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Entire Large Solutions to Elliptic Equations of Power Non-linearities with Variable Exponents

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Abstract

We give conditions on the variable exponent q that ensure the existence and nonexistence of a positive solution to the elliptic equation $\Delta u = u^{q(x)}$ on $\mathbb{R}^N$ ($N \geq 3$) which satisfies $\lim_{|x|\to\infty} u(x) = \infty$. The nonnegative function q is required to be locally Hölder continuous on $\mathbb{R}^N$. We prove existence for $q > 1$ provided $q(x)$ decays to unity rapidly as $|x| \to \infty$. We treat the case $q \leq 1$ as a special case of $q - 1$ changing signs and show that a solution exists provided q is asymptotically radial. In addition, we give an example to show that our results are nearly optimal.

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1 Introduction

We consider the question of the existence of a positive solution to the problem

$$\begin{cases} \Delta u = u^{q(x)} & \text{in } \mathbb{R}^N \\ u(x) \to \infty & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

which is called an entire large solution. Here $N \geq 3$, and we assume that $q : \mathbb{R}^N \to [0, \infty)$ is a locally Hölder continuous function.

It is well known that, if q is a positive constant, then (1.1) has no solution if $q > 1$ (see Theorem II of [4] or Theorem 1 of [14]). Conversely, if $q \leq 1$, then (1.1) has a positive solution, and in fact, a radial solution (see Theorem IV of [4] or Lemma 3 of [14]). (Although these authors do not include a proof that their positive radial solutions are large, this is easily established (see, e.g., Theorem 1 of [10]).)

The goal of this article is to develop existence of positive solutions when q is a positive function. It will become clear that functions q for which a solution exists are not so easily classified as for q constant. For example, we show that (1.1) will, in fact, have a solution when $q(x) > 1$ provided q decays rapidly to unity at infinity (Theorems 4.2, 4.3). On the other hand, if $\liminf_{|x|\to\infty} q(x) > 1$, no positive large solution exists (Theorem 4.1). For the purely sublinear case ($q(x) \leq 1$), which is treated only as a special case of q being sublinear outside some ball, we show that a solution exists if q is asymptotically radial (Theorems 3.1 and 3.2). To establish the near optimality of these two theorems, we include an example of a radial function q in which problem (1.1) has no solution even though q is sublinear outside some ball (Theorem 5.1).

Although we know of no article other than this one that addresses the existence of positive solutions of (1.1), such results have been established for a bounded domain $\Omega \subseteq \mathbb{R}^N$; i.e., solutions of

$$\begin{cases} \Delta u &= u^{q(x)} & \text{in } \Omega \\ u(x) &\to \infty & \text{as } x \to \partial\Omega. \end{cases}$$

In particular, for q constant, a solution exists if and only if $q > 1$ (see Theorem 1 of [9] and [6], or Theorem 5.1 of [1] with $\lambda = 0$.) This, of course, is just the opposite of the case for solutions of (1.1) where existence requires $q \leq 1$ as noted above. For q a function, López-Gómez [11] proved existence for $q > 1$ in Ω and $q = 1$ on $\partial\Omega$ (see Section 6 of [11] with $\lambda = 0$), while in [3] the authors prove existence when $q > 1$ near $\partial\Omega$ (see Theorem 2 of [3]). Finally, although large solutions are not considered in [12], the authors allow $q - 1$ to switch signs (as we do here) and apparently were the first to do so.

2 Preliminary Results

We will need to prove the existence of positive solutions of the initial value problem

$$u''(r) + \frac{N-1}{r}u'(r) = g(u(r), r), \qquad u(0) = a > 0, \qquad u'(0) = 0 \tag{2.1}$$

under various conditions on the function g. To do this, we will prove the existence of a positive solution to the integral equation

$$u(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} g(u(s), s)\, ds\, dt. \tag{2.2}$$

We assume $g : (0,\infty) \times [0,\infty) \to (0,\infty)$ is a continuous function that satisfies one or more of the following:

G-1 $g(t, r)$ is non-decreasing in t for each fixed r. Furthermore, for each $t > 0$, there is a positive constant c_t such that $g(t, r) \geq c_t$ for all $r \geq 0$.

G-2 There exists a non-decreasing, differentiable function $f : (0,\infty) \to (0,\infty)$ satisfying

$$\int_0^1 \frac{1}{f(t)}\,dt = \infty \quad \text{and} \quad \int_1^\infty \frac{1}{f(t)}\,dt < \infty \tag{2.3}$$

and constants $R \geq 0$ and $0 < \beta \leq 1$ such that

$$g(t,r) \leq \begin{cases} f(t) & \text{for } 0 \leq r < R, \quad t \geq 0 \\ t^\beta & \text{for } r \geq R, \quad t \geq 0. \end{cases}$$

G-3 There is a locally bounded function $\mu : (0,\infty) \times (0,\infty) \to (0,\infty)$ such that

$$|g(t,r) - g(s,r)| \leq \mu(t,s)|t-s|, \quad \forall\, t,s \in (0,\infty), \quad 0 \leq r \leq R,$$

where R comes from G-2.

Remark 2.1 In the event that $R = 0$ in G-2, then condition (2.3) and condition G-3 are vacuous.

Lemma 2.1 *Suppose that g satisfies G-1 and G-2, with $R = 0$ in G-2. Then, for any $a > 0$, the equation (2.2) has a positive solution u defined for all $r \geq 0$. Furthermore, $u(r) \to \infty$ as $r \to \infty$.*

Proof. Let $[0, R_a)$ be the maximal interval of existence of u. We wish to show that $R_a = \infty$. Thus suppose $R_a < \infty$. Since u is nondecreasing, we get, for any $r < R_a$,

$$\begin{aligned} u(r) &= a + \int_0^r t^{1-N} \int_0^t s^{N-1} g(u(s), s)\,ds\,dt \\ &\leq a + \int_0^r t^{1-N} \int_0^t s^{N-1} g(1 + u(s), s)\,ds\,dt \\ &\leq a + \int_0^r t^{1-N} \int_0^t s^{N-1} (1 + u(s))^\beta\,ds\,dt \\ &\leq a + \int_0^r t(1 + u(t))^\beta\,dt. \end{aligned}$$

Therefore, since $0 < \beta \leq 1$, Gronwall's inequality gives that u is bounded on $[0, R_a)$. Thus u' is bounded as well and hence u can be extended beyond R_a, contradicting the maximality of $[0, R_a)$. Thus $R_a = \infty$. Now it is clear that $u(r) \to \infty$ as $r \to \infty$, since, by G-1 we have

$$\begin{aligned} u(r) &= a + \int_0^r t^{1-N} \int_0^t s^{N-1} g(u(s), s)\,ds\,dt \\ &\geq a + \int_0^r t^{1-N} \int_0^t s^{N-1} g(a, s)\,ds\,dt \geq a + \frac{c_a}{2N} r^2. \end{aligned}$$

Lemma 2.2 *Suppose g satisfies conditions G-1 and G-2 for some $R > 0$. Then there exists $a > 0$ such that the integral equation (2.2) has a positive solution u defined for all $r \geq 0$. Furthermore, $u(r) \to \infty$ as $r \to \infty$.*

