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Estimates of the Green's Function and its Regular Part on Heisenberg Group Domains

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Abstract

This paper is a preliminary work on Heisenberg group domains, devoted to the study of the Green's function for the Kohn Laplacian on domains far away from the set of characteristic points. We give some estimates of the Green's function, its regular part and their derivatives analogous to those proved by A. Bahri, Y.Y. Li, O. Rey in [1], and O. Rey in [16] for Euclidean domains. While the study of such functions on the set of characteristic points of the given domain will be discussed in a forthcoming paper.

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1 Introduction and main results

The Green's function has an important role in the study of semi-linear Dirichlet equations involving critical Sobolev exponent. More precisely to search a solution of semi-linear equations for the Kohn Laplacian Δ_H on domains of the Heisenberg group $\mathbb{H}^n$, we have to study the Green's function of the domain, its regular part and especially the behavior of their derivatives near the boundary of the domain. The aim of this paper is to establish some estimates of those functions, in terms of the natural distance of $\mathbb{H}^n$.

For regular domains the Green's function is related to the existence of a classical solution of a particularly Dirichlet problem. The existence and the regularity of this function were studied in [4] for degenerate elliptic operators, for hypoelliptic operators one can see [12, 10]. From [14, 15, 6], it is known that the Green's function can be estimated in terms of the natural sub-Riemannian distance. In [13], other results were derived using a probabilistic interpretation. In [17], some estimates of the Green's function have been studied for the Kohn Laplacian, under suitable boundary regularity assumptions on a bounded domain of the Heisenberg group $\mathbb{H}^n$. The solution to the Dirichlet problem for the Kohn Laplacian in a smooth domain may not be smooth up to the boundary due to the presence of characteristic points, [10]. One can find in [5] for domains with smooth and non characteristic boundary, some estimates up to the boundary for the Green's functions associated to Hörmander operators.

The methods used in the present work are partially inspired from the techniques introduced in [1, 16, 2]. We consider a bounded domain Ω of the Heisenberg group $\mathbb{H}^n$, such that Ω and $\mathbb{H}^n \setminus \overline{\Omega}$ satisfy the uniform exterior ball property. This condition seems to be natural since the Koranyi balls of $\mathbb{H}^n$ satisfy such a property. In [9], Hansen and Hueber proved that this condition implies that the domain Ω is then regular: for φ in $C(\partial\Omega)$, the Dirichlet problem $\Delta_H u = 0$ in Ω, $u = \varphi$ in $\partial\Omega$ has a classical solution $u \in H(\Omega) \cap C(\overline{\Omega})$. In particular Ω has a Green's function, which we denote by G, and for every $\xi \in \Omega$, there exists a classical solution $H(\xi, .)$ of the Dirichlet problem:

$$\begin{cases} \Delta_H \quad H(\xi,.) = 0 & in \ \ \Omega \\ \qquad\ H(\xi,.) = \Gamma(\xi,.) & on \ \ \partial\Omega, \end{cases} \tag{1.1}$$

where $\Gamma(\xi, .)$ is a fundamental solution of $-\Delta_H$ with pole at ξ and $H(\xi, .)$ is the regular part of G. In such a case the Green's function is defined by

$$G(\xi,\xi') = \Gamma(\xi,\xi') - H(\xi,\xi'), \ \ \forall \xi,\xi' \in \Omega. \tag{1.2}$$

We recall that a point $\xi_0 \in \partial\Omega$ is called a characteristic point of Ω, if $\nabla_{\mathbb{H}^n}\varphi(\xi_0) = 0$, where φ is a smooth function which describes the boundary of Ω in a neighborhood of ξ_0 and $\nabla_{\mathbb{H}^n}$ denotes the horizontal gradient of the Heisenberg group $\mathbb{H}^n$.

For a non characteristic point $\xi \in \partial\Omega$, the intrinsic outer unit normal to $\partial\Omega$ at ξ is given by

$$\overrightarrow{\nu} = \frac{\nabla_{\mathbb{H}^n}\varphi(\xi)}{|\nabla_{\mathbb{H}^n}\varphi(\xi)|_H} \tag{1.3}$$

For any ξ near the boundary, let $d_\xi = d(\xi, \partial\Omega)$, and define a subdomain of Ω as follows:

$$\Omega_{d_\xi} = \{\eta \in \Omega \ such \ that \ d(\eta, \partial\Omega) \ > \ d_\xi\}.$$

The intrinsic outer unit normal at ξ means the intrinsic outer unit normal to $\partial\Omega_{d_\xi}$ at ξ.

For Heisenberg group domains Ω, such that Ω and $\mathbb{H}^n \setminus \overline{\Omega}$ satisfy the uniform exterior ball property, and far away for the set of characteristic points of Ω, our main results are:

Theorem 1.1 *For ξ, $\eta \in \Omega$, ξ near the boundary $\partial\Omega$, let $\overrightarrow{\nu}_\xi = \overrightarrow{\nu}$ be the intrinsic outer unit normal to $\partial\Omega$ at ξ, and H the regular part of the Green's function at this point. We have the following estimates*

i. $H(\xi,\xi) = \dfrac{c_q}{(2d_\xi)^2} + o(d_\xi^{-2})$

ii. $H(\eta,\xi) \leq c(\max(d_\xi, d_\eta))^{-2}$

iii. $|\frac{\partial H}{\partial \eta}|(\eta,\xi) \leq \frac{c}{d_\eta} H(\eta,\xi)$

iv. $|\frac{\partial H}{\partial \vec{\nu}}|(\xi,\xi) \leq \frac{c_q}{2d_\xi^3} + o(\frac{1}{d_\xi^3})$

v. $H(\eta,\eta) \geq c.$

Theorem 1.2 *For each ξ_1 and ξ_2 in Ω, near the boundary, we denote d_i the distance of ξ_i to $\partial\Omega$. Let $\vec{\nu}_1$ be the intrinsic outer unit normal to $\partial\Omega$ at ξ_1 and H the regular part of the Green's function. If $\frac{d_1}{d_2}$, $\frac{d_2}{d_1}$ and $\frac{d(\xi_1,\xi_2)}{d_1}$ are bounded, then*

$$(\frac{\partial H}{\partial \vec{\nu}_1})(\xi_1,\xi_2) \ > \ 0.$$

Theorem 1.3 *Let $(\xi_1,\xi_2) \in \Omega^2$ be such that $d_1 \leq d_2$ and $c_2 \ d_2 \leq d(\xi_1,\xi_2)$, where c_2 is a fixed constant. If d_1 is small enough, then*

$$(\frac{\partial G}{\partial \vec{\nu}_{\xi_1}})(\xi_1,\xi_2) \leq 0.$$

Theorem 1.4 *Let $h = \delta_{(a,\lambda)} - P\delta_{(a,\lambda)}$ and $H(\xi,.)$ be the regular part of the Green's function. We have the following estimates:*

i. $h(\xi) = \sigma \frac{H(a,\xi)}{c_q\lambda} + O(\frac{1}{\lambda^3 d_a^4})$

ii. $\frac{\partial h(\xi)}{\partial a} = \sigma \frac{1}{c_q\lambda}\frac{\partial H(a,\xi)}{\partial a} + O(\frac{1}{\lambda^3 d_a^5})$

iii. $\lambda\frac{\partial h(\xi)}{\partial \lambda} = -\sigma \frac{H(a,\xi)}{c_q\lambda} + O(\frac{1}{\lambda^3 d_a^4})$

iv. $|h|_{L^4(\Omega)} = O(\frac{1}{\lambda d_a})$

v. $|\frac{1}{\lambda}\frac{\partial h}{\partial a}|_{L^4(\Omega)} = O(\frac{1}{\lambda^2 d_a^2})$

vi. $|\lambda\frac{\partial h}{\partial \lambda}|_{L^4(\Omega)} = O(\frac{1}{\lambda d_a})$

where for $a \in \Omega$ and $\lambda > 0$ a strictly positive constant, $\delta_{(a,\lambda)}$ denotes the solution of the Yamabe problem for $\mathbb{H}^n$ centered in a with concentration λ, and $P\delta_{(a,\lambda)}$ is its projection on Ω, given by (2.1).

