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The Sturm-Liouville Hierarchy of Evolution Equations

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Abstract

We introduce a hierarchy of evolution equations based on the Sturm-Liouville equation $-(p\varphi')' + q\varphi = \lambda y\varphi$. Our hierarchy includes the Korteweg-de Vries (K-dV) and the Camassa-Holm (CH) hierarchy. We determine a class of solutions of the hierarchy which are of algebro-geometric type. The initial condition of such a solution is drawn from a finite-gap isospectral class of the Sturm-Liouville equation..

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1 Introduction

The goal of this paper is to introduce, describe and solve a new hierarchy of evolution equations which is derived from a zero-curvature condition and is based on the general Sturm-Liouville equation

$$-(p\varphi')' + q\varphi = \lambda y\varphi, \qquad ' = \frac{d}{dx}. \tag{1.1}$$

Here λ is a spectral parameter and the coefficients p, q, y determine a "potential" $a = (p, q, y)$ which consists of three functions of x. The hierarchy contains in a natural way the well-known Korteweg-de Vries (K-dV) and Camassa-Holm (CH) hierarchies.

In a previous paper [24], one of us studied the CH-hierarchy, beginning with the so-called acoustic equation

$$-\varphi'' + \varphi = \lambda y \varphi. \tag{1.2}$$

Clearly (1.2) arises from (1.1) by setting $p = q = 1$. In [24], a detailed study of the algebro-geometric solutions of the CH-hierarchy was carried out. Such a solution has as initial condition a stationary solution of a fixed equation in the hierarchy. Each of these stationary solutions may be described in terms of quantities related to a hyperelliptic Riemann surface, which in turn is determined in a natural way from the spectral theory of (1.2).

In this paper, we generalize in an essential way the results contained in [24]. We use a procedure which is formally similar to that exploited in [24] but which is more complex in its details. We depart from the equation (1.1), and construct a hierarchy of evolution equations. Each of these is determined from a certain zero-curvature relation which involves (1.1). The flows determined by these zero-curvature relations are mutually commutative. For each $r \geq 1$, we determine classes of algebro-geometric solutions of the r-th order evolution equation, as follows. For each fixed $g > r$, introduce the set $\mathfrak{M}_g$ of stationary solutions of the g-th order equation in the hierarchy. These can be described in terms of data related to a hyperelliptic Riemann surface $\mathcal{R}$ of genus g, which in turn can be defined directly from the spectral problem (1.1). The motion induced by the r-th order equation on $\mathfrak{M}_g$ can be expressed via the motion of g "poles" $P_1(t, x), \ldots, P_g(t, x)$ in $\mathcal{R}$ which satisfy an explicit system of ordinary differential equations. Further, the pole motion can be transferred via a generalized Abel map so as to take place in a generalized Jacobi variety of $\mathcal{R}$. This leads to a good understanding of the r-th order motion on $\mathfrak{M}_g$. We remark that isospectrality of the flow generated by this kind of equation was first discovered in [10] and then applied with success by other researchers (see, for instance [6, 4]).

We wish to emphasize the following points. First, the study of the hierarchy of evolution equations based on (1.1) requires a substantial generalization of the techniques used in [24] to study the hierarchy based on (1.2). Second, our approach to the study of the sets $\mathfrak{M}_g$ of "algebro-geometric potentials" is based firmly on elementary properties of the spectral problem (1.1). Especially, the Weyl m-functions play an essential role, both conceptually and in terms of facilitating calculations. In fact, the Riemann surface $\mathcal{R}$ is constructed directly from the m-functions when the spectral problem (1.1) satisfies the appropriate conditions (which are of course valid if the "potential" $a = (p, q, y)$ lies in $\mathfrak{M}_g$). Third, we work out the necessary facts concerning the spectral theory of (1.1) by making use of the methods of nonautonomous dynamical systems (see, e.g., [7, 13, 14]). Apart from their convenience in the present context, their use leads one to consider interesting questions regarding the recurrence properties of the algebro-geometric potentials, and the corresponding solutions of the evolution equations in the hierarchy. Such recurrence properties were studied in [24] in the context of the algebro-geometric potentials corresponding of the CH hierarchy, and for the solutions of the CH equation itself. The question of recurrence properties of a general algebro-geometric potential $a = (p, q, y)$ was taken up in [14]. However, the solutions of the various evolution equations of this paper have not yet been studied from the point of view of recurrence.

Fourth and finally, our hierarchy contains not only that of Camassa-Holm ($p = q = 1$) but also that of Korteweg-de Vries. In fact, if we set $p = y = 1$, then (1.1) becomes the Schrödinger equation.

It is well-known that the K-dV hierarchy can be constructed via a zero-curvature condition involving the Schrödinger equation [6]. We can do more: if we set $p = \varepsilon$ and $y = 1$, and then let $\varepsilon \to 0$ we obtain a "Lax-Levermore" hierarchy. This object may perhaps be fruitfully studied using the results and methods of this paper.

The paper is organized as follows. In Section 2, we discuss some preliminary material concerning equation (1.1). in particular, we repeat the necessary statements regarding the Weyl m-functions. In Section 3 we derive a hierarchy of stationary equations departing from the spectral problem (1.1). The solutions of each such stationary equation turn out to determine a class of algebro-geometric Sturm-Liouville potentials. In Section 4, we introduce our (nonstationary) hierarchy of evolution equations determined by (1.1), and discuss the solutions which arise when the initial condition is an algebro-geometric solution of an appropriate stationary equation. In Section 5, we describe the flow determined by the solutions of the r-th order equation of the hierarchy by transferring it to an appropriate generalized Jacobi variety related to the spectral problem (1.1).

2 Preliminaries

In this section we provide the background material which will be necessary throughout the paper. Basically, here we will describe a procedure which allows to solve an inverse problem for a general Sturm-Liouville problem. In passing, we will introduce an algebro-geometric structure which will be very important in the interpretation of many objects we will meet in the course of our story. Throughout all the paper, we will denote by D (or by the symbol $'$) the operator of differentiation with respect to x. Let $\mathcal{E}_3$ be the set $\mathcal{E}_3 = \{a = (p, q, y) : \mathbb{R} \to \mathbb{R}^3 \mid a$ is uniformly continuous and bounded, $p \in C^1(\mathbb{R})$, $0 < \delta = \inf \min\{p(x), y(x) \mid x \in \mathbb{R}\}\}$, equipped with the standard topology of convergence on compact subsets of $\mathbb{R}$. If $a \in \mathcal{E}_3$, let us consider the operator $L_a : \mathcal{D} \to L^2(\mathbb{R}, ydx)$ defined as

$$L_a(\varphi) = \frac{1}{y}\left[-DpD + q\right]\varphi,$$

with domain $\mathcal{D} = \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \in L^2(\mathbb{R}, ydx)$, φ is absolutely continuous and $\varphi'' \in L^2(\mathbb{R}, ydx)\}$. With a usual abuse of terminology, we will refer to the elements of $\mathcal{E}_3$ as *potentials*. Then L_a admits a unique self-adjoint extension to all $L^2(\mathbb{R}, ydx)$. We will slightly abuse notation, and continue to denote with L_a this extension. Let Σ_a be the spectrum of L_a. Then $\Sigma_a \subset \mathbb{R}$, Σ_a is bounded below and unbounded above, the set $R_a = \mathbb{R} \setminus \Sigma_a$ is at most a countable union of disjoint open (possibly unbounded) real intervals. Moreover, Σ_a is invariant under translations of $a \in \mathcal{E}_3$. In general, it is not easy to obtain more detailed information about the spectrum and the structure of the function $a \in \mathcal{E}_3$ when a is chosen to be a general element of $\mathcal{E}_3$. Hence, if one wants to have detailed knowledge of the spectral properties of L_a, it is necessary to restrict to appropriate subsets of $\mathcal{E}_3$. For example, if $\mathcal{P} \subset \mathcal{E}_3$ consists of the periodic potentials, then one is able to say "almost everything" about the spectra and the potentials which lie in $\mathcal{P}$.

We note that the special case $a = (1, q, 1)$ which defines the Schrödinger operator $L_q = -D + q$ has received detailed attention for many years. Certain aspects of the theory of the more general operator L_a have been intensively studied in recent times, the essential motivation being the relation between the Camassa-Holm hierarchy and the operator $L_y = \frac{1}{y}(-D^2 + 1)$ which we described in the Introduction. It is worthwhile however to apply for the analysis of the operators L_a because, even in the simplest cases, their structure is very much more variegated and complex than the case L_q (see, for instance [14], where it is shown that algebro-geometric ergodic processes may have properties which are far from being trivial).

There is a good deal of results which have been recently obtained about recurrent Sturm-Liouville operators (see [13, 7, 15]). There are a number of important concepts of recurrence. A general type is that of (positive and negative) Poisson recurrence. Consider for example the following Bebutov-type construction. Let us choose an element $a_0 \in \mathcal{E}_3$. Consider the set $\mathcal{A} = cls\ Hull(a_0)$ with respect to the translation flow: $\tau_x(a)(\cdot) = a(x + \cdot)$ for every $a \in \mathcal{E}_3$. Then $\mathcal{A}$ is a compact, translation invariant subset of $\mathcal{E}_3$. Let μ be an ergodic measure on $\mathcal{A}$ and, if necessary, redefine $\mathcal{A}$ to be the topological support of μ. Then there exists a set $\mathcal{A}_1 \subset \mathcal{A}$ with $\mu(\mathcal{A}_1) = 1$ such that each $a \in \mathcal{A}_1$ is (both positively and negatively) Poisson recurrent.

For the moment, we will introduce some terminology and concepts which have been revealed to be useful in the study of the operators L_a. If $a = (p, q, y) \in \mathcal{E}_3$, the eigenvalue equation

$$E_a(\varphi, \lambda) := -DpD\varphi + q\varphi - \lambda y\varphi = 0$$

can be expressed in matrix form as follows:

$$X' = \begin{pmatrix} 0 & 1/p \\ q - \lambda y & 0 \end{pmatrix} X,$$

where $X = \begin{pmatrix} \varphi \\ p\varphi' \end{pmatrix}$. From now on, we will use the identification

$$a(x) = \begin{pmatrix} 0 & 1/p(x) \\ q(x) - \lambda y(x) & 0 \end{pmatrix}$$

whenever $a \in \mathcal{E}_3$. The context will clarify if $a : \mathbb{R} \to \mathbb{R}^3$ or if a is a matrix-valeud function which depends on the parameter $\lambda \in \mathbb{C}$ as above.

Define a function $A : \mathcal{E}_3 \to \mathbb{M}(2, \mathbb{R}) : a \mapsto a(0)$. If $\{\tau_x\}_{x\in\mathbb{R}}$ is the translation flow defined on $\mathcal{E}_3$ and if $a \in \mathcal{E}_3$, then $a(x) = A(\tau_x(a))$. Now, let us apply the same construction of Bebutov type sketched in the example above, which will allow to use the instruments of dynamical systems to study of the operator L_a (and of the corresponding eigenvalue equation $E_a(\varphi, \lambda)$). Let $a_0 \in \mathcal{E}_3$ be fixed, and consider the set $\mathcal{A} = clsHull(a_0)$ with respect to the translation flow, as above. Being $\mathcal{A}$ a compact and translation-invariant subset of $\mathcal{E}_3$ (compactness of $\mathcal{A}$ follows from the fact that a is uniformly continuous only), it makes sense to speak of the family of differential systems

$$\begin{pmatrix} \varphi \\ p\varphi' \end{pmatrix}' = A(\tau_x(a)) \begin{pmatrix} \varphi \\ p\varphi' \end{pmatrix}, \qquad a \in \mathcal{A}. \tag{2.1}$$

The study of the operator L_a can be carried out by using the instruments of dynamical systems. The most important tool is that of the exponential dichotomy (see [22]. Let $\Phi_a(x)$ be the fundamental matrix solution of the family (2.1) satisfying $\Phi_a(0) = Id$ for every $a \in \mathcal{A}$.

Definition 2.1 The family (2.1) is said to have an *exponential dichotomy* over $\mathcal{A}$ if there are positive constants η, δ together with a continuous, projection-valued function $P : \mathcal{A} \to \mathbb{M}_2$ (thus $P(a)^2 = P(a)$ for all $a \in \mathcal{A}$) such that the following estimates hold:

(i) $|\Phi_a(t)P(a)\Phi_a(s)^{-1}| \le \eta e^{-\delta(t-s)}, \qquad t \ge s,$

(ii) $|\Phi_a(t)(I - P(a))\Phi_a(s)^{-1}| \leq \eta e^{\delta(t-s)}, \qquad t \leq s.$

■

The exponential dichotomy concept is very important in the study of the dynamical and spectral properties of the family (2.1). In particular, one can prove the following important result (see [12], also [7]):

Proposition 2.1 *Let $\mathcal{A}$ be defined as above and let us consider the family* (2.1). *Let $a \in \mathcal{A}$ have dense orbit. Then the spectrum Σ_a of the operator L_a equals the set*

$$\Sigma_{ed} := \{\lambda \in \mathbb{C} \mid \text{the family (2.1) does } \underline{\text{not}} \text{ admit an exponential dichotomy over } \mathcal{A}\}.$$

It is not difficult to prove that if $\Im\lambda \neq 0$, then the family (2.1) admits an exponential dichotomy over $\mathcal{A}$. Hence $\Sigma_{ed} \subset \mathbb{R}$. Moreover, the definition of exponential dichotomy allows to introduce in a very convenient way the Weyl m-functions and the Green's function.

Let us fix the Dirichlet boundary condition $\varphi(0) = 0$. Then one can define self-adjoint operators $L_a^+ : L^2(\mathbb{R}^+, ydx) \to L^2(\mathbb{R}^+, ydx)$ and $L_a^- : L^2(\mathbb{R}^-, ydx) \to L^2(\mathbb{R}^-, ydx)$. It is well-known that, if $\Im\lambda \neq 0$, the Weyl m-functions $m_\pm(a, \lambda)$ are defined as those complex numbers such that

$$m_+(a) = \frac{p(0)D\varphi_a^+(0)}{\varphi_a^+(0)}, \qquad m_-(a,\lambda) = \frac{p(0)D\varphi_a^-(0,\lambda)}{\varphi_a^-(0,\lambda)},$$

where $\varphi_a^+(x,\lambda)$ (resp. $\varphi_a^-(x,\lambda)$) is the unique (up to a constant multiple) solution of the equation $E_a(\varphi, \lambda) = 0$ which lies in $L^2(\mathbb{R}^+, ydx)$ (resp. $L^2(\mathbb{R}^-, ydx)$). There is a concrete way of relating the Weyl m-functions to the idea of exponential dichotomy. If $\Im\lambda \neq 0$ and $a \in \mathcal{A}$, then both Ker P(a) and Im P(a) are lines in $\mathbb{C}^2$ (this follows from the fact that $\det \Phi_a(x) = 1$ for all $x \in \mathbb{R}$ and $a \in \mathcal{A}$). It is easy to show that these lines can be parametrized by

$$\text{Im P(a)} = \text{Span}\begin{pmatrix} 1 \\ m_+(a,\lambda)\end{pmatrix}, \qquad \text{Ker P(a)} = \text{Span}\begin{pmatrix} 1 \\ m_-(a,\lambda)\end{pmatrix}.$$

We use the convention that $m_+(a,\lambda) = \infty$ (resp. $m_-(a,\lambda) = \infty$) if and only if Im P(a) = Span $\begin{pmatrix} 0 \\ 1\end{pmatrix}$ (resp. Ker P(a) = Span $\begin{pmatrix} 0 \\ 1\end{pmatrix}$): actually this possibility can only occur if $\Im\lambda = 0$, in particular if φ_a^+ (resp. φ_a^-) is an eigenfunction of L_a^+ (resp L_a^-) and hence λ is an eigenvalue of L_a^+ (resp. L_a^-). It is possible to prove that the functions $m_\pm(a,\lambda)$ admit nontangential limits $m_\pm(a,\eta) := \lim_{\varepsilon\to 0} m_\pm(a, \eta + i\varepsilon)$ for a.a. $\eta \in \mathbb{R}$. Moreover $Sign(\Im m_\pm(a,\lambda)\Im\lambda) = \pm 1$ for every λ with $\Im\lambda \neq 0$. Further, it is possible to relate the imaginary parts of $m_\pm(a,\eta)$ to the Radon-Nikodym derivative of the spectral measures $\rho_\pm(a,\eta)$, more precisely to their absolutely continuous parts $\rho_\pm^{ac}$. Indeed, if ℓ denotes the Lebesgue measure one has

$$\frac{d\rho_\pm^{ac}}{d\ell}(\eta) = \frac{1}{\pi}\Im m_\pm(a,\eta)$$

for a.a. $\eta \in \mathbb{R}$.