Proof. : It is obvious that if a positive solution u exists for all r, then $\lim_{r\to\infty} u(r) = \infty$, as demonstrated in the proof of Lemma 2.1 above. Thus we only need to show that (2.2) has a solution for all $r \geq 0$. To do this we note that for each $a > 0$, equation (2.2) has a solution u_a in a small interval $[0, R_a)$. Suppose that $[0, R_a)$ is the maximal interval of existence for the solution u_a. We claim that $R_a > R$ for some $a > 0$. Suppose this is not the case, so that $R_a \leq R$ for all $a > 0$. Since $[0, R_a)$ is a maximal interval of existence of u_a, and u_a is positive and non-decreasing it must be that $\lim_{r\to R_a^-} u_a(r) = \infty$. Note for $0 \leq s \leq R_a \leq R$, we have $g(u_a, s) \leq f(u_a)$. Differentiating (2.2) and using this fact gives

$$u_a'(r) = r^{1-N} \int_0^r s^{N-1} g(u_a(s), s)\, ds \leq r^{1-N} \int_0^r s^{N-1} f(u_a(s))\, ds \leq \frac{r}{N} f(u_a(r))$$

on $[0, R_a)$ so that

$$\frac{d}{dr} \int_a^{u_a(r)} \frac{d\xi}{f(\xi)} \leq \frac{r}{N}.$$

Integrating this over the interval $[0, \eta)$ for $\eta < R_a$ we get

$$\int_a^{u_a(\eta)} \frac{d\xi}{f(\xi)} \leq \frac{\eta^2}{2N} \leq \frac{R^2}{2N}.$$

Therefore

$$\int_a^{\infty} \frac{d\xi}{f(\xi)} = \lim_{\eta\to R_a} \int_a^{u_a(\eta)} \frac{d\xi}{f(\xi)} \leq \frac{R^2}{2N}.$$

Since this is true for all $a > 0$, we obtain

$$\lim_{a\to 0} \int_a^{\infty} \frac{d\xi}{f(\xi)} \leq \frac{R^2}{2N} < \infty$$

which is impossible since the limit on the left must diverge to ∞ in view of (2.3). Hence we obtain a contradiction and therefore (2.2) must have a solution valid on $[0, R]$ for some $a > 0$.

We now show that this solution is, in fact, valid for all $r \geq 0$. To do this, we use u instead of u_a, suppressing the dependence on a. Since u exists (and is continuous) on $[0, R]$, we have $u(R) < \infty$ so that $u'(R) < \infty$ and hence the solution u of (2.2) can be extended to some interval $[0, R + \varepsilon)$, $\varepsilon > 0$. Let

$$R_0 = \sup\{r : u \text{ is a solution of (2.2) on } [0, r)\}.$$

Clearly $R_0 > R$. If $R_0 = \infty$, we are done. Thus suppose $R_0 < \infty$ so that as above we must have $\lim_{r\to R_0^-} u(r) = \infty$. Elementary calculations now give, for all $r < R_0$,

$$
\begin{aligned}
u(r) &= a + \int_0^r t^{1-N} \int_0^t s^{N-1} g(u(s), s)\, ds\, dt \\
&= u(R) + \int_R^r t^{1-N} \int_0^R s^{N-1} g(u(s), s)\, ds\, dt + \int_R^r t^{1-N} \int_R^t s^{N-1} g(u(s), s)\, ds\, dt \\
&= u(R) + \int_0^R s^{N-1} g(u(s), s)\, ds \int_R^r t^{1-N}\, dt + \int_R^r t^{1-N} \int_R^t s^{N-1} g(u(s), s)\, ds\, dt \\
&= u(R) + \frac{1}{N-2}\left(1 - \left(\frac{R}{r}\right)^{N-2}\right) R^{2-N} \int_0^R s^{N-1} g(u(s), s)\, ds \\
&\quad + \int_R^r t^{1-N} \int_R^t s^{N-1} g(u(s), s)\, ds\, dt \\
&= u(R) + \frac{Ru'(R)}{N-2}\left(1 - \left(\frac{R}{r}\right)^{N-2}\right) + \int_R^r t^{1-N} \int_R^t s^{N-1} g(u(s), s)\, ds\, dt.
\end{aligned}
$$

We use the inequality

$$
\int_R^r t^{1-N} \int_R^t s^{N-1} g(v(s), s)\, ds\, dt \le \int_R^r s g(v(s), s)\, ds
$$

along with $g(v(s), s) \le (1 + v(s))^\beta$ for all $s \in [R, R_0)$ to get

$$
u(r) \le u^*(R) + \int_R^r s(1 + u(s))^\beta\, ds
$$

where

$$
u^*(R) \equiv u(R) + \frac{Ru'(R)}{N-2}. \tag{2.4}
$$

Since $0 < \beta \le 1$, Gronwall's inequality now gives that u is uniformly bounded on $[0, R_0)$, providing a contradiction. Therefore, we conclude that $R_0 = \infty$.

Lemma 2.3 *Suppose that g satisfies the conditions G-1, G-2, and G-3. Then, given any constant $M > 0$, there exists $a > 0$ such that the corresponding positive entire solution u_a of (2.2) satisfies $u_a(R) > M$.*

Proof. Let $A = \sup\{a > 0 : Eq.\ (2.2) \text{ has a positive entire solution}\}$. If $A = \infty$, then we simply choose $a > M$ to obtain the result easily.

On the other hand, suppose $A < \infty$. To establish the result, we first prove that there exists $R_A \le R$ such that $\lim_{r \to R_A^-} u_A(r) = \infty$. To do this, we note that for any $a > A$, the solution u_a must blow up at some $R_a > 0$, and since u_a is increasing in a; i.e., if $a < a'$, then $u_a < u_{a'}$, we must have $R_{a'} \le R_a$. In addition, $R_a \le R$ follows from the proof of Lemma 2.2 since, if $u_a(R) < \infty$, then u_a would be entire. Therefore, we get that $\lim_{a \to A^+} R_a$ exists. We let $R_A = \lim_{a \to A^+} R_a$. To establish our needed result, we shall show that $u_a(r) \to u_A(r)$ as $a \to A$ for all $r \in [0, R_A)$ and that this convergence is

uniform on compact subsets of $[0, R_A)$. We then show that $u_A(r) \to \infty$ as $r \to R_A^-$ before completing the proof.

By Condition G-3, we have

$$\begin{aligned}|u_a(r) - u_A(r)| &\le |a - A| + \int_0^r t^{1-N} \int_0^t s^{N-1} |(g(u_a(s), s) - g(u_A(s), s)| ds dt \\ &\le |a - A| + \frac{1}{N-2} \int_0^r s|(g(u_a(s), s) - g(u_A(s), s)| ds \\ &\le |a - A| + \int_0^r s\mu(u_a(s), u_A(s))|u_a(s) - u_A(s)| ds.\end{aligned}$$

Now let $\delta > 0$ be given and we show that $u_a \to u_A$ uniformly on the closed interval $[0, R_A - \delta]$. Thus, for $s \in [0, R_A - \delta]$, we see that

$$A \le u_A(s) \le u_a(s) \le u_a(R - \delta).$$

Since μ is locally bounded, there is a constant $M_\delta > 0$ such that

$$\mu(u_a(s), u_A(s)) \le M_\delta, \qquad 0 \le s \le R - \delta.$$

Therefore, the above observation yields

$$|u_a(r) - u_A(r)| \le |a - A| + M_\delta \int_0^r s|u_a(s) - u_A(s)| ds.$$

Gronwall's inequality thus produces

$$|u_a(r) - u_A(r)| \le |a - A| e^{\Lambda r^2}, \qquad \text{for } 0 \le r \le R_A - \delta, \tag{2.5}$$

where $\Lambda := \Lambda_\delta = M_\delta/2$.