The paper is organized as follows. In section 2, we give notations and recall some known facts about the Heisenberg group. In section 3, we prove a counterpart of the antimaximum principle for Δ_H. The last section is devoted to the proofs of our main results.

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2 Notations and known facts

The Heinserberg group $\mathbb{H}^n$ is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$, with coordinates $\xi = (x, y, t)$ and law group $\xi \cdot \xi' = (x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(< x, y' > - < x', y >))$, where $< ., . >$ denotes the inner product in $\mathbb{R}^n$.

We define a norm on $\mathbb{H}^n$ by $||\xi||_H = ||(x, y, t)||_H = ((|x|^2 + |y|^2)^2 + t^2)^{\frac{1}{4}}$, hence we have a natural distance defined by $d(\xi, \xi') = ||\xi'^{-1} \cdot \xi||_H$. The dilatations are the following transformations $\delta_\lambda : (x, y, t) \to (\lambda x, \lambda y, \lambda^2 t)$, $\lambda > 0$. The Jacobian determinant of δ_λ is λ^{2n+2}, it yields that the homogeneous dimension of $\mathbb{H}^n$ is given by $2n + 2$.

The CR structure on $\mathbb{H}^n$ is given by the left invariant vectors fields: $X_i = \partial_{x_i} + 2y_i\partial_t$ and $Y_i = \partial_{y_i} - 2x_i\partial_t$, $(1 \leq i \leq n)$ which are homogenous of degree -1 with respect to the dilations.

We denote by Δ_H the sublaplacian operator, $\Delta_H = \sum_{i=1}^n (X_i^2 + Y_i^2)$, the subelliptic gradient is given by $\nabla_{\mathbb{H}^n} = (X, Y) = (X_1, ..., X_n, Y_1, ..., Y_n)$.

A fundamental solution of $-\Delta_H$ with pole at ξ is $\Gamma(\xi, \xi') = \dfrac{c_q}{d(\xi, \xi')^{q-2}}$, where $q = 2n + 2$ and the constant c_q is given in [7].

D. Jerison and J.M. Lee showed in [11], that all solutions of the Yamabe problem on $\mathbb{H}^n$ are obtained from

$$\delta_{(0,1)}(z, t) = \frac{\sigma}{|1 + |z|^2 - it|^n}\ , \quad \sigma > 0, \quad z = (x, y),$$

by left translations and dilatations on $\mathbb{H}^n$. That is for $\xi_0 = (z_0, t_0)$, $\xi = (z, t)$ in $\mathbb{H}^n$ and $\lambda > 0$, we have

$$\delta_{(\xi_0,\lambda)}(\xi) = \sigma \frac{\lambda^n}{|\ 1 + \lambda^2(|Z|^2 - iT)\ |^n} \quad where \quad (Z, T) = \xi_0^{-1} \cdot \xi.$$

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality:

$$|\varphi|_{q^*}^2 \leq c|\nabla_{\mathbb{H}^n}\varphi|_2^2, \ \forall\ \varphi \in C_0^\infty(\mathbb{H}^n)$$

where $q^* = \dfrac{2q}{q-2}$. This inequality ensures in particular that for every domain Ω of $\mathbb{H}^n$, the function

$$|\varphi| = |\nabla_{\mathbb{H}^n}\varphi|_2$$

is a norm on $C_0^\infty(\Omega)$. We shall denote by $S^{1,2}(\Omega)$ the Banach space of the functions $u \in L^2(\Omega)$ such that the distributional derivative $X_i u$, $Y_i u \in L^2(\Omega)$, for $i = 1, ...n$. Let $S_0^{1,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to this norm, $S_0^{1,2}(\Omega)$ is a Hilbert space with the inner product:

$$< u, v >_{S_0^{1,2}} = \int_\Omega < \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} v > .$$

Thus there exists a natural orthogonal projection

$$\begin{aligned} P : S_0^{1,2}(\mathbb{H}^n) &\longrightarrow S_0^{1,2}(\Omega) \\ \phi &\longmapsto P\phi \end{aligned}$$

defined by the equation:

$$P\phi = \phi - h \;\; with \;\; \begin{cases} \Delta_H h &= 0 \;\; in \; \Omega \\ h &= \phi \;\; on \; \partial\Omega \end{cases} \Longleftrightarrow \begin{cases} \Delta_H P\phi &= \Delta_H \phi \;\; in \;\; \Omega \\ P\phi &= 0 \quad on \; \partial\Omega. \end{cases} \tag{2.1}$$

q^* is the critical Sobolev exponent for Δ_H since the embedding $S_0^{1,2}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ is continuous but not compact even when Ω is bounded.

Let Ω be a smooth bounded domain of $\mathbb{H}^n$ and $\xi_0 \in \partial\Omega$. Let φ be a smooth function which describes the boundary of Ω in a neighborhood of ξ_0:
$\varphi : B_d(\xi_0, r) \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} \varphi(\xi) &= 0 \;\; iff \;\; \xi \in \partial\Omega \cap \; B_d(\xi_0, r) \\ \varphi(\xi) &> 0 \;\; iff \;\; \xi \in \Omega \cap \; B_d(\xi_0, r) \\ \nabla\varphi &\neq 0 \end{aligned}$$

where ∇ denotes the Euclidean gradient. We say that the domain Ω satisfies the uniform exterior ball property if the following condition holds

$$(P)_0 \begin{cases} There \; exists \; r_0 > 0 \; such \; that \\ \forall \; \xi \; \in \; \partial\,\Omega, \;\; \forall \; r \; \in \;]0, r_0] \; \exists \; \eta \; \in \; \mathbb{H}^n \; such \; that \\ B_d(\eta, r) \cap \Omega = \emptyset \;\; and \;\; \xi \in \partial B_d(\eta, r). \end{cases} \tag{2.2}$$

3 Maximum principle

In our work a basic role is played by different versions of maximum principles for the Kohn Laplacian Δ_H. First we introduce two versions of the weak maximum principle:

Lemma 3.1 *Let Ω be an open bounded set of $\mathbb{H}^n$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ with $\Delta_H u(\xi) \geq 0$ (or ≤ 0) in Ω. Then*

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u \;\; (\; or \;\; \inf_{\overline{\Omega}} u = \inf_{\partial\Omega} u).$$

Lemma 3.2 *Let Ω be an open set of $\mathbb{H}^n$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $\xi_0 \in \overline{\Omega}$, such that $\sup_{\xi\in\Omega} u(\xi) = u(\xi_0)$. If $\Delta_H u(\xi) > 0$ for all ξ in Ω, then $\xi_0 \in \partial\Omega$.*

For the proof and more details one can see [4].

We now turn to the versions of strong maximum principles, we begin by

Theorem 3.1 *Let Ω be an open set of $\mathbb{H}^n$, $u \in C^2(\Omega)$ such that $\Delta_H u \geq 0$ in Ω. Let $\xi_0 \in \partial\Omega$ be a non characteristic point such that*

i. u is continuous at ξ_0;

ii. $u(\xi_0) > u(\xi)$ for all $\xi \in \Omega$;

iii. there exists a ball $B_d(\eta, R) \subset \Omega$, with $\xi_0 \in \partial B_d(\eta, R)$.

Then $\dfrac{\partial u}{\partial \overrightarrow{\nu}}(\xi_0) > 0$, where $\overrightarrow{\nu}$ is the unit outer normal to $\partial\Omega$ at ξ_0.

Proof. Let φ be a function describing the boundary $\partial\Omega$ near ξ_0. For the Euclidean gradient ∇ we denote $\nabla\varphi(\xi_0) = (N_1, .., N_{2n+1})$. Since ξ_0 is a non characteristic point

$$\nabla_{\mathbb{H}^n}\varphi(\xi_0) = (N_1 + 2y_1N_{2n+1}, .., N_n + 2y_nN_{2n+1}, N_{n+1} - 2x_1N_{2n+1}, .., N_{2n} - 2x_nN_{2n+1}) \neq 0.$$

In the frame $(\partial_{x_i}, \partial_{y_i}, \partial_t)$, we have

$$^t\nabla_{\mathbb{H}^n}\varphi(\xi_0) = \begin{pmatrix} N_1 + 2y_1N_{2n+1} \\ \vdots \\ N_n + 2y_nN_{2n+1} \\ N_{n+1} - 2x_1N_{2n+1} \\ \vdots \\ N_{2n} - 2x_nN_{2n+1} \\ \sum_{i=1}^n 2y_iN_i - 2x_iN_{n+i} + 4|z|^4N_{2n+1} \end{pmatrix}, \tag{3.1}$$

and

$$\overrightarrow{\nu} = \frac{\nabla_{\mathbb{H}^n}\varphi(\xi_0)}{|\nabla_{\mathbb{H}^n}\varphi(\xi_0)|_H} = (\nu_i)_{1\leq i\leq 2n+1} = (Z_N, T_N). \tag{3.2}$$

Since ξ_0 is a non characteristic point, we deduce that $Z_N \neq 0$.