Now we make use of the translation flow τ_x to determine functions $m_\pm : \mathbb{R} \times (\mathbb{C} \setminus \mathbb{R}) \to \mathbb{C}$: $(x,\lambda) \mapsto m_\pm(\tau_x(a),\lambda)$. We will simply write $m_\pm(x,\lambda)$ when no confusion arises. Thus, if $\Im\lambda \neq 0$,

$m_+(x,\lambda)$ and $m_-(x,\lambda)$ span the complex lines Im $\mathrm{P}(\tau_\mathrm{x}(\mathrm{a}))$ and Ker $\mathrm{P}(\tau_\mathrm{x}(\mathrm{a}))$ respectively. Let us set

$$\varphi_\pm(x,\lambda) = \exp\left(\int_0^x \frac{m_\pm(s,\lambda)}{p(s)}ds\right).$$

Then $\varphi_\pm(x,\lambda)$ solve the eigenvalue equation $E_a(\varphi,\lambda)$ =0, $\varphi_\pm(0,\lambda) = 1$, $D\varphi_\pm(0,\lambda) = m_\pm(a,\lambda)$. From the definition of $m_\pm(x,\lambda)$, it follows that $\varphi_+(\cdot,\lambda) \in L^2(\mathbb{R}^+, ydx)$ and $\varphi_-(\cdot,\lambda) \in L^2(\mathbb{R}^-, ydx)$.

The functions $m_\pm(x,\lambda)$ have singularities at the isolated eigenvalues of the operators $L^\pm_{\tau_x(a)}$. If we start from a fixed $a \in \mathcal{E}_3$, and if $P(a)$ is an eigenvalue of L^+_a (resp. L^-_a), then by acting on a by translation ($a \mapsto \tau_x(a)$) we obtain a *moving eigenvalue* $P(x,\lambda) := P(\tau_x(a))$. It is nowadays a matter of fact that in many cases one can recover the potential a from *two of its spectra*, i.e., one can express a by means of some data which are related to the whole-line spectrum and to the half-lines spectra (see, for instance, [21]). We are interested in this place in a particular class of this kind of potentials, which we introduce in the next lines.

The diagonal zero-value of the Green's function $\mathcal{G}_a(\lambda)$ for the operator L_a can be defined as

$$\mathcal{G}_a(\lambda) := \frac{p(0)}{m_-(a,\lambda) - m_+(a,\lambda)} \qquad (\Im\lambda \neq 0)$$

and corresponds to the value $\mathcal{G}_a(0,0,\lambda)$ of the classical Green's function for the operator L_a. Actually also $\mathcal{G}_a(\lambda)$ admits nontangential limits

$$\mathcal{G}_a(\eta) := \lim_{\varepsilon\to\infty} \mathcal{G}_a(\eta + i\varepsilon)$$

for a.a. $\eta \in \mathbb{R}$. Again, acting by the translation, we have the function

$$\mathcal{G}(x,\lambda) := \frac{p(x)}{m_-(x,\lambda) - m_+(x,\lambda)},$$

which represents the diagonal zero-value of the Green's function of the operator $L_{\tau_x(a)}$. Moreover, it is easy to show that $\mathcal{G}(x,\lambda)$ equals the diagonal Green's function $\mathcal{G}(x,x,\lambda)$ of the operator L_a.

The behavior of the Green's function at real values determines particularly important classes of potentials $a \in \mathcal{A}$ (see [13, 7]).

Assume now that $a \in \mathcal{E}_3$ satisfies the following conditions:

(H1) the spectrum Σ_a of the operator L_a is a finite union of g disjoint closed intervals plus an halfline: $\Sigma_a = [\lambda_0,\lambda_1] \cup [\lambda_2,\lambda_3],\ldots,[\lambda_{2g},\infty)$, $(0 < \lambda_0 < \lambda_1 < \ldots < \lambda_{2g})$.

(H2) $\Re\mathcal{G}_a(x,\eta) = 0$ for a.a. $\eta \in \Sigma_a$ and for all $x \in \mathbb{R}$.

If $a \in \mathcal{E}_3$ satisfies (H1) and (H2), then Σ_a is the spectrum of all the operators $\tau_x(a)$ ($x \in \mathbb{R}$). The isolated eigenvalues of the half-line restricted operators $L^\pm_a$ all lie in the so called *spectral gaps* $[\lambda_1,\lambda_2],\ldots,[\lambda_{2g-1},\lambda_{2g}]$. It can be proved (see [12], also [7]) that there is exactly one isolated eigenvalue $P_i(a)$ in each spectral gap $[\lambda_{2i-1},\lambda_{2i}]$ ($i = 1,\ldots,g$). If a certain $P_i(a)$ is an eigenvalue of L^+_a then it is not an eigenvalue of L^-_a and vice versa. Moreover the points $P_i(a)$ are alternatively eigenvalues of L^+_a and L^-_a. For $x \in \mathbb{R}$, the translated potentials $\tau_x(a)$ determine the eigenvalues $P_i(x) := P_i(\tau_x(a))$ ($i = 1,\ldots,g$). Clearly we are left with maps $P_i : \mathbb{R} \to [\lambda_{2i-1},\lambda_{2i}]$. The x-derivative of an eigenvalue $P_i(x)$ is zero when it reaches either λ_{2i-1} or λ_{2i}. In this case it *transforms* into an eigenvalue of L^-_a (resp. L^+_a) and starts moving towards the opposite endpoint in the interval.

A correct way of interpreting this motion is that of seeing each P_i as a map with values in a circle c_i which is determined from the interval $[\lambda_{2i-1}, \lambda_{2i}]$. The circle c_i is a subset of a certain Riemann surface $\mathcal{R}$. Thus P_i will be viewed as a map from $\mathbb{R}$ to $\mathcal{R}$.

We explain this in detail. Let $k(\lambda) = \sqrt{-(\lambda - \lambda_0)(\lambda - \lambda_1) \dots (\lambda - \lambda_{2g})}$. Then $k(\lambda)$ assumes real values in the spectral intervals $[\lambda_0, \lambda_1]$, $[\lambda_2, \lambda_3], \dots, [\lambda_{2g}, \infty)$ and has purely imaginary values in the spectral gaps $\mathbb{R} \setminus \Sigma_a$. Let $\mathcal{R}$ be the Riemann surface of the algebraic relation $w^2 = -(\lambda - \lambda_0)(\lambda - \lambda_1) \dots (\lambda - \lambda_{2g})$. Then $\mathcal{R}$ is obtained by cutting two copies of the Riemann sphere $\hat{\mathbb{C}}$ along the intervals $[\lambda_{2i}, \lambda_{2i+1}]$ $(i = 0, \dots, g-1)$ and $[\lambda_{2g}, \infty]$ and glueing together the spheres in a standard way. The resulting surface $\mathcal{R}$ is a torus with g holes corresponding to the spectral gaps $[\lambda_{2i-1}, \lambda_{2i}]$ $(i = 1, \dots, g)$, hence $\mathcal{R}$ has ramification points $\lambda_0, \lambda_1, \dots, \lambda_{2g}, \infty$. These holes are circles c_i $(i = 1, \dots, g)$.

Let π denote the standard projection from $\mathcal{R}$ to the Riemann sphere. Then π is 2-1, except at the ramification points where it is 1-1. Each point $P \in \mathcal{R}$ has a unique image under π, namely $\pi(P) = \lambda$, and $\pi^{-1}(\lambda) = \{P^+, P^-\}$ for every $\lambda \in \mathbb{C}$ except for the ramification points (one can define points in $\mathcal{R}$ as having two coordinates: $P^+ = (\lambda, k(\lambda))$ and $P^- = (\lambda, -k(\lambda))$, where $\pi^{-1}(\lambda) = \{P^+, P^-\}$). Let us extend the function $k(\lambda)$ to $\mathcal{R}$ in such a way that it is single-valued on $\mathcal{R}$. This can be done by defining $k(0^+)$ to be the positive square root of $\sqrt{\lambda_0 \lambda_1 \dots \lambda_{2g}}$, then defining a function $k(P)$ $(P \in \mathcal{R})$ by analytic continuation along paths from 0^+ to P in such a way that $k(P) = k(\pi(\lambda))$ is the appropriate value of the square root (if $\pi^{-1}(\lambda) = \{P^+, P^-\}$, then $k(P^+)$ is the positive square root $k(\lambda)$ and $k(P^-)$ is the negative one). One can pass from one sheet to another in $\mathcal{R}$ by the hyperelliptic involution $\sigma(P) : w(\pi(P)) \mapsto -w(\pi(P))$.

The circles c_i have the property that a point moving along them changes its coordinates each time it crosses the values $\lambda_{2i-1}, \lambda_{2i}$. We introduce an angular coordinate $\theta_i \in [0, 2\pi]$ in c_i in such a way that if a point P lies in c_i with angular coordinates between 0 and π, then $k(P)$ is positive, while if P has an angle between π and 2π, then $k(P)$ is negative.

Let us return now to the moving eigenvalues $P_i(x)$ of the spectral problem under consideration. It is understood that each time we speak of $P_i \in \mathcal{R}$, then P_i is the appropriate value of $\pi^{-1}(P_i)$, as explained above. If we make use of the angular coordinate θ_i in c_i which was introduced above, then the map $x \mapsto P_i(x) : \mathbb{R} \to \mathcal{R}$ can be coordinatized by the following relation

$$\pi(P_i(x)) = (\lambda_{2i-1} - \lambda_{2i}) \sin^2 \frac{\theta_i(x)}{2} + \lambda_{2i}, \qquad x \in \mathbb{R},\ \theta_i(x) \in [0, 2\pi]. \tag{2.2}$$

The equation (2.2) and the sign of $k(P_i(x))$ *define* the angular coordinate $\theta_i(x)$ of the eigenvalue $P_i(x) \in \mathcal{R}$.

From now on, we will denote by Σ_0 the set $[\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \dots \cup [\lambda_{2g}, \infty)$. We will write $\mathcal{S}_{\Sigma_0}$ for the subset of $\mathcal{E}_3$ consisting of those potentials $a \in \mathcal{E}_3$ which satisfy (H1) and (H2) with $\Sigma_a = \Sigma_0$. Let $a \in \mathcal{S}_{\Sigma_0}$. A detailed study of the properties of the operator L_a allows to reconstruct a in terms of the spectral parameters $\{\lambda_0, \lambda_1, \dots, \lambda_{2g}, P_1(0), \dots, P_g(0)\}$. The reader can find all the details of this construction in [13, 7]. Here, we will sketch briefly the procedure, focusing on those parts which will be useful in the development of the present paper. To begin with, let us assume that we are given a potential $a \in \mathcal{S}_{\Sigma_0}$. For such a potential, let us denote by $P_1(0), \dots, P_g(0)$ the eigenvalues of the half-lines restricted operators. The spectral parameters $\{\lambda_0, \lambda_1, \dots, \lambda_{2g}, P_1(0), \dots, P_g(0)\}$ are then fixed. The assumption (H2) has some fundamental consequences: we list them here. First of all, any translated potential $\tau_x(a)$ lies in $\mathcal{S}_{\Sigma_0}$; secondly, it turns out that both $m_+(x, \cdot)$ and $m_-(x, \cdot)$ extend holomorphically through every open interval contained in the spectrum Σ_0. If we denote these extensions by $h_\pm(x,\cdot)$ then we have

$$h_+(x,\lambda) = \begin{cases} m_+(x,\lambda), & \Im\lambda > 0 \\ m_-(x,\lambda), & \Im\lambda < 0 \end{cases} \qquad \text{and} \qquad h_-(x,\lambda) = \begin{cases} m_-(x,\lambda), & \Im\lambda > 0 \\ m_+(x,\lambda), & \Im\lambda < 0. \end{cases}$$

This shows that, when crossing an interval contained in the spectrum, the extension of m_+ is m_- and vice versa. This property allows us to define $m_\pm(x,\cdot) : \mathcal{R} \to \hat{\mathbb{C}}$ and a single meromorphic function $M(x,\cdot) : \mathcal{R} \to \hat{\mathbb{C}}$ in such a way that

$$M(x,P) = m_+(x,P), \qquad M(x,\sigma(P)) = m_-(x,P).$$

The (simple) poles of $M(x,\cdot)$ are exactly the points $P_i(x) \in \mathcal{R}$ determined by the moving eigenvalues of the spectral problem under consideration. From now on, we will allow ourselves to abuse notation: when there can be no motive for confusion we will confound $\pi(P_i)$ with P_i.

Now, using the properties stated above, one can reconstruct $a \in \mathcal{S}_{\Sigma_0}$ by means of the spectral data. In particular, let us keep fixed the function $p : \mathbb{R} \to \mathbb{R}$. Further, let us write $\mathcal{M}(x) = m_-(x,0) - m_+(x,0)$. Then $\mathcal{M}(x)$ is real valued and continuous. It can be proved (see [13, 7]) that the eigenvalues $P_i(x)$ move according to the following equation

$$P_{r,x}(x) = \frac{(-1)^g k(P_r(x))\mathcal{M}(x)\prod_{i=1}^{g} P_i(x)}{p(x)k(0^+)\prod_{s\neq r}(P_r(x) - P_s(x))}, \qquad 1 \le r \le g. \tag{2.3}$$

These equations must be interpreted as system of ordinary differential equations, where each solution $P_r(x)$ takes values in the interval $[\lambda_{2r-1}, \lambda_{2r}]$ $(r = 1,\dots,g)$. The quantity $k(P_r(x))$ must be considered to have a different sign according to the angular coordinate of $P_r(x)$. To avoid any ambiguity which may derive from this fact, it is convenient to take the coordinatization for $P_r(x)$ given in (2.2), from which the formula

$$P_{r,x}(x) = \frac{1}{2}(\lambda_{2i-1} - \lambda_{2i}) \sin\theta_r(x)\theta_{r,x}(x) \tag{2.4}$$

follows. From (2.3) it follows that critical points of P_r are simple, hence $\theta_{r,x}(x)$ cannot vanish. We thus obtain the following equation for the motion of θ_r:

$$\theta_{r,x}(x) = \frac{(-1)^{g+1}\mathcal{M}(x)}{p(x)k(0^+)}\sqrt{P_r(x) - \lambda_0}\prod_{s\neq r}\frac{\sqrt{(P_r(x) - \lambda_{2s-1})(P_r(x) - \lambda_{2s})}}{P_r(x) - P_s(x)}, \tag{2.5}$$

from which the motion of the eigenvalue P_r can be calculated directly from (2.4). Here the square roots are positive.

Once we have determined the functions $P_r(x)$ $(r = 1,\dots,g)$ we obtain the so-called trace formulas, which express the remaining elements of $a \in \mathcal{S}_{\Sigma_0}$ in terms of the spectral data:

$$y(x) = \frac{\mathcal{M}^2(x)\prod_{i=1}^{g} P_i^2(x)}{4p^2(x)k^2(0^+)}, \tag{2.6}$$

and

$$q(x) = y(x)\Big(\sum_{i=0}^{2g} \lambda_i - 2\sum_{i=1}^{g} P_i(x)\Big) + q_g'(x) + \frac{q_g(x)^2}{p(x)}, \tag{2.7}$$

where $q_g = -\dfrac{(py)_x}{4y}$.

