^s dx\Big)^{\frac{1}{s}} \leq \Big(\fint_{B_r} |v|^t dx\Big)^{\frac{1}{t}}$, with $v = |\nabla u|^{p(x)}$, $s = 1+\epsilon$ and $t = 1+\epsilon_0$, we get from the previous estimate

$$\begin{aligned}
\fint_{B_r} |\nabla u|^{p(x)(1+\epsilon)} &\leq \Big(\fint_{B_r} |\nabla u|^{p(x)(1+\epsilon_0)}\Big)^{\frac{1+\epsilon}{1+\epsilon_0}}\\
&\leq C_0^{\frac{1+\epsilon}{1+\epsilon_0}}\Big(\fint_{B_{2r}} (1+|\nabla u|^{p(x)})dx\Big)^{1+\epsilon}\\
&\leq C_0\Big(\fint_{B_{2r}} (1+|\nabla u|^{p(x)})dx\Big)^{1+\epsilon} \qquad \text{since } C_0 > 1.
\end{aligned}$$

Let us now prove (4.3). Let $r < t < s \leq 2r$ and $\eta \in C_0^\infty(B_s)$ be a cut-off function satisfying

$$\eta = 1 \quad \text{in} \quad B_t, \qquad 0 \leq \eta \leq 1 \qquad \text{and} \qquad |\nabla\eta| \leq \frac{2}{s-t} \quad \text{in} \quad B_{2r}.$$

Let $v = u_s + (1-\eta)(u-u_s)$. Since $v_{/\partial B_s} = u_{/\partial B_s}$, the function $w = v\chi_{B_s} + u\chi_{\Omega\setminus B_s}$ is an admissible test function for (P) and we have

$$\int_\Omega \Big(\frac{1}{p(x)}|\nabla u|^{p(x)} + Q(x)\chi_{[u>0]}\Big)dx \leq \int_\Omega \Big(\frac{1}{p(x)}|\nabla w|^{p(x)} + Q(x)\chi_{[w>0]}\Big)dx$$

which can be written as

$$\int_{B_s} \Big(\frac{1}{p(x)}|\nabla u|^{p(x)} + Q(x)\chi_{[u>0]}\Big)dx \leq \int_{B_s} \Big(\frac{1}{p(x)}|\nabla v|^{p(x)} + Q(x)\chi_{[v>0]}\Big)dx$$

or

$$\int_{B_s} |\nabla u|^{p(x)} dx \leq \frac{p_+}{p_-} \int_{B_s} |\nabla v|^{p(x)} dx + p_+ \int_{B_s} Q(x) dx. \tag{4.4}$$

Remark that we have in B_s: $\nabla v = ((1-\eta)\nabla u - (u - u_s)\nabla\eta)\chi_{B_s \setminus B_t}$. We deduce that

$$\begin{aligned} |\nabla v|^{p(x)} &\leq ((1-\eta)|\nabla u| + |u - u_s|.|\nabla\eta|)^{p(x)}\chi_{B_s\setminus B_t} \\ &\leq 2^{p(x)-1}((1-\eta)|\nabla u|^{p(x)} + (|u-u_s|.|\nabla\eta|)^{p(x)})\chi_{B_s\setminus B_t} \\ &\leq 2^{p_+-1}\Big(|\nabla u|^{p(x)} + \Big(\frac{2|u-u_s|}{s-t}\Big)^{p(x)}\Big)\chi_{B_s\setminus B_t}. \end{aligned}$$

Then we get from (4.4), for $c_1 = \frac{p_+ 2^{p_+-1}}{p_-}$

$$\int_{B_t} |\nabla u|^{p(x)} dx \leq c_1 \int_{B_s\setminus B_t} |\nabla u|^{p(x)} dx + c_1 2^{p_+} \int_{B_s\setminus B_t} \left(\frac{|u-u_s|}{s-t}\right)^{p(x)} dx + p_+ \int_{B_s} Q(x) dx.$$

Adding $c_1 \int_{B_t} |\nabla u|^{p(x)} dx$ to both sides of the previous inequality and dividing by $1 + c_1$, we get for $\vartheta = \frac{c_1}{1+c_1}$

$$\int_{B_t} |\nabla u|^{p(x)} dx \leq \vartheta \int_{B_s} |\nabla u|^{p(x)} dx + \frac{c_1 2^{p_+}}{1+c_1} \int_{B_s\setminus B_t} \left(\frac{|u-u_s|}{s-t}\right)^{p(x)} dx + \frac{p_+}{1+c_1} \int_{B_{2r}} Q(x) dx. \tag{4.5}$$

Since $0 < s - t < 1$ and $0 \leq u, u_s \leq M$, we have for $p_1 = \min_{x\in B_{2r}} p(x)$ and $p_2 = \max_{x\in B_{2r}} p(x)$

$$\begin{aligned} \int_{B_s\setminus B_t} \left(\frac{|u-u_s|}{s-t}\right)^{p(x)} dx &\leq \frac{1}{(s-t)^{p_2}} \int_{B_s} |u-u_s|^{p(x)-p_1} |u-u_s|^{p_1} dx \\ &\leq \frac{1}{(s-t)^{p_2}} \int_{B_s} M^{p(x)-p_1} |u-u_s|^{p_1} dx \\ &\leq \frac{\max(1, M^{p_+-p_-})}{(s-t)^{p_2}} \int_{B_s} |u-u_s|^{p_1} dx. \end{aligned} \tag{4.6}$$

Moreover we have by convexity of t^{p_1}

$$\begin{aligned} \int_{B_s} |u-u_s|^{p_1} dx &\leq \int_{B_s} (|u-u_{2r}| + |u_s - u_{2r}|)^{p_1} dx \\ &\leq \int_{B_s} 2^{p_1-1}(|u-u_{2r}|^{p_1} + |u_s-u_{2r}|^{p_1}) dx \\ &= 2^{p_1-1}\int_{B_s} |u-u_{2r}|^{p_1} dx + 2^{p_1-1}\int_{B_s} |u_s-u_{2r}|^{p_1} dx. \end{aligned}$$

By Hölder's inequality we have

$$\begin{aligned}
\int_{B_s} |u_s - u_{2r}|^{p_1} dx &= |B_s||u_s - u_{2r}|^{p_1} \\
&= |B_s|\Big|\frac{1}{|B_s|}\int_{B_s} u dx - u_{2r}\Big|^{p_1} \\
&= |B_s|^{1-p_1}\Big|\int_{B_s} (u - u_{2r}) dx\Big|^{p_1} \\
&\le |B_s|^{1-p_1}.|B_s|^{p_1(1-\frac{1}{p_1})}\int_{B_s} |u - u_{2r}|^{p_1} dx \\
&= \int_{B_s} |u - u_{2r}|^{p_1} dx.
\end{aligned}$$

We deduce that

$$\int_{B_s} |u - u_s|^{p_1} dx \le 2^{p_+}\int_{B_{2r}} |u - u_{2r}|^{p_1} dx. \tag{4.7}$$

Letting $Z(t) = \int_{B_t} |\nabla u|^{p(x)} dx$, we obtain from (4.5)-(4.7)

$$Z(t) \le [A(s-t)^{-p_2} + C] + \vartheta Z(s),$$

where $A = \dfrac{c_1 2^{2p_+}\max(1, M^{p_+-p_-})}{1+c_1}\displaystyle\int_{B_{2r}} |u - u_{2r}|^{p_1} dx$, and $C = \dfrac{p_+}{1+c_1}\displaystyle\int_{B_{2r}} Q(x) dx$.

Applying Lemma 6.1 [14], p. 191 with $\rho = r$ and $R = 2r$, we get $Z(r) \le c(p_2, \vartheta)[Ar^{-p_2} + C]$, where $c(p_2, \vartheta) = (1-\lambda)^{-p_2}(1 - \vartheta\lambda^{-p_2})^{-1} \le (1-\lambda)^{-p_+}(1 - \vartheta\lambda^{-p_+})^{-1} = c(p_+, \vartheta)$, and $\lambda \in (0, 1)$ is such that $\vartheta\lambda^{-p_2} < 1$ which is satisfied if $\vartheta\lambda^{-p_+} < 1$, or $\lambda > \vartheta^{\frac{1}{p_+}}$. It follows that we have for $c_2 = c(p_+, \vartheta)\dfrac{c_1 2^{2p_+}\max(1, M^{p_+-p_-})}{1+c_1}$ and $c_3 = \dfrac{p_+ c(p_+, \vartheta)}{1+c_1}$

$$\begin{aligned}
\int_{B_r} |\nabla u|^{p(x)} dx &\le \frac{c_2}{r^{p_2}}\int_{B_{2r}} |u - u_{2r}|^{p_1} dx + c_3\int_{B_{2r}} Q(x) dx \\
&= \frac{c_2}{r^{p_2-p_1}}\int_{B_{2r}} \left(\frac{|u - u_{2r}|}{r}\right)^{p_1} dx + c_3\int_{B_{2r}} Q(x) dx.
\end{aligned} \tag{4.8}$$

Since $r \le r_0 < \frac{1}{e^{1/\beta}}$, we have by (4.1)

$$\frac{1}{r^{p_2-p_1}} \le \Big(\frac{1}{r}\Big)^{L(4r)^\beta} = e^{-L(4r)^\beta \ln(r)} \le e^{\frac{L4^\beta}{e\beta}}.$$

Then (4.8) becomes

$$\int_{B_r} |\nabla u|^{p(x)} dx \le c_2 e^{\frac{L4^\beta}{e\beta}}\int_{B_{2r}} \left(\frac{|u - u_{2r}|}{r}\right)^{p_1} dx + c_3\int_{B_{2r}} Q(x) dx,$$

or

$$⨍_{B_r} |\nabla u|^{p(x)} dx \le c_2 2^n e^{\frac{L4^\beta}{e\beta}}⨍_{B_{2r}} \left(\frac{|u - u_{2r}|}{r}\right)^{p_1} dx + c_3 2^n⨍_{B_{2r}} Q(x) dx. \tag{4.9}$$

The next step consists in estimating the integral $\fint_{B_{2r}} \left(\frac{|u-u_{2r}|}{r}\right)^{p_1} dx$. Let $q = \frac{np_1}{n+p_1} < n$. We distinguish two cases:

<u>Case 1</u>: $q \geq 1 \Leftrightarrow p_1 \geq \frac{n}{n-1}$. Since $q^* = \frac{nq}{n-q} = p_1$, we get by applying the Sobolev-Poincaré inequality (see [18], Corollary 1.64 p. 38, for $q = p_1$ and $p = q$)

$$\left(\fint_{B_{2r}} |u-u_{2r}|^{p_1} dx\right)^{\frac{1}{p_1}} \leq C(n,p_1) r \left(\fint_{B_{2r}} |\nabla u|^q dx\right)^{\frac{1}{q}},$$

which can be written as

$$\fint_{B_{2r}} \left(\frac{|u-u_{2r}|}{r}\right)^{p_1} dx \leq C(n,p_-,p_+)\left(\fint_{B_{2r}} |\nabla u|^q dx\right)^{\frac{p_1}{q}}. \tag{4.10}$$

Using (4.9)-(4.10), we get for $c_4 = c_2 e^{\frac{LA^\beta}{e\beta}} C(n,p_-,p_+) 2^n$

$$\fint_{B_r} |\nabla u|^{p(x)} dx \leq c_4 \left(\fint_{B_{2r}} |\nabla u|^{\frac{np_1}{n+p_1}} dx\right)^{\frac{n+p_1}{n}} + c_3 2^n \fint_{B_{2r}} Q(x) dx. \tag{4.11}$$

Now we have by the convexity of $t^{\frac{n+p_1}{n}} = t^{\frac{p_1}{n}+1}$

$$\begin{aligned}
\left(\fint_{B_{2r}} |\nabla u|^{\frac{np_1}{n+p_1}} dx\right)^{\frac{n+p_1}{n}} &= \left(\fint_{B_{2r}\cap[|\nabla u|\leq 1]} |\nabla u|^{\frac{np_1}{n+p_1}} dx + \fint_{B_{2r}\cap[|\nabla u|>1]} |\nabla u|^{\frac{np_1}{n+p_1}} dx\right)^{\frac{n+p_1}{n}} \\
&\leq \left(1 + \fint_{B_{2r}\cap[|\nabla u|>1]} |\nabla u|^{\frac{np(x)}{n+p_1}} dx\right)^{\frac{n+p_1}{n}} \\
&\leq 2^{p_1/n}\left(1 + \left(\fint_{B_{2r}\cap[|\nabla u|>1]} |\nabla u|^{\frac{np(x)}{n+p_1}} dx\right)^{\frac{n+p_1}{n}}\right) \\
&\leq 2^{p_+/n}\left(1 + \left(\fint_{B_{2r}} |\nabla u|^{\frac{np(x)}{n+p_1}} dx\right)^{\frac{n+p_1}{n}}\right).
\end{aligned} \tag{4.12}$$

Using the inequality $\left(\fint_{B_{2r}} |v|^s dx\right)^{\frac{1}{s}} \leq \left(\fint_{B_{2r}} |v|^t dx\right)^{\frac{1}{t}}$, for $0 < s < t$ and $v \in L^t(B_{2r})$, with $v = |\nabla u|^{p(x)}$, $s = \frac{n}{n+p_1}$ and $t = \frac{n}{n+p_-}$, we get

$$\left(\fint_{B_{2r}} |\nabla u|^{\frac{np(x)}{n+p_1}}\right)^{\frac{n+p_1}{n}} \leq \left(\fint_{B_{2r}} |\nabla u|^{\frac{np(x)}{n+p_-}}\right)^{\frac{n+p_-}{n}}.$$

It follows from (4.11)-(4.12), that we have

$$\begin{aligned}
\fint_{B_r} |\nabla u|^{p(x)} dx &\leq 2^{p_+/n} c_4 \left(\fint_{B_{2r}} |\nabla u|^{\frac{np(x)}{n+p_-}}\right)^{\frac{n+p_-}{n}} + 2^{p_+/n} c_4 + c_3 2^n \fint_{B_{2r}} Q(x) dx \\
&\leq \max(c_3 2^n, 2^{p_+/n} c_4)\left(\left(\fint_{B_{2r}} |\nabla u|^{\frac{np(x)}{n+p_-}} dx\right)^{\frac{n+p_-}{n}} + \fint_{B_{2r}} (1+Q(x)) dx\right).
\end{aligned}$$

This proves (4.3) in this case with $m = \frac{n}{n+p_-}$ and $C = \max(c_3 2^n, 2^{p_+/n} c_4)$.