For a smooth function f, we define the corresponding Taylor expansion of second order at the point η_0 in the direction of the vector field $\nabla_{\mathbb{H}^n} f$ as:

$$f(\xi) = f(\eta_0) + uX_if(\eta_0) + vY_if(\eta_0) + w\partial_tf(\eta_0) + \frac{1}{2}q_Hf(\eta_0)(u, v) + o(d^2(\xi, \eta_0))$$

where (u, v, w) are the components of $\tau_{\eta_0^{-1}}(\xi)$ and "." is the scalar product in $\mathbb{R}^n$.

Then for $\xi_0 = \xi \cdot t\overrightarrow{\nu}$, where $t\overrightarrow{\nu} = (u, v, w) = (t\nu_i, t\nu_{i+n}, t^2\nu_{2n+1})$, we have

$$\begin{aligned} \frac{\partial f}{\partial\overrightarrow{\nu}}(\xi_0) &= \lim_{t\to 0}\frac{f(\xi\cdot t\overrightarrow{\nu}) - f(\xi)}{t} \\ &= \lim_{t\to 0}\frac{1}{t}(u.X_if(\xi) + v.Y_if(\xi) + t^2\nu_{2n+1}\partial_tf(\xi) + \frac{t^2}{2}q_Hf(\xi)(\nu_i, \nu_{n+i}) \\ &+ t^2o(1)) \\ &= \sum_{i=1}^n \nu_iX_if(\xi_0) + \nu_{n+i}Y_if(\xi_0). \end{aligned}$$

Therefore

$$\frac{\partial f}{\partial\overrightarrow{\nu}}(\xi_0) = \nabla_{\mathbb{H}^n}f(\xi_0).Z_N. \tag{3.3}$$

In the sequel, we have to show that $\frac{\partial u}{\partial\overrightarrow{\nu}}(\xi_0) > 0$. For $0 < \rho < R$, we introduce an auxiliary function v, given by

$$v = e^{-\alpha r^2} - e^{-\alpha R^2}$$

where $r = d(\xi,\eta) > \rho$ and α is a positive constant, yet to be determined. We denote $\eta^{-1}\cdot\xi$ by (Z,T) then $r = (|Z|^4 + T^2)^{\frac{1}{4}}$. Direct computation gives

$$X_j(d(\xi,.)) = \frac{(x_j - \cdot_j)|Z|^2 + (y_j - \cdot_j)T}{d^3(\xi,.)}, \tag{3.4}$$

and

$$Y_j(d(\xi,.)) = \frac{(y_j - \cdot_j)|Z|^2 + (\cdot_j - x_j)T}{d^3(\xi,.)}. \tag{3.5}$$

Therefore, the subelliptic Laplacian is given by

$$\Delta_H r = \frac{2n+1}{r}|\nabla_{\mathbb{H}^n} r|^2$$

where

$$\nabla_{\mathbb{H}^n} r = \frac{1}{r^3}\Big[|Z|^2\Big(\begin{array}{c}(x_j - \cdot_j)\\(y_j - \cdot_j))\end{array}\Big) + T\Big(\begin{array}{c}(y_j - \cdot_j)\\-(x_j - \cdot_j)\end{array}\Big)\Big]. \tag{3.6}$$

For $\xi \to \xi_0$ in the direction of $\overrightarrow{\nu}$ and for R small enough, we can write ξ as $\eta \cdot r\overrightarrow{\nu}$ and

$$Z = rZ_N. \tag{3.7}$$

Then

$$|\nabla_{\mathbb{H}^n} r|^2 = \frac{|Z|^2}{r^2} = |Z_N|^2 \neq 0 \; and \; \frac{r\Delta_H r}{|\nabla_{\mathbb{H}^n} r|^2} = 2n+1.$$

We obtain

$$\begin{aligned}\Delta_H v &= e^{-2\alpha r^2}(4\alpha^2 r^2 |\nabla_{\mathbb{H}^n} r|^2 - 2\alpha(|\nabla_{\mathbb{H}^n} r|^2 + r\Delta_H r))\\ &= e^{-2\alpha r^2}|Z_N|^2(4\alpha^2 r^2 - (4n+4)\alpha).\end{aligned}$$

Hence, α may be chosen large enough so that $\Delta_H v \geq 0$ throughout the annular region $A = B\backslash B_d(\eta,\rho)$. Since $u - u(\xi_0) < 0$ on $\partial B_d(\eta,\rho)$, there is a constant $\epsilon > 0$ for which $u - u(\xi_0) + \epsilon v \leq 0$ on $\partial B_d(\eta,\rho)$. This inequality is also satisfied on $\partial B_d(\eta, R)$, where $v = 0$. Thus, we have $\Delta_H(u - u(\xi_0) + \epsilon v) \geq 0$ in A, and $u - u(\xi_0) + \epsilon v \leq 0$ on ∂A. The weak maximum principle implies that

$$u - u(\xi_0) + \epsilon v \leq 0 \quad in \;\; A. \tag{3.8}$$

Taking the normal derivative at ξ_0, we obtain

$$\frac{\partial u}{\partial \overrightarrow{\nu}}(\xi_0) \geq -\epsilon \frac{\partial v}{\partial \overrightarrow{\nu}}(\xi_0).$$

A computation gives $\nabla_{\mathbb{H}^n} v = -2\alpha r e^{-\alpha r^2}\nabla_{\mathbb{H}^n} r$, and using (3.3), (3.6) and (3.7), we derive

$$\frac{\partial v}{\partial \overrightarrow{\nu}}(\xi_0) = \nabla_{\mathbb{H}^n} v(\xi_0).Z_N = -2\alpha R|Z_N|^4 e^{-\alpha R^2}.$$

Since $|Z_N|^4 \neq 0$, it yields

$$\frac{\partial u}{\partial \overrightarrow{\nu}}(\xi_0) \geq 2\epsilon\alpha R|Z_N|^4 e^{-\alpha R^2} \;\; > 0.$$

We have also the following result

Theorem 3.2 *[3] Let Ω be a domain of $\mathbb{H}^n$ and $u \in C^2(\Omega)$ such that $\Delta_H u \geq 0$ (≤ 0). If u achieves its maximum (minimum) in the interior of Ω, then u is a constant.*

As a direct consequence of these results, we deduce the following

Corollary 3.1 *(The generalized strong maximum principle) If the condition (P_0) is satisfied by a bounded domain Ω of $\mathbb{H}^n$ and by $\mathbb{H}^n \setminus \overline{\Omega}$, for given $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and non positive function f, such that*

$$\Delta_H u(\xi) = f \leq 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

we have two cases.
a) If $f = 0$ in Ω, then $u = 0$ ($\inf_{\overline{\Omega}} = \sup_{\overline{\Omega}} = 0$).
b) If $f \neq 0$ in Ω, then $u > 0$ and $\frac{\partial u}{\partial \overrightarrow{\nu}}(\xi) < 0$ for all $\xi \in \partial\Omega$.