We finish this part by observing that the procedure can be inverted as well. We will not go into the details, and offer only a brief explanation [13]. Let us fix two strictly positive functions $p(x), \mathcal{M}(x)$ with some additional regularity properties. Let us choose real numbers

$$\lambda_0 < \lambda_1 < \tilde{P}_1 < \lambda_2 < \lambda_3 < \tilde{P}_2 < \lambda_4 < \ldots < \lambda_{2g}.$$

Let $P_1(x), \ldots, P_g(x)$ be the solution of the system (2.3) satisfying $P_r(0) = \tilde{P}_r$ for every $r = 1, \ldots, g$. Define $y(x)$ and $q(x)$ as in (2.6) and (2.7) respectively. Then the potential $a = (p, q, y)$ lies in $\mathcal{S}_{\Sigma_0}$.

We conclude this section by explaining briefly the idea which lies behind the concept of the zero curvature. We look for the solution of the system

$$\begin{cases} \Phi_x(t,x) = A(t,x)\Phi(t,x), \\ \Phi_t(t,x) = B(t,x)\Phi(t,x) \end{cases} \tag{2.8}$$

together with some initial data $\Phi(0,0)$. Here $A(t,x)$ and $B(t,x)$ are $n \times n$ complex-valued matrices depending of t, x and other parameters as well, and $\Phi(t,x)$ is the fundamental matrix solution of the equation $X' = A(t,x)X$. For the system (2.8) to have a solution, it is necessary and sufficient that a compatibility condition is fulfilled, namely that

$$A_t - B_x + [A, B] = 0, \tag{2.9}$$

where $[A, B] = AB - BA$ is the commutator of A and B. The equation (2.9) is the *zero-curvature equation*.

We will take the matrix $A(t,x)$ to be of the type

$$A(t,x) = \begin{pmatrix} 0 & 1/p(t,x) \\ q(t,x) - \lambda y(t,x) & 0 \end{pmatrix},$$

while $B(t,x)$ is a 2×2 matrix whose entries are polynomial in the complex parameter λ of degree $r > 0$, and whose coefficients depend smoothly on t and x. This choice produces a compatibility condition for the equation (2.9) consisting of a single nonlinear evolution equation for one among the functions $q(t,x)$ and $y(t,x)$. It is not always possible to determine explicitly the solution of (2.9). However, there is at least one case in which it is possible, i.e., when the matrix $A(0,x)$ satisfies the so-called *stationary zero-curvature equation*

$$-B_{g,x}(x) + [A(0,x), B_g(x)] = 0,$$

where $x \mapsto B_g(x)$ is a matrix valued function whose entries are some polynomials as for $B(t,x)$. This is exactly what we will explain in detail in the rest of the paper.

3 The stationary hierarchy

As previously anticipated, in this section we introduce the stationary hierarchy which will determine the initial conditions for which the time-dependent hierarchy will be solved. We assume the validity of the stationary zero-curvature condition

$$-B_x + [A, B] = 0 \tag{3.1}$$

where

$$B = \begin{pmatrix} -T & \lambda^{-k}\dfrac{U}{p} \\ \lambda^{-k}(q-\lambda y)V & T \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & \dfrac{1}{p} \\ q-\lambda y & 0 \end{pmatrix},$$

$k \in \mathbb{Z}$, p is a positive bounded C^1 function, y is a strictly positive bounded continuous function, q is bounded and continuous, U, V are polynomials of degree g in $\lambda \in \mathbb{C}$ whose coefficients depend on x, and T given by λ^{-k} times a polynomial of degree g in λ and coefficients depending on x (k is a positive integer whose range will be specified later).

The equation (3.1) translates to the following compatibility conditions

$$T_x + \frac{\lambda^{-k}}{p}(q-\lambda y)(V-U) = 0 \tag{3.2}$$

$$-\lambda^{-k}\left(\frac{U}{p}\right)_x + \frac{2}{p}T = 0 \tag{3.3}$$

$$-2T(q-\lambda y) - \lambda^{-k}\left((q-\lambda y)V\right)_x = 0 \tag{3.4}$$

Differentiation of (3.2) with respect to x and summation with (3.4) gives

$$(pT_x)_x - \lambda^{-k}((q-\lambda y)U)_x - 2T(q-\lambda y) = 0. \tag{3.5}$$

For simplicity, we now set $\tilde{U} = \dfrac{U}{p}$, so that $T = \lambda^{-k}\dfrac{p}{2}\tilde{U}_x$, hence (3.5) translates to

$$\frac{\lambda^{-k}}{2}\left((p(p\tilde{U}_x)_x)_x - 2(p(q-\lambda y))_x\tilde{U} - 4p(q-\lambda y)\tilde{U}_x\right) = 0 \tag{3.6}$$

The formula (3.6) is of fundamental importance, because it determines the compatibility condition for which the equation (3.1) can be solved. Indeed, if we are able to determine the coefficients of $\tilde{U}$ (and hence of U), then we determine T by the formula $T = \lambda^{-k}\dfrac{p}{2}\tilde{U}_x$, and finally we determine the coefficients of V by using the relation (3.4), which can be rewritten as

$$\lambda^{-k}\left(p(q-\lambda y)\tilde{U}_x + ((q-\lambda y)V)_x\right) = 0 \tag{3.7}$$

It is immediate to observe that in the stationary formulation the quantity λ^{-k} does not play any role. It will be important in the connection with the time-dependent hierarchy.

We now prove an easy result concerning a property of the zero-curvature equation (3.1). This result is strictly connected to the property of the spectrum of the Sturm-Liouville operator.

Lemma 3.1 *Let* $A = \begin{pmatrix} 0 & \frac{1}{p} \\ q-\lambda y & 0 \end{pmatrix}$ *and let* $B = \begin{pmatrix} -a & b \\ c & a \end{pmatrix}$*, where the coefficients* a, b *and* c *depend smoothly on* x*. Assume that the zero-curvature relation* (3.1) *holds for* A *and* B*, that is,* $-B_x + [A, B] = 0$*. Then*

$$\frac{d}{dx} \det B = 0. \tag{3.8}$$

Proof. The proof is an easy computation. The relation (3.1) gives

$$\begin{cases} a_x = \dfrac{c}{p} - b(q-\lambda y) \\ -b_x = \dfrac{2a}{p} \\ c_x = 2a(q-\lambda y) \end{cases}.$$

It follows that $-\dfrac{d}{dx}\det B = 2aa_x + b_x c + bc_x = 0$ by using the above relations. ■

The equation (3.8) translates in our case to the relation

$$\frac{d}{dx}\left(p^2 \frac{\tilde{U}_x^2}{4} + \frac{1}{p}(q-\lambda y)UV\right) = 0,$$

so $\det B$ depends only on λ. We write

$$p^2 \frac{\tilde{U}_x^2}{4} + \frac{1}{p}(q-\lambda y)UV = k^2(\lambda) := -\prod_{i=0}^{2g}(\lambda - \lambda_i), \tag{3.9}$$

where $0 < \lambda_0 < \lambda_1 < \ldots < \lambda_{2g}$. Note that (3.9) represent a first real restriction on the possible choices of the matrix B. Following the notation of Section 2, we write $\Sigma_0 = [\lambda_0, \lambda_1] \cup \cdots \cup [\lambda_{2g}, \infty)$.

Now we turn to the recursion formulas for the coefficients of $\tilde{U}$. Write

$$\tilde{U}(x,\lambda) = \sum_{j=0}^{g} \tilde{u}_j(x)\lambda^j, \qquad U(x,\lambda) = \sum_{j=0}^{g} u_j(x)\lambda^j, \qquad V(x,\lambda) = \sum_{j=0}^{g} v_j(x)\lambda^j.$$

To simplify the notation, we will write D for the operator of differentiation with respect to x, $\mathcal{D}_{py}$ for the operator $pyD + Dpy$ and $\mathcal{D}_{pq}$ for $pqD + Dpq$ acting on the space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$. From the compatibility condition (3.6) one obtains the following recursion formulas for the coefficients $\tilde{u}_j(x)$

$$\begin{cases} 2\mathcal{D}_{py}\tilde{u}_g = 0 \\ -2\mathcal{D}_{py}\tilde{u}_{j-1} = DpDpD\tilde{u}_j - 2\mathcal{D}_{pq}\tilde{u}_j, \quad j = 1, \ldots, g\ . \\ DpDpD\tilde{u}_0 - 2\mathcal{D}_{pq}\tilde{u}_0 = 0 \end{cases} \tag{3.10}$$

We call (3.10) a *recursion system*.

The system (3.6) contains $g + 2$ relations. We interpret the situation as follows. One fixes p and one among q and y: in this way one has exactly $g + 2$ unknowns, $\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_g, q$ (or y). According to if one chooses q or y, we move upwards or downwards in the recursion. That is, if we fix q (and p obviously), then we first determine $\tilde{u}_0$ (almost uniquely), then $\tilde{u}_1$ and so on, until we determine $\tilde{u}_g$. We are left with exactly one equation, which determines y in such a way that $2\mathcal{D}_{py}\tilde{u}_g = 0$. Vice-versa, if we fix y (and p) we first determine $\tilde{u}_g$, then $\tilde{u}_{g-1}$ and then we go downwards until $\tilde{u}_0$. Again, we are left with a single equation, which determines q in such a way that $DpDpD\tilde{u}_0 - 2\mathcal{D}_{pq}\tilde{u}_0 = 0$.

We remark that the usage of the coefficients $\tilde{u}_j$ is only for notational convenience. The relations (3.10) must be interpreted as a system for the unknown $u_0, u_1, \ldots, u_g$ and one between q and y.

We give some concrete examples of initial conditions which are furnished by the machinery of the stationary hierarchy. Suppose $p \equiv 1$. Then $\tilde{U} = U$. The last equation in (3.10) is $D^3u_0 - 2\mathcal{D}_q u_0 = 0$. If $g = 1$ and $y \equiv 1$ we obtain $Du_1 = 0$, $-4Du_0 = -2(2qD + q_x)u_1 = -2u_1q_x$. Putting all this information together we have $u_1 = c_1$, $u_0 = \dfrac{c_1}{2}q + c_2$, and the last equation is

$$q_{xxx} - 6qq_x + cq_x = 0,$$

where c is a constant. If $c = 0$, then this equation is the classical stationary K-dV equation. If $p \equiv \varepsilon > 0$ and $y \equiv 1$, then one obtain the stationary equation

$$\varepsilon q_{xxx} - 6qq_x = 0$$

which is the stationary version of the K-dV equation studied by Lax, Levermore, and Venakides in [16, 17, 18, 19]. If $p \equiv 1$ and $q \equiv \varepsilon > 0$, then one gets (if $u_0 = 1$) $2y = 4\varepsilon u_1 - u_{1,xx}$ and the corresponding stationary equation reads

$$2u_{1,xx}u_{1,x} + u_1u_{1,xxx} - 12\varepsilon u_1u_{1,x} = 0,$$

which can be viewed as the small dispersion limit of the stationary Camassa-Holm equation, in analogy with the small dispersion limit of the K-dV equation described above. If $p = q \equiv 1$, then one obtains the stationary Camassa-Holm equation

$$2u_{1,xx}u_{1,x} + u_1u_{1,xxx} - 12u_1u_{1,x} = 0.$$

Note, however, that the constants of integration appearing in the previous examples must be chosen in accordance with the (a priori) chosen constants $\lambda_0, \lambda_1, \ldots, \lambda_{2g}$ in (3.9). ■

Let us now introduce certain functions in terms of the polynomial $\tilde{U}$. We will see that they are none other than the Weyl m-functions of the triple $a = (p, q, y)$. This will be very important in the following.

Let us take a look at the hierarchy. The coefficient $\tilde{u}_0(x)$ satisfies $DpDpD\tilde{u}_0 - 2\mathcal{D}_{pq}\tilde{u}_0 = 0$. Let $k(0^+)$ denote the positive value of the square root of the product $\lambda_0\lambda_1\ldots\lambda_{2g}$. Define a function $\mathcal{M}(x)$ in such a way that

$$\frac{-2k(0^+)}{\mathcal{M}(x)} = \tilde{u}_0(x).$$

We will see later the real nature (the "Weyl" nature) of the function $\mathcal{M}$. Note that $\mathcal{M}$ exists because $\tilde{u}_0$ exists. Now, set

$$\tilde{U}(x,\lambda) = \frac{2(-1)^{g+1}k(0^+)}{\mathcal{M}(x)\prod_{i=1}^{g}\mu_i(x)}\prod_{i=1}^{g}(\lambda - \mu_i(x)),$$

where $\mu_i(x)$ are continuous functions defined in $\mathbb{R}$ and with values in the interval $[\lambda_{2i-1}, \lambda_{2i}]$. To see that such points $\mu_i(x)$ exist, we determine their evolution. Compute (3.9) at $\lambda = \mu_i(x)$: we obtain

$$\frac{p^2}{4}\tilde{U}_x^2(\mu_i(x)) = k^2(\mu_i(x)).$$

We will need in the following to give a correct interpretation of the value $k(\mu_i(x))$. We argue as in the preceding section, i.e., we take the positive or the negative value of this quantity according to the angular coordinate of the point $\mu_i(x)$. In this way we establish the fact that the motion of each point $\mu_i(x)$ takes place in the Riemann surface $\mathcal{R}$ described in the previous section. Since

$$\tilde{U}_x(\mu_i(x)) = \frac{2(-1)^{g}k(0^+)}{\mathcal{M}(x)\prod_{i=1}^{g}\mu_i(x)}\prod_{j\neq i}(\mu_i(x) - \mu_j(x))\mu_{i,x}(x),$$

we have

$$\mu_i'(x) = \frac{(-1)^g k(\mu_i(x))\mathcal{M}(x)\prod_{i=1}^{g}\mu_i(x)}{p(x)k(0^+)\prod_{j\neq i}(\mu_i - \mu_j)}. \tag{3.11}$$

Note that the equations in (3.11) are well defined and unambiguous. It is understood that a set of initial conditions $\mu_1(0), \ldots, \mu_g(0)$ is given. Now, if $P \in \mathcal{R}$, set

$$\tilde{m}_+(x,P) = \frac{k(P) + p(x)\tilde{U}_x(x,\lambda)/2}{\tilde{U}(x,\lambda)}, \qquad \pi(P) = \lambda.$$

Then $\tilde{m}_+(x,\cdot)$ is a meromorphic function on $\mathcal{R}$ having simple poles at the zeroes $\mu_i(x)$ of $\tilde{U}(x,\lambda)$. Let us further set

$$\tilde{m}_-(x,\cdot) : \mathcal{R} \to \mathbb{C} : P \mapsto \tilde{m}_+(x,\sigma(P)).$$

Note that

$$\tilde{m}_-(x,0^+) - \tilde{m}_+(x,0^+) = -\frac{2k(0^+)}{\tilde{U}(x,0)} = \mathcal{M}(x).$$

Now, if $\Im\lambda \neq 0$, let us choose the value $P_\lambda \in \pi^{-1}(\lambda)$ for which $\Im\lambda\Im k(P_\lambda) > 0$. Set

$$\tilde{m}_+(x,\lambda) = \tilde{m}_+(x,P_\lambda), \quad \tilde{m}_-(x,\lambda) = \tilde{m}_+(x,\sigma(P_\lambda)).$$

If $\Im\lambda \neq 0$, it follows that

$$\tilde{m}_\pm(x,\lambda) = \frac{\pm k(P_\lambda) + p(x)\tilde{U}_x(x)/2}{\tilde{U}(x)}.$$

We show now that the functions $m_\pm(x,\lambda)$ are the Weyl m-functions for the operator $L = \frac{1}{y}(-DpD+q)$ acting on $L^2(\mathbb{R}^\pm, ydx)$ respectively (again, D denotes the operator of differentiation with respect to x). For, let us assume that we have determined all the coefficients $\tilde{u}_i(x)$ ($i = 1, \ldots, g$), together with the triple $a = (p, q, y)$ in such a way that the stationary hierarchy (3.1) can be solved. All the procedure is subject to the restriction (3.9) which fixes the numbers $\lambda_0, \lambda_1, \ldots, \lambda_{2g}$, together with the initial data $\mu_1(0), \ldots, \mu_g(0)$ for the system (3.11).