From (2.5), it is clear that $\lim_{a \to A} u_a(r) = u_A(r)$ for all $r \in [0, R_A)$ with the convergence being uniform on compact subsets of $[0, R_A)$.

To show that $u_A(r) \to \infty$ as $r \to R_A^-$, we note first that, by Condition G-2, $g(u_a, r) \le C f(u_a)$ for all $0 \le r \le R_A \le R$ and define the function $F : (0, \infty) \to (0, \infty)$ by

$$F(z) = \int_z^\infty \frac{ds}{f(s)}.$$

Observe that $F' < 0$ and $F'' \ge 0$, and for any $a > A$ and $r < R_A$, we get

$$\Delta F(u_a) = F''(u_a)|\nabla u_a|^2 + F'(u_a)\Delta u_a \ge F'(u_a)\Delta u_a = \frac{-g(u_a, r)}{f(u_a)} \ge -1.$$

Hence

$$\frac{d}{dr}\left(r^{N-1} \frac{d}{dr} F(u_a(r))\right) \ge -r^{N-1},$$

which, by integrating over $[0, r]$, gives

$$\frac{d}{dr}F(u_a(r)) \geq r^{1-N}(-r^N/N) = -r/N.$$

Integrating this over $[r, R_a]$ and noting that $u_a(r) \to \infty$ as $r \to R_a^-$, we get

$$F(u_a(r)) \leq \frac{R_a^2 - r^2}{2N} \leq R_a^2 - r^2.$$

Since F is decreasing, this produces

$$u_a(r) \geq F^{-1}\left(R_a^2 - r^2\right).$$

Letting $a \to A^+$, we get

$$u_A(r) \geq F^{-1}\left(R_A^2 - r^2\right).$$

Therefore, since $F^{-1}(z) \to \infty$ as $z \to 0^+$, we get $u_A(r) \to \infty$ as $r \to R_A^-$.

To complete the proof, we choose $\delta_0 > 0$ small so that $u_A(r) > M+1$ for $r > R_A - \delta_0$. Specifically, observe that $u_A(R_A - \delta_0/2) > M + 1$. Using (2.5) with $\delta = \delta_0/4$ and $r = R_A - \delta_0/2$, we choose $\epsilon > 0$ small so that $(A - a)e^{\Lambda_0 R_A^2} < 1$ for $0 < A - a < \epsilon$ to get

$$0 < u_A(R_A - \delta_0/2) - u_a(R_A - \delta_0/2) < (A - a)e^{\Lambda_0 R_A^2} < 1$$

so that $u_a(R_A - \delta_0/2) > M$ for $A - \epsilon < a < A$. Here $\Lambda_0 := \Lambda_{\delta_0}$ is the constant in (2.5) corresponding to the choice δ_0. Therefore, $u_a(R) > M$ since u_a is an increasing function.

The following lemma provides an estimate of solutions to (2.2) under appropriate conditions.

Lemma 2.4 *Suppose g satisfies G-1, G-2 and G-3. If u is a solution of (2.2) for all $r \geq 0$ such that $v(R) \geq 1$, then for $r \geq R$ the solution u satisfies*

$$u(r) \leq \begin{cases} \left(\frac{Ru'(R)}{N-2} + u(R) + r^2\right)^{1/(1-\beta)} & \text{if } \ 0 < \beta < 1 \\ u(R)\exp\left(\frac{Ru'(R)}{N-2} + \frac{r^2}{2N}\right) & \text{if } \ \beta = 1. \end{cases} \tag{2.6}$$

Proof. From the proof of Lemma 2.2, we get for all $r \geq R$,

$$u(r) = u(R) + \frac{Ru'(R)}{N-2}\left(1 - \left(\frac{R}{r}\right)^{N-2}\right) + \int_R^r t^{1-N}\int_R^t s^{N-1}g(u(s), s)\,ds\,dt.$$

Therefore, for $r \geq R$, we note that $u(r) \geq 1$ by G-2, and therefore we have

$$\begin{aligned} u'(r) &= u'(R)\left(\frac{R}{r}\right)^{N-1} + r^{1-N}\int_R^r s^{N-1}g(u(s), s)\,ds \\ &\leq u'(R)\left(\frac{R}{r}\right)^{N-1} + r^{1-N}\int_R^r s^{N-1}u(s)^\beta\,ds \\ &\leq u'(R)\left(\frac{R}{r}\right)^{N-1} + \frac{r}{N}u(r)^\beta. \end{aligned}$$

Multiplying both sides of the last inequality by $u^{-\beta}$ and recalling that $u(r) \geq 1$ for all $r \geq R$, we find that

$$u(r)^{-\beta}u'(r) \leq u'(R)\left(\frac{R}{r}\right)^{N-1} + \frac{r}{N}, \qquad r \geq R.$$

Now suppose that $0 < \beta < 1$. We integrate both sides on the interval (R, r) for any $r > R$ to get

$$\begin{aligned} u(r)^{1-\beta} - u(R)^{1-\beta} &\leq (1-\beta)\left[\frac{u'(R)R^{N-1}}{N-2}(R^{2-n} - r^{2-n}) + \frac{1}{2N}(r^2 - R^2)\right] \\ &\leq \frac{Ru'(R)}{N-2} + \frac{1}{2N}r^2. \end{aligned}$$

Consequently, we have

$$u(r) \leq \left[\frac{Ru'(R)}{N-2} + u(R)^{1-\beta} + \frac{1}{2N}r^2\right]^{1/(1-\beta)} \leq \left[\frac{Ru'(R)}{N-2} + u(R) + r^2\right]^{1/(1-\beta)}.$$

The proof for the case $\beta = 1$ is similar and therefore omitted.

3 The case when $q(x) \leq 1$ at infinity

We define the functions g_* and g^* from $(0,\infty) \times [0,\infty)$ into $(0,\infty)$ by

$$g_*(z,r) := \min\{z^{q_*(r)}, z^{q^*(r)}\}, \qquad g^*(z,r) := \max\{z^{q_*(r)}, z^{q^*(r)}\} \tag{3.1}$$

where

$$q^*(r) := \max_{|x|=r} q(x), \qquad q_*(r) := \min_{|x|=r} q(x)$$

and q comes from (1.1). We require q to satisfy one or both of the following:

Q-1 $q : \mathbb{R}^N \to (0,\infty)$ is locally Hölder continuous.

Q-2 There exists $R \geq 0$ such that $q(x) \geq 1$ for $|x| \leq R$, and $0 < q(x) \leq 1$ for $|x| \geq R$.

When $R = 0$, condition Q-2 is to be understood in the sense that $0 < q(x) \leq 1$ on $\mathbb{R}^N$.

Lemma 3.1 *Suppose q satisfies Q-1 and Q-2. Then the functions g_* and g^* defined in (3.1) satisfy G-1, G-2 and G-3.*

Proof. It is obvious that both $g_*(t,r)$ and $g^*(t,r)$ are continuous on $(0,\infty) \times [0,\infty)$ and each is non-decreasing in t for each $r \geq 0$. If $R = 0$, and $t \geq 1$, then clearly $g^*(t,r) \leq t$ for all $r \geq 0$. So, suppose

$R > 0$, and hence we suppose that $q(x) > 1$ for some $|x| \le R$. Then, for $0 \le r \le R$, we see that $q^*(r) \ge q_*(r) \ge 1$, and therefore

$$g^*(t,r) \le t \quad \text{for } 0 < t \le 1, \quad \text{and} \quad g^*(t,r) \le t^\zeta, \quad \text{for } t \ge 1,$$

where $1 < \zeta := \max\{q(x) : |x| \le R\}$. Therefore $g^*(t,r) \le t + t^\zeta$ for all $(t,r) \in (0,\infty) \times [0,\infty)$. Consequently, both g_* and g^* satisfy G-2 with $f(t) = t + t^\zeta$. Furthermore, for $r \ge R$, we note that

$$g^*(t,r) \le t^\beta \quad \forall\, t \ge 1 \quad \text{where} \quad \beta := \max\{q(x) : |x| \ge R\}.$$

Finally, we note that for $0 \le r \le R$

$$|g_*(s,r) - g_*(t,r)| \le \max\{|s^{q_*(r)} - t^{q_*(r)}|, |s^{q^*(r)} - t^{q^*(r)}|\} \le \zeta(1+s+t)^{\zeta-1}|s-t|.$$

The same inequality holds with g_* replaced by g^*. Therefore, both g_* and g^* satisfy condition G-3 with $\mu(s,t) = \zeta(1+s+t)^{\zeta-1}$.