4 Proofs of main results

Proof of Theorem 1.1. **(i)** Following the work of [16], $d = d(\xi, \partial\Omega) < d_0$, where d_0 is a fixed positive constant, and let ξ' be the unique point on the boundary which satisfies $\xi' = \xi \cdot d\overrightarrow{\nu}$, and $\xi'' = \xi.2d\overrightarrow{\nu}$ the "symmetric" point of ξ with respect to the boundary of Ω. For d_0 small enough, ξ'' is not in Ω. Let us define the following function on Ω

$$f : \eta \longmapsto \frac{c_q}{d^2(\xi'', \eta)}.$$

It is an harmonic function, and we have

$$\frac{1}{d^2(\xi, \eta)} - \frac{1}{d^2(\xi'', \eta)} = o(\frac{1}{d^2(\xi', \xi)}) \text{ on } \partial\Omega \text{ as } d \to 0.$$

We have

$$H(\xi, \eta) = \frac{c_q}{d^2(\xi, \eta)} = f(\eta) + o(\frac{1}{d^2(\xi', \xi)}) \text{ on } \partial\Omega$$

thus, the maximum principle yields

$$H(\xi, .) = f(.) + o(\frac{1}{d^2(\xi', \xi)}) \text{ on } \Omega. \tag{4.1}$$

Therefore

$$H(\xi, \xi) = f(\xi) + o(\frac{1}{d^2}) = \frac{c_q}{(2d)^2} + o(\frac{1}{d^2}). \tag{4.2}$$

(ii) We denote by ζ the point of Ω which realizes $d_{max} = \max(d_\xi, d_\eta)$. The function H_ζ is harmonic in Ω and since this function is symmetric, we obtain:

$$H(\eta, \xi) = H(\xi, \eta) \leq \sup_{\alpha \in \Omega} H(\zeta, \alpha)$$

and using the maximum principle, we derive

$$H(\xi,\eta) \leq \sup_{\alpha\in\partial\Omega} H(\zeta,\alpha) = \frac{c}{inf_{\alpha\in\partial\Omega} d^2(\zeta,\alpha)} = \frac{c}{d_{max}^2}.$$

Hence, $H(\xi,\eta) \leq c\ \max(d_\xi, d_\eta)^{-2}$.

(iii) Let $B_\eta = B(\eta, \frac{1}{2}d_\eta)$. According to Proposition 2.1 of [17], we have

$$|\nabla^\eta_{\mathbb{H}^n} H_\xi(\eta)| \leq \frac{c}{d_\eta} \sup_{B_\eta} |H_\xi|.$$

H_ξ is an harmonic positive function in Ω and in particular in B_η, then the maximum principle shows that

$$\sup_{B_\eta} |H_\xi| = \sup_{B_\eta} H_\xi = \sup_{\overline{B}_\eta} H_\xi.$$

The Harnack inequality yields

$$\sup_{\overline{B}_\eta} H_\xi \leq c'\ \inf_{\overline{B}_\eta} H_\xi.$$

We derive

$$|\nabla^\eta_{\mathbb{H}^n} H_\xi(\eta)| \leq \frac{c''}{d_\eta} \inf_{\overline{B}_\eta} H_\xi \leq \frac{c''}{d_\eta} H_\xi(\eta).$$

(iv) The function $X_i(H)(\xi,.)$ (resp $Y_i(H)(\xi,.)$) is harmonic on Ω and

$$X_i(\frac{c_q}{d^2(\xi,.)}) = -2c_q \frac{X_i(d(\xi,.))}{d(\xi,.)^3} \ on\ \partial\Omega.$$

For $\xi = (z,t) = (x_i, y_i, t)$, $\eta^{-1}\cdot\xi = (Z,T)$; using (3.4) and (3.5), on $\partial\Omega$, a simple computation gives

$$\nabla_{\mathbb{H}^n} H(\xi,.) = \frac{-2c_q}{d^6(\xi,.)}\Big[|Z|^2\begin{pmatrix} (x_j - \cdot_j) \\ (y_j - \cdot_j)) \end{pmatrix} + T\begin{pmatrix} (y_j - \cdot_j) \\ -(x_j - \cdot_j) \end{pmatrix}\Big]$$

And using (3.3), we derive

$$\frac{\partial H}{\partial \overrightarrow{\nu}}(\xi,.) \quad = \quad \nabla_{\mathbb{H}^n} H(\xi).Z_N.$$

Let $\xi'' = \xi\cdot 2d\overrightarrow{\nu}$, we consider for $\eta^{-1}\cdot\xi'' = (Z'', T'')$ the functions defined on Ω by:

$$\begin{aligned} f_\xi^X(\eta) \quad &= \quad \frac{-2c_q}{d(\xi'',\eta)^6}[(x'' - \cdot)|Z''|^2 + (y'' - \cdot)T'' \\ &\quad -2\frac{\nu_i}{|Z_N|^2} < |Z''|^2\begin{pmatrix} (x'' - \cdot) \\ (y'' - \cdot) \end{pmatrix} + T''\begin{pmatrix} (y'' - \cdot) \\ -(x'' - \cdot) \end{pmatrix}, Z_N >] \end{aligned}$$

$$\begin{aligned} f_\xi^Y(\eta) \quad &= \quad \frac{-2c_q}{d(\xi'',\eta)^6}[(y'' - \cdot)|Z''|^2 - (x'' - \cdot)T'' \\ &\quad -2\frac{\nu_{n+i}}{|Z_N|^2} < |Z''|^2\begin{pmatrix} (x'' - \cdot) \\ (y'' - \cdot) \end{pmatrix} + T''\begin{pmatrix} (y'' - \cdot) \\ -(x'' - \cdot) \end{pmatrix}, Z_N >] \end{aligned}$$

where X denotes X_i, and Y denotes Y_i, $i \in 1, 2, ...n$.

Since we have $\xi^{'-1} \cdot \xi^{''} = \xi^{-1} \cdot \xi^{'}$, then

$$\begin{aligned} X_j H(\xi,\eta) - f_\xi^X(\eta) &= -2c_q[(x - \cdot)(\frac{|Z|^2}{d(\xi^{'-1}\xi, \xi^{'-1}\eta)^6} - \frac{|Z^{''}|^2}{d(\xi^{-1}\xi^{'}, \xi^{'-1}\eta)^6}) \\ &+ (y - \cdot)(\frac{T}{d(\xi^{'-1}\xi, \xi^{'-1}\eta)^6} - \frac{T^{''}}{d(\xi^{-1}\xi^{'}, \xi^{'-1}\eta)^6}) \\ &+ \frac{1}{d(\xi^{-1}\xi^{'}, \xi^{'-1}\eta)^6}[-2d\nu_j|Z^{''}|^2 - 2d\nu_{j+n}T^{''} \\ &+ 2\frac{\nu_j}{|Z_N|^2}| < V_1, Z_N >], \end{aligned}$$

where $V_1 = \left(\ |Z^{''}|^2 Z^{''} \quad + \quad T^{''} z_{-i\eta^{-1}\cdot\xi^{''}} \ \right)$.

If d is small enough, $\xi^{''}$ is not in Ω, and f_ξ^X (resp f_ξ^Y) is an harmonic function. Moreover

$$X_j H(\xi,\eta) = f_\xi^X(\eta) + o(\frac{1}{d^3}) \quad (\textit{ respectively } Y_j H(\xi,\eta) = f_\xi^Y(\eta) + o(\frac{1}{d^3}))$$

uniformly in $\eta \in \partial\Omega$ as d goes to zero, and we obtain the following estimates for the function $\widehat{H}(\xi) = H(\xi,\xi)$:

$$\begin{aligned} X_j\widehat{H}(\xi) &= 2X_j H(\xi,\xi) \\ &= \frac{-4c_q}{(2d)^6}[(2d)^3(|Z_N|^2\nu_j + \nu_{2n+1}\nu_{n+j}) - 2\frac{\nu_j}{|Z_N|^2} < V_2, Z_N >] \\ &+ o(\frac{1}{d^3}), \end{aligned}$$

where

$$V_2 = \begin{pmatrix} (2d)^3|Z_N|^2\nu_i + \nu_{2n+1}\nu_{n+i} \\ (2d)^3|Z_N|^2\nu_{n+i} - \nu_{2n+1}\nu_i \end{pmatrix}.$$