Now, we can directly obtain from (3.7) and (3.9) expressions for the quantities $\sqrt{p(x)y(x)}$ and $q(x)$ in terms of the functions $\mu_i(x)$ (which evolve according to (3.11)) and $\mathcal{M}(x)$. In fact, one has,

$$\sqrt{p(x)y(x)} = \frac{(-1)^{g+1}\mathcal{M}(x)\prod_{i=1}^{g}\mu_i(x)}{2k(0^+)},$$

$$q(x) = y(x)\left(\lambda_0 + \sum_{i=0^g}[\lambda_{2i} + \lambda 2i - 1 - 2\mu_i(x)]\right) + q_g'(x) + \frac{q_g^2(x)}{p(x)},$$

where $q_g(x) = -\frac{(p(x)y(x))_x}{4y(x)}$. From the discussion in the preceding Section, we can observe that all the quantities involved in the hierarchy equal the corresponding quantities which one can obtain from an algebro-geometric triple $a = (p, q, y)$. If we prove that $\tilde{m}_\pm(x, \lambda)$ are the Weyl m-functions for the operator L defined above, then we will have shown that a lies in the set $\mathcal{S}_{\Sigma_0}$ defined in Section 2. We put this result in the following

Proposition 3.1 *Let* (3.1) *be satisfied for a set of constants* $\{\lambda_0, \lambda_1, \ldots, \lambda_{2g}\}$. *Let* $\mu_i : \mathbb{R} \to [\lambda_{2i-1}, \lambda_{2i}]$ *be defined so as to satisfy the system* (3.11) *with arbitrarily chosen initial data* $\{\mu_1(0), \ldots, \mu_g(0)\}$. *Then the functions*

$$\tilde{m}_\pm(x, \lambda) = \frac{\pm k(P_\lambda) + p(x)\tilde{U}_x(x)/2}{\tilde{U}(x)}$$

are the Weyl m-functions for the operator L.

Proof. We will give the proof only for $\tilde{m}_+$. The first step is to show that $\tilde{m}_+(x, P)$ satisfies the Riccati equation

$$M'(x) + \frac{1}{p(x)}M^2(x) = q(x) - \lambda y(x), \qquad \pi(P) = \lambda$$

for every $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$. This, together with the definitions of $\tilde{m}_+(x, \lambda)$ and the fact that the system (3.11) is exactly the same as the system (2.3), will prove the proposition.

We have, using (3.2), (3.3) and (3.9) and simplifying the notation,

$$\begin{aligned} \tilde{m}_+' + \frac{1}{p}\tilde{m}_+^2 &= \frac{1}{\tilde{U}^2}\left[\frac{(p\tilde{U}_x)_x}{2}\tilde{U} + \frac{k^2(\lambda)}{p} - \frac{p\tilde{U}_x}{4}\right] = \\ &= \frac{1}{\tilde{U}^2}\left[\lambda^k\tilde{U}T_x + \frac{1}{p}(q - \lambda y)\tilde{U}V\right] = q - \lambda y. \end{aligned}$$

■

Summing up, we have shown that a solution of the stationary hierarchy (3.1) which satisfies (3.9) gives rise to a potential $a = (p, q, y)$ in $\mathcal{S}_{\Sigma_0}$, where $\mathcal{S}_{\Sigma_0} = [\lambda_0.\lambda_1] \cup \cdots \cup [\lambda_{2g}, \infty)$. Vice versa, every potential $a = (p, q, y) \in \mathcal{S}_{\Sigma_0}$ is the solution of an appropriate hierarchy of the type (3.1).

We will use the names *generalized stationary K-dV equation of order g* for the equation

$$DpDpD\tilde{u}_0 - 2\mathcal{D}_{pq}\tilde{u}_0 = 0,$$

and *generalized stationary CH equation of order g* for the equation

$$2\mathcal{D}_{py}\tilde{u}_g = 0.$$

The solutions of these equations (which will be the initial conditions for which the time-dependent hierarchies will be solved in a few lines) are the functions $q(x)$ and $y(x)$ respectively. The terms $\tilde{u}_g$ and $\tilde{u}_0$ are determined by recursion according to (3.10).

4 The time-dependent hierarchy

We now move our attention to the time-dependent hierarchy. Our aim is that of solving a certain evolution equation when the initial datum is given by a solution of the stationary hierarchy of order g. To write down the equations we will be able to solve, we assume a priori that we are given the following setup.

(i) We fix three integers $k \le r < g$.

(ii) For each $t \in \mathbb{R}$, we introduce a matrix function $A = \begin{pmatrix} 0 & 1/p \\ q - \lambda y & 0 \end{pmatrix}$ together with a polynomial matrix function B of the type encountered in Section 3: $B = \begin{pmatrix} -T & \lambda^{-k}U/p \\ \lambda^{-k}(q-\lambda y)V & T \end{pmatrix}$. Here $p = p(t,x)$, $q = q(t,x)$, $y = y(t,x)$, $U = \sum_{j=0}^{g} u_j(t,x)\lambda^j$, $V = \sum_{j=0}^{g} v_j(t,x)\lambda^j$, T is obtained by means of the formula (3.3), and it is required that, for each $t \in \mathbb{R}$, the stationary zero-curvature relation holds:

$$-B_x + [A, B] = 0, \qquad \text{for all } t, x, \lambda.$$

(iii) We introduce polynomials $U_r(t,x,\lambda)$ and $V_r(t,x,\lambda)$ of degree r, and a polynomial $T_r(t,x,\lambda)$ so that the matrix function

$$B_r = \begin{pmatrix} -T_r & \lambda^{-k}U_r/p \\ \lambda^{-k}(q-\lambda y)V_r & T_r \end{pmatrix}$$

satisfies the non-stationary zero-curvature relation

$$A_t - B_{r,x} + [A, B_r] = 0 \qquad \text{for all } t, x, \lambda.$$

We wish to determine whether the relations

$$-B_x(t,x,\lambda) + [A(t,x,\lambda), B(t,x,\lambda)] = 0 \tag{4.1}$$

$$A_t(t,x,\lambda) + B_{r,x}(t,x,\lambda) + [A(t,x,\lambda), B_r(t,x,\lambda)] = 0 \tag{4.2}$$

define a flow with respect to t, whose trajectories lie in the fixed spectral class $\mathcal{S}_{\Sigma_0}$ of Sections 2 and 3.

We will see that equations (4.1) and (4.2) each give rise to a recursion system. We make the following convention. We fix either the pair (p, q) or the pair (p, y). In the first case we agree to move downward in both the recursion systems. In the second case we move upward in both the recursion systems.

Writing down the relation (4.2) explicitly we have

$$\begin{cases} T_{r,x} + \dfrac{\lambda^{-k}}{p}(q - \lambda y)(V_r - U_r) = 0 \\ \left(\dfrac{1}{p}\right)_t - \lambda^{-k}\left(\dfrac{U_r}{p}\right)_x + \dfrac{2}{p}T_r = 0 \\ (q - \lambda y)_t - 2T_r(q - \lambda y) - \lambda^{-k}((q - \lambda y)V_r)_x = 0 \end{cases} \tag{4.3}$$

Setting $\tilde{U} = \dfrac{U}{p}$ as in the stationary case, and doing some computations, the system (4.3) gives us

$$\begin{aligned} &2(q - \lambda y)_t - \lambda^{-k}\frac{p_t}{p}(q - \lambda y) + 2\left(p\left(\frac{p_t}{2p}\right)_x\right)_x = \\ &= \lambda^{-k}\left[2(p(q - \lambda y))_x\tilde{U}_r + 4p(q - \lambda y)\tilde{U}_{r,x} - (p(p\tilde{U}_{r,x})_x)_x\right] \end{aligned} \tag{4.4}$$

The equation (4.4), when the expression U_r/p is used in place of $\tilde{U}_r$, contains all the equations of the time-dependent hierarchy. As we will see in a while, there is a compatibility condition for the equation (4.4) to be solvable, $\tilde{U}_r(t, x)$ being a polynomial of degree r in λ and with coefficients depending smoothly on t and x. This compatibility condition will be an evolution equation for only one between the coordinates of the triple $a(t, x) = (p(t, x), q(t, x), y(t, x))$, which we will call *the r-th order equation of the Sturm-Liouville hierarchy*. We consider some concrete examples.

Assume $k = 0$, $p(t, x) = y(t, x) = 1$. Then $\tilde{U}_r = U_r$. (4.4) reads

$$2q_t = 2q_x U_r + 4(q - \lambda)U_{r,x} - U_{r,xxx}.$$

This is the standard K-dV hierarchy ([6]). For $r = 1$, set $U_1(t, x) = f_1(t, x)\lambda + f_0(t, x)$. Then

$$\begin{cases} f_{1,x}(t, x) = 0; \\ 2q_x(t, x)f_1(t, x) - 4f_{0,x}(t, x) = 0; \\ 2q_t(t, x) = 2q_x(t, x)f_0(t, x) + 4q(t, x)f_{0,x} = (t, x) - f_{0,xxx}(t, x). \end{cases}$$

If $f_1(t, x) = c_1$, we obtain $c_1 q_x(t, x) = 2f_{0,x}(t, x)$, which implies $f_0(t, x) = \dfrac{c_1}{2}q(t, x) + c_2$. Hence the last relation in the system above translates to

$$q_t(t, x) = \frac{3}{2}c_1 q(t, x)q_x(t, x) - \frac{c_1}{4}q_{xxx}(t, x) + c_2 q_x(t, x)$$

which is a generalized version of the classical K-dV equation, and in fact we obtain it if the constant c_2 vanishes, i.e., we have after a time-scaling

$$q_t(t,x) = \frac{3}{2}q(t,x)q_x(t,x) - \frac{1}{4}q_{xxx}(t,x).$$

As another example, let us assume that $k = 1$ and $p(t,x) = q(t,x) = 1$. Then (4.4) translates to

$$2\lambda^2 y_t(t,x) = 2\lambda y_x(t,x)U_r(t,x) - 4(1-\lambda y(t,x))U_{r,x}(t,x) + U_{r,xxx}(t,x).$$

This is a version of the Camassa-Holm hierarchy (another one can be obtained by setting $k = r$ as in [11]). If $r = 1$, a possible solution is given by

$$\begin{cases} f_0 = c_1; \\ c_1 y(t,x) + c_2 = f_1(t,x) - \frac{1}{2}f_{1,xx}(t,x); \\ y_t(t,x) = y_x(t,x)f_1(t,x) + 2y(t,x)f_{1,x}(t,x). \end{cases}$$

This system is a generalized version of the Camassa-Holm equation. The classical Camassa-Holm equation is obtained by setting $c_1 = 1$ and $c_2 = 0$ (see [5]).

Note that the constants in these constructions can be chosen at will.

We describe two more examples, which in our opinion could be the starting point for further developments concerning the relations between the K-dV equation and the Burger's equation and between the Camassa-Holm equation and the Hunter-Saxton equation.

Let us set $p(t,x) \equiv \varepsilon$, $y(t,x) \equiv 1$, $k = 0$ and $r = 1$. Then $\tilde{U}_1 = U_1/\varepsilon$, and the equation (4.4) translates to the system

$$\begin{cases} f_1 = c_1 \\ c_1 q_x = 2f_0 \\ q_t = \frac{3}{2}c_1 q q_x - \frac{c_1\varepsilon}{4}q_{xxx} + c_2 q_x. \end{cases}$$

If $c_1 = 4$ and $c_2 = 0$, then the compatibility condition is given by

$$q_t = 6qq_x - \varepsilon q_{xxx},$$

which is a well-known and important generalization of the K-dV equation, used by many authors ([16, 17, 18, 19]) in connection to the Burger's equation, which is indeed the limit as $\varepsilon \to 0$ of such a K-dV generalization.

If $p(t,x) = 1$, $q(t,x) \equiv \varepsilon$ $k = 1$ and $g = 1$, then the compatibility condition reads (for suitable chosen constants c_1 and c_2)

$$4\varepsilon u_{1,t} - u_{1,xxt} = 12\varepsilon u_1 u_{1,x} - u_1 u_{1,xxx} - 2u_{1,x}u_{1,xx}.$$

This equation is a generalization of the CH equation. Its limit (whenever it exists) as $\varepsilon \to 0$ is the Hunter-Saxton equation

$$u_{1,xxt} = u_1 u_{1,xxx} + 2u_{1,x}u_{1,xx}.$$

■

Again, the relation

$$\frac{d}{dx}\det B = 0, \tag{4.5}$$

holds, because of (4.1). Actually, the equation (4.5) only tells us that $\det B$ is a function of the variable t . We make a restriction on the admissible functions U, V and T by requiring that

$$p^2(t,x)\frac{\tilde{U}_x^2(t,x,\lambda)}{4} + \frac{1}{p(t,x)}(q(t,x) - \lambda y(t,x))U(t,x,\lambda)V(t,x,\lambda) = k^2(\lambda). \tag{4.6}$$

holds *for every* $t \in \mathbb{R}$. It will turn out that this condition is consistent with the other compatibility condition (4.2). So in what follows we impose the conditions (4.1), (4.2) and (4.6).

From (4.1) and (4.6), the triples $(p(t,\cdot), q(t,\cdot), y(t,\cdot))$ are algebro-geometric Sturm-Liouville potentials with spectrum $\Sigma_0 = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2g}, \infty)$. If we set $m_\pm(t,x) = \dfrac{\pm k(\lambda) + \frac{p}{2}\tilde{U}_x(t,x)}{\tilde{U}(t,x)}$, then $m_\pm(t,x)$ satisfy

$$M' + \frac{1}{p}M^2 = q - \lambda y.$$

As in the stationary case, define $\mathcal{M}(t,x) = \dfrac{-2k(0^+)}{\tilde{u}_0(t,x)}$, and set

$$\tilde{U}(x,t,\lambda) = \frac{2(-1)^{g+1}k(0^+)}{\mathcal{M}(t,x)\prod_{i=1}^g \mu_i(t,x)} \prod_{i=1}^g (\lambda - \mu_i(t,x)).$$

Then $\mathcal{M}(t,x) = m_-(t,x,0) - m_+(t,x,0)$.

One easily obtains

$$\begin{cases} m_+ + m_- = \dfrac{p\tilde{U}_x}{\tilde{U}} \\ m_+ - m_- = \dfrac{2k(\lambda)}{\tilde{U}} \\ m_+ m_- = \dfrac{1}{\tilde{U}^2}\left(\dfrac{p^2}{4}\tilde{U}_x^2 - k^2(\lambda)\right) = \dfrac{1}{\tilde{U}^2}\left(\dfrac{1}{p}(q - \lambda y)UV\right). \end{cases} \tag{4.7}$$

The first result we prove concerns an equation of Riccati type for the Weyl m-functions with respect to $t \in \mathbb{R}$.