Case 2: $q < 1 \Leftrightarrow p_1 < \frac{n}{n-1}$. Since $p_1 < \frac{n}{n-1} = 1^*$, we get by applying the Sobolev-Poincaré inequality (see [18], Corollary 1.64 p. 38, for $q = p_1$ and $p = 1$)

$$\Big(\fint_{B_{2r}} |u - u_{2r}|^{p_1} dx\Big)^{\frac{1}{p_1}} \leq C(n, p_1) r \fint_{B_{2r}} |\nabla u| dx,$$

which can be written as

$$\fint_{B_{2r}} \Big(\frac{|u - u_{2r}|}{r}\Big)^{p_1} dx \leq C(n, p_-, p_+) \Big(\fint_{B_{2r}} |\nabla u| dx\Big)^{p_1}. \tag{4.13}$$

Using (4.9) and (4.13), we get for $c_4 = c_2 C(n, p_-, p_+) e^{\frac{L4^\beta}{e\beta}} 2^n$

$$\fint_{B_r} |\nabla u|^{p(x)} dx \leq c_4 \Big(\fint_{B_{2r}} |\nabla u| dx\Big)^{p_1} + c_3 2^n \fint_{B_{2r}} Q(x) dx. \tag{4.14}$$

As in the previous case, we have by the convexity of t^{p_1}

$$\begin{aligned} \Big(\fint_{B_{2r}} |\nabla u| dx\Big)^{p_1} &= \Big(\fint_{B_{2r}\cap[|\nabla u|\leq 1]} |\nabla u| dx + \fint_{B_{2r}\cap[|\nabla u|>1]} |\nabla u| dx\Big)^{p_1} \\ &\leq \Big(1 + \fint_{B_{2r}\cap[|\nabla u|>1]} |\nabla u|^{\frac{p(x)}{p_1}} dx\Big)^{p_1} \\ &\leq 2^{p_1-1}\Big(1 + \Big(\fint_{B_{2r}\cap[|\nabla u|>1]} |\nabla u|^{\frac{p(x)}{p_1}} dx\Big)^{p_1}\Big) \\ &\leq 2^{p_+-1}\Big(1 + \Big(\fint_{B_{2r}} |\nabla u|^{\frac{p(x)}{p_1}} dx\Big)^{p_1}\Big). \end{aligned} \tag{4.15}$$

Using the inequality $\Big(\fint_{B_{2r}} |v|^s dx\Big)^{\frac{1}{s}} \leq \Big(\fint_{B_{2r}} |v|^t dx\Big)^{\frac{1}{t}}$, with $v = |\nabla u|^{p(x)}$, $s = \frac{1}{p_1}$ and $t = \frac{1}{p_-}$, we get

$$\Big(\fint_{B_{2r}} |\nabla u|^{\frac{p(x)}{p_1}} dx\Big)^{p_1} \leq \Big(\fint_{B_{2r}} |\nabla u|^{\frac{p(x)}{p_-}} dx\Big)^{p_-}.$$

It follows from (4.14)-(4.15), that we have

$$\begin{aligned} \fint_{B_r} |\nabla u|^{p(x)} dx &\leq 2^{p_+-1} c_4 \Big(\fint_{B_{2r}} |\nabla u|^{\frac{p(x)}{p_-}} dx\Big)^{p_-} + c_3 2^n \fint_{B_{2r}} Q(x) dx + 2^{p_+-1} c_4 \\ &\leq \max(c_3 2^n, 2^{p_+-1} c_4)\Big(\Big(\fint_{B_{2r}} |\nabla u|^{\frac{p(x)}{p_-}} dx\Big)^{p_-} + \fint_{B_{2r}} (1 + Q(x)) dx\Big). \end{aligned}$$

This proves (4.3) in this case with $m = \frac{1}{p_-}$ and $C = \max(c_3 2^n, 2^{p_+-1} c_4)$. □

Proof of Theorem 4.1. We use some ideas from [4]. Let ϵ_0, r_0 be as in Lemma 4.1, $\epsilon \in (0, \min(\epsilon_0, \beta/n))$ and let r_1 such that $L(2r_1)^\beta \leq \frac{\epsilon p_-}{\epsilon+2}$. For each r, we set

$$p_1(r) = \min_{x\in B_r} p(x) \quad \text{and} \quad p_2(r) = \max_{x\in B_r} p(x).$$

Let now $u \in \mathcal{S}(g,\Omega)$ and let $B_r = B_r(x_0)$ be a ball of center $x_0 \in \Omega$ and radius $r \in (0, \min(1, 2r_0, r_1))$ such that $B_r \subset\subset \Omega$. Then one can easily verify that

$$p_2 < \Big(1+\frac{\epsilon}{2}\Big)p_2 \leq (1+\epsilon)p_1, \quad u \in W^{1,(1+\frac{\epsilon}{2})p_2}(B_r). \tag{4.16}$$

Indeed we have by (4.1)

$$\begin{aligned}
\Big(1+\frac{\epsilon}{2}\Big)p_2 - (1+\epsilon)p_1 &= p_2 - p_1 + \frac{\epsilon}{2}(p_2-p_1) - \frac{\epsilon}{2}p_1 \\
&= (1+\frac{\epsilon}{2})(p_2-p_1) - \frac{\epsilon}{2}p_1 \\
&\leq (1+\frac{\epsilon}{2})L(2r)^\beta - \frac{\epsilon}{2}p_1 \\
&\leq (1+\frac{\epsilon}{2})L(2r_1)^\beta - \frac{\epsilon}{2}p_1 \\
&\leq (1+\frac{\epsilon}{2})\frac{\epsilon p_-}{\epsilon+2} - \frac{\epsilon}{2}p_1 \\
&\leq (1+\frac{\epsilon}{2})\frac{\epsilon p_1}{\epsilon+2} - \frac{\epsilon}{2}p_1 \\
&= \frac{\epsilon}{2}p_1 - \frac{\epsilon}{2}p_1 = 0.
\end{aligned}$$

We deduce from (4.16) that $\Big(1+\frac{\epsilon}{2}\Big)p_2 \leq (1+\epsilon)p(x)$ for all $x \in B_r$. By Lemma 4.1, it follows that $u \in W^{1,(1+\frac{\epsilon}{2})p_2}(B_r)$.

Let now v be the unique solution of the Dirichlet problem

$$\Delta_{p_2} v = 0 \quad \text{in} \quad B_{r/4} \qquad \text{and} \qquad v - u \in W_0^{1,p_2}(B_{r/4}). \tag{4.17}$$

Then there exists (see [8] Lemma 2.7) two positive constants C_1 and $\mu < \frac{\epsilon}{2}$ depending only on p_-, p_+ and n such that

$$\Big(⨍_{B_{r/4}} |\nabla v|^{(1+\mu)p_2} dx\Big)^{\frac{1}{1+\mu}} \leq C_1 \Big(⨍_{B_{r/2}} \Big(1+|\nabla u|^{(1+\frac{\epsilon}{2})p_2}\Big) dx\Big)^{\frac{1}{1+\frac{\epsilon}{2}}}. \tag{4.18}$$

Since $(1+\frac{\epsilon}{2})p_2 \leq (1+\epsilon)p(x)$ for $x \in B_{r/2}$, we have

$$\begin{aligned}
⨍_{B_{r/2}} (1+|\nabla u|^{(1+\frac{\epsilon}{2})p_2}) dx &= 1 + ⨍_{B_{r/2}\cap[|\nabla u|\leq 1]} |\nabla u|^{(1+\frac{\epsilon}{2})p_2} dx + ⨍_{B_{r/2}\cap[|\nabla u|>1]} |\nabla u|^{(1+\frac{\epsilon}{2})p_2} dx \\
&\leq 2 + ⨍_{B_{r/2}} |\nabla u|^{(1+\epsilon)p(x)} dx.
\end{aligned}$$

Using (4.2) and (4.18), we get

$$\begin{aligned}\Big(\fint_{B_{r/4}} |\nabla v|^{(1+\mu)p_2}dx\Big)^{\frac{1}{1+\mu}} &\leq C_1\Big(2+\fint_{B_{r/2}} |\nabla u|^{(1+\epsilon)p(x)}dx\Big)^{\frac{2}{2+\epsilon}}\\ &\leq C_1\Big(2+C_0\Big(\fint_{B_r}(1+|\nabla u|^{p(x)})dx\Big)^{1+\epsilon}\Big)^{\frac{2}{2+\epsilon}}\\ &\leq C_1(2\max(2,C_0))^{\frac{2}{2+\epsilon}}\Big(\fint_{B_r}\Big(1+|\nabla u|^{p(x)}\Big)dx\Big)^{\frac{1+\epsilon}{1+\frac{\epsilon}{2}}}\\ &\leq C_2\Big(\fint_{B_r}\Big(1+|\nabla u|^{p(x)}\Big)dx\Big)^{\frac{1+\epsilon}{1+\frac{\epsilon}{2}}},\ C_2=2C_1\max(2,C_0). \qquad (4.19)\end{aligned}$$

Note that the function $w = v\chi_{B_{r/4}} + u\chi_{\Omega\setminus B_{r/4}}$ is an admissible test function for the problem (P). So we have

$$\int_\Omega\Big(\frac{1}{p(x)}|\nabla u|^{p(x)} + Q(x)\chi_{[u>0]}\Big)dx \leq \int_\Omega\Big(\frac{1}{p(x)}|\nabla w|^{p(x)} + Q(x)\chi_{[w>0]}\Big)dx$$

which can be written as

$$\int_{B_{r/4}}\Big(\frac{1}{p(x)}|\nabla u|^{p(x)} + Q(x)\chi_{[u>0]}\Big)dx \leq \int_{B_{r/4}}\Big(\frac{1}{p(x)}|\nabla v|^{p(x)} + Q(x)\chi_{[v>0]}\Big)dx$$

or

$$\int_{B_{r/4}}\frac{1}{p(x)}(|\nabla u|^{p(x)} - |\nabla v|^{p(x)})dx \leq \int_{B_{r/4}} Q(x)dx \leq C(n)Q_+r^n. \qquad (4.20)$$

Let $u_t = tu + (1-t)v$ for $t \in [0,1]$. Using (4.17) and the inequalities for $q > 1$ and $\xi, \zeta \in \mathbb{R}^n$ (see [3])

$$\begin{cases} |\xi|^q - |\zeta|^q \geq q|\zeta|^{q-2}\zeta.(\xi-\zeta) + \frac{|\xi-\zeta|^q}{2^{q-1}-1} & \text{if} \quad q \geq 2\\ |\xi|^q - |\zeta|^q \geq q|\zeta|^{q-2}\zeta.(\xi-\zeta) + \frac{3q(q-1)}{16}|\xi-\zeta|^2.(|\xi|+|\zeta|)^{q-2} & \text{if} \quad 1<q<2\\ \text{where } |\xi-\zeta|^2.(|\xi|+|\zeta|)^{q-2} = 0 \text{ if } \xi=\zeta=0, \end{cases}$$

we obtain if $p_2 \geq 2$

$$\begin{aligned}\frac{1}{2^{p_2-1}-1}\int_{B_{r/4}}|\nabla(u-v)|^{p_2}dx &\leq \int_{B_{r/4}}(|\nabla u|^{p_2} - |\nabla v|^{p_2})dx - p_2\int_{B_{r/4}}|\nabla v|^{p_2-2}\nabla v.\nabla(u-v)dx\\ &= \int_{B_{r/4}}(|\nabla u|^{p_2} - |\nabla v|^{p_2})dx,\end{aligned}$$

which can be written, with $c(p_+) = 2^{p_+-1} - 1$, as

$$\int_{B_{r/4}}|\nabla(u-v)|^{p_2} \leq c(p_+)\int_{B_{r/4}}(|\nabla u|^{p_2} - |\nabla v|^{p_2})dx. \qquad (4.21)$$