For $z = (x, y) \in \Omega$, we denote by $\widetilde{z} = (y, -x)$. Since $z^{''} = z + 2dz_N$ and $\widetilde{z}^{''} = \widetilde{z} + 2d\widetilde{Z}_N$, it yields

$$\begin{aligned} \nabla_{\mathbb{H}^n} H(\xi,\eta) - f_\xi(\eta) &= -2c_q[Z(\frac{|Z|^2}{d(\xi^{'-1}\xi, \xi^{'-1}\eta)^6} - \frac{|Z^{''}|^2}{d(\xi^{-1}\xi^{'}, \xi^{'-1}\eta)^6}) \\ &+ \widetilde{Z}(\frac{T}{d(\xi^{'-1}\xi, \xi^{'-1}\eta)^6} - \frac{T^{''}}{d(\xi^{-1}\xi^{'}, \xi^{'-1}\eta)^6}) \\ &+ \frac{1}{d(\xi^{-1}\xi^{'}, \xi^{'-1}\eta)^6}[-2|Z^{''}|^2 dZ_N - 2T^{''}d\widetilde{Z}_N \\ &+ 2\frac{Z_N}{|Z_N|^2} < V_1, Z_N >]. \end{aligned}$$

If d goes to zero and for $\eta \in \partial\Omega$, we have:

$$d^3[\nabla_{\mathbb{H}^n} H(\xi,\eta) - f_\xi(\eta)] \longrightarrow 0.$$

Then, the maximum principle shows that

$$
\begin{aligned}
&\nabla_{\mathbb{H}^n} H(\xi,\eta) = f_\xi(\eta) + o(\frac{1}{d^3})\\
= \ & \frac{-2c_q}{d(\xi'',\eta)^6}[|Z''|^2 Z'' + T''\widetilde{Z}'' - \frac{2Z_N}{|Z_N|^2} < V_1, Z_N >] + o(\frac{1}{d^3})\\
= \ & \frac{-2c_q}{d(\xi'',\eta)^6}[|Z''|^2 Z + \widetilde{Z}Z'' + 2d(Z_N|Z''|^2 + T''\widetilde{Z}_N)\\
& -\frac{2Z_N}{|Z_N|^2} < |Z''|^2(Z + 2dZ_N) + T''(\widetilde{Z} + 2d\widetilde{Z}_N), Z_N >] + o(\frac{1}{d^3}).
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
\nabla_{\mathbb{H}^n} H(\xi,\xi) &= 2f_\xi(\xi) + o(\frac{1}{d^3})\\
&= \frac{-4c_q}{(2d)^3}[(|Z_N|^2 Z_N + T_N\widetilde{Z}_N) - \frac{2Z_N}{|Z_N|^2} < V_2, Z_N >] + o(\frac{1}{d^3}).
\end{aligned}
$$

Finally, we obtain :

$$
\begin{aligned}
\frac{\partial H}{\partial \overrightarrow{\nu}}(\xi,\xi) &= \frac{-4c_q}{(2d)^6}[(2d)^3(|Z_N|^2 < Z_N, Z_N > + T_N < \widetilde{Z}_N, Z_N >)\\
&\quad -2 < V_2, Z_N >] + o(\frac{1}{d^3})\\
&= \frac{-4c_q}{(2d)^3}[|Z_N|^4 - 2|Z_N|^4] + o(\frac{1}{d^3})\\
&= \frac{4c_q|Z_N|^4}{(2d)^3} + o(\frac{1}{d^3}).
\end{aligned} \tag{4.4}
$$

Since $\overrightarrow{\nu}$ is a unit vector, $|Z_N|^4 \leq 1$, therefore

$$
|\frac{\partial H}{\partial \overrightarrow{\nu}}(\xi,\xi)| = \frac{c_q|Z_N|^4}{2d^3} + o(\frac{1}{d^3}) \quad \leq \quad \frac{c_q}{2d^3} + o(\frac{1}{d^3}). \tag{4.5}
$$

(v) Let d_0 be a small positive constant. If $d_\eta < d_0$, we have

$$
H(\eta,\eta) = \frac{c_q}{(2d_\eta)^2}(1 + o(1)) \geq \frac{c}{d_\eta^2} \geq \frac{c}{d_0^2}.
$$

If $d_\eta \geq d_0$, then the function $\widehat{H}$ defined in $\overline{\Omega}_{d_0} \subset \Omega$ by $\widehat{H}(\eta) = H(\eta,\eta)$ is continuous. Since $\overline{\Omega}_{d_0}$ is a compact set, we derive

$$
c = \inf_{\overline{\Omega}_{d_0}} \widehat{H} \leq H(\eta,\eta).
$$

Therefore Claim (v) follows.

Proof of Theorem 1.2. We argue by contradiction, and assume the existence of a sequence $(\xi_1^m, \xi_2^m) \in \Omega^2$ such that

$$
\frac{d_{\xi_1^m}}{d_{\xi_2^m}}\ ,\ \frac{d_{\xi_2^m}}{d_{\xi_1^m}}\ \textit{and}\ \frac{d(\xi_1^m,\xi_2^m)}{d_{\xi_1^m}}\ \textit{are bounded and}\ (\frac{\partial H}{\partial \overrightarrow{\nu}_{\xi_1^m}})(\xi_1^m,\xi_2^m) \ \leq\ 0.
$$

According to (4.3), we have

$$\frac{\partial H}{\partial \overrightarrow{\nu}_{\xi_1^m}}(\xi_1^m,\xi_2^m) = \frac{2c_q}{d^6(\xi_1^{m''},\xi_2^m)}[|Z^{''}|^2 < Z^{''}, Z_N > +T^{''} < \widetilde{Z}, Z_N >] + o(\frac{1}{d^3}),$$

where (Z_N, T_N) denotes $\overrightarrow{\nu}_{\xi_1^m}$. Since $\xi_1^{m''} = \xi_1^m \cdot 2d_{\xi_1^m}\overrightarrow{\nu}_{\xi_1^m}$, it yields

$$\begin{aligned} < Z^{''}, Z_N > &= < z_{\xi_1^{m''}} - z_{\xi_2^m}, Z_N > \\ &= < z_{\xi_1^m} - z_{\xi_2^m} + 2d_1^m Z_N, Z_N > \\ &= < z_{\xi_1^m} - z_{\xi_2^m}, Z_N > +2d_1^m |Z_N|^2. \end{aligned}$$

Let $\overrightarrow{\nu}_{\xi_1} = (Z_N^1, T_N^1)$, we have $Z_N^1 = Z_N(\xi_2^m) + o(1)$. Then

$$< z_{\xi_1^m} - z_{\xi_2^m}, Z_N >= |Z_N|^2(d_2^m - d_1^m + o(d_1^m)).$$

Therefore

$$\begin{aligned} < Z^{''}, Z_N > &= |Z_N|^2(d_2^m - d_1^m + o(d_1^m)) + 2d_1^m|Z_N|^2 \\ &= |Z_N|^2(d_1^m + d_2^m + o(d_1^m)), \end{aligned}$$

and

$$< \widetilde{Z}, Z_N > = < (y_1^m - y_2^m, -(x_1^m - x_2^m)), Z_N > .$$

We have

$$y_1^m - y_2^m = \nu_2(d_2^m - d_1^m + o(d_1^m))$$

and

$$x_1^m - x_2^m = \nu_1(d_2^m - d_1^m + o(d_1^m)).$$

Therefore

$$< \widetilde{Z}, Z_N >= (d_2^m - d_1^m + o(d_1^m)) < \widetilde{Z_N}, Z_N >= 0.$$

On the other hand

$$d^4(\xi_1^{m''},\xi_2^m) = |Z^{''}|^4 + T^{''2}.$$

We have

$$\begin{aligned} |Z^{''}|^2 &= |Z + 2d_1^m Z_N|^2 \\ &= |Z|^2 + |Z_N|^2(4d_1^m d_2^m + o(d_1^2)), \end{aligned}$$

hence

$$\begin{aligned} |Z^{''}|^4 &= |Z|^4 + |Z_N|^4(4d_1^m d_2^m + o(d_1^{m^2}))^2 + 2|Z|^2|Z_N|^2(4d_1^m d_2^m + o(d_1^{m^2})) \\ &= |Z|^4 + 2|Z|^2|Z_N|^2(4d_1^m d_2^m + o(d_1^{m^2})) + |Z_N|^4(16 d_1^{m^2} d_2^{m^2} + o(d_1^{m^4}) \\ &\quad + o(d_1^{m^3} d_2^m)) \\ &= |Z|^4 + 2|Z|^2|Z_N|^2(4d_1^m d_2^m + o(d_1^{m^2})) + |Z_N|^4(16 d_1^{m^2} d_2^{m^2} + o(d_1^{m^4})). \end{aligned}$$