Proposition 4.1 *The functions $m_\pm(t,x,\lambda)$ defined above satisfy the equation*

$$M_t = 2T_r M - \lambda^{-k}\frac{U_r}{p}M^2 + \lambda^{-k}(q - \lambda_y)V_r. \tag{4.8}$$

Proof. We prove the proposition only for $M = m_+$. We use arguments similar to those in [11, 24]. By using (4.3), (4.1) and (4.7), we have

$$\left(\frac{\partial}{\partial x} + \frac{2M}{p}\right)(M_t - 2T_r M + \lambda^{-k}\tilde{U}_r M^2 - \lambda^k(q - \lambda y)V_r) = 0.$$

This means that

$$M_t - 2T_r M + \lambda^{-k}\tilde{U}_r M^2 - \lambda^{-k}(q - \lambda y)V_r = C\exp\left(-\int_0^x 2M\,ds\right). \tag{4.9}$$

Now, the left side of (4.9) is meromorphic at ∞, while the right side is not. Hence we must have $C = 0$. This concludes the proof. ∎

Then next step is to obtain some relations between the time derivatives of the polynomials $\tilde{U}, V, T$ and $\tilde{U}_r, V_r, T_r$. For, we will use heavily the Riccati-type equation (4.8). We prove now these relations.

Proposition 4.2 *Let the zero-curvature structure be satisfied, i.e., let us assume that* (4.1)*,* (4.2) *and* (4.6) *are valid with polynomials* U, V, T *and* U_r, V_r, T_r*. Then*

$$\tilde{U}_t(t,x) = \frac{2}{p(t,x)}\left[T(t,x)U_r(t,x) - T_r(t,x)U(t,x)\right], \tag{4.10}$$

$$T_t(t,x) = \frac{1}{\lambda^{2k}}\frac{q(t,x)-\lambda y(t,x)}{p(t,x)}\left[V_r(t,x)U(t,x) - U_r(t,x)V(t,x)\right], \tag{4.11}$$

$$\begin{aligned}&[(q(t,x)-\lambda y(t,x))V(t,x)]_t = \\ &= 2(q(t,x)-\lambda y(t,x))\left[T_r(t,x)V(t,x) - T(t,x)V_r(t,x)\right].\end{aligned} \tag{4.12}$$

Equations (4.10)–(4.12) *in turn are equivalent to*

$$-B_t + [B_r, B] = 0 \tag{4.13}$$

Proof. It is straightforward to prove the equivalence between (4.10)–(4.12) and (4.13). In the following we will heavily simplify the notation. This, however, will (hopefully) not cause any confusion to the reader. To prove (4.10), we differentiate the formula for $m_+ - m_-$ in (4.7) to obtain $(m_+ - m_-)_t = -\frac{2k(\lambda)\tilde{U}_t}{\tilde{U}^2}$. Moreover, the equation (4.8) holds both for m_+ and m_-, so we have

$$\begin{aligned}(m_+ - m_-)_t &= 2T_r(m_+ - m_-) - \lambda^{-k}\frac{U_r}{p}(m_+ - m_-)(m_+ + m_-) = \\ &= \frac{4k(\lambda)T_r}{\tilde{U}} - \frac{2}{p}TU_r\frac{2k(\lambda)}{\tilde{U}^2}.\end{aligned}$$

Putting together the formulas for $(m_+ - m_-)_t$ we have just obtained, (4.10) follows easily.

To prove (4.11), we first differentiate with respect to t the formula for $m_+ + m_-$ in (4.7) and use (4.10):

$$(m_+ + m_-)_t = 2\frac{\lambda^{2k}}{\tilde{U}^2}\left[T_t\tilde{U} - \frac{2}{p}T^2U_r + \frac{2}{p}TT_rU\right].$$

Then we use (4.1) to compute $(m_+ + m_-)_t$ as well:

$$(m_+ + m_-)_t = \frac{4\lambda^k TT_r}{\tilde{U}} - \frac{1}{\lambda^k}\frac{U_r}{p}\left[\frac{4k^2(\lambda)}{\tilde{U}^2} + \frac{2}{\tilde{U}}(q-\lambda y)\tilde{U}V\right].$$

Putting together these two formulas for $(m_+ + m_-)_t$ and using (3.9), (4.11) follows.

Finally, we prove (4.12). The method is the same as those used just before. This time we have $(\tilde{U}m_+m_-)_t = [(q-\lambda y)V]_t$ (see (4.7)). Moreover, if we use (4.8) to compute the same quantity, we get

$$\begin{aligned}(\tilde{U}m_+m_-)_t &= \tilde{U}_t m_+m_- + \tilde{U}m_{+,t}m_- + \tilde{U}m_+m_{-,t} = \\ &= \frac{2}{p}(q-\lambda y)(TU_r - UT_r)\frac{V}{\tilde{U}} + 4(q-\lambda y)T_rV - \\ &- \frac{1}{\lambda^k}(q-\lambda y)\left[\frac{U_r\tilde{U}_x}{\tilde{U}} - p\tilde{U}_xV_r\right].\end{aligned}$$

(4.12) follows by putting together these formulas again. ■

Actually, the most important formula is (4.10). Indeed, by using it, we determine the time motion of the points $\mu_i(t,x)$. In particular:

Theorem 4.1 *Let us assume that* (4.1), (4.2) *and* (4.6) *are valid with polynomials* U, V, T *and* U_r, V_r, T_r *and* $k^2(\lambda) = -(\lambda-\lambda_0)\ldots(\lambda-\lambda_{2g})$ *as above. Then*

$$\begin{cases}\mu_{i,x}(t,x) = \dfrac{(-1)^g k(\mu_i(t,x))\mathcal{M}(t,x)\prod_{i=1}^g \mu_i(t,x)}{p(t,x)k(0^+)\prod_{j\neq i}(\mu_i(t,x)-\mu_j(t,x))}. \\ \\ \mu_{i,t}(t,x) = \dfrac{U_r(t,x,\mu_i)}{\mu_i^k(t,x)}\mu_{i,x}(t,x).\end{cases} \tag{4.14}$$

The solution of the r-th order equation can be expressed as follows:

(i) If the functions $p(t,x)$ *and* $q(t,x)$ *are given, then* $y(t,x)$ *is the solution of the r-th order equation, and*

$$y(t,x) = \frac{\mathcal{M}^2(t,x)}{4p(t,x)\lambda_0}\prod_{i=1}^g \frac{\mu_i^2(t,x)}{\lambda_{2i-1}\lambda_{2i}};$$

(ii) if $p(t,x)$ *and* $y(t,x)$ *are given, then* $q(t,x)$ *is the solution of the r-th order equation. It can be expressed by*

$$q(t,x) = y(t,x)\left(\lambda_0 + \sum_{i=1}^g[\lambda_{2i-1}+\lambda_{2i}-2\mu_i(t,x)]\right) + q_{g,x}(t,x) + \frac{q_g^2(t,x)}{p(t,x)},$$

where $q_g(t,x) = -\dfrac{(p(t,x)y(t,x))_x}{4y(t,x)}.$

So far, we have shown that, if the relations (4.1) and (4.2) are satisfied, then the g-tuple $\underline{\mu}(t,x) := (\mu_1(t,x),\ldots,\mu_g(t,x))$ satisfies the system (4.14). Now it is time to go backwards, i.e., we will assume now that the g-tuple $\underline{\mu}(t,x)$ is given by the system (4.14), $U_t(t,x,\lambda)$ being a polynomial of degree r in λ whose coefficients are functions of $\underline{\mu}(t,x)$, $p(t,x)$, $q(t,x)$ and $y(t,x)$. We must show that a solution of the system (4.14) exists. In particular U_r must be determined. Clearly for the system (4.14) to have a solution, it must satisfy a compatibility condition concerning mixed derivatives. Below, we will use a particular expression for $U_r(t,x,\lambda)$. To explain better the reason of this choice, we make a preliminary observation. Let us define a function $T_r(t,x)$ by

$$T_r(t,x) = \frac{p(t,x)}{2}\left[\frac{\tilde{U}_{r,x}(t,x)}{\lambda^k} - \left(\frac{1}{p(t,x)}\right)_t\right].$$

Consider $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ as parameters. Assume that there is a function $U_r(t,x,\lambda)$ of the variable x which satisfies the following relation:

$$\tilde{U}_t(t,x) = \frac{2}{p(t,x)}\left[T(t,x)U_r(t,x) - T_r(t,x)U(t,x)\right], \tag{4.15}$$

at *all* the values of the parameters $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. Note that the form of the polynomial $U(t,x)$ is determined completely from the choice of the triples $a(t,\cdot) \in \mathcal{S}_{\Sigma_0}$. It follows that $U_r(t,x,\lambda)$ is determined once $U(t,x,\lambda)$ is known. A priori, $U_r(t,x,\lambda)$ can have a very complicated dependence on the parameters t and λ. We will show that there is a particular choice for $U_r(t,x,\lambda)$ in such a way that it is a polynomial of degree r in λ whose coefficients depend smoothly on t and x only. Moreover, it will turn out that $U_r(t,x,\lambda)$ can be defined for every choice of the parameters $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, even at those values which we excluded because of the singularities.

If a solution $U_r(t,x,\lambda)$ of (4.15) exists with those described properties, then it has to satisfy the second equation in (4.14). Set

$$H(t,x) = \mathcal{M}(t,x)\prod_{i=1}^{g}\mu_i(t,x).$$

We have, simplifying the notation,

$$\tilde{U}_t = -\tilde{U}\left[\sum_{i=1}^{g}\frac{\mu_{i,t}}{\lambda-\mu_i} + \frac{H_t}{H}\right], \tag{4.16}$$

$$\tilde{U}_x = -\tilde{U}\left[\sum_{i=1}^{g}\frac{\mu_{i,x}}{\lambda-\mu_i} + \frac{H_x}{H}\right]. \tag{4.17}$$

We required in (4.15) that

$$\tilde{U}_t = \frac{2}{p}[TU_r - UT_r],$$

that is

$$\tilde{U}_t = \frac{p}{\lambda^k}\tilde{U}_x\tilde{U}_r - \frac{p}{\lambda^k}\tilde{U}\tilde{U}_{r,x} + p\left(\frac{1}{p}\right)_t\tilde{U}. \tag{4.18}$$

Putting (4.16)–(4.18) together, we obtain

$$\begin{aligned}\tilde{U}_{r,x}(\lambda) = \frac{\lambda^k}{p}\left[\frac{H_t}{H} + p\left(\frac{1}{p}\right)_t\right] - \frac{H_x}{H}\tilde{U}_r(\lambda) + \\ + \sum_{i=1}^{g}\left[\frac{\lambda^k}{\mu_i^k}\tilde{U}_r(\mu_i) - \tilde{U}_r(\lambda)\right]\frac{\mu_{i,x}}{\lambda-\mu_i}.\end{aligned} \tag{4.19}$$

The equation (4.19) is the key to solve our problem. Indeed, let us make the ansatz that

$$\tilde{U}_r(t,x,\lambda) = \sum_{j=0}^{r} f_j\,(t,x)\,\lambda^j.$$

Then it suffices to determine the coefficients f_j. It follows easily that they can be determined from (4.19) uniquely (up to additive constants). Indeed, note that the last summation in (4.19) is a determined polynomial of degree $r-1$, since μ_i is a root of the expression into parentheses. Clearly, the form of $\tilde{U}_r$ depends deeply on the exponent $k \leq r$. This shows that $U_r(t,x,\lambda)$ actually is a polynomial in λ. To give a more direct formula for $\tilde{U}_r$, we define

$$P_i(\lambda) := \frac{1}{\lambda-\mu_i}\left[\frac{\lambda^k}{\mu_i^k}\tilde{U}_r(\mu_i) - \tilde{U}_r(\lambda)\right].$$

Clearly $P_i(t,x,\lambda)$ is a polynomial of degree $r-1$, i.e., $P_i(t,x,\lambda) = \sum_{j=0}^{r-1} p_{ij}(t,x)\lambda^j$. The last summand in the formula (4.19) is given by

$$S(t,x,\lambda) = \sum_{i=1}^{g} P_i(t,x,\lambda)\mu_{i,x}(t,x).$$

Hence $S(t,x,\lambda)$ is a polynomial of degree $r-1$ as well, say $S(t,x,\lambda) = \sum_{j=0}^{r-1} s_j(t,x)\lambda^j$. Equation (4.19) can be rewritten as

$$\sum_{j=0}^{r} f_{j,x}(t,x)\lambda^j = \frac{\lambda^k}{p(t,x)}\left[\frac{H_t(t,x)}{H(t,x)} + p(t,x)\left(\frac{1}{p(t,x)}\right)_t\right] - \\ - \frac{H_x(t,x)}{H(t,x)}\left(\sum_{j=0}^{r} f_j(t,x)\lambda^j\right) + \sum_{j=0}^{r-1} s_j(t,x)\lambda^j.$$

Comparing the coefficients of the same powers of λ one can determine recursively the coefficients f_j, and hence the polynomial $\tilde{U}_r$. Actually, the power of λ^k has a great influence in this part of the story. Each of the coefficients $s_j(t,x)$ is a sum of the coefficients $f_j(t,x)$ multiplied by a power of μ_j. All these facts allow us to conclude that (4.19) provides a system of $r+1$ linear differential equations for $f_0,\dots,f_r$ with coefficients depending on $H(t,x)$, $p(t,x)$ and powers of the points $\mu_i(t,x)$ (i.e., they *do not* depend on λ). Such a system is solvable by standard ODE methods, and it gives us a solution which is unique up to $r+1$ additive functions depending only on t. Note that if $U_r(t,x,\lambda)$ is defined by (4.19), then the relation (4.15) is valid for every choice of $\lambda \in \mathbb{C}$, even at those points where $U(t,x,\lambda)$ vanishes.

We will write down explicitly the polynomial $U_r(t,x,\lambda)$ in some particular cases in a few lines, and we will completely determine it in the general case in the next section. But first we continue to move towards the determination of the hierarchy (4.3). We summarize the steps we made: we assumed that a solution of (4.15) exists for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ and we determined recursively the coefficients $f_j(t,x)$ of the polynomial $\tilde{U}_r(t,x,\lambda)$ under the condition (4.15).

Now, define a polynomial $V_r(t, x, \lambda)$ such that

$$T_{r,x}(t, x, \lambda) + \frac{1}{p(t, x)\lambda^k} \left(V_r(t, x) - U_r(t, x)\right) = 0.$$

Note that this last equation is exactly the first equation in (4.3). The second equation in (4.3) is the definition of $T_r(t, x, \lambda)$ given above. We want to retrieve the last equation in (4.3). To this aim, we need some tedious computations. We sketch them here.

First of all we find the formula (4.11), i.e.,

$$T_t(t, x) = \frac{1}{\lambda^{2k}} \frac{q(t, x) - \lambda y(t, x)}{p(t, x)} \left[V_r(t, x)U(t, x) - U_r(t, x)V(t, x)\right].$$

For, we differentiate (4.15) with respect to x and use the definition of V_r, obtaining

$$\tilde{U}_{tx} = 2[T\tilde{U}_r - T_r\tilde{U}]_x = 2\left[\frac{q - \lambda y}{p\lambda^k}\left(\tilde{U}V_r - \tilde{U}_r V\right) + \frac{p}{2}\left(\frac{1}{p}\right)_t \tilde{U}_x\right] \tag{4.20}$$

On the other hand, from the definition of T given in (3.3), we obtain

$$\tilde{U}_{xt} = 2\lambda^k \left(\frac{1}{p}\right)_t T + 2\lambda^k \frac{1}{p} T_t. \tag{4.21}$$

Putting (4.20) into (4.21), one finds the desired expression (4.11).

Next, we show that (4.12) holds, i.e.,

$$\begin{aligned}
&[(q(t, x) - \lambda y(t, x))V(t, x)]_t = \\
&= 2(q(t, x) - \lambda y(t, x))\left[T_r(t, x)V(t, x) - T(t, x)V_r(t, x)\right].
\end{aligned}$$

For, we differentiate the expression (4.5) with respect to t and use (4.10) and (4.11), obtaining

$$\begin{aligned}
(k^2(\lambda))_t = 0 = 2\lambda^{2k} T T_t + \left[(q - \lambda y)V\tilde{U}\right]_t = \\
= 2(q - \lambda y)TV_r\tilde{U} + [(q - \lambda y)V]_t\tilde{U} + 2(q - \lambda y)T_r V\tilde{U},
\end{aligned}$$

from which (4.12) follows easily.