When $p_2 < 2$, we obtain

$$\begin{aligned}\frac{3p_2(p_2-1)}{16}\int_{B_{r/4}}|\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p_2-2}dx &\leq \int_{B_{r/4}}(|\nabla u|^{p_2}-|\nabla v|^{p_2})dx\\ &\quad - p_2\int_{B_{r/4}}|\nabla v|^{p_2-2}\nabla v.\nabla(u-v)dx\\ &= \int_{B_{r/4}}(|\nabla u|^{p_2}-|\nabla v|^{p_2})dx.\end{aligned}$$

This leads, with $c(p_-) = \frac{16}{3p_-(p_--1)}$, to

$$\int_{B_{r/4}}|\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p_2-2}dx \leq c(p_-)\int_{B_{r/4}}(|\nabla u|^{p_2}-|\nabla v|^{p_2})dx.$$

Using Hölder's inequality for the pair $(\frac{2}{p_2}, \frac{2}{2-p_2})$, the convexity of t^{p_2}, and the minimality of v for the integral $\int_{B_{r/4}}|\nabla w|^{p_2}dx$ among all $w \in u + W_0^{1,p_2}(B_{r/4})$

$$\begin{aligned}\int_{B_{r/4}}|\nabla(u-v)|^{p_2}dx &= \int_{B_{r/4}}|\nabla(u-v)|^{p_2}.(|\nabla u|+|\nabla v|)^{\frac{(p_2-2)p_2}{2}}.(|\nabla u|+|\nabla v|)^{\frac{(2-p_2)p_2}{2}}dx\\ &\leq \left(\int_{B_{r/4}}|\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p_2-2}dx\right)^{\frac{p_2}{2}}.\left(\int_{B_{r/4}}(|\nabla u|+|\nabla v|)^{p_2}dx\right)^{\frac{2-p_2}{2}}\\ &\leq \left(\int_{B_{r/4}}|\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p_2-2}dx\right)^{\frac{p_2}{2}}.\left(\int_{B_{r/4}}2^{p_2-1}(|\nabla u|^{p_2}+|\nabla v|^{p_2})dx\right)^{\frac{2-p_2}{2}}\\ &\leq c(p_-)^{\frac{p_2}{2}}\left(\int_{B_{r/4}}(|\nabla u|^{p_2}-|\nabla v|^{p_2})dx\right)^{\frac{p_2}{2}}.\left(2^{p_2}\int_{B_{r/4}}|\nabla u|^{p_2}dx\right)^{\frac{2-p_2}{2}}\\ &\leq c(p_-,p_+)\left(\int_{B_{r/4}}(|\nabla u|^{p_2}-|\nabla v|^{p_2})dx\right)^{\frac{p_2}{2}}.\left(\int_{B_{r/4}}|\nabla u|^{p_2}dx\right)^{\frac{2-p_2}{2}}. \qquad (4.22)\end{aligned}$$

Now it is easy to verify that $\forall s > 0,\ \forall t \geq 1 \quad \ln(t) \leq \dfrac{t^s}{se}$.

Taking $s = \mu p_2$, we get $\forall t \geq 1 \quad \ln(t) \leq \dfrac{t^{\mu p_2}}{\mu p_2 e} \leq \dfrac{t^{\mu p_2}}{\mu p_- e}$. This leads to

$$\forall t \geq 1 \quad (\ln(t))t^{p_2} \leq \frac{1}{\mu p_- e}t^{(1+\mu)p_2}.$$

Using (4.1), we obtain for some positive constant $C(\mu,\beta,L,p_-)$

$$\forall t \geq 1 \quad |t^{p_2}-t^{p(x)}| \leq |p_1-p_2||\ln(t)|t^{p_2} \leq Cr^\beta t^{(1+\mu)p_2}. \qquad (4.23)$$

It follows by (4.2), (4.16), (4.19), (4.20), (4.23) and since $\mu < \frac{\epsilon}{2}$, that

$$\begin{aligned}
\int_{B_{r/4}} (|\nabla u|^{p_2} - |\nabla v|^{p_2})dx &= \int_{B_{r/4}} (|\nabla u|^{p(x)} - |\nabla v|^{p(x)})dx \\
&+ \int_{B_{r/4}} (|\nabla u|^{p_2} - |\nabla u|^{p(x)})dx + \int_{B_{r/4}} (|\nabla v|^{p(x)} - |\nabla v|^{p_2})dx \\
&= \int_{B_{r/4}} (|\nabla u|^{p(x)} - |\nabla v|^{p(x)})dx \\
&+ \int_{B_{r/4}\cap[|\nabla u|\leq 1]} (|\nabla u|^{p_2} - |\nabla u|^{p(x)})dx + \int_{B_{r/4}\cap[|\nabla u|\leq 1]} (|\nabla v|^{p(x)} - |\nabla v|^{p_2})dx \\
&+ \int_{B_{r/4}\cap[|\nabla u|> 1]} (|\nabla u|^{p_2} - |\nabla u|^{p(x)})dx + \int_{B_{r/4}\cap[|\nabla u|> 1]} (|\nabla v|^{p(x)} - |\nabla v|^{p_2})dx \\
&\leq Cr^n + Cr^\beta \int_{B_{r/4}\cap[|\nabla u|>1]} |\nabla u|^{(1+\mu)p_2}dx + Cr^\beta \int_{B_{r/4}\cap[|\nabla v|>1]} |\nabla v|^{(1+\mu)p_2}dx \\
&\leq Cr^n + Cr^\beta \int_{B_{r/4}} |\nabla u|^{(1+\epsilon)p(x)}dx + Cr^\beta \int_{B_{r/4}} |\nabla v|^{(1+\mu)p_2}dx \\
&\leq Cr^n + Cr^{\beta+n}\Big(\fint_{B_{r/2}} \big(1 + |\nabla u|^{p(x)}\big)dx\Big)^{1+\epsilon} + Cr^{\beta+n}\Big(\fint_{B_r} \big(1 + |\nabla u|^{p(x)}\big)dx\Big)^{\frac{(1+\epsilon)(1+\mu)}{1+\frac{\epsilon}{2}}} \qquad (4.24)
\end{aligned}$$

where C is a constant independent of r.

Note that $\delta = \frac{(1+\epsilon)(1+\mu)}{1+\frac{\epsilon}{2}} - 1 = \frac{\epsilon+2\mu(1+\epsilon)}{2+\epsilon} > 0$. Moreover since $2\mu < \epsilon$, we have $\delta < \epsilon$. It follows then from (4.24), since $r < 1$ and $\beta - n\epsilon > 0$

$$\begin{aligned}
\int_{B_{r/4}} (|\nabla u|^{p_2} - |\nabla v|^{p_2})dx &\leq Cr^n + Cr^{\beta-n\epsilon}\Big(\int_{B_r} |\nabla u|^{p(x)}dx\Big)^{1+\epsilon} + Cr^{\beta-n\delta}\Big(\int_{B_r} |\nabla u|^{p(x)}dx\Big)^{1+\delta} \\
&\leq Cr^n + Cr^{\beta-n\epsilon}\Big(\Big(\int_{B_r} |\nabla u|^{p(x)}dx\Big)^{\epsilon} + \Big(\int_{B_r} |\nabla u|^{p(x)}dx\Big)^{\delta}\Big)\Big(\int_{B_r} |\nabla u|^{p(x)}dx\Big) \\
&\leq Cr^n + CM_1 r^{\beta-n\epsilon} \int_{B_r} |\nabla u|^{p(x)}dx, \quad M_1 = (p_+J(g))^\epsilon + (p_+J(g))^\delta \\
&= Cr^n + CM_1 r^{\beta-n\epsilon} \int_{B_r\cap[|\nabla u|\leq 1} |\nabla u|^{p(x)}dx + CM_1 r^{\beta-n\epsilon} \int_{B_r\cap[|\nabla u|>1} |\nabla u|^{p(x)}dx \\
&= Cr^n + CM_1 r^{\beta-n\epsilon}|B_1|r^n + CM_1 r^{\beta-n\epsilon} \int_{B_r\cap[|\nabla u|>1} |\nabla u|^{p_2}dx \\
&\leq Cr^n + Cr^{\beta-n\epsilon} \int_{B_r} |\nabla u|^{p_2}dx. \qquad (4.25)
\end{aligned}$$

If $p_2 \geq 2$, we obtain from (4.21) and (4.25)

$$\int_{B_{r/4}} |\nabla(u-v)|^{p_2}dx \leq Cr^n + Cr^{\beta-n\epsilon} \int_{B_r} |\nabla u|^{p_2}dx.$$

This leads for each $\rho \in (0, r/4)$ to

$$
\begin{aligned}
\int_{B_\rho} |\nabla u|^{p_2} dx &\leq \int_{B_\rho} (|\nabla(u-v)| + |\nabla v|)^{p_2} dx \\
&\leq 2^{p_2-1} \int_{B_\rho} (|\nabla(u-v)|^{p_2} + |\nabla v|^{p_2}) dx \\
&\leq 2^{p_+-1} \int_{B_\rho} |\nabla(u-v)|^{p_2} dx + 2^{p_+-1} \int_{B_\rho} |\nabla v|^{p_2} dx \\
&\leq Cr^n + C \int_{B_\rho} |\nabla v|^{p_2} dx + Cr^{\beta - n\epsilon} \int_{B_r} |\nabla u|^{p_2} dx.
\end{aligned}
$$

Using the estimate (see [16] Lemma 1.1)

$$
\int_{B_\rho} |\nabla v|^{p_2} dx \leq C(n, p_-, p_+) \Big(\frac{\rho}{r}\Big)^n \int_{B_{r/4}} |\nabla v|^{p_2} dx \tag{4.26}
$$

and the minimality of v, we get

$$
\int_{B_\rho} |\nabla u|^{p_2(r)} dx \leq Cr^n + C\Big(\frac{\rho}{r}\Big)^n \int_{B_r} |\nabla u|^{p_2(r)} dx + Cr^{\beta - n\epsilon} \int_{B_r} |\nabla u|^{p_2(r)} dx. \tag{4.27}
$$

Now (4.27) clearly holds also for $\rho \in [r/4, r]$. Therefore if we set $\phi(\rho) = \int_{B_\rho} |\nabla u|^{p_2(\rho)} dx$, we obtain from (4.27) for each $\rho \in (0, r]$

$$
\begin{aligned}
\phi(\rho) &= \int_{B_\rho \cap [|\nabla u| \leq 1]} |\nabla u|^{p_2(\rho)} dx + \int_{B_\rho \cap [|\nabla u| > 1]} |\nabla u|^{p_2(\rho)} dx \\
&\leq C\rho^n + \int_{B_\rho} |\nabla u|^{p_2(r)} dx \\
&\leq Cr^n + C\Big[\Big(\frac{\rho}{r}\Big)^n + r^{\beta - n\epsilon}\Big]\phi(r).
\end{aligned} \tag{4.28}
$$

If $p_2 < 2$, we obtain from (4.22) and (4.25)

$$
\begin{aligned}
\int_{B_{r/4}} |\nabla(u-v)|^{p_2} dx &\leq C(p_-, p_+)\Big(Cr^n + Cr^{\beta - n\epsilon} \int_{B_r} |\nabla u|^{p_2} dx\Big)^{\frac{p_2}{2}} . \Big(\int_{B_{r/4}} |\nabla u|^{p_2} dx\Big)^{\frac{2-p_2}{2}} \\
&\leq Cr^{\frac{(\beta - n\epsilon)p_2}{2}} \Big(r^{n-\beta+n\epsilon} + \int_{B_r} |\nabla u|^{p_2} dx\Big)^{\frac{p_2}{2}} . \Big(\int_{B_r} |\nabla u|^{p_2} dx\Big)^{\frac{2-p_2}{2}} \\
&\leq Cr^{\frac{(\beta - n\epsilon)p_2}{2}} \Big(r^{n-\beta+n\epsilon} + \int_{B_r} |\nabla u|^{p_2} dx\Big)^{\frac{p_2}{2}} . \Big(r^{n-\beta+n\epsilon} + \int_{B_r} |\nabla u|^{p_2} dx\Big)^{\frac{2-p_2}{2}} \\
&= Cr^{\frac{(\beta - n\epsilon)p_2}{2}} \Big(r^{n-\beta+n\epsilon} + \int_{B_r} |\nabla u|^{p_2} dx\Big) \\
&\leq Cr^{n - \frac{(\beta - n\epsilon)(2-p_2)}{2}} + Cr^{\frac{(\beta - n\epsilon)p_2}{2}} \int_{B_r} |\nabla u|^{p_2} dx.
\end{aligned}
$$

Taking into account (4.26) and the minimality of v, this leads for each $\rho \in (0, r/4)$ to

$$\int_{B_\rho} |\nabla u|^{p_2} dx \le C r^{n - \frac{(\beta - n\epsilon)(2-p_2)}{2}} + C\Big(\frac{\rho}{r}\Big)^n \int_{B_r} |\nabla u|^{p_2} dx + C r^{\frac{(\beta - n\epsilon)p_2}{2}} \int_{B_r} |\nabla u|^{p_2} dx. \tag{4.29}$$