The following corollary follows easily from Lemmas 2.1, 2.2 and 2.3.

Corollary 3.1 *Suppose q satisfies conditions Q-1 and Q-2, and take g in (2.2) to be either g_* or g^* as defined in (3.1). Then the following hold:*

1. *If $R = 0$, then for any given $a > 0$ equation (2.2) has a solution defined for all $r \ge 0$ and $v(r) \to \infty$ as $r \to \infty$.*

2. *If $R > 0$ and M is any positive constant, then there exists $a > 0$ such that (2.2) has a solution u defined for all $r \ge 0$ for which $u(R) > M$ and $u(r) \to \infty$ as $r \to \infty$.*

3. *If u is a solution of (2.2) defined for all $r \ge 0$ and satisfying $v(R) \ge 1$, then u satisfies (2.6) for $r \ge R$.*

The next theorem gives a sufficient condition on q for (1.1) to admit a positive solution.

Theorem 3.1 *Suppose q satisfies conditions Q-1 and Q-2. If*

$$\int_0^\infty r \exp(\lambda r^2) q_{osc}(r)\, dr < \infty, \tag{3.2}$$

for some λ such that $2N\lambda > 1$, then problem (1.1) admits infinitely many solutions.

Proof. Consider the equations

$$\Delta v = g^*(v(r), r) \quad \text{and} \quad \Delta w = g_*(w(r), r) \quad \text{in} \quad \mathbb{R}^N \tag{3.3}$$

where the functions g_* and g^* are defined in (3.1). Given positive constants a and b, then $v(x) = v_a(|x|)$ and $w(x) := w_b(|x|)$, where v_a and w_b are solutions of the integral equation (2.2) with $v_a(0) = a$ and $w_b(0) = b$, respectively. We assume that the positive constant a has been chosen so that $v(R) \ge 1$, as given in (2) of Corollary 3.1.

We now show that b can be chosen so that $v_a \le w_b$ in $(0,\infty)$. Unless stated otherwise, we will suppress the subscripts in v_a and w_b, and simply write v and w, respectively. Note that $\beta = \max\{q(x) : |x| \ge R\} = 1$. Therefore, using (3) of Corollary 3.1 with $\beta = 1$, we have

$$v(r) \le \zeta(r) \quad r \ge R, \tag{3.4}$$

where

$$\zeta(r) := v(R) \exp\left(\frac{Rv'(R)}{N-2} + \frac{r^2}{2N}\right).$$

Let

$$R_0 := \sup\{r > 0 \,|\, v(t) \le w(t) \quad \text{for all} \quad 0 \le t \le r\}.$$

Since $g^*(z,r) \ge g_*(z,r)$ and $v(R) < w(R)$, it is an easy consequence of the maximum principle that $v(r) \le w(r)$ for all $0 \le r \le R$ so that $R_0 > R$. If $R_0 = \infty$, then the desired inequality $v \le w$ holds for all $r \ge 0$. Thus suppose $R_0 < \infty$. Then since $v(r) \ge 1$ for $r \ge R$, we get

$$\begin{aligned}
v(R_0) &= a + \int_0^{R_0} t^{1-N} \int_0^t s^{N-1} g^*(v(s), s)\, ds\, dt \\
&= v(R) + \frac{Rv'(R)}{N-2}\left(1 - \left(\frac{R}{R_0}\right)^{N-2}\right) \\
&+ \int_R^{R_0} t^{1-N} \int_R^t s^{N-1} [v(s)^{q^*(s)} - v(s)^{q_*(s)}]\, ds\, dt \\
&+ \int_R^{R_0} t^{1-N} \int_{R_0}^t s^{N-1} v(s)^{q_*(s)}\, ds\, dt \\
&\le v^*(R) + \int_R^{R_0} t^{1-N} \int_R^t s^{N-1} q_{\mathrm{osc}}(s) v(s) \log v(s)\, ds\, dt \\
&+ \int_R^{R_0} t^{1-N} \int_R^t s^{N-1} w(s)^{q_*(s)}\, ds\, dt,
\end{aligned}$$

where

$$v^*(R) := v(R) + \frac{Rv'(R)}{N-2}.$$

In the last inequality, we have used the hypothesis $0 < q(x) \le 1$ for $|x| \ge R$ and the inequality

$$c^\gamma - c^\alpha \le (\gamma - \alpha) c \log c$$

which is valid for any $c \ge 1$ and any $0 < \alpha \le \gamma \le 1$. Now using the inequality (3.4) and integration by parts we get

$$\begin{aligned}
&\int_R^{R_0} t^{1-N} \int_R^t s^{N-1} q_{\mathrm{osc}}(s) v(s) \log v(s)\, ds\, dt \\
&\le \int_R^{R_0} t^{1-N} \int_R^t s^{N-1} q_{\mathrm{osc}}(s) \zeta(s) \log \zeta(s)\, ds\, dt \\
&\le \int_0^\infty r q_{\mathrm{osc}}(r) \zeta(r) \log \zeta(r)\, dr.
\end{aligned}$$

Using (2) of Corollary 3.1, we choose the constant b in (2.2) so that

$$w(R) > v^*(R) + \int_0^\infty r q_{\mathrm{osc}}(r)\zeta(r)\log\zeta(r)\,dr \tag{3.5}$$

and note that since

$$\zeta(r)\log\zeta(r) \le \zeta(r)^{2N\lambda} = v(R)^{2N\lambda}\exp\left(\frac{2N\lambda R v'(R)}{N-2} + \lambda r^2\right)$$

for sufficiently large r, the integral in (3.5) is finite as a consequence of (3.2). Therefore, we have shown that, by choosing b so that (3.5) holds, we have

$$\begin{aligned} v(R_0) &\le v^*(R) + \int_0^\infty r\, q_{\mathrm{osc}}(r)\zeta(r)\log\zeta(r)\ dr + \int_R^{R_0} t^{1-N}\int_R^t s^{N-1}w(s)^{q_*(s)}\,ds\,dt \\ &< w(R) + \frac{Rw'(R)}{N-2}\left(1-\left(\frac{R}{R_0}\right)^{N-2}\right) + \int_R^{R_0} t^{1-N}\int_R^t s^{N-1}w(s)^{q_*(s)}\,ds\,dt \\ &= w(R_0). \end{aligned}$$

That is, $v(R_0) < w(R_0)$, which contradicts the definition of R_0. Therefore, $v \le w$ in $\mathbb{R}^N$ and

$$\Delta v = g^*(v(r), r) \ge v^{q(x)} \quad \text{and} \quad \Delta w = g_*(w(r), r) \le w^{q(x)} \quad \text{in} \quad \mathbb{R}^N.$$

Consequently, there is a solution $u_a \in C^2(\mathbb{R}^N)$ of $\Delta u = u^{q(x)}$ in $\mathbb{R}^N$ such that $v_a \le u_a \le w_b$, (see [13, Theorem 2.10]). Now, according to (2) of Corollary 3.1, we can choose $a' > 0$ such that $v_{a'}(R) > u_a(R)$, and as in the proof above we pick $b' > 0$ such that $v_{a'} \le w_{b'}$. Then again, we find a solution $u_{a'}$ such that $v_{a'} \le u_{a'} \le w_{b'}$. We note that $u_a \ne u_{a'}$ since $u_a(R) < v_{a'}(R) \le u_{a'}(R)$. We proceed in this manner to produce an infinite number of solutions.