The last step is to use the formulas (4.10)–(4.12) we just derived to obtain the last equation in (4.3). Let us differentiate (4.11) with respect to t:

$$\begin{aligned}
T_{tx} = \frac{1}{\lambda^{2k}}\Big[&[(q - \lambda y)V_r]_x\tilde{U} - [(q - \lambda y)V]_x\tilde{U}_r + \\
&+ (q - \lambda y)V_r\tilde{U}_x - (q - \lambda y)V\tilde{U}_{r,x}\Big]
\end{aligned} \tag{4.22}$$

On the other hand, if we differentiate (3.2) with respect to t, we have

$$\begin{aligned}
T_{xt} = -\frac{1}{\lambda^k}\Big[&\left(\frac{1}{p}\right)_t (q - \lambda y)V + \frac{1}{p}[(q - \lambda y)V]_t - \\
&- (q - \lambda y)_t\tilde{U} - (q - \lambda y)\tilde{U}_t\Big]
\end{aligned} \tag{4.23}$$

Putting (4.22) into (4.23), we obtain

$$\begin{aligned}&\frac{1}{\lambda^k}\left[(q-\lambda y)V_r\right]_x\tilde{U}-(q-\lambda y)_t\tilde{U}+\\&+2(q-\lambda y)T_r\tilde{U}=0\end{aligned},$$

from which the desired expression follows by eliminating $\tilde{U}$.

We proved the second part of the following

Theorem 4.2 *Let $\lambda_0<\lambda_1<\ldots<\lambda_{2g}$ be given real numbers. Let $\mathcal{S}_{\Sigma_0}$ be the set of all the triples $a=(p,q,y)\in\mathcal{E}_3$ such that the operator $L_a=\frac{1}{y}(DpD-q)$ satisfies: (i) the spectrum Σ_a of L_a equals the set $\Sigma_0=[\lambda_0,\lambda_1]\cup[\lambda_2,\lambda_3]\cup\cdots\cup[\lambda_{2g},\infty)$; (ii) $\Re\mathcal{G}(\eta)=0$ for a.a. $\eta\in\Sigma_0$. Let us define a polynomial $U_r(t,x,\lambda)=p(t,x)\tilde{U}_r(t,x,\lambda)$ recursively by* (4.19). *Then there exists a curve $a:\mathbb{R}\to\mathcal{S}_{\Sigma_0}:t\mapsto a(t,\cdot)=(p(t,\cdot),q(t,\cdot),y(t,\cdot))$ such that the second equation in the system* (4.14) *holds.*

Define polynomials T_r and V_r by the first and the second equations in (4.3) *respectively. Then* (4.2) *is valid (note that the first part of the theorem ensures that the system* (4.14) *has a solution).*

Proof. It is only to be proved that the system (4.14) with $U_r(t,x,\lambda)=p(t,x)\tilde{U}(t,x,\lambda)$ defined recursively by (4.19) has a solution. We first show a general fact concerning a system of differential equations of the form (4.14). Let $\mathbb{T}^g$ be the real torus $\mathbb{T}^g=\pi^{-1}([\lambda_1\lambda_2])\times\pi^{-1}([\lambda_3\lambda_4])\times\cdots\times\pi^{-1}([\lambda_{2g-1}\lambda_{2g}])$.

Let us denote with the symbol $\underline{\mu}$ a vector $(\mu_1,\ldots,\mu_g)$. Let $F,G:\mathbb{R}^2\times\mathbb{T}^g\to\mathbb{R}^g$ be two smooth functions.

Suppose we are given a system of differential equations

$$\begin{cases}\underline{\mu}_x(t,x)=F(t,x,\underline{\mu}(t,x));\\ \underline{\mu}_t(t,x)=G(t,x,\underline{\mu}(t,x)),\end{cases}\tag{4.24}$$

with a fixed initial condition $\underline{\mu}(0,0)=\underline{\mu}_0$. Let F,G satisfy the following relation

$$F_t-G_x+F_{\underline{\mu}}\cdot G-G_{\underline{\mu}}\cdot F=0,\tag{4.25}$$

where $F_{\underline{\mu}}$ and $G_{\underline{\mu}}$ are the Jacobian matrices of F and G with respect to the vector $\underline{\mu}$, and $\cdot$ is the multiplication by the column vectors obtained from F and G.

We claim that if F and G satisfy the relation (4.25), then the system (4.24) is solvable. For, let us consider for the moment the variable x as a parameter, and let $\underline{\mu}(t,x)$ be a solution of the second equation in (4.24) satisfying $\underline{\mu}(0,x)=H(x)$, where $H_x(x)=F(0,x,H(x))$ and $H(0)=\underline{\mu}_0$.

From (4.24), it is possible to determine a differential equation for the vector $\underline{\mu}_x(t,x)$, namely

$$(\underline{\mu}_x)_t=G_x(t,x,\underline{\mu})+G_{\underline{\mu}}\cdot\underline{\mu}_x,\tag{4.26}$$

together with the initial condition $\underline{\mu}_x(0,x)=H_x(x)=F(0,x,H(x))$. Let us compare $\underline{\mu}_x(t,x)$ with the function $t\mapsto F(t,x,\underline{\mu}(t,x))$. We have

$$\frac{d}{dt}F(t,x,\underline{\mu}(t,x))=F_t(t,x,\underline{\mu})+F_{\underline{\mu}}\cdot\underline{\mu}_t=F_t(t,x,\underline{\mu})+F_{\underline{\mu}}\cdot G.\tag{4.27}$$

Now we use the relation (4.25) to obtain from (4.27)

$$\frac{d}{dt}F(t,x,\underline{\mu}(t,x)) = G_x + G_{\underline{\mu}} \cdot F, \tag{4.28}$$

together with the initial condition $F(0,x,\underline{\mu}(0,x)) = F(0,x,\underline{H}(x)) = H_x(x)$. By uniqueness, we conclude that $F(t,x,\underline{\mu}(t,x)) = \underline{\mu}_x(t,x)$, and this proves the claim.

Now, let us write $G = (G_1,\ldots,G_g)$ and $F = (F_1,\ldots,F_g)$. To conclude the proof, we must show that, if $F_i(t,x,\underline{\mu}(t,x))$ is given by the righthand side of the first equation in (4.14) and $G_i(t,x,\underline{\mu}(t,x))$ is given by the righthand side of the second equation in (4.14) (where $U_r(t,x,\lambda)$ is obtained by means of the recursion formulas (4.19)), then (4.25) is satisfied. This can be proved by a direct computation, which we omit there. ∎

We promised the reader the explicit expression of the polynomial U_r in some particular cases. We do this in what follows.

(1) Let $p \equiv y = 1$ for all $t, x \in \mathbb{R}$ and let $k = 0$. Then the hierarchy one obtains is just the K-dV hierarchy (see [4, 25]). The equation (4.4)reads:

$$2q_t = 2q_x\tilde{U}_r + 4(q-\lambda)\tilde{U}_{r,x} - \tilde{U}_{r,xxx}.$$

If $r = 1$, then the compatibility condition for the above equation is (an equivalent form of) the classical K-dV equation, namely we obtain:

$$q_t = c_1(6qq_x - q_{xxx}) + c_2q_x,$$

where $c_1, c_2 \in \mathbb{R}$ are constants of integration. If $r = 1$ and $k = 0$, then it is easy to show that the polynomial $\tilde{U}_{r=1}(t,x,\lambda) = f_1(t,x)\lambda + f_0(t,x)$ satisfies $f_1 = c_1$, and $f_0(t,x) = -c_1\left(\sum_{i=1}^{g}\mu_i(t,x)\right) + c_2$, where c_1 and c_2 are constants.

In the general case $r > 1$, then $U_r(t,x,\lambda)$ can be taken as a symmetric polynomial of the points $\mu_i(t,x)$. Namely, let us denote by $\underline{\mu}(t,x)$ the g-tuple $\underline{\mu}(t,x) = (\mu_1(t,x)\ldots,\mu_g(t,x))$. Set

$$\varsigma_i(\underline{\mu}(t,x)) = (-1)^i\sum_{\ell\in\Lambda_i}\mu_{\ell_1}(t,x)\mu_{\ell_2}(t,x)\ldots\mu_{\ell_i}(t,x), \qquad 1 \le i \le g,$$

where

$$\Lambda_i = \left\{\ell \in \mathbb{N}^i \mid 1 \le \ell_1 < \ldots < \ell_i \le g\right\},$$

and

$$\sigma_i^{(j)}(\underline{\mu}(t,x)) = (-1)^i\sum_{\ell\in\Lambda_i^{(j)}}\mu_{\ell_1}(t,x)\mu_{\ell_2}(t,x)\ldots\mu_{\ell_i}(t,x), \quad 1 \le i \le g-1,$$

where

$$\Lambda_i^{(j)} = \left\{\ell \in \mathbb{N}^i \mid 1 \le \ell_1 < \ldots < \ell_i \le g,\ \ell_k \ne j\right\}.$$

Then

$$U_r(t,x,\lambda) = \sum_{i=0}^{r}\varsigma_{r-i}(\underline{\mu}(t,x))\lambda^i.$$

(2) Let $p \equiv q = 1$ for all $t, x \in \mathbb{R}$. Some observations concerning the Sturm-Liouville operator imply that $\mathcal{M}(t,x) \equiv 2$. The hierarchy thus obtained is a generalization of the Camassa-Holm hierarchy. In [11] the authors studied the case when $k = r$, while in [24] the case when $k = 1$ is considered. In both cases one obtains a solution, which we now retrieve without difficulties.

First of all, consider the case when $r = k = 1$. Then the obtained time-dependent hierarchy (4.3) (or (4.4)) gives as compatibility condition the standard Camassa-Holm equation (see also [4]). In [24] an algebro-geometric approach is considered for the general case when $r \geq 1$ and $k = 1$. After some computation, the equation (4.19) defining the polynomial $\tilde{U}_r$ can be rewritten as follows (we simplify the notation):

$$\tilde{U}_{r,x}(\lambda) = \sum_{i=1}^{g} \left[\frac{\lambda^k}{\mu_i^k} \tilde{U}_r(\mu_i) - \tilde{U}_r(\lambda) \right] \frac{\lambda \mu_{i,x}}{\mu_i(\lambda - \mu_i)}.$$

In the case $r = k = 1$, we easily obtain $\tilde{U}_r(\lambda) = f_1(t,x)\lambda + f_0(t,x)$, where

$$f_0(t,x) = c_1, \qquad f_1(t,x) = -c_1 \left(\sum_{i=1}^{g} \frac{1}{\mu_i(t,x)} \right) + c_2,$$

where c_1 and c_2 are real constants. When $r > 1$, then it is not difficult to obtain the coefficients f_j of the polynomial $\tilde{U}_r$. Indeed, assuming that all the integration constants vanish (except for $c_0 = f_0 = 1$), we obtain the following formulas:

$$\begin{cases} f_r = \dfrac{1}{\prod_i \mu_i}, \\ f_s = f_r \varsigma_{r-s}(\mu), \quad s = 1, \dots, r-2, \\ f_1 = -\sum_i \dfrac{1}{\mu_i} \\ f_0 = 1 \end{cases}$$

If $k = r$, then one one obtains the Camassa-Holm hierarchy in [11]. Due to a different differential operator considered there, we find with different recursion formulas for the coefficients f_j; in particular the coefficient $f_1(t,x)$ assumes the form (again, we assume that all the (additive) integration constants vanish)

$$f_1 = -\sum_{i=1}^{g} \frac{1}{\mu_i(t,x)},$$

which differs substantially from that in [11]. This is due to the fact that those authors consider a different spectral problem.

5 The pole motion and the generalized Jacobian

We have already noticed that the pole motion naturally takes place in the Riemann surface $\mathcal{R}$ of the algebraic relation $w^2 = -(\lambda - \lambda_0)(\lambda - \lambda_1)\dots(\lambda - \lambda_{2g})$. We have briefly described its construction in Section 2. We will use now some notions and terminology of the theory of Riemann surfaces.

The reader can be addressed to texts as [23] for more information. The Riemann surface $\mathcal{R}$ is homeomorphic to a 2-dimensional torus with g holes. Let a_i, b_i ($i = 1, \ldots, g$) be a standard basis for the homology of $\mathcal{R}$. Then a_i, b_i are cycles. Let $dw_1, \ldots, dw_g$ be a basis of normalized holomorphic differentials on $\mathcal{R}$. By the term normalized we mean, as usual, that

$$\int_{a_i} dw_j = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Let $\mathcal{J}(\mathcal{R})$ be the Jacobian variety obtained from $\mathcal{R}$. To describe $\mathcal{J}(\mathcal{R})$ in a little more detail, we use the notion of divisor. Let $D^g(\mathcal{R})$ be the set of unordered g-tuples of points in $\mathcal{R}$. The elements of $D^g(\mathcal{R})$ are called divisors. On $D^g(\mathcal{R})$ we introduce an equivalence relation as follows: two divisors $\mathfrak{D}$ and $\mathfrak{G}$ are identified if $\mathfrak{D} - \mathfrak{G}$ is the divisor of a meromorphic function on $\mathcal{R}$. One can define a map $I : D^g(\mathcal{R}) \to \mathbb{C}^g$ as follows: if $\mathfrak{D} = (P_1, \ldots, P_g)$ then

$$I(\mathfrak{D}) = \sum_{i=1}^{g} \int_{P^*}^{P_i} (dw_1, \ldots, dw_g),$$

where $P^* \in \mathcal{R}$ is a fixed initial point different from the ramification points. Two vectors $u, v \in \mathbb{C}^g$ are identified if $u - v = n + mZ$, where $n, m \in \mathbb{Z}^g$ and Z is the so called period matrix of the differentials $dw_1, \ldots, dw_g$. The map I sends equivalence classes in $D^g(\mathcal{R})$ into equivalence classes in $\mathbb{C}^g$. The quotient set of $\mathbb{C}^g$ under the equivalence above is called the Jacobian variety $\mathcal{J}(\mathcal{R})$ of $\mathcal{R}$. The map $I : D^g(\mathcal{R}) \to \mathcal{J}(\mathcal{R})$ is called the Abel map and is a birational isomorphism. The Riemann theta-function $\Theta : \mathbb{C}^g \to \mathbb{C}$ is used to solve the classical Jacobi inversion problem. We will not go into the details here (see [23] for more information).

Let us introduce the following differentials on $\mathcal{R}$:

$$dv_i = \frac{\lambda^{i-1}}{k(\lambda)}, \qquad \lambda = \pi(P), \quad i = 1, \ldots, g.$$

The pole motion $\mu_j(t, x)$ takes place on $\mathcal{R}$. let us define functions

$$v_i : \mathbb{R}^2 \to \mathcal{J}(\mathcal{R}) : (t, x) \mapsto \sum_{j=1}^{g} \int_{P^*}^{\mu_j(t,x)} dv_i, \qquad i = 1, \ldots, g.$$

In order to study the nature of the motion of the poles, we try to determine the motion of the maps $v_i(t, x)$ on $\mathcal{J}(\mathcal{R})$. To do this we differentiate $v_i(t, x)$. We have

$$v_{i,x} = \sum_{j=1}^{g} \frac{\mu_j^{i-1} \mu_{j,x}}{k(\mu_j)} = \frac{(-1)^g \mathcal{M} \prod_{s=1}^{g} \mu_s}{pk(0^+)} \left(\sum_{j=1}^{g} \frac{\mu_j^{i-1}}{\prod_{s \neq j} (\mu_j - \mu_s)} \right).$$

Now, we use a formula, called the Lagrange interpolation formula.