Note that (4.29) clearly holds also for $\rho \in [r/4, r]$. Therefore by arguing as in (4.28), we obtain for each $\rho \in (0, r]$

$$\phi(\rho) \le C r^{n - \frac{(\beta - n\epsilon)(2-p_2)}{2}} + C\Big[\Big(\frac{\rho}{r}\Big)^n + r^{\frac{(\beta - n\epsilon)p_2}{2}}\Big]\phi(r). \tag{4.30}$$

Now since $r < 1$, we deduce from (4.28) and (4.30) that we have in both cases for each $\rho \in (0, r]$

$$\phi(\rho) \le C r^{n - (\beta - n\epsilon)} + C\Big[\Big(\frac{\rho}{r}\Big)^n + r^{\frac{(\beta - n\epsilon)}{2}}\Big]\phi(r). \tag{4.31}$$

Using Lemma 2.7 of [6], we conclude that for each $0 < \lambda < n - (n - (\beta - n\epsilon)) = \beta - n\epsilon$, there exists $r_\lambda > 0$ and $B = B(C, \lambda, n) > 0$ such that for each $r \le r_\lambda$, we have

$$\begin{aligned} \forall \rho \in (0, r) \quad \int_{B_\rho} |\nabla u|^{p_2(\rho)} dx &\le (1 + C)\Big(\frac{\rho}{r}\Big)^{n - (\beta - n\epsilon) - \lambda} \int_{B_r} |\nabla u|^{p_2(r)} dx + B\rho^{n - (\beta - n\epsilon)} \\ &= (1 + C)\Big(\frac{\rho}{r}\Big)^{n - \beta + n\epsilon - \lambda} \int_{B_r} |\nabla u|^{p_2(r)} dx + B\rho^{n - \beta + n\epsilon}. \end{aligned}$$

Taking $0 < \lambda < \min(\beta - n\epsilon, n\epsilon)$ and remarking that

$$\begin{aligned} \int_{B_\rho} |\nabla u|^{p_-} dx &= \int_{B_\rho \cap [|\nabla u| \le 1]} |\nabla u|^{p_-} dx + \int_{B_\rho \cap [|\nabla u| > 1]} |\nabla u|^{p_-} dx \\ &\le C(n)\rho^n + \int_{B_\rho} |\nabla u|^{p_2(\rho)} dx, \end{aligned}$$

we obtain for some constant C possibly depending on r, but independent on ρ

$$\forall \rho \in (0, r) \quad \int_{B_\rho} |\nabla u|^{p_-} dx \le C\rho^{n - \beta}. \tag{4.32}$$

Now let $\alpha \in (0, 1)$ and notice that since p is β-Hölder continuous, it is also γ-Hölder continuous for each $\gamma \le \beta$. We can therefore assume that $\beta < p_-(1 - \alpha)$, which leads to

$$\rho^{n - \beta} = \rho^{n - p_- + \alpha p_- + p_-(1 - \alpha) - \beta} \le \rho^{n - p_- + \alpha p_-} \quad \forall \rho \in (0, r).$$

Hence (4.32) becomes

$$\int_{B_\rho} |\nabla u|^{p_-} dx \le C\rho^{n - p_- + \alpha p_-} \quad \forall \rho \in (0, r). \tag{4.33}$$

Using (4.33) and Hölder's inequality, we obtain for some constant C independent on ρ

$$\begin{aligned} \forall \rho \in (0, r) \quad \int_{B_\rho} |\nabla u| dx &\le |B_\rho|^{1 - \frac{1}{p_-}} \Big(\int_{B_\rho} |\nabla u|^{p_-} dx\Big)^{\frac{1}{p_-}} \\ &\le C\rho^{n - \frac{n}{p_-}} \rho^{\frac{n - p_- + \alpha p_-}{p_-}} \\ &\le C\rho^{n - 1 + \alpha}. \end{aligned}$$

We deduce ([18] Theorem 1.53 (Morrey) p. 30) that $u \in C^{0,\alpha}_{loc}(\Omega)$, and this holds for any $\alpha \in (0, 1)$. □

Remark 4.1 Consider the functional

$$J(u) = \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + Q_1(x)\chi_{[u<0]} + Q_2(x)\chi_{[u>0]} \right) dx$$

with Q_1 and Q_2 satisfying the same assumption as Q. For each $g \in W^{1,p(x)}(\Omega)$, let $\mathcal{S}(g,\Omega)$ be the set of all minimizers u of J under the condition $u - g \in W_0^{1,p(x)}(\Omega)$. Then it is not difficult to extend most of the results of the sections 2, 3 and 4 in the following way

i) $\mathcal{S}(g,\Omega) \neq \emptyset$.

ii) If $|g|_\infty \leq M$, then $\quad |u|_\infty \leq M \quad \forall u \in \mathcal{S}(g,\Omega)$.

iii) $\forall u \in \mathcal{S}(g,\Omega) \quad \Delta_{p(x)} u = 0 \quad$ in $[u \neq 0]$.

iv) If $p, Q_1, Q_2 \in C^1(\Omega)$, then we have for each $u \in \mathcal{S}(g,\Omega)$ and $\eta \in \mathcal{D}(\Omega, \mathbb{R}^n)$

$$\lim_{\epsilon \to 0} \int_{\partial[u<\epsilon]} \Big(\frac{p(x)-1}{p(x)} |\nabla u|^{p(x)} - Q_1 \Big) \eta \cdot \nu d\sigma(x) = 0$$
$$\lim_{\epsilon \to 0} \int_{\partial[u>\epsilon]} \Big(\frac{p(x)-1}{p(x)} |\nabla u|^{p(x)} - Q_2 \Big) \eta \cdot \nu d\sigma(x) = 0.$$

v) If $p \in C^{0,\beta}(\Omega)$, then we have $\mathcal{S}(g,\Omega) \subset C^{0,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$.

5 Lipschitz continuity

The main result of this section is the Lipschitz Continuity of the minimizers.

Theorem 5.1 *We have* $\mathcal{S}(g,\Omega) \subset C^{0,1}_{loc}(\Omega)$.

Lemma 5.1 *If u is a minimizer in $B_r(x_0)$ of the functional*

$$J(u) = \int_{B_r(x_0)} \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + Q(x)\chi_{[u>0]} \right) dx,$$

then v defined by $v(y) = \frac{u(x_0+ry)}{r}$ is a minimizer in B_1 for the functional

$$\widetilde{J}(v) = \int_{B_1} \left(\frac{1}{\widetilde{p}(y)} |\nabla v|^{\widetilde{p}(y)} + \widetilde{Q}(y)\chi_{[v>0]} \right) dy$$

where $\widetilde{p}(y) = p(x_0 + ry)$ and $\widetilde{Q}(y) = Q(x_0 + ry)$.

Proof. Indeed let $\widetilde{w} \in W^{1,\widetilde{p}(y)}(B_1)$ be such that $\widetilde{w} = v$ on ∂B_1. Then w defined by $w(x) = r\widetilde{w}(\frac{x-x_0}{r})$

belongs to $W^{1,p(x)}(B_r(x_0))$ and satisfies $w = u$ on $\partial B_r(x_0)$. Moreover we have

$$\begin{aligned}
\widetilde{J}(v) &= \int_{B_1}\left(\frac{1}{\widetilde{p}(y)}|\nabla v|^{\widetilde{p}(y)} + \widetilde{Q}(y)\chi_{[v>0]}\right)dy \\
&= \int_{B_1}\left(\frac{1}{p(x_0+ry)}|\nabla u(x_0+ry)|^{p(x_0+ry)} + Q(x_0+ry)\chi_{[u(x_0+ry)>0]}\right)dy \\
&= r^{-n}\int_{B_r(x_0)}\left(\frac{1}{p(x)}|\nabla u(x)|^{p(x)} + Q(x)\chi_{[u>0]}\right)dx \\
&\leq r^{-n}\int_{B_r(x_0)}\left(\frac{1}{p(x)}|\nabla w(x)|^{p(x)} + Q(x)\chi_{[w>0]}\right)dx \\
&= \int_{B_1}\left(\frac{1}{\widetilde{p}(y)}|\nabla \widetilde{w}|^{\widetilde{p}(y)} + \widetilde{Q}(y)\chi_{[\widetilde{w}>0]}\right)dy = \widetilde{J}(\widetilde{w}).
\end{aligned}$$

□

Lemma 5.2 *There exists a constant depending only on n, p_-, p_+, β, L, M, and Q_+ such that for each $u \in \mathcal{S}(g,\Omega)$ and each ball $B_{2r}(x_0) \subset\subset \Omega$ with $u(x_0) = 0$, we have*

$$\max_{\overline{B}_{r/3}(x_0)} u \leq Cr.$$

Proof. First note that since u is bounded and continuous in $\overline{B}_{r/3}(x_0)$, u achieves its maximum on the compact set $\overline{B}_{r/3}(x_0)$. Now to prove Lemma 5.2, we argue by contradiction as in [9] and [17]. However we don't work on the unit ball because otherwise, the constant in the Harnack inequality (5.8) may depend on the L^∞ norm of u_k. We point out that instead of the Harnack inequality $\max_{\overline{B}_R} u \leq C\min_{\overline{B}_R} u$ satisfied by nonnegative $p-$Harmonic functions when p is constant, we only have the inequality $\max_{\overline{B}_R} u \leq C\big(\min_{\overline{B}_R} u + R\big)$ for nonnegative $p(x)-$Harmonic functions (see [3]).

Assume that there exists a sequence $(u_k)_k \in \mathcal{S}(g,\Omega)$, a sequence $(r_k)_k$ of real numbers and a sequence of points $(x_{0k})_k$ such that

$$\forall k \geq 1 \quad B_{2r_k}(x_{0k}) \subset\subset \Omega, \quad u_k(x_{0k}) = 0 \quad \text{and} \quad \max_{\overline{B}_{r_k/3}(x_{0k})} u_k > kr_k. \tag{5.1}$$

Let

$$d_k(x) = d\big(x, [u_k = 0] \cap \overline{B}_{r_k}(x_{0k})\big) \quad \text{and} \quad F_k = \left\{x \in \overline{B}_{r_k}(x_{0k}) \ / \ d_k(x) \leq \frac{r_k - |x - x_{0k}|}{2}\right\}.$$

Because the function $x \to \left(\frac{r_k-|x-x_{0k}|}{r_k}\right)u_k(x)$ is continuous in $\overline{B}_{r_k}(x_{0k})$, it achieves its maximum M_k on the compact set F_k. Hence there exists $x_k \in F_k$ such that

$$M_k = \frac{r_k - |x_k - x_{0k}|}{r_k}u_k(x_k).$$

We claim that

$$\forall k \geq 1 \quad u_k(x_k) > \frac{2}{3}kr_k. \tag{5.2}$$

First remark that $\overline{B}_{r_k/3}(x_{0k}) \subset F_k$. Indeed if $|x - x_{0k}| \leq r_k/3$, then we have

$$\frac{r_k - |x - x_{0k}|}{2} \geq \frac{r_k}{3} \geq |x - x_{0k}| \geq d_k(x).$$

It follows by (5.1) that

$$M_k = \max_{x \in F_k} \Big(\frac{r_k - |x - x_{0k}|}{r_k}\Big) u_k(x) \geq \max_{x \in \overline{B}_{r_k/3}(x_{0k})} \Big(\frac{r_k - |x - x_{0k}|}{r_k}\Big) u_k(x) > \frac{2}{3} k r_k.$$

We deduce that we have necessarily $|x_k - x_{0k}| < r_k$ and then

$$u_k(x_k) = \frac{r_k M_k}{r_k - |x_k - x_{0k}|} \geq M_k > \frac{2}{3} k r_k.$$

Let now $\delta_k = d_k(x_k) = d\Big(x_k, [u_k = 0] \cap \overline{B}_{r_k}(x_{0k})\Big) = |x_k - y_k|$ for some $y_k \in [u_k = 0] \cap \overline{B}_{r_k}(x_{0k})$. Clearly we have $y_k \in (\partial[u_k > 0]) \cap \overline{B}_{r_k}(x_{0k})$. Since $x_k \in F_k$, we have

$$\delta_k \leq \frac{r_k - |x_k - x_{0k}|}{2} \iff |x_k - x_{0k}| \leq r_k - 2\delta_k. \tag{5.3}$$

Moreover we claim that

$$B_{\delta_k}(y_k) \subset B_{r_k}(x_{0k}). \tag{5.4}$$

$$B_{\frac{\delta_k}{3}}(y_k) \subset F_k. \tag{5.5}$$

$$r_k - |y - x_{0k}| \geq \frac{r_k - |x_k - x_{0k}|}{3} \quad \forall y \in B_{\frac{\delta_k}{3}}(y_k). \tag{5.6}$$

Indeed

(i) If $y \in B_{\delta_k}(y_k)$, then we have by the triangle inequality and (5.3)

$$|y - x_{0k}| \leq |y - y_k| + |y_k - x_k| + |x_k - x_{0k}| < 2\delta_k + |x_k - x_{0k}| \leq r_k.$$

(ii) If $y \in B_{\frac{\delta_k}{3}}(y_k)$, then we have again by the triangle inequality and (5.3)

$$|y - x_{0k}| \leq |y - y_k| + |y_k - x_k| + |x_k - x_{0k}| < \frac{4\delta_k}{3} + |x_k - x_{0k}| \leq r_k - \frac{2\delta_k}{3}$$

which leads to

$$\frac{r_k - |y - x_{0k}|}{2} > \frac{\delta_k}{3} > |y - y_k| \geq d_k(y).$$

Hence $y \in F_k$.