Since

$$|Z|^2 = |Z_N|^2(d_1^{m^2} + d_2^{m^2} - 2d_1^m d_2^m + o(d_1^{m^2})),$$

we derive

$$|Z''|^4 = |Z|^4 + 8|Z_N|^4(d_1^{m^3} d_2^m + d_1^m d_2^{m^3} + o(d_1^{m^4})).$$

For the last component, we obtain

$$\begin{aligned} T'' &= T + (2d_1^m)^2 T_N + 4d_1^m < \widetilde{Z}, Z_N > \\ &= T + 4d_1^{m^2} T_N. \end{aligned}$$

Since $|Z_N|^4 + T_N^2 = 1$ and $\dfrac{d(\xi_1^m, \xi_2^m)}{d_1^m}$ is bounded, we obtain $|T| \leq c d_1^{m^2}$ and $|T_N| \leq 1$. Therefore

$$\begin{aligned} T''^2 &= T^2 + 16 d_1^{m^4} T_N^2 + 8 T T_N d_1^{m^2} \\ &\leq T^2 + c d_1^{m^4}. \end{aligned}$$

$$|Z''|^4 + T''^2 \quad \leq \quad |Z|^4 + T^2 + c d_1^{m^4} + 8|Z_N|^4 (d_1^{m^3} d_2^m + d_1^m d_2^{m^3}) + o(d_1^{m^4}).$$

It yields

$$\begin{aligned} \frac{\partial H(\xi_1^m - \xi_2^m)}{\partial \overrightarrow{\nu}_{\xi_1^m}} &\geq \frac{|Z_N|^4 (4 d_1^m d_2^m + o(d_1^{m^2}))(d_1^m + d_2^m + o(d_1^m))}{(|Z|^4 + T^2 + c d_1^{m^4} + 8|Z_N|^4 (d_1^{m^3} d_2^m + d_1^m d_2^{m^3}) + o(d_1^{m^4})))^{6/4}} \\ &\quad + o(\frac{1}{d_1^{m^3}}) \\ &= \frac{|Z_N|^4 (4 d_1^{m^2} d_2^m + 4 d_1^m d_2^{m^2} + o(d_1^{m^3}))}{(|Z|^4 + T^2 + c d_1^{m^4} + 8|Z_N|^4 (d_1^{m^3} d_2^m + d_1^m d_2^{m^3}) + o(d_1^{m^4})))^{6/4}} \\ &\quad + o(\frac{1}{d_1^{m^3}}), \end{aligned}$$

for m large enough, we derive $\dfrac{\partial H(\xi_1^m, \xi_2^m)}{\partial \overrightarrow{\nu}_{\xi_1^m}} > 0$.

Proof of Theorem 1.3. We argue by contradiction, and we assume that there exists a sequence $(\xi_1^m, \xi_2^m) \in \Omega^2$, such that

$$d_1^m \to 0, \;\; d_2^m \geq d_1^m, \;\; c_2 d_2^m \leq d(\xi_1^m, \xi_2^m) \;\; \textit{and} \;\; \frac{\partial}{\partial \overrightarrow{\nu}_{\xi_1^m}} G(\xi_1^m, \xi_2^m) > 0.$$

We have two cases:

- If $\lim \xi_1^m \neq \lim \xi_2^m$, then the strong maximum principle yields a contradiction.
- Suppose now that $\lim \xi_1^m = \lim \xi_2^m$. Three cases may occur:

First case: $\dfrac{d_2^m}{d(\xi_1^m,\xi_2^m)}$ goes to zero Let $f:\Omega\to f(\Omega)=\widehat{\Omega}$, $\xi\longmapsto\widehat{\xi}=\dfrac{\xi}{d(\xi_1^m,\xi_2^m)}$. We have

$$G(\xi,\eta)=\frac{1}{d^2(\xi_1^m,\xi_2^m)}\widehat{G}(\widehat{\xi},\widehat{\eta}),$$

where G and $\widehat{G}$ are the Green's functions of Ω and $\widehat{\Omega}$ respectively. We remark that $Z_N(\overrightarrow{\nu}_{\xi_1^m}) = Z_N(\overrightarrow{\nu}_{\widehat{\xi}_1^m})$, thus

$$\frac{\partial}{\partial\overrightarrow{\nu}_1}G(\xi_1^m,\eta_1^m)=\frac{1}{d^3(\xi_1^m,\xi_2^m)}\frac{\partial}{\partial\overrightarrow{\nu}_{\xi_1}}\widehat{G}(\widehat{\xi}_1^m,\widehat{\eta}_2^m).$$

Let $\overline{\xi}_j^m$ be such that $\overline{\xi}_j^m\in\partial\widehat{\Omega}$ and $\widehat{d}_j^m=d(\widehat{\xi}_j^m,\overline{\xi}_j^m)$. For a fixed constant R ($0<R<\frac{1}{2}$), and $\eta_1\in\widehat{\Omega}$, let $B_1(\eta_1,R)$ be a ball around η_1 contained in $\widehat{\Omega}$, which intersects the boundary $\partial\widehat{\Omega}$ in $\overline{\xi}_1^m$. We introduce the functional

$$v(\xi)=e^{-\alpha|\xi-\eta_1|_H^2}-e^{-\alpha R^2},$$

where α is chosen such that $\Delta_H v\geq 0$ on $B_1/B_2(\eta_1,r)$, where $0<r<\frac{R}{2}$. The functionnal v satisfies

$$v=0 \textit{ on } \partial B_1 \textit{ and } v>0 \textit{ in } B_1.$$

Let A be a set independent of m, such that

$$B_2(\eta_1,r)\subset A\subset\widehat{\Omega},\ \ \widehat{\xi}_2^m\in A \textit{ and } \overline{\xi}_2^m\in\partial A.$$

Observe that, for $\xi\in\partial B_2$, we have $d(\overline{\xi}_2^m,\xi)\geq 1/4$ and for each $\varsigma\in[\widehat{\xi},\overline{\xi}_2^m]$, ($\varsigma=\widehat{\xi}\cdot k\overrightarrow{\nu}_2$ with $0\leq k\leq\widehat{d}_2^m$), $d(\varsigma,\xi)\geq 1/4$. Thus

$$G_A(\xi,\widehat{\xi}_2^m)=G_A(\xi,\overline{\xi}_2^m)+\frac{\partial G_A}{\partial\overrightarrow{\nu}_2}(\xi,\overline{\xi}_2^m)\widehat{d}_2^m+O(\sup_{\varsigma\in[\widehat{\xi}_2^m,\overline{\xi}_2^m]}\frac{\partial^2 G_A}{\partial\overrightarrow{\nu}_2^2}(\xi,\varsigma)\widehat{d}_2^{m^2}).$$

Since $\overline{\xi}_2^m\in\partial A$, then $G_A(\xi,\overline{\xi}_2^m)=0$. Using the generalized maximum principle, we obtain

$$G_A(\xi,\widehat{\xi}_2^m)\geq c(A)\widehat{d}_2^m+O(\widehat{d}_2^{m^2})\geq c\widehat{d}_2^m,$$

where c is independent of m. Since $A\subset\widehat{\Omega}$, then $\widehat{G}\geq G_A$.