Our next result shows that for sufficiently small $R > 0$, we can allow $1 - q$ to change sign in $|x| \le R$ and still get existence of solution to (1.1). In preparation, we need a lemma about the existence of a solution to (2.1) with $g(u, r) = u^{q^*(r)}$; i.e.,

$$(r^{N-1}v')' = r^{N-1}v^{q^*(r)}, \quad r > 0, \qquad v(0) = a, \qquad v'(0) = 0. \tag{3.6}$$

In the lemma below we assume that $q \in L^\infty(\mathbb{R}^N)$, and we use the notation

$$\|q\|_\infty := \max\{q(x) : x \in \mathbb{R}^N\}.$$

Lemma 3.2 *Suppose that $q(x) \le 1$ for $|x| \ge R$ for some $0 < R < \sqrt{2^{1-\|q\|_\infty}}$. Then problem (3.6) admits a solution $v_a \ge a$ for some $a \ge 1$, and $v_a(r) \to \infty$ as $r \to \infty$. In particular, if $0 < \|q\|_\infty \le 1$, problem (3.6) admits such a solution v_a for any $a \ge 1$.*

Proof. We fix any $a \ge 1$ such that $0 < R < \sqrt{(2a)^{1-\|q\|_\infty}}$. Note that if $0 < \|q\|_\infty \le 1$, then the hypothesis implies that $0 < R < 1$ and hence $0 < R < \sqrt{(2a)^{1-\|q\|_\infty}}$ holds for any $a \ge 1$.

Let $v_0 = a$, and define a sequence $\{v_j\}$ inductively as follows.

$$v_j(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} \left(v_{j-1}(s)\right)^{q^*(s)} \, ds\, dt, \qquad r \geq 0. \tag{3.7}$$

Choose r_0 small enough such that

$$R < r_0 < \sqrt{(2a)^{1-\|q\|_\infty}}\,.$$

Then, by induction, we can show that

$$a \leq v_j \leq 2a, \qquad r \in [0, r_0], \qquad j = 0, 1, 2, \cdots .$$

Moreover, we note that $v_j \leq v_{j+1}$ in $[0, r_0]$. Consequently v_j converges to some v in $[0, r_0]$ and v satisfies the equation

$$v(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} v(s)^{q^*(s)} \, ds\, dt, \qquad 0 < r < r_0. \tag{3.8}$$

Thus we see that v satisfies (3.6) for $0 < r < r_0$. Let $[0, r_1]$, for some $r_1 \geq r_0$, be the maximum interval of existence of v. We claim that $r_1 = \infty$, for otherwise $v(r) \to \infty$ as $r \to r_1^-$. (Note that if v is bounded on $[0, r_1]$, so is v', and hence v can be extended beyond $[0, r_1]$.) Since $q^*(|x|) \leq 1$ for $R < |x| < r_1$, this would imply that v is a large solution of

$$\Delta v = v^{q^*(|x|)}, \qquad x \in B(0, r_1).$$

But this is impossible (see [3, Theorem 1]), and therefore $r_1 = \infty$ as claimed.

In Theorem 3.1, we required that $q(x) \geq 1$ for $|x| \leq R$. In that theorem, there was no restriction on how big R may be. However, if R is sufficiently small, depending on $\|q\|_\infty$, we can allow $1 - q(x)$ to change sign for $|x| \leq R$, and still get existence of entire large solution to problem (1.1). This is the content of the next theorem. We will need the following notation. Given $0 < \beta \leq 1$ we set

$$\vartheta_\beta(r) := \begin{cases} r^{2/(1-\beta)} & \text{if} \quad 0 < \beta < 1 \\ \\ \exp(r^2/(2N)) & \text{if} \quad \beta = 1. \end{cases} \tag{3.9}$$

Theorem 3.2 *Let q satisfy Q-1 and suppose there are constants $0 < R < 2^{1-\|q\|_\infty}$ and $0 < \beta \leq 1$ such that $0 \leq q(x) \leq \beta$ for all $|x| \geq R$. Furthermore, assume that $q_*(r) \leq 1$ for $r > 0$. If*

$$\int_1^\infty r\, \vartheta_\beta(r)\, q_{osc}(r)\, dr < \infty, \tag{3.10}$$

then problem (1.1) admits a positive solution.

Proof. Consider the equations

$$\Delta v = v^{q^*(r)} \qquad \text{and} \qquad \Delta w = w^{q_*(r)} \qquad \text{in} \quad \mathbb{R}^N. \tag{3.11}$$

Fix any $a \geq 1$ such that $R < \sqrt{(2a)^{1-\|q\|_\infty}}$. Note that if $\|q\|_\infty = 1$, then we can take $R < 1$, and any choice of $a \geq 1$ will do. Lemma 3.2 guarantees the existence of a solution v_a to problem (3.6) such

that $v_a \geq a$ and $v_a(r) \to \infty$ as $r \to \infty$. Thus $v(x) = v_a(|x|)$, where v_a is a solution of the following integral equation, will provide a solution of (3.11); i.e.,

$$v_a(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} v_a(s)^{q^*(s)}\, ds\, dt, \qquad r \geq 0. \tag{3.12}$$

We will assume that $0 < \beta < 1$, and omit the case $\beta = 1$ since the proof is similar. By Lemma 2.4, we see that

$$v(r) \leq \varpi(r), \qquad r \geq R, . \tag{3.13}$$

where

$$\varpi(r) := \left(\frac{Rv'(R)}{N-2} + v(R) + r^2 \right)^{1/(1-\beta)}, \qquad r \geq 0.$$

Note that since $q_*(r) \leq 1$ for $r \geq 0$, a large solution w_b exists for

$$w_b(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1} w_b(s)^{q_*(s)}\, ds\, dt, \qquad r \geq 0. \tag{3.14}$$

for any $b \geq 1$. This follows from Lemma 2.1 with $g(z, r) = z^{q_*(r)}$.

We will show that b can be chosen in such a way that $1 \leq v_a \leq w_b$ in $(0, \infty)$. Since the proof is similar to that of Theorem 3.1, we will be brief.
To this end, we take b such that

$$b > v(R) + \frac{Rv'(R)}{N-2} + \frac{1}{N-2} \int_0^\infty r q_{\mathrm{osc}}(r) \varpi^\beta(r)\, \log \varpi(r)\, dr. \tag{3.15}$$

Since $0 < \beta < 1$, we note that $\varpi(r)^\beta \log \varpi(r) \leq \varpi(r)$ for sufficiently large r. Therefore, by virtue of (3.10), the integral in (3.15) is finite.