(iii) Let $y \in B_{\frac{\delta_k}{3}}(y_k)$. In ii) we established that $|y - x_{0k}| \leq \frac{4}{3}\delta_k + |x_k - x_{0k}|$. So by (5.3)

$$r_k - |y - x_{0k}| > r_k - |x_k - x_{0k}| - \frac{4}{3}\delta_k \geq r_k - |x_k - x_{0k}| - \frac{4}{3}\frac{r_k - |x_k - x_{0k}|}{2} = \frac{r_k - |x_k - x_{0k}|}{3}.$$

Using (5.5) and (5.6), we deduce that

$$\begin{aligned}\max_{x\in F_k}\Big(\frac{r_k-|x-x_{0k}|}{r_k}\Big)u_k(x) &\geq \max_{x\in \overline{B}_{\frac{\delta_k}{3}}(y_k)}\Big(\frac{r_k-|x-x_{0k}|}{r_k}\Big)u_k(x)\\ &\geq \frac{r_k-|x_k-x_{0k}|}{3r_k}\max_{x\in \overline{B}_{\frac{\delta_k}{3}}(y_k)} u_k(x)\end{aligned}$$

which leads to $\Big(\frac{r_k-|x_k-x_{0k}|}{r_k}\Big)u_k(x_k)\geq \frac{r_k-|x_k-x_{0k}|}{3r_k}\max_{x\in \overline{B}_{\frac{\delta_k}{3}}(y_k)} u_k(x)$, or

$$\max_{\overline{B}_{\frac{\delta_k}{3}}(y_k)} u_k(x) \leq 3u_k(x_k). \tag{5.7}$$

Now since $B_{\delta_k}(x_k)\subset[u_k>0]$, we have $\Delta_{p(x)}\, u_k=0$ in $B_{\delta_k}(x_k)$. Moreover since $\delta_k\leq r_k/2$, we have $\frac{10\delta_k}{6}\leq\frac{5r_k}{6}$ and $\overline{B}_{\frac{10\delta_k}{6}}(x_k)\subset\overline{B}_{\frac{11r_k}{6}}(x_{0k})\subset\overline{B}_{2r_k}(x_{0k})\subset\Omega$. Applying the Harnack inequality in [3], we get for a positive constant C depending only on n, p and M

$$\max_{\overline{B}_{\frac{5\delta_k}{6}}(x_k)} u_k\leq C\Big(\min_{\overline{B}_{\frac{5\delta_k}{6}}(x_k)} u_k+\frac{5\delta_k}{6}\Big). \tag{5.8}$$

Since

$$\overline{B}_{\frac{5\delta_k}{6}}(x_k)\cap\overline{B}_{\frac{\delta_k}{6}}(y_k)\neq\phi,\ \exists\, x_*\in\overline{B}_{\frac{5\delta_k}{6}}(x_k)\cap\overline{B}_{\frac{\delta_k}{6}}(y_k),$$

and we obtain from (5.8) and (5.2)

$$\begin{aligned}u_k(x_*)\geq\min_{\overline{B}_{\frac{5\delta_k}{6}}(x_k)} u_k &\geq C^{-1}\max_{\overline{B}_{\frac{5\delta_k}{6}}(x_k)} u_k-\frac{5\delta_k}{6}\\ &\geq C^{-1}\,u_k(x_k)-\frac{5\delta_k}{6}\\ &= \frac{C^{-1}}{2}u_k(x_k)+\frac{C^{-1}}{2}u_k(x_k)-\frac{5\delta_k}{6}\\ &\geq \frac{1}{2C}\,u_k(x_k)+\frac{1}{3C}\,kr_k-\frac{5\delta_k}{6}\\ &\geq \frac{1}{2C}\,u_k(x_k)\ \text{ for } k \text{ large enough since } \delta_k\leq r_k.\end{aligned}$$

It follows that

$$\max_{\overline{B}_{\frac{\delta_k}{6}}(y_k)} u_k\geq u_k(x_*)\geq\frac{1}{2C}u_k(x_k). \tag{5.9}$$

Let now $v_k(x)=\dfrac{u_k\Big(y_k+\frac{\delta_k}{6}x\Big)}{u_k(x_k)}$. Then $v_k(0)=0$ and we have by (5.7) and (5.9)

$$\max_{\overline{B}_1} v_k\geq\frac{1}{2C}\ \text{ and }\ \max_{\overline{B}_2} v_k\leq 3. \tag{5.10}$$

Let $\epsilon_k = \frac{\delta_k}{6u_k(x_k)}$. We have by (5.2), since $\delta_k \le r_k$,

$$0 < \epsilon_k < \frac{\delta_k}{6}.\frac{3}{2kr_k} = \frac{1}{4k}.\frac{\delta_k}{r_k} \le \frac{1}{4k} \to 0,\ k \to \infty.$$

From Lemma 5.1, we know that $\dfrac{1}{\epsilon_k}\, v_k$ is a minimizer of

$$J_k(v) = \int_{B_2} \frac{1}{p_k(x)} |\nabla v|^{p_k(x)} + Q_k(x)\chi_{[v>0]},$$

over all functions of $W^{1,\,p_k(x)}(B_2)$ with $v - \frac{1}{\epsilon_k} v_k \in W_0^{1,\,p_k(x)}(B_2)$, where

$$p_k(x) = p(y_k + \frac{\delta_k}{6} x) \quad \text{and} \quad Q_k(x) = Q(y_k + \frac{\delta_k}{6} x).$$

Now let w_k be defined by

$$\begin{cases} \Delta_{p_k(x)} w_k = 0 \quad \text{in} \quad B_2 \\ w_k - v_k \in W_0^{1,\,p_k(x)}(B_2). \end{cases}$$

Note that since $0 \le v_k \le 3$, we easily derive that $0 \le w_k \le 3$. We deduce that $w_k \in C^{1,\gamma}_{loc}(B_2)$ for some $\gamma \in (0,1)$ and $|w_k|_{1,\gamma,B_1} \le C(p_-, p_+, n, L, \beta)$ (see [7], [10]). In particular we have, up to a subsequence,

$$w_k \longrightarrow w \quad \text{in} \quad C^1(\overline{B_1}). \tag{5.11}$$

Since $\frac{1}{\epsilon_k} v_k - \frac{1}{\epsilon_k} w_k \in W_0^{1,\,p_k(x)}(B_2)$, we have

$$J_k\left(\frac{1}{\epsilon_k} v_k\right) \le J_k\left(\frac{1}{\epsilon_k} w_k\right)$$

or

$$\int_{B_2} \frac{1}{\epsilon_k^{p_k(x)}} \frac{1}{p_k(x)} \left(|\nabla v_k|^{p_k(x)} - |\nabla w_k|^{p_k(x)}\right) dx \le \int_{B_2} Q_k(x) \left(\chi_{[w_k>0]} - \chi_{[v_k>0]}\right) dx$$

Since $\epsilon_k \to 0$, we can assume that $\frac{1}{\epsilon_k^{p_k(x)}} \ge \frac{1}{\epsilon_k^{p-}}$, which leads to

$$\int_{B_2} \frac{1}{p_k(x)} \left(|\nabla v_k|^{p_k(x)} - |\nabla w_k|^{p_k(x)}\right) dx \le \int_{B_2} \epsilon_k^{p-}\, Q_k(x) dx. \tag{5.12}$$

Note that for $v_k^s = s v_k + (1-s) w_k$, $s \in [0,1]$, we have

$$\begin{aligned} \int_{B_2} \frac{1}{p_k(x)} \left(|\nabla v_k|^{p_k(x)} - |\nabla w_k|^{p_k(x)}\right) dx &= \int_0^1 ds \int_{B_2} |\nabla v_k^s|^{p_k(x)-2} \nabla v_k^s \cdot \nabla (v_k - w_k)\, dx \\ &= \int_0^1 ds \int_{B_2} \left(\left|\nabla v_k^s\right|^{p_k(x)-2} \nabla v_k^s - |\nabla w_k|^{p_k(x)-2} \nabla w_k\right) \cdot \nabla (v_k - w_k)\, dx \\ &= \int_0^1 \frac{ds}{s} \int_{B_2} \left(\left|\nabla v_k^s\right|^{p_k(x)-2} \nabla v_k^s - |\nabla w_k|^{p_k(x)-2} \nabla w_k\right) \cdot \nabla \left(v_k^s - w_k\right) dx. \end{aligned}$$

Let us recall the following well know inequalities for $q > 1$ and $\xi, \zeta \in \mathbb{R}^n$

$$(|\xi|^{q-2}\xi - |\zeta|^{q-2}\zeta).(\xi - \zeta) \geq c(n,q) \begin{cases} |\xi - \zeta|^q & \text{if} \quad q \geq 2 \\ |\xi - \zeta|^2(|\xi| + |\zeta|)^{q-2} & \text{if} \quad 1 < q < 2. \end{cases} \tag{5.13}$$

In particular we deduce from (5.13) the monotonicity of the operator $|\xi|^{q-2}\xi$. Denoting by E_k the set $\{x \in B_1 \ : \ p_k(x) \geq 2\}$, we obtain

$$\begin{aligned} &\int_{B_2} \frac{1}{p_k(x)}(|\nabla v_k|^{p_k(x)} - |\nabla w_k|^{p_k(x)})dx \\ &\geq \int_0^1 \frac{ds}{s} \int_{B_1} (|\nabla v_k^s|^{p_k(x)-2}\nabla v_k^s - |\nabla w_k|^{p_k(x)-2}\ \nabla w_k) \cdot \nabla(v_k^s - w_k)dx \\ &= \int_0^1 \frac{ds}{s} \int_{E_k} (|\nabla v_k^s|^{p_k(x)-2}\nabla v_k^s - |\nabla w_k|^{p_k(x)-2}\ \nabla w_k) \cdot \nabla(v_k^s - w_k)dx \\ &+ \int_0^1 \frac{ds}{s} \int_{B_1 \setminus E_k} (|\nabla v_k^s|^{p_k(x)-2}\nabla v_k^s - |\nabla w_k|^{p_k(x)-2}\ \nabla w_k) \cdot \nabla(v_k^s - w_k)dx. \end{aligned} \tag{5.14}$$

Using the inequalities (5.13), we get

$$\begin{aligned} &\int_0^1 \frac{ds}{s} \int_{E_k} (|\nabla v_k^s|^{p_k(x)-2}\nabla v_k^s - |\nabla w_k|^{p_k(x)-2}\ \nabla w_k) \cdot \nabla(v_k^s - w_k)dx \\ &\geq C(n,p) \int_0^1 \frac{ds}{s} \int_{E_k} |\nabla(v_k^s - w_k)|^{p_k(x)}dx. \end{aligned} \tag{5.15}$$

Setting $q_k(x) = \frac{p_k(x)}{p_k(x)-1}$, $V_k = |\nabla v_k^s|^{p_k(x)-2}\nabla v_k^s$ and $W_k = |\nabla w_k|^{p_k(x)-2}\nabla w_k$, it is easy to see that we have $\nabla v_k^s = |V_k|^{q_k(x)-2}V_k$ and $\nabla w_k = |W_k|^{q_k(x)-2}W_k$.