Lemma 5.1 *Let $\underline{P} = (P_1, \ldots, P_g) \in \mathbb{C}^g$ be a g-tuple of distinct complex numbers. For every $i = 1, \ldots, g$ we have*

$$\sum_{j=1}^{g} \frac{P_j^{i-1}}{\prod_{s \neq j} (P_j - P_s)} = \delta_{i,g}.$$

Using the Lemma above we have:

$$v_{i,x}(t,x) = \begin{cases} 0, & i = 1, \dots g-1 \\ \dfrac{(-1)^g \mathcal{M}(t,x)}{p(t,x)k(0^+)} \displaystyle\prod_{j=1}^{g} \mu_j(t,x), & i = g, \end{cases} \tag{5.1}$$

or equivalently

$$v_i(t,x) = \begin{cases} \gamma_i(t), & i = 1, \dots g-1 \\ \gamma_g(t) + \displaystyle\int_0^x \dfrac{(-1)^g \mathcal{M}(t,s)}{p(t,s)k(0^+)} \prod_{j=1}^{g} \mu_j(t,s)ds, & i = g. \end{cases} \tag{5.2}$$

In the general case, the Abel map does not linearize the pole motion with respect to x, unless the quantity

$$\frac{(-1)^g \mathcal{M}(t,x)}{p(t,x)k(0^+)} \prod_{j=1}^{g} \mu_j(t,x)$$

is a constant. This happens if and only if $\sqrt{\dfrac{y(t,x)}{p(t,x)}}$ is a constant. A well-known situation in which this occurs is the standard K-dV hierarchy, where $p(t,x) = y(t,x) \equiv 1$, hence the x-motion of the poles $\mu_i(t,x)$ is given by

$$v_i(t,x) = \begin{cases} \gamma_i(t), & i = 1, \dots g-1, \\ \gamma_g(t) - 2x, & i = g. \end{cases}.$$

However, in the case when this motion in nonlinear, i.e. when $\sqrt{\dfrac{y(t,x)}{p(t,x)}}$ is not constant, it is easily seen that it is of an intrinsic implicit nature. Indeed, the pole motion on $\mathcal{J}(\mathcal{R})$ depends implicitly on the product of the poles. In many cases a different construction provide much insight into this problem. The idea is to describe the motion in a larger space than $\mathcal{J}(\mathcal{R})$, i.e., in a generalized Jacobian variety $\mathcal{J}_0(\mathcal{R})$. In the next lines we introduce briefly this object.

Let us consider the Riemann surface $\mathcal{R}$. Let us denote by $\mathcal{R}_0$ the surface obtained by identifying the points 0^+ and 0^- and cutting $\mathcal{R}$ along a path b joining 0^+ and 0^-. In view of the above identification, the path b is closed in $\mathcal{R}_0$. On $\mathcal{R}_0$ there is and additional "hole", hence roughly speaking the standard basis of holomorphic differentials on $\mathcal{R}$ is no longer sufficient to describe the analytic structure of $\mathcal{R}_0$. To this aim, we introduce a differential dw_0 as follows: dw_0 is a differential of the third kind on $\mathcal{R}$ having simple poles at 0^+ and 0^- with residues 1 and -1 respectively, and normalized so as to have

$$\int_{a_i} dw_0 = 0 \qquad (i = 1, \dots g).$$

As in the case of the standard Riemann surface $\mathcal{R}$, there exists a Jacobi variety which we can associate to $\mathcal{R}_0$. It can be constructed by considering the set $D^{g+1}(\mathcal{R})$ of unordered $(g+1)$-tuples of points in $\mathcal{R}$. Again, the elements of $D^{g+1}(\mathcal{R})$ are called divisors. We identify elements in $D^{g+1}(\mathcal{R})$ as follows: two divisors $\mathfrak{D}$, $\mathfrak{G}$ are identified if $\mathfrak{D} - \mathfrak{G}$ is the divisor of a meromorphic function on $\mathcal{R}_0$.

Two vectors $u, v \in \mathbb{C}^{g+1}$ are identified if $u - v = n + mZ_0$, where $n, m \in \mathbb{Z}^{g+1}$ and Z_0 is the "generalized" period matrix of the differentials $dw_0, dw_1, \ldots, dw_g$. It turns out that Z_0 is the matrix $Z_0 = \begin{pmatrix} Z & C \\ 0 & 0 \end{pmatrix}$, where

$$C = \left(\int_{0^-}^{0^+} (dw_1, \ldots, dw_g) \right)^T \in \mathbb{C}^g.$$

The set of the equivalence classes in $\mathbb{C}^{g+1}$ under the equivalence above is called the *generalized Jacobian variety* $\mathcal{J}(\mathcal{R}_0)$. Equivalently, $\mathcal{J}(\mathcal{R}_0) = \mathbb{C}^{g+1}/\Lambda_0$, where Λ_0 is the lattice of rank $2g + 1$ spanned by the matrix $\begin{pmatrix} I \\ Z_0 \end{pmatrix}$ (see [8, 20] for further details concerning the generalized Jacobian variety). There is a birational isomorphism, called the generalized Abel map $I_0 : D^{g+1}(\mathcal{R}) \to \mathcal{J}(\mathcal{R}_0)$, which can be defined as follows:

$$I_0(P_0, P_1, \ldots, P_g) = \sum_{i=0}^{g} \int_{P^*}^{P_i} \left(dw_0, dw_1, \ldots, dw_g \right) \in \mathbb{C}^{g+1}/\Lambda_0,$$

where P^* is a fixed point different from the ramification points of $\mathcal{R}$ and from $0^\pm$.

There is also a generalized Riemann Theta function Θ_0 defined as follows:

$$\Theta_0(z) = e^{z_0/2}\Theta(s + q/2) + e^{-z_0/2}\Theta(s - q/2), \qquad z = (z_0, s) \in \mathbb{C} \times \mathbb{C}^g,$$

where $\Theta(s)$ is the Riemann Theta function. We will make use of this function before long.

Now, consider the holomorphic differentials $dv_1, \ldots, dv_g$ defined above, and let $dv_0 = \dfrac{d\lambda}{\lambda k(\lambda)}$. Then dv_0 is a differential of the third kind in $\mathcal{R}$ having simple poles at $0^\pm$ with residue $\dfrac{1}{k(0^\pm)} = \dfrac{\pm 1}{\sqrt{\prod_{i=0}^{2g} \lambda_i}}$ respectively. As before we define functions $v_0, v_1, \ldots, v_g : \mathbb{R}^2 \to \mathcal{J}(\mathcal{R}_0)$ as follows:

$$v_i(t, x) = \sum_{j=1}^{g} \int_{P^*}^{\mu_j(t,x)} dv_i, \qquad i = 0, 1, \ldots, g.$$

There exists a constant matrix C_0 such that

$$\begin{pmatrix} dv_0 \\ dv_1 \\ \vdots \\ dv_g \end{pmatrix} = C_0 \begin{pmatrix} dw_0 \\ dw_1 \\ \vdots \\ dw_g \end{pmatrix}.$$

Note that the vector $C_0^{-1}(v_0(t, x), v_1(t, x), \ldots, v_g(t, x))^T$ is not the image under the generalized Abel map of a point in $D^{g+1}(\mathcal{R})$, because only g points $\mu_1(t, x), \ldots, \mu_g(t, x)$ are considered. Rather, it is the image of a point of the domain of the restriction, obtained simply by setting $\mu_0 = P^*$, of the generalized Abel map to the set $D^g(\mathcal{R}_0) \subset D^{g+1}(\mathcal{R})$, where $D^g(\mathcal{R}_0)$ is the set of unordered g-tuples of points in $\mathcal{R} \setminus \{0^+, 0^-\}$.

We are now interested in expressing the flow determined by the r-th order equation of the time dependent hierarchy in $\mathcal{J}(\mathcal{R}_0)$. For the moment, let us forget the coordinate $v_g(t,x)$, and derive formulas for $v_0(t,x),\ldots,v_{g-1}(t,x)$. Clearly, if $i=1,\ldots,g-1$, $v_{i,x}(t,x)=0$ as proved above. Moreover,

$$v_{0,x}(t,x)=\frac{(-1)^g\mathcal{M}(t,x)}{p(t,x)k(0^+)}\left(\sum_{i=1}^{g}\frac{\sigma_{g-1}^{(i)}(\underline{\mu})}{\prod_{j\neq i}(\mu_i(t,x)-\mu_j(t,x))}\right)=\frac{(-1)^g\mathcal{M}(t,x)}{p(t,x)k(0^+)}.$$

We thus obtain [7, 13],

$$v_0(t,x)=\gamma_0(t)-\int_0^x\frac{\mathcal{M}(t,s)}{p(t,s)k(0^+)}ds. \tag{5.3}$$

Actually this formula is more pleasant because the x-motion depends on $\mathcal{M}$ and p (which depend only on the initial choice of p and one between q and y). However, a priori, to express the motion on $\mathcal{J}(\mathcal{R}_0)$ the coordinate $v_g(t,x)$ is needed. This coordinate still contains the dependence on the products $\prod_{i=1}^g\mu_i(t,x)$.

To overcome this difficulty, we must understand where the flow takes place in $\mathcal{R}_0$. The theories of the generalized Jacobi variety and of the generalized Abel map help us in answering this question.

When we restrict the Abel map I_0 to the set $D^g(\mathcal{R})$, its image is a noncompact subvariety of $\mathcal{J}(\mathcal{R}_0)$. This subvariety is the locus of the zeros of the generalized Riemann Theta function $\Theta_0(z)$. In other words, I_0 maps the set $D^g(\mathcal{R}_0)$ into complex vectors z satisfying $\Theta_0(z-\Delta_0)=0$, where Δ_0 is a fixed constant vector of Riemann constants. Clearly this subvariety has codimension 1, and we denote it by Υ_0.

Now, for every $i=1,\ldots,g$, the point μ_i lie in the circles $c_i=\pi^{-1}([\lambda_{2i-1},\lambda_{2i}])$ (see Section 2). The g-tuple $\underline{\mu}(t,x)=(\mu_1(t,x),\ldots,\mu_g(t,x))$ lies in the torus $T=c_1\times\cdots\times c_g$. Let us consider the restriction of the Abel map to the torus T. Clearly $I_0(T)\subset\Upsilon_0$. If $\underline{P}=(P_1,\ldots,P_g)\in T$, we use the notation $I_0(\underline{P})=(w_0(\underline{P}),\ldots,w_g(\underline{P}))$, where

$$w_i(\underline{P})=\sum_{j=1}^{g}\int_{P^*}^{P_j}dw_i,\qquad i=0,\ldots g.$$

The motion of the g-tuple $(\mu_1(t,x),\ldots,\mu_g(t,x))$ can be transferred to $\mathcal{J}(\mathcal{R}_0)$. However, it is not directly transferred into Υ_0, but rather into an affine translation Υ of Υ_0. It can be re-mapped to Υ_0 via the linear transformation $C_0^{-1}[I_0(\mu_1,\ldots,\mu_g)]-\Delta_0$.

Now, it is important to underline the fact that often (but not always!) $I_0(T)$ is the graph of a function of the g variables $w_0(\underline{P}),\ldots,w_{g-1}(\underline{P})$, where $\underline{P}$ lies in T [14]. In fact, introduce the map $\eta: I_0(T)\to\mathbb{C}^g/\Gamma$ as follows

$$\eta\left(\sum_{i=1}^{g}\int_{P^*}^{P_i}(dw_0,\ldots,dw_g)\mod\Lambda_0\right)\mapsto\left(\sum_{i=1}^{g}\int_{P^*}^{P_i}(dw_0,\ldots,dw_{g-1})\mod\Lambda\right).$$

This map depends only on the Riemann surface, hence on the parameters $\lambda_0,\lambda_1,\ldots,\lambda_{2g}$. It can be shown (see [14]) that except for a closed nowhere dense subset of $F=\{(\lambda_0,\lambda_1,\ldots,\lambda_{2g})\in\mathbb{R}^{2g+1}\mid 0<\lambda_0<\ldots<\lambda_{2g}\}$, the map η is a diffeomorphism. In this case, then, the motion $(t,x)\mapsto(\mu_1(t,x),\ldots,\mu_g(t,x))$ can be fully described on $\mathcal{J}(\mathcal{R}_0)$ by using the differentials $dw_0,\ldots,dw_{g-1}$ only.

It remains to determine the t-motion on $\mathcal{J}(\mathcal{R}_0)$ of $I_0(\underline{\mu}(t,x))$. We have

$$\begin{aligned} v_{i,t}(t,x) &= \sum_{j=1}^{g} \frac{\mu_j^{i-1}}{k(\mu_j)}\mu_{j,t} = \sum_{j=1}^{g} \frac{\mu_j^{i-1-k}U_r(\mu_j)}{k(\mu_j)}\mu_{j,t} = \\ &= \frac{(-1)^g \mathcal{M}\prod_{s=1}^{g}\mu_s}{k(0^+)p}\left(\sum_{j=1}^{g}\frac{\mu_j^{i-1-k}U_r(\mu_j)}{\prod_{s\neq j}(\mu_j-\mu_s)}\right) \end{aligned} \tag{5.4}$$

Incidentally, the relations (5.2) and (5.3) tell us that the $v_{i,t}(t,x)$ do not depend on x for $i = 1,\dots,g-1$, while $v_{0,t}(t,x)$ and $v_{g,t}$ are more complicated in the sense that they depend on both t and x.

We determine the t-motion for all g holomorphic differentials $v_1,\dots,v_g$ and for the non-holomorphic one v_0. To do this, we need to understand better the form of the coefficients of the polynomial $U_r(t,x)$. Let us write α for the value $\dfrac{1}{p}\left(\dfrac{H_t}{H}+p\left(\dfrac{1}{p}\right)_t\right)$ in (4.19). Fix $i = 1,\dots,g$ and compute the coefficients of the polynomial of degree $r-1$ given by

$$\left[\frac{\lambda^k}{\mu_i^k}\tilde{U}_r(\mu_i)-\tilde{U}_r\right]\frac{\mu_{i,x}}{\lambda-\mu_i}.$$

We use the notation

$$\sum_{s=1}^{r} p_{is}(t,x)\lambda^{s-1}$$

for this polynomial. A direct computation shows that

$$p_{is} = \begin{cases} -\displaystyle\sum_{l=s}^{r} f_l\mu_i^{l-s}\mu_{i,x}, & s = k+1,\dots,r \\ \displaystyle\sum_{l=0}^{s-1} f_l\mu_i^{l-s}\mu_{i,x}, & s = 1,\dots,k \end{cases}$$

Hence the polynomial in the last summand of (4.19) is given by

$$\sum_{s=1}^{r}\left(\sum_{i=1}^{g} p_{is}(t,x)\right)\lambda^{s-1}.$$

In the case $k = r$ one has only the second formula for the coefficients p_{is}.

Let us compute the coefficients of the polynomial $\tilde{U}_r = \displaystyle\sum_{j=0}^{r} f_j\lambda^j$. We do this in the case when $k < r$. The other case $k = r$ is analogous. We start from the coefficients from f_{k+1} to f_r. For $j = k+1,\dots,r-1$ we have

$$f_{j,x} = -\frac{H_x}{H}f_j + \sum_{i=1}^{g} p_{i,j+1},$$

while

$$f_{r,x} = -\frac{H_x}{H}f_r.$$

Hence $f_r = \dfrac{c_r}{H}$, where c_r is a constant.