Since $w(R) > v(R)$, by the maximum principle, we see that $v_a(r) \leq w_b(r)$ for all $0 \leq r \leq R$. Now let

$$R_0 := \sup\{r > 0 \,|\, v_a(t) \leq w_b(t) \quad \text{for all} \;\; 0 \leq t \leq r\}.$$

Since $v_a(R) > w_b(R)$, we have $R_0 > R$. If $R_0 = \infty$, then the desired inequality $v_a \leq w_b$ holds for all $r \geq 0$. So let us suppose that $R_0 < \infty$ and proceed to draw a contradiction. By an argument similar to the one used in the proof of Theorem 3.1 we have

$$\begin{aligned} v(R_0) \leq v(R) &+ \frac{Rv'(R)}{n-2} \left(1 - \left(\frac{R}{R_0} \right)^{n-2} \right) \\ &+ \int_R^{R_0} t^{1-n} \int_R^t s^{n-1} q_{\mathrm{osc}}(s) v(s)^\beta \log v(s)\, ds\, dt \\ &+ \int_R^{R_0} t^{1-n} \int_R^t s^{n-1} w(s)^{q_*(s)}\, ds\, dt. \end{aligned}$$

In this last inequality, we have used the hypothesis $0 < q(x) \leq \beta$ for $|x| \geq R$ and the inequality

$$c^\gamma - c^\alpha \leq (\gamma - \alpha) c^\beta \log c$$

which is valid for any $c \geq 1$ and any $0 < \alpha \leq \gamma \leq 1$. Now, recalling the estimate (3.13) we get, after integration by parts,

$$\int_R^{R_0} t^{1-N} \int_R^t s^{N-1} q_{\mathrm{osc}}(s) v(s)^\beta \log v(s)\, ds\, dt$$

$$\leq \frac{1}{N-2} \int_0^\infty r q_{\mathrm{osc}}(r) \varpi(r)^\beta \log \varpi(r)\, dr.$$

Therefore, our choice of b as in (3.15) shows that

$$v(R_0) < w(R) + \frac{R w'(R)}{N-2} \left(1 - \left(\frac{R}{R_0}\right)^{N-2}\right) + \int_R^{R_0} t^{1-n} \int_R^t s^{n-1} w(s)^{q_*(s)}\, ds\, dt$$

$$= w(R_0).$$

But the last inequality contradicts the definition of R_0. Therefore, $v \leq w$ in $\mathbb{R}^N$ and

$$\Delta v = v(r)^{q^*(r)} \geq v^{q(x)} \quad \text{and} \quad \Delta w = w(r)^{q_*(r)} \leq w^{q(x)} \quad \text{in} \quad \mathbb{R}^N.$$

Consequently, [13, Theorem 2.10] shows that problem (1.1) admits a solution u such that $v \leq u \leq w$ in $\mathbb{R}^N$.

Corollary 3.2 *If $q_* \leq 1$ on $[0, \infty)$ and q is radial at infinity, that is there is $R > 0$ such that $q_*(|x|) = q^*(|x|)$ for $|x| \geq R$, then problem (1.1) admits infinitely many positive solutions.*

4 The case when $q(x) \geq 1$ at infinity

We now investigate the case $q(x) > 1$ outside some ball in $\mathbb{R}^N$.

Theorem 4.1 *If $\liminf_{|x|\to\infty} q(x) = \beta_0 > 1$, then problem (1.1) has no nonnegative entire large solution.*

Proof. Suppose the theorem is not true; i.e., suppose (1.1) has a nonnegative entire large solution, u. Choose γ to be a number in the open interval $(1, \beta_0)$, and choose R such that $q(x) \geq \gamma > 1$ for $|x| \geq R$. Let v_n be a nonnegative large solution of $\Delta v_n = v_n^\gamma$ on $|x| < n$. We will be done if we can show that the limit function v of the decreasing sequence $\{v_n\}$ is large since it would then be a nonnegative entire large solution of $\Delta v = v^\gamma$ which is impossible. Thus let $K_R = \max_{|x|\leq R} u(x)$, and we assume with no loss of generality that $K_R > 1$. We show that, for n large, we have $u - K_R \leq v_n$ on $|x| < n$. If this is, in fact, true, then this shows v is large and we're done. Thus suppose $\max_{|x|\leq n}(u(x) - K_R - v_n(x)) > 0$ and note that since $u - K_R - v_n < 0$ near the boundary $|x| = n$, this maximum must be attained in the open ball $|x| < n$. Suppose in particular that it occurs at x_0. Note that since $u(x_0) > K_R + v_n(x_0) \geq K_R$, it must be that $|x_0| > R$ and hence $q(x_0) \geq \gamma$. Consequently we get $0 \geq \Delta(u - K_R - v_n))|_{x=x_0} = u(x_0)^{q(x_0)} - v_n^\gamma(x_0) > (v_n(x_0) + K_R)^{q(x_0)} - v_n^\gamma(x_0) > (v_n(x_0) + K_R)^\gamma - v_n^\gamma(x_0) > 0$, which provides the needed contradiction.

Theorem 4.2 *Suppose q is radial and $q(r) = 1 + h(r)$ where the nonnegative function h is bounded and satisfies*

$$\int_0^\infty \exp\left(\frac{Mr^2}{2(N-2)}\right) rh(r)dr < \infty \tag{4.1}$$

where $M = \sup_{r\geq 0} h(r)$. Then problem (1.1) admits a positive solution.

Proof. It is easy to show that $(1+z)^\varepsilon \leq 1 + \varepsilon z$ for $z \geq 0$ and $0 \leq \varepsilon \leq 1$, and we use this to get

$$z^{q(r)} = zz^{M(h(r)/M)} \leq z(1+z^M)^{h(r)/M} \leq z\left(1 + \frac{h(r)}{M} z^M\right). \tag{4.2}$$

If, therefore, we can show that the problem

$$\Delta v = v + \left(\frac{h(r)}{M}\right) v^{M+1} \tag{4.3}$$

has a radial positive entire large solution, then it is easy to show that (1.1) has a positive radial solution. However, since h satisfies (4.1), we know that (4.3) has a positive entire large solution (see Lemma in [8]), and hence problem (1.1) has a positive solution.

Theorem 4.3 *Suppose q is bounded, locally Hölder continuous, and $q > 1$, all on $\mathbb{R}^N$. If q^* satisfies*

$$\int_0^\infty \exp\left(\frac{Mr^2}{2(N-2)}\right) r(q^*(r) - 1)dr < \infty \tag{4.4}$$

where $M = \sup_{r\geq 0}(q^*(r) - 1)$, then problem (1.1) admits a positive solution.

Proof. For $k \in \mathbb{N}$, let u_k be a positive solution of (See Theorem 2 of [3].)

$$\begin{aligned} \Delta u_k &= u_k^{q(x)} && \text{in } |x| < k \\ u_k(x) &\to \infty && \text{as } |x| \to k. \end{aligned}$$

It is easy to show that $u_{k+1} \leq u_k$ on $|x| \leq k$. Since $u_k \geq 0$, we know then that the sequence $\{u_k\}$ converges to a nonnegative function u. We now show that u is positive and $\lim_{|x|\to\infty} u(x) = \infty$. To do this, note that as in (4.2), we get, for any nonnegative z,

$$z^{q(x)} = zz^{M((q(x)-1)/M)} \leq z(1+z^M)^{(q(x)-1)/M} \leq z(1+z^M)^{(q^*(|x|)-1)/M} \tag{4.5}$$

$$\leq z\left(1 + \frac{(q^*(r)-1)}{M} z^M\right) = z + \frac{(q^*(r)-1)}{M} z^{M+1}.$$

As with (4.3), there exists a postive entire large radial solution of

$$\Delta v = v + \frac{(q^*(r)-1)}{M} v^{M+1}$$

since q^* satisfies (4.4). We now show that $v \leq u_k$ on $|x| \leq k$ for all k which, in turn, yields $v \leq u$ on $\mathbb{R}^n$, giving the desired result. Thus, suppose that for some k, it is not the case that $v \leq u_k$ on $|x| \leq k$;

i.e., $\max_{|x|\le k}(v(|x|) - u_k(x)) > 0$. Since this maximum cannot occur when $|x| = k$, it must occur in $|x| < k$. At that point then we have, using inequality (4.5),

$$0 \ge \Delta(v - u_k) = v + \frac{(q^* - 1)}{M} v^{M+1} - u_k^q \ge v^q - u_k^q > 0$$

which provides the needed contradiction.