Since $q_k(x) > 2$ in $B_1 \setminus E_k$, we obtain by using again the inequalities (5.13)

$$\begin{aligned} &\int_0^1 \frac{ds}{s} \int_{B_1 \setminus E_k} (|\nabla v_k^s|^{p_k(x)-2}\nabla v_k^s - |\nabla w_k|^{p_k(x)-2}\ \nabla w_k) \cdot \nabla(v_k^s - w_k)dx \\ &= \int_0^1 \frac{ds}{s} \int_{B_1 \cap [q_k(x) > 2]} (|V_k|^{q_k(x)-2}V_k - |W_k|^{q_k(x)-2}W_k).(V_k - W_k)dx \\ &\geq C(n,p) \int_0^1 \frac{ds}{s} \int_{B_1 \setminus E_k} |V_k - W_k|^{q_k(x)}dx \\ &= C(n,p) \int_0^1 \frac{ds}{s} \int_{B_1 \setminus E_k} ||\nabla v_k^s|^{p_k(x)-2}\nabla v_k^s - |\nabla w_k|^{p_k(x)-2}\nabla w_k|^{q_k(x)}dx. \end{aligned} \tag{5.16}$$

Combining (5.12) and (5.14)-(5.16), we obtain

$$\begin{aligned} &C(n,p) \int_0^1 \frac{ds}{s} \int_{E_k} |\nabla(v_k^s - w_k)|^{p_k(x)}dx \\ &+C(n,p) \int_0^1 \frac{ds}{s} \int_{B_1 \setminus E_k} \left||\nabla v_k^s|^{p_k(x)-2}\nabla v_k^s - |\nabla w_k|^{p_k(x)-2}\ \nabla w_k\right|^{q_k(x)}dx \leq C(n,Q_+)\epsilon_k^{p-}. \end{aligned} \tag{5.17}$$

Using the fact that ∇w_k is uniformly bounded in B_1, the convexity of $t^{p_k(x)}$ and $t^{q_k(x)}$, and since $\frac{1}{s} \geq 1$, we deduce from (5.17) that

$$\int_0^1 \int_{E_k} |\nabla v_k^s|^{p_k(x)} dx + \int_0^1 \int_{B_1 \setminus E_k} \left| |\nabla v_k^s|^{p_k(x)-2} \nabla v_k^s \right|^{q_k(x)} dx \leq C(n, p, Q_+) \epsilon_k^{p-}$$

$$\text{or} \quad \int_0^1 ds \int_{B_1} |\nabla v_k^s|^{p_k(x)} dx \leq C \quad \text{since } \left| |\nabla v_k^s|^{p_k(x)-2} \nabla v_k^s \right|^{q_k(x)} = |\nabla v_k^s|^{p_k(x)}.$$

Since $v_k^s - (1-s)w_k = sv_k$, we have

$$\begin{aligned}
\int_0^1 s^{p_k(x)} ds \int_{B_1} |\nabla v_k|^{p_k(x)} dx &= \int_0^1 ds \int_{B_1} |s \nabla v_k|^{p_k(x)} dx \\
&= \int_0^1 ds \int_{B_1} |\nabla v_k^s - (1-s) \nabla w_k|^{p_k(x)} dx \\
&\leq \int_0^1 ds \int_{B_1} 2^{p_k(x)-1} (|\nabla v_k^s|^{p_k(x)} + |\nabla w_k|^{p_k(x)}) dx \quad \text{convexity of } t^{p_k(x)} \\
&\leq 2^{p_+-1} \int_0^1 ds \int_{B_1} |\nabla v_k^s|^{p_k(x)} dx + 2^{p_+-1} \int_0^1 ds \int_{B_1} |\nabla w_k|^{p_k(x)}) dx.
\end{aligned}$$

Then we get for some constant C

$$\int_{B_1} |\nabla v_k|^{p_k(x)} dx \leq C. \tag{5.18}$$

Using again (5.12), (5.14), and the inequalities (5.13), we get

$$\begin{aligned}
&\int_0^1 \frac{ds}{s} \int_{E_k} |\nabla (v_k^s - w_k)|^{p_k(x)} dx \\
&\quad + \int_0^1 \frac{ds}{s} \int_{B_1 \setminus E_k} |\nabla (v_k^s - w_k)|^2 (|\nabla v_k^s| + |\nabla w_k|)^{p_k(x)-2} dx \leq C(n, p, Q_+) \epsilon_k^{p-}.
\end{aligned} \tag{5.19}$$

Note that since $|\nabla v_k^s| \leq |\nabla v_k| + |\nabla w_k|$, we have for $x \in B_1 \setminus E_k$

$$\begin{aligned}
(|\nabla v_k^s| + |\nabla w_k|)^{p_k(x)-2} &\geq (|\nabla v_k| + 2|\nabla w_k|)^{p_k(x)-2} \\
&\geq 2^{p_k(x)-2} (|\nabla v_k| + |\nabla w_k|)^{p_k(x)-2} \\
&\geq 2^{p_--2} (|\nabla v_k| + |\nabla w_k|)^{p_k(x)-2}.
\end{aligned} \tag{5.20}$$

Moreover one has $v_k^s - w_k = s(v_k - w_k)$. It follows then from (5.19)-(5.20) that

$$\begin{aligned}
&\int_0^1 s^{p_k(x)-1} ds \int_{E_k} |\nabla (v_k - w_k)|^{p_k(x)} dx \\
&\quad + \int_0^1 s\, ds \int_{B_1 \setminus E_k} |\nabla (v_k - w_k)|^2 (|\nabla v_k| + |\nabla w_k|)^{p_k(x)-2} dx \leq C(n, p, Q_+) \epsilon_k^{p-}
\end{aligned}$$

and then

$$\int_{E_k} |\nabla(v_k - w_k)|^{p_k(x)}\, dx + \int_{B_1\setminus E_k} |\nabla(v_k - w_k)|^2 \left(|\nabla v_k| + |\nabla w_k|\right)^{p_k(x)-2}\, dx \le C(n, p, Q_+)\epsilon_k^{p-}. \tag{5.21}$$

Let $\alpha = p_-/2$ and

$$E_k^1 = \left\{ x \in B_1 \setminus E_k \ : \ |\nabla(v_k - w_k)(x)| \le \epsilon_k^{\frac{\alpha}{2-p_k(x)}} \left(|\nabla v_k(x)| + |\nabla w_k(x)|\right) \right\}$$

$$E_k^2 = \left\{ x \in B_1 \setminus E_k \ : \ |\nabla(v_k - w_k)(x)| > \epsilon_k^{\frac{\alpha}{2-p_k(x)}} \left(|\nabla v_k(x)| + |\nabla w_k(x)|\right) \right\}$$

Note that

$$\begin{aligned} x \in E_k^2 &\Rightarrow |\nabla(v_k - w_k)(x)|^{2-p_k(x)} > \epsilon_k^{\alpha}\left(|\nabla v_k(x)| + |\nabla w_k(x)|\right)^{2-p_k(x)} \\ &\Rightarrow |\nabla(v_k - w_k)(x)|^{p_k(x)} < \epsilon_k^{-\alpha}|\nabla(v_k - w_k)(x)|^2 \left(|\nabla v_k(x)| + |\nabla w_k(x)|\right)^{|p_k(x)-2}. \end{aligned}$$

Hence we get from (5.21)

$$\int_{E_k^2} |\nabla(v_k - w_k)|^{p_k(x)} dx \le C\epsilon_k^{p_- - \alpha} = C\epsilon_k^{p_-/2}. \tag{5.22}$$

$$x \in E_k^1 \Rightarrow |\nabla(v_k - w_k)(x)|^{p_k(x)} \le \epsilon_k^{\frac{\alpha p_k(x)}{2-p_k(x)}} \left(|\nabla v_k(x)| + |\nabla w_k(x)|\right)^{p_k(x)}$$

$$\left.\begin{array}{c} \alpha p_k(x) \ge \alpha p_- = p_-^2/2 \\ 2 - p_k(x) \le 2 - p_- \Rightarrow \frac{1}{2-p_k(x)} \ge \frac{1}{2-p_-} \end{array}\right\} \Rightarrow \frac{\alpha p_k(x)}{2 - p_k(x)} \ge \frac{p_-^2}{2(2 - p_-)}.$$

Since $0 < \epsilon_k < 1$, we have

$$\epsilon_k^{\frac{\alpha p_k(x)}{2-p_k(x)}} \le \epsilon_k^{\frac{p_-^2}{2(2-p_-)}}.$$

Then we obtain by using (5.18) and the fact that ∇w_k is uniformly bounded in B_1

$$\begin{aligned} \int_{E_k^1} |\nabla(v_k - w_k)|^{p_k(x)} dx &\le \epsilon_k^{\frac{p_-^2}{2(2-p_-)}} \int_{E_k^1} \left(|\nabla v_k| + |\nabla w_k|\right)^{p_k(x)} dx \\ &\le 2^{p_+-1}\epsilon_k^{\frac{p_-^2}{2(2-p_-)}} \int_{E_k^1} \left(|\nabla v_k|^{p_k(x)} + |\nabla w_k|^{p_k(x)}\right) dx \\ &\le 2^{p_+-1}\epsilon_k^{\frac{p_-^2}{2(2-p_-)}} \int_{B_1} \left(|\nabla v_k|^{p_k(x)} + |\nabla w_k|^{p_k(x)}\right) dx \\ &\le C\epsilon_k^{\frac{p_-^2}{2(2-p_-)}}. \end{aligned} \tag{5.23}$$

We deduce from (5.22) and (5.23) that

$$\int_{B_1\setminus E_k} |\nabla(v_k - w_k)|^{p_k(x)} dx \le C\Big(\epsilon_k^{p-/2} + \epsilon_k^{\frac{p_-^2}{2(2-p_-)}}\Big).$$

Since $\epsilon_k < 1$, we obtain by taking into account (5.21)

$$\int_{B_1} |\nabla(v_k - w_k)|^{p_k(x)} dx \leq C\epsilon_k^{p_-/2}.$$

This leads by Poincaré's inequality and (5.11) to

$$v_k \longrightarrow w \quad \text{in} \quad W^{1,p_-}(B_1). \tag{5.24}$$

Note that we have from (5.1) and (5.3), $\delta_k \leq r_k \leq \frac{M}{k}$. We deduce that $\delta_k \to 0$ as $k \to \infty$, and we get from (5.10)-(5.11)

$$\begin{cases} \Delta_{p_0} w = 0 & \text{in} \quad B_1 \\ \frac{1}{2C} \leq \max_{\overline{B_1}} w \leq 3 & \text{in} \quad B_1 \end{cases}$$

where $p_0 = \lim_{k\to\infty} p_k(x) = \lim_{k\to\infty} p(y_k + \dfrac{\delta_k}{6} x) = p(y_*)$, and where up to a subsequence $y_* = \lim_{k\to\infty} y_k$. Now since $v_k - w_k \in W^{1,1}(B_1) \cap C^{0,\alpha}(\overline{B_1})$, we can apply Lemma 1.50 of [18] p. 29. We get since $v_k(0) = 0$

$$0 \leq w_k(0) \leq \frac{1}{|B_1|} \int_{B_1} |v_k - w_k| dx + C(n) \int_{B_1} \frac{|\nabla(v_k - w_k)|}{|x|^{n-1}} dx.$$

Letting $k \to \infty$ and using (5.11) and (5.24), we obtain $w(0) = 0$. Given that w is nonnegative and p_0-Harmonic in B_1, we deduce from the strong maximum principle (see [19]) that $w \equiv 0$ in B_1. But this contradicts the fact that $\max_{\overline{B_1}} w \geq \frac{1}{2C}$. □

Proof of Theorem 5.1. Let $\Omega_\epsilon = \{x \in \Omega \,/\, d(x, \partial\Omega) > \epsilon\}$. We shall prove that for $0 < \epsilon < 1$ small enough, ∇u is bounded in $\Omega_{8\epsilon}$ by a constant depending only on n, p_-, p_+, M, L, β and ϵ. Let $x_0 \in \Omega_{8\epsilon}$. We distinguish two cases :

i) $\underline{B_\epsilon(x_0) \subset [u > 0]}$: Let v be defined in B_1 by $v(y) = \frac{u(x_0+\epsilon y)}{\epsilon}$. We easily verify that v satisfies $\Delta_{q(y)} v = 0$ in B_1, with $q(y) = p(x_0 + \epsilon y)$ satisfying (1.1) and (4.1) with the same constants p_-, p_+, β and L. Moreover v is uniformly bounded by $\frac{M}{\epsilon}$ in B_1. We deduce (see [7], [10]) that we have for a positive constant $C = C(n, p_-, p_+, M/\epsilon, L, \beta)$

$$\sup_{B_{1/2}} |\nabla v| \leq C$$

which leads to

$$|\nabla u(x_0)| \leq \sup_{B_{\epsilon/2}(x_0)} |\nabla u| \leq C.$$

ii) $\underline{B_\epsilon(x_0) \cap [u = 0] \neq \emptyset}$: Assume that $u(x_0) > 0$ and let $r_0 = d(x_0, [u = 0])$ be the distance between x_0 and the set $[u = 0]$. Clearly we have $B_{r_0}(x_0) \subset [u > 0]$. Moreover we have $r_0 \leq \epsilon$. Now let $x_1 \in \partial B_{r_0}(x_0) \cap [u = 0]$. Then we have for each $x \in B_{6r_0}(x_1)$

$$8\epsilon < d(x_0, \partial\Omega) \leq |x_0 - x_1| + |x_1 - x| + d(x, \partial\Omega) < 7\epsilon + d(x, \partial\Omega)$$

which leads to

$$d(x, \partial\Omega) > \epsilon \quad \forall x \in B_{6r_0}(x_1).$$

Hence

$$B_{6r_0}(x_1) \subset \Omega_\epsilon.$$

It follows from Lemma 5.2 that we have for some positive constant C_0 depending only on n, p_-, p_+, L, β, Q_+ and M

$$\max_{\overline{B}_{r_0}(x_1)} u \le C_0 r_0.$$

Consequently the function defined in B_1 by

$$v(y) = \frac{u(x_0 + r_0 y)}{r_0}$$

is uniformly bounded by C_0 in B_1. Moreover, it satisfies

$$\Delta_{q(y)} v = 0 \quad \text{in} \quad B_1, \quad \text{with} \quad q(y) = p(x_0 + r_0 y).$$

Obviously the function $q(y)$ satisfies (1.1) and (4.1) with the same constants p_-, p_+, β and L. We then deduce (see [7], [10]) that we have for a positive constant $C = C(n, p_-, p_+, M, L, \beta, Q_+)$

$$\sup_{B_{1/2}} |\nabla v| \le C$$

which leads to

$$|\nabla u(x_0)| \le \sup_{B_{r_0/2}(x_0)} |\nabla u| \le C.$$

Since $\nabla u(x) = 0$ a.e. in $\Omega_{8\epsilon} \cap [u = 0]$, it follows that ∇u is uniformly bounded in $\Omega_{8\epsilon}$. □

6 Nondegeneracy and Lebesgue measure of the free boundary

In this section, we assume that the constant Q_- in (1.3) is positive. We prove the nondegeneracy of the minimizers at their free boundaries and local uniform positive density of the sets $[u > 0]$ and $[u = 0]$. As a consequence we obtain that the free Boundary $(\partial[u > 0]) \cap \Omega$ has Lebesgue measure zero.