Next

$$\begin{aligned} f_{r-1} &= -\frac{H_x}{H} f_{r-1} + \sum_{i=1}^{g} p_{i,r} = -\frac{H_x}{H} f_{r-1} - \sum_{i=1}^{g} f_r \mu_{i,x} = \\ &= \frac{1}{H}\left(c_{r-1} - c_r\left(\sum_{i=1}^{g}\mu_i\right)\right). \end{aligned}$$

Continuing this way, we obtain

$$f_j = \frac{1}{H}\left(c_j + c_{j+1}\varsigma_1(\underline{\mu}) + c_{j+2}\varsigma_2(\underline{\mu}) + \cdots + c_r\varsigma_{r-j}(\underline{\mu})\right). \tag{5.5}$$

Things change when we look for formulas for the coefficients $f_0, f_1, \ldots, f_{k-1}$. In this case we have

$$\begin{aligned} f_0 &= \frac{c_0}{\mathcal{M}}, \\ f_1 &= \frac{1}{\mathcal{M}}\left(c_1 - c_0 \sum_{i=1}^{g}\frac{1}{\mu_i}\right), \\ &\vdots \\ f_{k-1} &= \frac{1}{\mathcal{M}}\left(c_{k-1} - c_{k-2}\sum_{i=1}^{g}\frac{1}{\mu_i} + c_{k-3}\left(\sum_{i,j=1}^{g}\frac{1}{\mu_i\mu_j} - \sum_{i=1}^{g}\frac{1}{\mu_i^2}\right) + \ldots\right). \end{aligned}$$

Let us denote by $1/\underline{\mu}$ the vector $(1/\mu_1, \ldots, 1/\mu_g)$. Formulas for $f_0, \ldots, f_{k-1}$ may be rewritten as

$$\begin{aligned} f_0 &= \frac{c_0}{\mathcal{M}}, \\ f_1 &= \frac{1}{\mathcal{M}}\left(c_1 + c_0\varsigma_1(1/\underline{\mu})\right), \\ f_2 &= \frac{1}{\mathcal{M}}\left(c_2 + c_1\varsigma_1(1/\underline{\mu}) + c_0\varsigma_2(1/\underline{\mu})\right), \\ &\vdots \\ f_{k-1} &= \frac{1}{\mathcal{M}}\left(c_{k-1} + c_{k-2}\varsigma_1(1/\underline{\mu}) + \cdots + c_0\varsigma_{k-1}(1/\underline{\mu})\right). \end{aligned}$$

The most important term is f_k. We have

$$f_k(t,x) = \frac{1}{H}\left(c_k + \rho_k(t,x) + \int_0^x H(t,s)\alpha(t,s)ds\right) \tag{5.6}$$

where $\alpha = \dfrac{1}{p}\left(\dfrac{H_t}{H} + p\left(\dfrac{1}{p}\right)_t\right)$, and

$$\rho_k(t,x) = \left(c_{k+1}\varsigma_1(\underline{\mu}(t,x)) + c_{k+2}\varsigma_2(\underline{\mu}(t,x)) + \cdots + c_r\varsigma_{r-k}(\underline{\mu}(t,x))\right).$$

We now need some additional formulas. For a proof, see i.e., [11]. The first one is a generalization of that in Lemma 5.1, namely

Lemma 5.2 *Let $\underline{P} = (P_1, \ldots, P_g) \in \mathbb{C}^g$ be a g-tuple of distinct complex numbers. For every $i = 1, \ldots, g$ and $n = 0, \ldots, g-1$ we have*

$$\sum_{j=1}^{g} \frac{P_j^{i-1}\sigma_n^{(j)}(\underline{P})}{\prod_{s\neq j}(P_j - P_s)} = \delta_{i,g-n} - \varsigma_{n+1}(\underline{P})\delta_{i,g+1}.$$

The formula in Lemma 5.1 is a particular case of the above one obtained by setting $n = 0$.

Lemma 5.3 *For every $r, j = 0, \ldots, g$ there holds*

$$\sum_{i=0}^{r} \varsigma_{r-i}(\underline{P})P_j^i = \sigma_r^{(j)}(\underline{P}).$$

Let us return to the investigation of the derivatives $v_{i,t}$ ($i = 1, \ldots, g$). Formula (5.4) can be rewritten as

$$v_{i,t} = \frac{(-1)^g \mathcal{M}\prod_{s=1}^{g}\mu_s}{k(0^+)}\left(\sum_{j=1}^{g}\frac{\mu_j^{i-1}}{\prod_{s\neq j}(\mu_j - \mu_s)}\frac{\tilde{U}_r(\mu_j)}{\mu_j^k}\right).$$

We split $\dfrac{\tilde{U}_r(\mu_j)}{\mu_j^k}$ into two summands, namely

$$U_r(\mu_i) = U_r^{(1,i)} + U_r^{(2,i)} = \left(\sum_{l=k}^{r} f_l\mu_i^{l-k}\right) + \left(\sum_{l=0}^{k-1} f_l\mu_i^{l-k}\right).$$

Then each $v_{i,t}$ splits into two summands, $v_{i,t} = v_{i,t}^{(1)} + v_{i,t}^{(2)}$, depending on $U_r^{(1,i)}$ and $U_r^{(1,2)}$ respectively. Let us write them explicitly.

We have

$$v_{i,t}^{(1)} = \frac{(-1)^g}{k(0^+)}\sum_{s=0}^{r-k} c_{k+s}\left(\sum_{j=1}^{g}\sigma_s^{(j)}(\underline{\mu})\frac{\mu_j^{i-1}}{\prod_{l\neq i}(\mu_i - \mu_l)} + \int_0^x \left(\frac{H(t,s)}{p(t,s)}ds\right)\frac{\mu_j^{i-1}}{\prod_{l\neq j}(\mu_j - \mu_l)}\right)$$

Using Lemma 5.2 together with the change of variables $\eta_j = \dfrac{1}{\mu_j}$ (which is needed only for the coordinate v_0) we obtain

$$v_{i,t}^{(1)} = \begin{cases} \dfrac{(-1)^g}{k(0^+)}\displaystyle\int_0^x \left(\frac{H(t,s)}{p(t,s)}\right)_t ds, & i = g \\[2ex] \dfrac{(-1)^g}{k(0^+)} c_{k+g-i}, & g - r + k \le i \le g-1, \\[2ex] 0, & 1 \le i < g - r + k, \\[2ex] \dfrac{-1}{k(0^+)\prod_{j=1}^{g}\mu_j}\left(\displaystyle\sum_{s=0}^{r-k} c_{k+s}\varsigma_s(\underline{\mu}) + \int_0^x \left(\frac{H(t,s)}{p(t,s)}\right)_t ds\right), & i = 0. \end{cases}$$

Concerning $v_{i,t}^{(2)}$, we have

$$v_{i,t}^{(2)} = \frac{1}{\mathcal{M}} \sum_{j=1}^{g} \left(c_0 \sigma_{k-1}^{(j)}(1/\underline{\mu}) + c_1 \sigma_{k-2}^{(j)}(1/\underline{\mu}) + \cdots + c_{k-1} \right) \frac{\mu_j^{i-2}}{\prod_{l \neq j}(\mu_j - \mu_l)}.$$

Again, the change of variables $\eta_j = 1/\mu_j$ and Lemma 5.2 imply that

$$v_{i,t}^{(2)} = \begin{cases} \dfrac{(-1)^{g+1}}{k(0^+)} c_{k-i}, & i = 1, \dots, k \\ 0, & i = k+1, \dots, g \\ \dfrac{-1}{k(0^+) \prod_{j=1}^{g} \mu_j} \left(\displaystyle\sum_{s=0}^{k-1} c_s \varsigma_{g-k+s}(\underline{\mu}) \right), & i = 0. \end{cases}$$

Putting together the formulas for $v_{i,t}^{(1)}$ and $v_{i,t}^{(2)}$ above we obtain the equations for the motion with respect to t of the coordinates $v_0, \dots, v_g$. Note that the formula we obtain here for $v_{0,t} = v_{0,t}^{(1)} + v_{0,t}^{(2)}$ is not obviously equal to that which we obtain from (5.3). However, some additional computations show that the formulas are the same. For further results concerning the flow on the generalized Jacobi variety induced by the classical Camassa-Holm equation, see, for instance, [1, 2, 3, 9].

We finish the paper by expressing the solution of the r-th order equation of the hierarchy by means of the generalized Riemann Theta function Θ_0 we introduced before. We repeat here a good deal of results which appear in [24] (see also [11]).

Let $\underline{P} = (P_1, \dots, P_g) \in \mathcal{J}(\mathcal{R})$. Introduce additional variables $\alpha_1, \dots, \alpha_g \in \mathbb{C}$ in such a way that $P_j = P_j(\alpha_1, \dots, \alpha_g)$ and

$$\frac{\partial P_j}{\partial \alpha_k} = \frac{\sigma_{g-k}^{(j)}(\underline{P}) k(P_j)}{\prod_{l \neq j}(P_j - P_l)}.$$

It follows that

$$\frac{\partial}{\partial \alpha_s} \sum_{j=1}^{g} \int_{P^*}^{P_j(\alpha_1, \dots, \alpha_g)} dv_k = \delta_{s,k} \qquad s, k = 1, \dots g,$$

hence

$$\alpha_k = v_k(\underline{P}) - v_k^0,$$

where $v_k^0 \in \mathbb{C}$ is a constant ($k = 1, \dots, g$).

Let us consider the coordinate

$$v_0(\underline{P}) = \sum_{j=1}^{g} \int_{P^*}^{P_j(\alpha_1, \dots, \alpha_g)} dv_0 = \sum_{j=1}^{g} \int_{P^*}^{P_j(\alpha_1, \dots, \alpha_g)} \frac{d\lambda}{\lambda k(\lambda)}.$$

Differentiating $v_0(\underline{P})$ with respect to α_k (and hence with respect to v_k), we have

$$\frac{\partial v_0}{\partial \alpha_k}(\underline{P}) = \sum_{i=1}^{g} \frac{\sigma_{g-k}^{(i)}(\underline{P})}{P_j \prod_{l \neq j}(P_j - P_l)} = -\frac{\varsigma_{g-k}(\underline{P})}{\varsigma_g(\underline{P})}$$

Next, observe that if we choose to move upward in the recursions systems obtained from (4.1) and (4.2), then the solution of the r-th equation of the hierarchy in $y(t,x)$, which can be expressed by means of a symmetric function of the points $\mu_1(t,x),\dots,\mu_g(t,x)$:

$$y(t,x) = \frac{\mathcal{M}^2(t,x)\varsigma_g^2(\underline{\mu})}{4p(t,x)k^2(0^+)}.$$

If we move downward the solution is given by $q(t,x)$ which has the expression

$$q(t,x) = y(t,x)\Big(\sum_{i=0}^{2g}\lambda_i + 2\varsigma_1(\underline{\mu})\Big) + q_g'(t,x) + \frac{q_g(t,x)^2}{p(t,x)},$$

where $q_g = -\dfrac{(py)_x}{4y}$.

Recall that the vector $(v_0(t,x),\dots,v_g(t,x))$ lies in the subvariety $\Upsilon \subset \mathcal{J}(\mathcal{R}_0)$, which is an affine translation of the zero-locus Υ_0 of the generalized Riemann Theta function. This implies that all the symmetric functions of $\mu_1(t,x),\dots,\mu_g(t,x)$ can be interpreted as maps from Υ into $\mathbb{R}$, and hence from Υ_0 into $\mathbb{R}$ via the linear map C_0 (which express the non-normalized differential $dv_0,\dots,dv_g$ as linear combination of the normalized differentials $dw_0,\dots,dw_g$).

If we write down the equation

$$\Theta_0(z) = e^{z_0/2}\Theta(s+q/2) + e^{-z_0/2}\Theta(s-q/2) = 0,$$

where $z = (z_0, s) \in \mathbb{C}\times\mathbb{C}^g$, we obtain

$$z_0 = -\ln\frac{\Theta\Big(s - \int_{P^*}^{0^+}(dw_1,\dots,dw_g)\Big)}{\Theta\Big(s - \int_{P^*}^{0^-}(dw_1,\dots,dw_g)\Big)}.$$

Now, if a $\underline{P} = (P_1,\dots,P_g) \in Symm^g(\mathcal{R})$ and $(w_0(\underline{P}), w_1(\underline{P}),\dots,w_g(\underline{P}))$ lies in Υ_0, then

$$w_0(\underline{P}) = -\ln\frac{\Theta\Big(s - \int_{P^*}^{0^+}(dw_1,\dots,dw_g)\Big)}{\Theta\Big(s - \int_{P^*}^{0^-}(dw_1,\dots,dw_g)\Big)},$$

where $s = I(\underline{P})$, and $I : D^g(\mathcal{R}) \to \mathcal{J}(\mathcal{R})$ is the standard Abel map.

It follows that each symmetric function of the points $\mu_1(t,x),\dots,\mu_g(t,x)$ can be expressed as a theta quotient which involves the Riemann Theta function Θ. In general, a symmetric function of $g+1$ distinct points $P_0,\dots,P_g$ in $\mathcal{R}$ defines a meromorphic function defined in $\mathcal{J}(\mathcal{R}_0)$. The restriction of this map obtained by setting $P_0 = P^*$ can be viewed as a function defined on Υ_0. This restriction, however need not be meromorphic anymore. If $\underline{P} \in \mathcal{J}(\mathcal{R})$, let us use the notation

$$\Theta(s,\pm) = \Theta\left(s - \int_{P^*}^{0^\pm}(dw_1,\dots,dw_g)\right), \qquad s = I(\underline{P}).$$

There are constants c_{ij} $(i = 0, \dots, g,\ j = 1, \dots g)$ such that

$$dv_k = \sum_{j=1}^{g} c_{kj} dw_j$$

$$dv_0 = \frac{1}{k(0^+)} dw_0 + \sum_{j=1}^{g} c_{0j} dw_j.$$

By inversion

$$dw_k = \sum_{j=1}^{g} \tilde{c}_{kj} dv_j.$$

Then

$$dw_0 = k(0^+)\left(dv_0 - \sum_{j,l=1}^{g} c_{0j}\tilde{c}_{jl} dv_l \right).$$

It follows easily that

$$\frac{\partial w_0}{\partial v_k}(\underline{P}) = k(0^+)\left(\frac{\partial v_0}{\partial v_k}(\underline{P}) - \sum_{j=1}^{g} c_{0j}\tilde{c}_{jk} \right) = -k(0^+)\left(\frac{\varsigma_{g-k}(\underline{P})}{\varsigma(\underline{P})} + \sum_{j=1}^{g} c_{0j}\tilde{c}_{jk} \right).$$

A further simple computation gives the proof of the following

Proposition 5.1 *Let $\underline{P} = (P_1, \dots, P_g) \in \mathcal{J}(\mathcal{R})$. Then for every $k = 1, \dots, g$ we have*

$$\frac{\varsigma_{g-k}(\underline{P})}{\varsigma_g(\underline{P})} = \frac{1}{k(0^+)}\left[\sum_{j=1}^{g} \tilde{c}_{jk}\left(\frac{\partial}{\partial w_j} \ln \frac{\Theta(I(\underline{P}), +)}{\Theta(I(\underline{P}), -)} - c_{0j} \right) \right].$$

Now, let us take $\underline{P} = \underline{\mu}(t, x)$. Then $y(t, x)$ can be expressed by means of $\varsigma_g(\underline{\mu}(t, x))$, and in fact

$$y(t, x) = \frac{\mathcal{M}^2(t, x)}{4p(t, x)k^2(0^+)\beta_g^2} k^2(0^+)\left[\sum_{j=1}^{g} \tilde{c}_{jg}\left(\frac{\partial}{\partial w_j} \ln \frac{\Theta(I(\underline{\mu}(t, x)), +)}{\Theta(I(\underline{\mu}(t, x)), -)} - c_{0g} \right) \right]^{-2}.$$

An expression for $q(t, x)$ can be similarly obtained.

We plan to derive in a future work more detailed expressions for the solutions in terms of the logarithmic derivative of the Theta function.

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