In the theorem below C stands for the following positive constant.

$$2\sqrt{N} \int_1^\infty \frac{1}{\sqrt{t^{\alpha+1} - 1}}\, dt \le \frac{C}{\alpha - 1} \tag{4.6}$$

independently of $1 < \alpha \le 7$. We also need the following lemma.

Lemma 4.1 *Suppose $1 < p(r) \le q(r)$ and w, z are radial solutions, respectively, of*

$$\Delta w = w^{p(r)}, \quad w(0) = 1,\ w'(0) = 0 \quad \text{and} \quad \Delta z = z^{q(r)}, \quad z(0) = 1,\ z'(0) = 0.$$

Then $w \le z$ on any interval $[0, R)$ on which both solutions exist.

Outline of Proof. Define the sequence $\{z_k\}$ as the (unique) radial solution of

$$\Delta z_k = z_k^{q(r)}, \quad z_k(0) = 1 + 1/k,\ z_k'(0) = 0$$

and note that it is easy to prove that $w \le z_k$ on every interval on which both solutions exist since $w(0) < z_k(0)$. We can then use a standard argument to get $w(r) < z_k(r)$ for all r for which both solutions exist. Then, by the continuous dependence on initial data, we know $z(r) = \lim_{k\to\infty} z_k(r)$ and hence $w \le z$.

Theorem 4.4 *Suppose that q is a non-increasing function such that $q(r) \ge 1 + Cr^{-1}$ for $r \ge R$ and some $R > 0$, and $q(r) \to 1$ as $r \to \infty$. Any solution u of problem (1.1) satisfies*

$$u \le v, \qquad r \ge 0,$$

where v is a solution of (1.1) such that $v(0) = 1$ with $q \equiv 1$.

Proof. For each $j \in \mathbb{N}$, let v_j be the solution to

$$v j \Delta v = v^{q(j)}, \qquad v(0) = 1, \quad v'(0) = 0. \tag{4.7}$$

Let $[0, R_j)$ be the maximal interval of existence for v_j. Integrating the equation $(r^{N-1} v')' = r^{N-1} v_j^{q(j)}$ on $(0, r)$ for any $0 < r < R_j$, we find that

$$v_j'(r) \le \frac{r}{N} v_j^{q(j)}, \quad 0 \le r < R_j.$$

Using this in the equation

$$v_j'' + \frac{N-1}{r} v_j' = v_j^{q(j)},$$

we get

$$v_j'' \geq \frac{1}{N} v_j^{q(j)}.$$

Multiplying both sides by v_j' and integrating this inequality on $(0, r)$ we obtain

$$\frac{v_j'(r)}{\left(v_j^{(q(j)+1)} - 1\right)^{1/2}} \geq \sqrt{\frac{2}{N(q(j)+1))}} \geq \frac{1}{2\sqrt{N}} \qquad \text{for } j \text{ large enough that } q(j) \leq 7.$$

We now integrate this last inequality on $(0, R_j)$ and get

$$\frac{R_j}{2\sqrt{N}} \leq \int_1^\infty \frac{dt}{\sqrt{t^{q(j)+1} - 1}}.$$

Taking (4.6) into account, and recalling that $q(r) \geq 1 + Cr^{-1}$ for $r \geq R$, we conclude that

$$R_j \leq \frac{C}{q(j) - 1} \leq j, \qquad \text{provided that } \ j \geq R.$$

Therefore, we see that $q(r) \geq q(j)$ for $r \in [0, R_j)$. Furthermore, $R_j \to \infty$ as $j \to \infty$. To see this, first we integrate the inequality $v_j'' v_j' \leq v_j^{q(j)} v_j'$ on $(0, r)$ to get

$$\frac{v_j'}{\sqrt{v_j^{q(j)+1}}} \leq \sqrt{2}.$$

Next we integrate the last inequality on $(0, R_j)$ to find that

$$\frac{2}{q(j) - 1} = \int_1^\infty \frac{dt}{\sqrt{t^{q(j)+1}}} \leq \sqrt{2} R_j.$$

Therefore, indeed, $R_j \to \infty$ as $j \to \infty$.

Going back to the solutions of (3.7), we note that, by Lemma 4.1,

$$1 \leq v_{j+1}(r) \leq v_j(r) \qquad r \in [0, R_j),$$

and hence $R_j \leq R_{j+1}$. Consequently, we also have

$$v_{j+1}' \leq v_j', \qquad r \in [0, R_j).$$

Let

$$v := \lim_{j \to \infty} v_j(r), \qquad r \geq 0.$$

Then it follows that

$$v'(r) = \lim_{j \to \infty} v_j'(r), \qquad r \geq 0.$$

In fact we see that v is a solution of

$$\Delta v = v, \quad r > 0, \qquad v(0) = 1, \quad v'(0) = 0.$$

Now let u be any non-negative solution of (1.1). We wish to show that $u \leq v$ for $r \geq 0$. We consider two cases.

Case 1. $u(0) \geq 1$. Then by Lemma 4.1, we have $u(r) \geq v_j(r)$ in $[0, R_j)$, which is not possible.

Case 2. $u(0) < 1 = v(0)$. Suppose that $u(t) > v(t)$ for some $t > 0$. Let

$$\hat{r} := \inf\{\varrho : u(\eta) > v(\eta) \ \ \forall\ \eta \in (\varrho, t]\}.$$

Then $0 < \hat{r},\ u(\hat{r}) = v(\hat{r})$ and $u(\eta) > v(\eta)$ for $\eta \in (\hat{r}, t]$. Let $r_0 \in (\hat{r}, t)$ such that $u'(r_0) > v'(r_0)$. Then $u(r_0) > v_j(r_0)$ and $u'(r_0) > v_j'(r_0)$ for some $j \geq j_0$, and some $j_0 \in \mathbb{N}$. Without loss of generality, we may take $j_0 \geq R$, and let us fix $j \geq j_0$. We claim that $u(r) \geq v_j(r)$ for $r_0 \leq r < R_j$. This would provide the needed contradiction. Suppose, then

$$\sigma := \sup\{T : u(r) > v_j(r) \ \ \forall\ r \in [r_0, T)\}.$$

Clearly $\sigma > r_0$. We claim that $\sigma = R_j$. Assume that $\sigma < R_j$. Then proceeding as in Lemma 2.4, we find that

$$\begin{aligned}
u'(\sigma) &= u'(r_0)\left(\frac{r_0}{\sigma}\right)^{N-1} + \sigma^{1-N}\int_{r_0}^{\sigma} s^{N-1}u(s)^{q(s)}\,ds \\
&\geq u'(r_0)\left(\frac{r_0}{\sigma}\right)^{N-1} + \sigma^{1-N}\int_{r_0}^{\sigma} s^{N-1}u(s)^{q(j)}\,ds \\
&\geq u'(r_0)\left(\frac{r_0}{\sigma}\right)^{N-1} + \sigma^{1-N}\int_{r_0}^{\sigma} s^{N-1}v_j(s)^{q(j)}\,ds \\
&= u'(r_0)\left(\frac{r_0}{\sigma}\right)^{N-1} - v_j'(r_0)\left(\frac{r_0}{\sigma}\right)^{N-1} + v_j'(\sigma) > v_j'(\sigma),
\end{aligned}$$

and this last conclusion contradicts the definition of σ.