For each $\xi\in\partial B_2$, $v(\xi)\leq M$. Thus, for each $\xi\in\partial(B_1\setminus B_2)$

$$\widehat{G}(\xi,\widehat{\xi}_2^m)-\frac{c\widehat{d}_2^m}{M}v(\xi)\geq 0.$$

Observe that

$$-\Delta_H(\widehat{G}(\cdot,\widehat{\xi}_2^m)-\frac{c\widehat{d}_2^m}{M}v(\cdot))\geq 0 \textit{ in } B_1\setminus B_2.$$

Thus, using the maximum principle, we derive

$$\widehat{G}(\cdot,\widehat{\xi}_2^m)-\frac{c\widehat{d}_2^m}{M}v(\cdot)\geq 0 \textit{ in } B_1\setminus B_2.$$

We have $\overline{\xi}_1^m \in \partial\widehat{\Omega}$ and $v(\overline{\xi}_1^m) = 0$. Thus

$$\widehat{G}(\overline{\xi}_1^m, \widehat{\xi}_2^m) - \frac{c\widehat{d}_2^m}{M} v(\overline{\xi}_1^m) = 0,$$

and

$$\frac{\partial}{\partial \overrightarrow{v}_1}(\widehat{G}(\overline{\xi}_1^m, \widehat{\xi}_2^m) - \frac{c\widehat{d}_2^m}{M} v(\overline{\xi}_1^m)) \leq 0.$$

We derive

$$\frac{\partial \widehat{G}}{\partial \overrightarrow{v}_1}(\overline{\xi}_1^m, \widehat{\xi}_2^m) \leq \frac{c\widehat{d}_2^m}{M} \frac{\partial v}{\partial \overrightarrow{v}_1}(\overline{\xi}_1^m) \leq -c\widehat{d}_2^m. \tag{4.6}$$

Observe that

$$\frac{\partial \widehat{G}}{\partial \overrightarrow{v}_1}(\widehat{\xi}_1^m, \widehat{\xi}_2^m) = \frac{\partial \widehat{G}}{\partial \overrightarrow{v}_1}(\overline{\xi}_1^m, \widehat{\xi}_2^m) + O(\sup_{\varsigma \in [\widehat{\xi}_1^m, \overline{\xi}_1^m]} \frac{\partial^2 \widehat{G}}{\partial \overrightarrow{v}_1^2}(\varsigma, \widehat{\xi}_2^m)\widehat{d}_1^m). \tag{4.7}$$

For $\varsigma \in [\widehat{\xi}_1^m, \overline{\xi}_1^m]$, we have

$$\frac{\partial^2 \widehat{G}}{\partial \overrightarrow{v}_1^2}(\varsigma, \widehat{\xi}_2^m) = \frac{\partial^2 \widehat{G}}{\partial \overrightarrow{v}_1^2}(\varsigma, \overline{\xi}_2^m) + O(\sup_{\zeta \in [\widehat{\xi}_2^m, \overline{\xi}_2^m]} \frac{\partial^3 \widehat{G}}{\partial \overrightarrow{v}_1^2 \partial \overrightarrow{v}(\zeta)}(\varsigma, \zeta)\widehat{d}_2^m).$$

Now, for $\zeta \in [\widehat{\xi}_2^m, \overline{\xi}_2^m]$ and $\varsigma \in [\widehat{\xi}_1^m, \overline{\xi}_1^m]$, we have to estimate $\dfrac{\partial^3 \widehat{G}(\varsigma, \zeta)}{\partial \overrightarrow{v}_1^2 \partial \overrightarrow{v}(\zeta)}$. Then, we introduce the sets

$$B_3(\overline{\xi}_1^m, 1/4) \cap \widehat{\Omega} \textit{ and } B_4(\overline{\xi}_2^m, 1/4) \cap \widehat{\Omega}.$$

For each $\xi \in B_3 \cap \widehat{\Omega}$ and $\eta \in B_4 \cap \widehat{\Omega}$, $\widehat{G}(\xi, \eta)$ is an harmonic function in $B_3 \cap \widehat{\Omega}$. Using the divergence theorem, we obtain

$$\widehat{G}(\xi, \eta) = -\int_{\partial(B_3 \cap \widehat{\Omega})} \frac{\partial G_3}{\partial \overrightarrow{v}}(\varsigma, \xi)\widehat{G}(\varsigma, \eta)d\varsigma.$$

For every $Z_1, ..., Z_\alpha \in \{X_1, ..., X_n, Y_1, ..., Y_n\}$, we have

$$Z_1...Z_\alpha \widehat{G}_\eta(\xi) = -\int_{\partial(B_3 \cap \widehat{\Omega})} (\frac{\partial}{\partial \overrightarrow{v}}(Z_1...Z_\alpha G_3(\varsigma, \xi)))\widehat{G}(\varsigma, \eta)d\varsigma = O(1).$$

We have also

$$\widehat{G}(\xi, \eta) = -\int_{\partial(B_4 \cap \widehat{\Omega})} \frac{\partial G_4}{\partial \overrightarrow{v}}(\varsigma, \eta)\widehat{G}(\varsigma, \xi)d\varsigma.$$

Therefore

$$Z_1...Z_\alpha(Z_1...Z_\beta \widehat{G}(\xi, \eta)) = \int_{\partial(B_4 \cap \widehat{\Omega})} \frac{-\partial}{\partial \overrightarrow{v}}(Z_1...Z_\beta G_4(\varsigma, \eta))(Z_1...Z_\alpha \widehat{G}(\varsigma, \xi))d\varsigma \ = O(1).$$

It yields

$$\frac{\partial^3\widehat{G}(\varsigma,\zeta)}{\partial\vec{\nu}_1^2\partial\vec{\nu}(\zeta)} = O(1). \tag{4.8}$$

Therefore (4.6), (4.7) and (4.8) give $\frac{\partial\widehat{G}}{\partial\vec{\nu}_1}(\widehat{\xi_1^m},\widehat{\xi_2^m}) \leq 0$, a contradiction hence the result follows.

Second case: $\frac{d(\xi_1^m,\xi_2^m)}{d_2^m}$ is bounded and $\frac{d_1^m}{d(\xi_1^m,\xi_2^m)}$ goes to zero.

Let $f:\Omega\to f(\Omega)=\widehat{\Omega}$, $\xi\longmapsto\widehat{\xi}=\frac{\xi}{d_2^m}$ then,

$\widehat{d_2^m}=1$, $\widehat{d_1^m}=\frac{d_1^m}{d_2^m}=o(1)$ and $d(\widehat{\xi_1^m},\widehat{\xi_2^m})=\frac{d(\xi_1^m,\xi_2^m)}{d_2^m}\geq c_2$.

We denote by G and $\widehat{G}$ the Green's functions of Ω and $\widehat{\Omega}$ respectively. We have

$$G(\xi,\eta)=\frac{1}{d_2^{m^2}}\widehat{G}(\widehat{\xi},\widehat{\eta}).$$

As in the first case, we introduce a set A such that

$$A\subset\widehat{\Omega},\quad \overline{\xi}_1^m\in\partial A,\quad and\ \widehat{\xi_2^m}\in A$$

(A is a compact set independent of m). Observe that

$$\widehat{G}-G_A\geq 0\ in\ A\ and\ \widehat{G}(\overline{\xi}_1^m,\widehat{\xi_2^m})-G_A(\overline{\xi}_1^m,\widehat{\xi_2^m})=0.$$

Hence

$$\frac{\partial}{\partial\vec{\nu}_1}(\widehat{G}(\overline{\xi}_1^m,\widehat{\xi_2^m})-G_A(\overline{\xi}_1^m,\widehat{\xi_2^m}))\leq 0.$$

Therefore

$$\frac{\partial\widehat{G}}{\partial\vec{\nu}_1}(\overline{\xi}_1^m,\widehat{\xi_2^m})\leq -c$$

(c is independent of m). Hence, we derive

$$\begin{aligned}\frac{\partial}{\partial\vec{\nu}_1}(\widehat{G}(\widehat{\xi_1^m},\widehat{\xi_2^m}) &= \frac{\partial}{\partial\vec{\nu}_1}(\widehat{G}(\overline{\xi}_1^m,\widehat{\xi_2^m})+O(\sup_{\varsigma\in[\widehat{\xi_1^m},\overline{\xi}_1^m]}\frac{\partial^2\widehat{G}}{\partial\vec{\nu}_1^2}(\varsigma,\widehat{\xi_2^m})\widehat{d_1^m})\\ &\leq -c+O(\widehat{d_1^m})\leq -c+O(\widehat{d_1^m})\leq -\frac{c}{2},\end{aligned}$$

a contradiction and the result follows in this case.