Lemma 6.1 *Let $u \in \mathcal{S}(g, \Omega)$, $D \subset\subset \Omega$ be a domain and C a Lipschitz constant of u over $\overline{D}$. If $c_1 > 2C$, $B_r \subset D$, then we have*

$$\max_{\overline{B}_r} u \ge c_1 r \quad \Rightarrow \quad u > 0 \quad in \quad B_r.$$

Proof. We prove the contrapositive of the assertion. Let us assume that $u \not> 0$ in B_r. Then there exists a point $x_0 \in B_r \cap [u = 0]$, and we have for each $x \in B_r$

$$u(x) = |u(x) - u(x_0)| \le C|x - x_0| \le 2Cr$$

which leads to $\max_{\overline{B}_r} u \le 2Cr < c_1 r$. □

Lemma 6.2 *Let $D \subset\subset \Omega$ be a domain. For each $\kappa \in (0,1)$, there exists a positive constant $c = c(\kappa, n, p_-, p_+, \beta, L, M, Q_-, Q_+, d(D, \partial\Omega))$ such that for any $u \in \mathcal{S}(g,\Omega)$ and any ball $B_r(x_0) \subset D$ with $u(x_0) = 0$, we have*

$$\max_{\overline{B}_{\sqrt{\kappa}r}(x_0)} u < c_\kappa r \quad \Rightarrow \quad u \equiv 0 \quad in \quad B_{\kappa r(x_0)}.$$

Proof. Let r and x_0 be as in the lemma, and let v be defined by $v(y) = \dfrac{u(x_0 + ry)}{r}$. Note that to prove the lemma, it is enough to prove that for each $\kappa \in (0,1)$, there exists a positive constant c_κ such that

$$\max_{\overline{B}_{\sqrt{\kappa}}} v < c_\kappa \quad \Rightarrow \quad v \equiv 0 \quad \text{in} \quad B_\kappa.$$

By Lemma 5.1, the function $v(y) = \dfrac{u(x_0 + ry)}{r}$ is a minimizer for the functional $\widetilde{J}$ in B_1 over all functions $w \in v + W_0^{1,p(x_0+ry)}(B_1)$. Moreover v is bounded in $\overline{B}_1$ independently of r. Indeed let C be the Lipschitz constant of u over D. Since $u(x_0) = 0$, we have for each $x \in \overline{B}_r(x_0)$, $u(x) = |u(x) - u(x_0)| \leq C|x - x_0| \leq Cr$. We deduce that $\max_{\overline{B}_r(x_0)} u \leq Cr$, or $\max_{\overline{B}_1} v \leq C(n, p_-, p_+, \beta, L, M, Q_+, d(D, \partial\Omega))$. Hence it is enough to prove the lemma when $r = 1$ and $x_0 = 0$. To do that let $\epsilon = \max_{\overline{B}_{\sqrt{\kappa}}} u$ and consider the function v_ϵ defined by

$$\begin{cases} \Delta_{p(x)} v_\epsilon = 0 & \text{in} \quad B_{\sqrt{\kappa}} \setminus B_\kappa \\ v_\epsilon = 0 & \text{on} \quad \partial B_\kappa \\ v_\epsilon = \epsilon & \text{on} \quad \partial B_{\sqrt{\kappa}}. \end{cases} \tag{6.1}$$

We extend v_ϵ by 0 to B_κ and remark that $v_\epsilon \geq u$ on $\partial B_{\sqrt{\kappa}}$. Therefore $w_\epsilon = \min(u, v_\epsilon)\chi_{B_{\sqrt{\kappa}}} + u\chi_{B_1 \setminus B_{\sqrt{\kappa}}}$ is an admissible function for the functional $J(u)$. Hence we have $J(u) \leq J(w_\epsilon)$ which leads to

$$\int_{B_{\sqrt{\kappa}}} \Big(\frac{1}{p(x)}|\nabla u|^{p(x)} + Q(x)\chi_{[u>0]}\Big)dx \leq \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \Big(\frac{1}{p(x)}|\nabla w_\epsilon|^{p(x)} + Q(x)\chi_{[w_\epsilon>0]}\Big)dx.$$

or

$$\begin{aligned} \int_{B_\kappa} \Big(\frac{1}{p(x)}|\nabla u|^{p(x)} + Q(x)\chi_{[u>0]}\Big)dx \quad &\leq \quad \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \Big(\frac{1}{p(x)}(|\nabla w_\epsilon|^{p(x)} - |\nabla u|^{p(x)}\Big)dx \\ &\quad + \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} Q(x)(\chi_{[w_\epsilon>0]} - \chi_{[u>0]})dx \\ &= \quad \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \frac{1}{p(x)}(|\nabla w_\epsilon|^{p(x)} - |\nabla u|^{p(x)})dx \end{aligned}$$

since we have in $B_{\sqrt{\kappa}} \setminus B_\kappa$, $\quad w_\epsilon = 0 \;\Leftrightarrow\; \min(u, v_\epsilon) = 0 \;\Leftrightarrow\; u = 0$, due to the fact that we have by the maximum principle $v_\epsilon > 0$ in $B_{\sqrt{\kappa}} \setminus B_\kappa$ (see [11]).

Using the inequality $|\zeta|^p - |\xi|^p \leq p|\zeta|^{p-2}\zeta.(\zeta - \xi)$, due to the convexity of $\zeta \to |\zeta|^p$, we obtain

$$\begin{aligned}\int_{B_\kappa}\Big(\frac{1}{p(x)}|\nabla u|^{p(x)}+Q(x)\chi_{[u>0]}\Big)dx &\le \int_{B_{\sqrt{\kappa}}\setminus B_\kappa}\frac{1}{p(x)}(|\nabla w_\epsilon|^{p(x)}-|\nabla u|^{p(x)})dx\\ &\le \int_{B_{\sqrt{\kappa}}\setminus B_\kappa}|\nabla w_\epsilon|^{p(x)-2}\nabla w_\epsilon.\nabla(w_\epsilon-u)dx\\ &\le -\int_{B_{\sqrt{\kappa}}\setminus B_\kappa}|\nabla v_\epsilon|^{p(x)-2}\nabla v_\epsilon.\nabla(u-v_\epsilon)^+dx\\ &= -\int_{\partial B_\kappa}u|\nabla v_\epsilon|^{p(x)-2}\nabla v_\epsilon.\nu d\sigma(x)\ \text{ since }\ u^+=u. \end{aligned}\tag{6.2}$$

Setting $M_\epsilon = \max_{x\in\partial B_\kappa}|\nabla v_\epsilon(x)|^{p(x)-1} = |\nabla v_\epsilon(x_1)|^{p(x_1)-1}$ with $x_1\in\partial B_\kappa$, we deduce from (6.2) that

$$\int_{B_\kappa}\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+Q(x)\chi_{[u>0]}\right)dx \le M_\epsilon\int_{\partial B_\kappa}ud\sigma(x).\tag{6.3}$$

Now we have by (1.3), since $Q_->0$, and by using Young's inequality

$$\begin{aligned}\int_{\partial B_\kappa}ud\sigma(x) &\le C(n,\kappa)\Big(\int_{B_\kappa}udx+\int_{B_\kappa}|\nabla u|dx\Big)\\ &= C(n,\kappa)\Big(\int_{B_\kappa}u.\chi_{[u>0]}dx+\int_{B_\kappa}|\nabla u|.\chi_{[u>0]}dx\Big)\\ &\le C(n,\kappa)\Big(\int_{B_\kappa}\frac{\epsilon}{Q_-}Q(x)\chi_{[u>0]}dx+\int_{B_\kappa}\Big(\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{1}{q(x)}\frac{Q(x)}{Q_-}\chi_{[u>0]}\Big)dx\Big)\\ &\le C_1(1+\epsilon)\int_{B_\kappa}\Big(\frac{1}{p(x)}|\nabla u|^{p(x)}+Q(x)\chi_{[u>0]}\Big)dx,\end{aligned}$$

where $C_1=C(n,\kappa,p_-,p_+,Q_-)$ is a constant that we can obviously assume to be such that $C_1>1$. This leads by (6.3) to

$$\int_{B_\kappa}\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+Q(x)\chi_{[u>0]}\right)dx \le C_1(1+\epsilon)M_\epsilon\int_{B_\kappa}\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+Q(x)\chi_{[u>0]}\right)dx.\tag{6.4}$$

We will show that for ϵ small enough, we have $C_1(1+\epsilon)M_\epsilon<1$. Let us first estimate $|\nabla v_\epsilon(x_1)|$ in terms of ϵ. From (6.1), we know that $v_\epsilon\in C^{1,\alpha}(\overline{B_{\sqrt{\kappa}}\setminus B_\kappa})$ and for $\epsilon<1$, that $|v_\epsilon|_{1,\alpha,\overline{B_{\sqrt{\kappa}}\setminus B_\kappa}}\le C(p_-,p_+,n,L,\beta,\kappa)=C_2$ (see [10]). Integrating by part and using (5.1), we get

$$\begin{aligned}\int_{B_{\sqrt{\kappa}}\setminus B_\kappa}|\nabla v_\epsilon|^{p(x)}dx &= \int_{B_{\sqrt{\kappa}}\setminus B_\kappa}|\nabla v_\epsilon|^{p(x)-2}\nabla v_\epsilon.\nabla(v_\epsilon-\epsilon)dx\\ &= -\epsilon\int_{\partial B_\kappa}|\nabla v_\epsilon|^{p(x)-2}\nabla v_\epsilon.\nu d\sigma(x)\le\epsilon\int_{\partial B_\kappa}|\nabla v_\epsilon|^{p(x)-1}d\sigma(x)\\ &\le \epsilon M_\epsilon|\partial B_\kappa|\le\epsilon C_3,\quad\text{where }\ C_3=C(p_-,p_+,n,L,\beta,\kappa).\end{aligned}\tag{6.5}$$

Let $\delta=\epsilon^{\frac{1}{n+\alpha p_+}}$ and assume that $\delta<\dfrac{\sqrt{\kappa}-\kappa}{2}$. We claim that

$$|\nabla v_\epsilon(x_1)|\le C_4\delta^\alpha=C_4\epsilon^{\frac{\alpha}{n+\alpha p_+}},\quad\text{where }\ C_4=C_2+\max\Big(\Big(\frac{C_3}{|B_1|}\Big)^{1/p_-},\Big(\frac{C_3}{|B_1|}\Big)^{1/p_+}\Big).\tag{6.6}$$

Indeed let us consider the ball $B_{\delta/2}(x_2)$ contained in $B_{\sqrt{\kappa}} \setminus B_\kappa$ such that $x_1 \in \partial B_{\delta/2}(x_2)$.

-If $|\nabla v_\epsilon(x_1)| \leq C_2\delta^\alpha$, we are done.

-If $|\nabla v_\epsilon(x_1)| > C_2\delta^\alpha$, we obtain by using the Hölder continuity of ∇v_ϵ in $\overline{B_{\sqrt{\kappa}}} \setminus B_\kappa$,

$$\begin{aligned}\int_{B_{\sqrt{\kappa}}\setminus B_\kappa} |\nabla v_\epsilon|^{p(x)}dx &\geq \int_{B_{\delta/2}(x_2)} |\nabla v_\epsilon|^{p(x)}dx \\ &\geq \int_{B_{\delta/2}(x_2)} (|\nabla v_\epsilon(x_1)| - C_2\delta^\alpha)^{p(x)}dx \\ &\geq \delta^n|B_{1/2}| \min\left((|\nabla v_\epsilon(x_1)| - C_2\delta^\alpha)^{p_-}, (|\nabla v_\epsilon(x_1)| - C_2\delta^\alpha)^{p_+}\right).\end{aligned} \tag{6.7}$$

Combining (6.5) and (6.7), we obtain

$$\min\left((|\nabla v_\epsilon(x_1)| - C_2\delta^\alpha)^{p_-}, (|\nabla v_\epsilon(x_1)| - C_2\delta^\alpha)^{p_+}\right) \leq \frac{C_3}{|B_{1/2}|\delta^n}\epsilon = \frac{C_3}{|B_{1/2}|}\delta^{\alpha p_+}.$$

Using the fact that $\delta < 1$ and discussing the cases $|\nabla v_\epsilon(x_1)| - C_2\delta^\alpha < 1$ and $|\nabla v_\epsilon(x_1)| - C_2\delta^\alpha \geq 1$, we easily get

$$|\nabla v_\epsilon(x_1)| \leq C_2\delta^\alpha + \max\Big(\Big(\frac{C_3}{|B_{1/2}|}\Big)^{1/p_-}, \Big(\frac{C_3}{|B_{1/2}|}\Big)^{1/p_+}\Big)\delta^\alpha = C_4\delta^\alpha.$$

Hence (6.6) holds. It follows that for $\delta < \dfrac{\sqrt{\kappa}-\kappa}{2}$ or equivalently $\epsilon < \Big(\dfrac{\sqrt{\kappa}-\kappa}{2}\Big)^{n+\alpha p_+} < 1$, one has, assuming $C_4 > 1$ if necessary

$$M_\epsilon C_1(1+\epsilon) < 2C_1M_\epsilon = 2C_1|\nabla v_\epsilon(x_1)|^{p(x_1)-1} \leq 2C_1(C_4\epsilon^{\frac{\alpha}{n+\alpha p_+}})^{p(x_1)-1} \leq 2C_1C_4^{p_+-1}\epsilon^{\frac{\alpha(p_--1)}{n+\alpha p_+}}.$$

Therefore if $\epsilon < \min\Big(\Big(\dfrac{\sqrt{\kappa}-\kappa}{2}\Big)^{n+\alpha p_+}, \dfrac{1}{(2C_1C_4^{p_+-1})^{\frac{n+\alpha p_+}{\alpha(p_--1)}}}\Big)$, we have $M_\epsilon C_1(1+\epsilon) < 1$.