5 An Example

In Section 3 we showed that problem (1.1) admits a positive solution provided that q is sublinear outside some ball. In particular, Theorem 3.1 establishes existence for q sublinear outside a given ball, provided $q \geq 1$ inside the ball. Similarly, Theorem 3.2 proves existence if q is sublinear outside some small ball, but inside $q - 1$ is allowed to change signs. In this section, we provide an example to show the near optimality of these results by constructing a class of radial exponents q for which Problem (1.1) fails to admit a positive solution in spite of the fact that q is sublinear outside some ball.

Suppose $p : [0, \infty) \to (0, \infty)$ is a continuous function having the following properties.

1. There is $\delta > 0$ such that $0 < p(r) < 1$ for $0 \leq r \leq \delta$.
2. There is $r_0 > \delta$ such that $p(r_0) > 1$.

For each $R > r_0$, we introduce a continuous function $\rho_R : [0, \infty) \to (0, \infty)$ as follows.

$$\rho_R(r) = \begin{cases} p(r) & 0 \le r \le r_0 \\ p(r_0) & r_0 \le r \le R \\ p(r - R + r_0) & r \ge R. \end{cases}$$

Theorem 5.1 *There is $R^* > 0$ such that if $R > R^*$, then problem (1.1) has no solution when $q(x) = \rho_R(|x|)$ on $\mathbb{R}^N$.*

Proof. For each $0 < \sigma < 1$ we let

$$\phi_\sigma(r) = a_\sigma r^{\frac{2}{1-\sigma}}, \quad r \ge 0, \qquad \text{where} \quad a_\sigma := \left[\frac{2}{1-\sigma}\left(\frac{2}{1-\sigma} + N - 2\right)\right]^{-1/(1-\sigma)}.$$

We note that $0 < \sigma_1 \le \sigma_2 < 1$ implies that $a_{\sigma_1} \ge a_{\sigma_2}$, and therefore

$$\phi_{\sigma_2}(r) \le \phi_{\sigma_1}(r) \qquad \text{for } r \in [0, 1]. \tag{5.1}$$

Moreover, a direct computation shows that

$$(r^{N-1}\phi'_\sigma)' = r^{N-1}\phi_\sigma^\sigma, \qquad r > 0, \qquad \phi_\sigma(0) = 0, \quad \phi'_\sigma(0) = 0.$$

For notational simplicity, in what follows we will write ρ for ρ_R. Let

$$\alpha := \min\{\rho(s) : 0 \le s \le \delta\} \qquad \text{and} \qquad \beta := \max\{\rho(s) : 0 \le s \le \delta\}.$$

Since $0 < \alpha \le \beta < 1$, we note that for $r \in [0, \varepsilon]$, where $\varepsilon = \min(1, \delta)$,

$$\phi_\alpha^\alpha(r) \ge \phi_\alpha^{\rho(r)} \quad \text{and} \qquad \phi_\beta^\beta(r) \le \phi_\beta^{\rho(r)}(r).$$

Let us also note (see (5.1)) that

$$\phi_\beta \le \phi_\alpha, \qquad r \in [0, 1].$$

Let $\psi_0 = \phi_\alpha$, and for $j \ge 1$, define a sequence $\{\psi_j\}$ on $[0, \varepsilon]$, iteratively, as follows.

$$\psi_j(r) = \int_0^r t^{1-N} \int_0^t s^{N-1}\psi_{j-1}(s)^{\rho(s)}\, ds\, dt.$$

By induction it is easy to show that

$$\phi_\beta \le \psi_j \le \psi_{j-1} \le \phi_\alpha \qquad \text{on} \quad [0, \varepsilon].$$

Therefore, on letting $\psi(r) = \lim_{j\to\infty} \psi_j(r)$ for $0 \le r \le \varepsilon$, we see that ψ is a solution to the following initial value problem

$$(r^{N-1}\psi'(r))' = r^{N-1}\psi(r)^{\rho(r)} \qquad \text{for } 0 < r < \varepsilon, \quad \text{and} \quad \psi(0) = 0, \quad \psi'(0) = 0$$

such that $\phi_\beta \le \psi \le \phi_\alpha$ on $[0, \varepsilon]$. Now define $\widehat{p} : [0, \infty) \to (0, \infty)$ as follows.

$$\widehat{p}(r) = \begin{cases} p(r) & \text{if } 0 \le r \le r_0 \\ p(r_0) & \text{if } r \ge r_0. \end{cases} \tag{5.2}$$

Since $\widehat{p}(r) = p(r_0) > 1$ for $r \geq r_0$, notice that the problem

$$\Delta w = w^{\widehat{p}(|x|)}, \quad w(0) = 0, \quad w'(0) = 0 \tag{5.3}$$

has no non-trivial and non-negative entire solution since, if it did, then the solution would necessarily be large, and according to Theorem 4.1, this is not possible. Recall that ψ is a solution of problem (5.3) on the ball $B(0,\varepsilon)$. Since $\psi(\varepsilon) \leq \phi_\alpha(\varepsilon)$, it is clear that ψ can be extended to a solution of (5.3) beyond the ball $B(0,\varepsilon)$. Let w be such an extension, and let $[0, R_0)$ be the maximal interval of existence of w. Since, as observed above, problem (5.3) cannot have a non-trivial solution it is clear that $R_0 < \infty$. Since w is non-negative and non-decreasing on $[0, R_0)$ we see that $w(r) \to \infty$ as $r \to R_0^-$. We now show that for any $R > R^* := \max\{r_0, R_0\}$, problem (1.1), with $q(x) := \rho_R(|x|)$, has no positive entire radial solution. Indeed, suppose u is such a solution. Observe that

$$\widehat{p}(r) = \rho_R(r), \quad 0 \leq r \leq R.$$

Clearly $w(0) = 0 < u(0)$, so that there must exist an interval $[0, b]$, maybe quite small, on which $w \leq u$. Let

$$B := \sup\{b : b < R_0 \text{ and } w(r) \leq u(r) \text{ for all } r \in [0, b]\}.$$

Then $B \leq R_0$. If $B = R_0$, we are done. Suppose then $B < R_0$. Then we have

$$\begin{aligned} w(B) &= \int_0^B t^{1-N} \int_0^t s^{N-1} w(s)^{\widehat{p}(s)}\, ds\, dt = \int_0^B t^{1-N} \int_0^t s^{N-1} w(s)^{\rho_R(s)}\, ds\, dt \\ &\leq \int_0^B t^{1-N} \int_0^t s^{N-1} u(s)^{\rho_R(s)}\, ds\, dt \\ &< u(0) + \int_0^B t^{1-N} \int_0^t s^{N-1} u(s)^{\rho_R(s)}\, ds\, dt = u(B). \end{aligned}$$

Therefore $w \leq u$ on the interval $[0, B + \delta)$ for some $\delta > 0$. This contradicts the definition of B, and thus $w \leq u$ on $[0, R_0)$. But then u cannot be entire since $R_0 < \infty$ and $w(r) \to \infty$ for $r \to R_0$.

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