Third case: $\frac{d(\xi_1^m,\xi_2^m)}{d_1^m}$ is bounded. We have

$$\frac{\partial}{\partial\vec{\nu}}G(\xi_1^m,\xi_2^m)=\frac{\partial}{\partial\vec{\nu}}(\frac{c_q}{d^2(\xi_1^m,\xi_2^m)})-\frac{\partial}{\partial\vec{\nu}}H(\xi_1^m,\xi_2^m).$$

According to Theorem 1.2, the second term of the right hand side of the previous equality is strictly positive. Turning to the first term, we have

$$\frac{\partial}{\partial \overrightarrow{\nu}}\left(\frac{c_q}{d^2(\xi_1^m,\xi_2^m)}\right) = \frac{-2c_q}{d^6(\xi_1^m,\xi_2^m)} < |Z|^2\begin{pmatrix} (x_1^m - x_2^m) \\ (y_1^m - y_j^m) \end{pmatrix} + T\begin{pmatrix} (y_1^m - y_j^m) \\ -(x_1^m - x_j^m) \end{pmatrix}, Z_N >$$

$$= \frac{-2c_q}{d^6(\xi_1^m,\xi_2^m)}|Z|^2(< Z, Z_N > +T < \widetilde{Z}, Z_N >).$$

Recall that $< \widetilde{Z}, Z_N >= 0$ and $< Z, Z_N >= |Z_N|^4(d_2^m - d_1^m + o(d_1^m))$, a simple computation yields

$$|Z|^2 < Z, Z_N > +T < \widetilde{Z}, Z_N >= -(d_1^m - d_2^m)^3|Z_N|^4.$$

We derive

$$\frac{\partial}{\partial \overrightarrow{\nu}}G(\xi_1^m,\xi_2^m) \quad = \quad \frac{2c_q(d_1^m - d_2^m)^3|Z_N|^4}{d^6(\xi_1^m,\xi_2^m)} - \frac{\partial}{\partial \overrightarrow{\nu}}H(\xi_1^m,\xi_2^m) < 0$$

for m large enough, a contradiction and the result follows. The proof of Theorem 1.3 is thereby complete.

Proof of Theorem 1.4. Recall that for $a \in \Omega$ and $\lambda > 0$ a strictly positive constant,

$$\delta_{(a,\lambda)}(\xi) = \sigma\frac{\lambda^n}{|\ 1 + \lambda^2(|Z|^2 - iT)\ |^n} \quad where\ (Z,T) = a^{-1}\cdot\xi$$

the constant σ is chosen so that $\delta_{(a,\lambda)}$ satisfies the Yamabe Equation on $\mathbb{H}^n$

$$\begin{cases} -\Delta_H u = u^{1+\frac{2}{n}} & on\ \ \mathbb{H}^n \\ \qquad u > 0. \end{cases} \tag{4.9}$$

(i) The function h is harmonic and $h = \delta_{(a,\lambda)}$ *in* $\partial\Omega$. Let $\xi \in \partial\Omega$, $(z,t) = a^{-1}\cdot\xi$ and $d_a = d(a,\partial\Omega)$. Then

$$h(\xi) \quad = \quad (\sigma\frac{\lambda}{|\ 1 + \lambda^2(|z|^2 - it)\ |}) = \sigma\frac{1}{\lambda||z|^2 - it|}(1 + O(\frac{1}{\lambda^2 d_a^2}))$$

by applying the maximum principle, we obtain

$$h(\xi) = \sigma\frac{H(a,\xi)}{c_q\lambda} + O(\frac{1}{\lambda^3 d_a^4}),$$

(ii) Let $h(\xi) := h_{(a,\lambda)}(\xi) = \dfrac{c_q}{\sigma\lambda}H(a,\xi) + f_{(a,\lambda)}(\xi)$. Then

$$f_{(a,\lambda)}(\xi) = \delta_{(a,\lambda)}(\xi) - P\delta_{(a,\lambda)}(\xi) - \frac{c_q}{\sigma\lambda}H(a,\xi).$$

We have

$$\frac{\partial}{\partial a}f_{(a,\lambda)}(\xi) = O(\frac{1}{\lambda^3 d_a^5}).$$

Then

$$\begin{aligned}\frac{\partial h_a}{\partial a}(\xi) &= \frac{\partial}{\partial a}(\sigma\frac{H(a,\xi)}{c_q\lambda}) + O(\frac{1}{\lambda^3 d_a^5})\\ &= \frac{\sigma}{c_q\lambda}\frac{\partial H(a,\xi)}{\partial a} + O(\frac{1}{\lambda^3 d_a^5}).\end{aligned}$$

(iii) We have

$$\frac{\partial}{\partial\lambda}f_{(a,\lambda)}(\xi) = O(\frac{1}{\lambda^4 d_a^4}).$$

Then

$$\begin{aligned}\frac{\partial h}{\partial\lambda}(\xi) &= \frac{\partial}{\partial\lambda}(\sigma\frac{H(a,\xi)}{c_q\lambda}) + O(\frac{1}{\lambda^4 d_a^4})\\ &= -\sigma\frac{H(a,\xi)}{c_q\lambda^2} + O(\frac{1}{\lambda^4 d_a^4})\end{aligned}$$

and therefore

$$\lambda\frac{\partial h}{\partial\lambda}(\xi) = -\sigma\frac{H(a,\xi)}{c_q\lambda} + O(\frac{1}{\lambda^3 d_a^4}).$$

(iv) Using the Sobolev-type inequality

$$|\varphi|_{q^*}^2 \le c|\nabla_{\mathbb{H}^n}\varphi|_2^2,\ \ \forall\varphi\in C_0^\infty(\mathbb{H}^n)$$

for $n = 1, \widetilde{h} = h\ on\ \Omega,\ \ and\ \widetilde{h} = \delta_{(a,\lambda)}\ on\ \mathbb{H}^1\backslash\Omega$, we obtain

$$(\int_{\mathbb{H}^1}\widetilde{h}^4)^{2/4} \le c\int_{\mathbb{H}^1}|\nabla_{\mathbb{H}^1}\widetilde{h}|^2. \tag{4.10}$$

Then

$$\int_{\mathbb{H}^1} -\Delta_H\delta_{(a,\lambda)}.\delta_{(a,\lambda)} = \int_{\mathbb{H}^1}\delta_{(a,\lambda)}^4 = \int_{\mathbb{H}^1}|\nabla_{\mathbb{H}^1}\delta_{(a,\lambda)}|^2 = S^2,$$

$$\begin{aligned}\int_{\mathbb{H}^1\backslash\Omega}\delta_{(a,\lambda)}^4 &= \int_{\mathbb{H}^1\backslash\Omega}\frac{\lambda^4}{(1+\lambda^2 d^2(a,\xi))^4} \le \int_{\mathbb{H}^1\backslash B_i}\frac{\lambda^4}{(1+\lambda^2 d^2(a,\xi))^4}\\ &= O(\frac{1}{(\lambda d_a)^4})\end{aligned}$$

and

$$\int_{\mathbb{H}^1\backslash\Omega}|\nabla_{\mathbb{H}^1}\delta_{(a,\lambda)}|^2 = O(\int_{\lambda d_a}^{+\infty}\frac{r^5}{(1+r^2)^4}dr) = O(\frac{1}{\lambda^2 d_a^2}).$$

It yields

$$\int_\Omega |\nabla_{\mathbb{H}^1}\delta_{(a,\lambda)}|^2 = \int_{\mathbb{H}} |\nabla_{\mathbb{H}^1}\delta_{(a,\lambda)}|^2 - \int_{\mathbb{H}^1\setminus\Omega} |\nabla_{\mathbb{H}^1}\delta_{(a,\lambda)}|^2$$
$$= S^2 + O(\frac{1}{(\lambda d_a)^2})$$

$$\int_\Omega |\nabla_{\mathbb{H}^1} P\delta_{(a,\lambda)}|^2 = \int_\Omega -\Delta_H\delta_{(a,\lambda)}P\delta_{(a,\lambda)} = \int_\Omega \delta^4_{(a,\lambda)} - \int_\Omega \delta^3_{(a,\lambda)}h$$
$$= S^2 + O(\frac{1}{\lambda^2 d_a^2}).$$

Since S^2 is a positive constant and $\widetilde{h} = h$ in Ω, we have

$$\int_\Omega |\nabla_{\mathbb{H}^1} h|^2 = O(\frac{1}{(\lambda d_a)^2}),$$

and using (4.10), we derive

$$|h|_{L^4(\Omega)} = O(\frac{1}{\lambda d_a}).$$

Similar computations give estimates **(v)** and **(vi)**.

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