We conclude from (6.4) that

$$\int_{B_\kappa}\left(\frac{1}{p(x)}|\nabla u|^{p(x)} + Q(x)\chi_{[u>0]}\right)dx = 0$$

which leads to $u \equiv 0$ in B_κ. □

As a corollary of Lemma 6.2, we obtain the following result.

Corollary 6.1 *Let $u \in \mathcal{S}(g,\Omega)$, $D \subset\subset \Omega$ a domain, $x_0 \in D \cap \partial[u>0]$. Then for any ball $B_r(x_0) \subset D$, we have*

$$\max_{x\in\overline{B}_r(x_0)} u(x) \geq cr,$$

where c is the constant in Lemma 6.2 corresponding to $\kappa = \frac{1}{2}$.

The following theorem shows that the sets $[u>0]$ and $[u=0]$ have local uniform positive densities.

Theorem 6.1 *For each domain $D \subset\subset \Omega$, there exists a constant $c \in (0,1)$ depending on n, p_-, p_+, β, L, Q_-, Q_+, M and D such that, for any $u \in \mathcal{S}(g,\Omega)$, for any $x_0 \in D \cap \partial[u > 0]$, and for any $r \in (0,1)$ with $B_r(x_0) \subset D$, we have*

$$c \le \frac{|B_r(x_0) \cap [u > 0]|}{|B_r(x_0)|} \le 1 - c.$$

Proof. i) By Lemma 6.2 there exists $y \in \overline{B}_{r/2}(x_0)$ such that $u(y) > c_1 r$, where c_1 is a constant depending on n, p_-, p_+, β, L, Q_-, Q_+, M and D. Now let C be a Lipschitz constant of u over D which may depend on n, p_-, p_+, Q_+, M, L, β and D. We claim that $u > 0$ in $B_{\kappa r}(y) \subset B_r(x_0)$, for each $\kappa \in \left(0, \min\left(\frac{1}{2}, \frac{c_1}{2C}\right)\right)$. Indeed let $\kappa \in \left(0, \min\left(\frac{1}{2}, \frac{c_1}{2C}\right)\right)$ and $x \in B_{\kappa r}(y)$. We have $|x - x_0| \le |x - y| + |y - x_0| < \kappa r + r/2 < r/2 + r/2 = r$. So $B_{\kappa r}(y) \subset B_r(x_0)$.

Since we have $\max\limits_{\overline{B}_{\kappa r}(y)} u \ge u(y) > c_1 r = \dfrac{c_1}{\kappa}(\kappa r)$, and $\frac{c_1}{\kappa} > 2C$, we deduce from Lemma 6.1 that $u > 0$ in $B_{\kappa r}(y)$. It follows that

$$\frac{|B_r(x_0) \cap [u > 0]|}{|B_r(x_0)|} \ge \frac{|B_{\kappa r}(y)|}{|B_r(x_0)|} = \kappa^n = c.$$

ii) Arguing by contradiction, we deduce that there exists a domain $D_0 \subset\subset \Omega$ such that

$$\forall k \in \mathbb{N} \quad \exists u_k \in \mathcal{S}(g,\Omega) \quad \exists x_{0k} \in D_0 \cap \partial[u_k > 0] \quad \exists r_k \in (0,1) :$$
$$B_{r_k}(x_{0k}) \subset D_0 \ \text{ and } \ \frac{|B_{r_k}(x_{0k}) \cap [u_k > 0]|}{|B_{r_k}(x_{0k})|} > 1 - \frac{1}{k+1}.$$

For each k, we define the function $v_k(y) = \frac{u_k(x_0 + r_K y)}{r_k}$. By Lemma 5.1, v_k is a minimizer in B_1 over all functions $w \in v_k + W_0^{1,p_k(y)}(B_1)$, for the functional

$$J_k(v) = \int_{B_1} \left(\frac{1}{p_k(y)} |\nabla v|^{p_k(y)} + Q_k(y)\chi_{[v>0]} \right) dy, \quad p_k(y) = p(x_0 + r_k y) \text{ and } Q_k(y) = Q(x_0 + r_k y).$$

Since $u_k(x_{0k}) = 0$, we have as seen in the proof of Lemma 6.2

$$\max_{\overline{B}_1} v_k \le C(n, p_-, p_+, \beta, L, M, Q_+, d(D, \partial\Omega)).$$

Moreover we have $0 \in \partial[v_k > 0]$ and

$$\begin{aligned}
|[v_k = 0] \cap B_1| &= \int_{B_1} \chi_{[v_k=0]}(y)dy = \int_{B_{r_k}(x_{0k})} \chi_{[u_k=0]}(x) r_k^{-n} dx \\
&= |B_1| \frac{|B_{r_k}(x_{0k}) \cap [u_k = 0]|}{|B_{r_k}(x_{0k})|} \\
&= |B_1| \frac{|B_{r_k}(x_{0k})| - |B_{r_k}(x_{0k}) \cap [u_k > 0]|}{|B_{r_k}(x_{0k})|} \\
&= |B_1| \left(1 - \frac{|B_{r_k}(x_{0k}) \cap [u_k > 0]|}{|B_{r_k}(x_{0k})|} \right) \\
&< |B_1| \left(1 - \left(1 - \frac{1}{k+1} \right) \right) = \frac{|B_1|}{k+1}.
\end{aligned}$$

Since the functions p_k and Q_k satisfy the same assumptions as p and Q with the same constants, we can assume that we have a sequence of uniformly bounded minimizers in B_1 for the functional J, that we shall denote by $(u_k)_k$, such that $0 \in \partial[u_k > 0]$ and $\lim_{k\to\infty} |[u_k = 0] \cap B_1| = 0$. Let $v_k \in W^{1,p(x)}(B_1)$ such that

$$\begin{cases} \Delta_{p(x)} v_k = 0 & \text{in} \quad B_1 \\ v_k = u_k & \text{on} \quad \partial B_1. \end{cases}$$

Since v_k is an admissible function for the functional J, we have $J(u_k) \le J(w_k)$ i.e.

$$\int_{B_1} \Big(\frac{1}{p(x)}|\nabla u_k|^{p(x)} + Q(x)\chi_{[u_k>0]}\Big)dx \le \int_{B_1} \Big(\frac{1}{p(x)}|\nabla v_k|^{p(x)} + Q(x)\chi_{[v_k>0]}\Big)dx.$$

Since by the strong maximum principle $v_k > 0$ in B_1, we obtain for $\epsilon_k = |[u_k = 0] \cap B_1|$

$$\begin{aligned} \int_{B_1} \frac{1}{p(x)} \Big(|\nabla u_k|^{p(x)} - |\nabla v_k|^{p(x)}\Big)\,dx &\le \int_{B_1} Q(x)(\chi_{[v_k>0]} - \chi_{[u_k>0]})dx \\ &= \int_{B_1} Q(x)(1 - \chi_{[u_k>0]})dx \\ &= \int_{B_1} Q(x)\chi_{[u_k=0]}dx \\ &\le Q_+ \int_{B_1} \chi_{[u_k=0]}dx \\ &= Q_+|[u_k = 0] \cap B_1| = C\epsilon_k. \end{aligned} \tag{6.8}$$

Using the fact that $\epsilon_k \to 0$ and arguing as in the proof of Lemma 5.2, we derive from (6.8) that

$$\lim_{k\to\infty} \int_{B_{1/2}} |\nabla(u_k - v_k)|^{p(x)} = 0. \tag{6.9}$$

Note that we have $\Delta_{p(x)} v_k = 0$ in B_1 and $|v_k|_{\infty,B_1} \le |u_k|_{\infty,B_1} \le C$, where C is a constant depending only on n, p_-, p_+, β, L, M, Q_+ and D_0. We deduce that $v_k \in C^{1,\alpha}(\overline{B_{1/2}})$ and $|v_k|_{1,\alpha,B_{1/2}} \le C(n, p_-, p_+, \beta, L, M, Q_+, D_0)$ (see [10]). In particular, we have up to a subsequence

$$v_k \longrightarrow v \quad \text{in} \quad C^1(\overline{B_{1/2}}). \tag{6.10}$$

Consequently we obtain

$$\Delta_{p(x)} v = 0 \text{ in } B_{1/2}. \tag{6.11}$$

Using (6.9) and the fact that $(u_k)_k$ is uniformly Lipschitz continuous in B_1, we deduce that there exists a subsequence and a function u such that

$$u_k \longrightarrow u \quad \text{uniformly in} \quad B_{1/2} \tag{6.12}$$

$$u_k \longrightarrow u \quad \text{in} \quad W^{1,p(x)}(B_{1/2}). \tag{6.13}$$

Now comparing (6.9)-(6.10) and (6.12)-(6.13), we deduce that $u = v + c_0$ in $B_{1/2}$, which leads by (6.11) to

$$\begin{cases} \Delta_{p(x)} u = 0 & \text{in} \quad B_{1/2} \\ 0 \le u \le C & \text{in} \quad B_{1/2} \\ u(0) = 0. \end{cases}$$

Hence we get by the strong maximum principle (see [11]) that $u \equiv 0$ in $B_{1/2}$.
But since $0 \in \partial[u_k > 0]$, we have by Corollary 6.1 $\max_{\overline{B}_{1/4}} u_k \geq c$, where c is independent of k. There exists therefore a sequence $x_k \in \overline{B}_{1/4}$ such that $u_k(x_k) \geq c$ for all k. Using (6.12), we get $u(x_*) \geq c$, where $x_* \in \overline{B}_{1/4}$ is the limit of a subsequence of $(x_k)_k$. We have reached a contradiction. □

As a consequence of Theorem 6.1, we obtain the following result regarding the Lebesgue measure of the free boundary.

Corollary 6.2 *The Lebesgue measure of the free boundary $\partial[u > 0] \cap \Omega$ is zero.*

Proof. Note that $\partial[u > 0] \cap \Omega = \bigcup_{k=1}^{\infty} \partial[u > 0] \cap \Omega_{\frac{1}{k}}$, where $\Omega_{\frac{1}{k}}$ has been defined in the proof of Theorem 5.1. Therefore it is enough to show that $\partial[u > 0] \cap \Omega_{\frac{1}{k}}$ has Lebesgue measure zero.
Let $\mathcal{L}_k$ be the set of Lebesgue points of the characteristic function $\chi_{[u>0]}$ in $\Omega_{\frac{1}{k}}$. We know that $|\Omega_{\frac{1}{k}} \setminus \mathcal{L}_k| = 0$. Moreover we claim that $\mathcal{L}_k \cap \partial[u > 0] = \emptyset$. Indeed let us assume that $\mathcal{L}_k \cap \partial[u > 0] \neq \emptyset$, and let $x_0 \in \mathcal{L}_k \cap \partial[u > 0]$. By definition we have

$$\lim_{r\to 0} \fint_{B_r} \chi_{\partial[u>0]}(x)dx = \partial[u > 0](x_0) = 1. \tag{6.14}$$

Now using the left hand-side estimate in Theorem 6.1 for $D = \Omega_{\frac{1}{k}}$, we have for r small enough

$$\begin{aligned}
\fint_{B_r} \chi_{\partial[u>0]}(x)dx &= \frac{|B_r(x_0) \cap (\partial[u > 0])|}{|B_r(x_0)|} \\
&= \frac{|B_r(x_0)| - |B_r(x_0) \cap [u > 0]| - |B_r(x_0) \cap Int([u = 0])|}{|B_r(x_0)|} \\
&= 1 - \frac{|B_r(x_0) \cap [u > 0]|}{|B_r(x_0)|} - \frac{|B_r(x_0) \cap Int([u = 0])|}{|B_r(x_0)|} \\
&\leq 1 - \frac{|B_r(x_0) \cap [u > 0]|}{|B_r(x_0)|} \\
&\leq 1 - c < 1.
\end{aligned} \tag{6.15}$$

Comparing (6.14) and (6.15), we see that we have reached a contradiction. Hence we have $\mathcal{L}_k \cap \partial[u > 0] = \emptyset$. We deduce that $\partial[u > 0] \cap \Omega_{\frac{1}{k}} \subset \Omega_{\frac{1}{k}} \setminus \mathcal{L}_k$ and therefore $|\partial[u > 0] \cap \Omega_{\frac{1}{k}}| = 0$. We conclude that $|\partial[u > 0] \cap \Omega| = 0$. □

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