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Multiplicity and Existence Results for a Nonlinear Elliptic Equation With Sobolev Exponent.

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Abstract

In this paper we consider the following nonlinear elliptic equation with Dirichlet boundary conditions: $-\Delta u = K(x)u^p,\ u > 0$ in $\Omega,\ u = 0$ on $\partial\Omega$, where Ω is a smooth domain in $\mathbb{R}^n$, $n \geqslant 4$ and $p + 1 = \frac{2n}{n-2}$ is the critical Sobolev exponent. Using dynamical and topological methods involving the study of critical points at infinity we establish, under generic conditions on K, some existence and multiplicity results.

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1 Introduction and main results.

In this paper, we prove some existence and multiplicity results for the following nonlinear problem under Dirichlet-boundary conditions

$$\begin{cases} -\Delta u = K(x)u^{\frac{n+2}{n-2}}, & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^n$, $n \geqslant 4$, and K is a $\mathcal{C}^3$-positive function on $\bar{\Omega}$. One motivation to study this equation comes from its resemblance to the prescribed scalar curvature problem in conformal geometry, which consists in finding suitable conditions on a given function K defined on M to be the scalar curvature of a metric $\widetilde{g}$ conformally equivalent to g, where (M, g) is an n-dimensional Riemannian manifold without boundary. The special nature of problem (1.1) appears once considered from a variational viewpoint. Indeed, although this problem enjoys a variational structure in the sense that its solutions can interpreted as critical points of some functional, its associated Euler-Lagrange functional does not satisfy the Palais-Smale condition. This means that there exist noncompact sequences along which, the functional is bounded and its gradient goes to zero. This is due to the non compactness of the embedding $H^1_0(\Omega)$ into $L^{\frac{2n}{n-2}}(\Omega)$. In the case of manifolds without boundary, this problem has been widely studied in various works (see for example the monograph [1] and the reference therein). In contrast to the extensive literature regarding the prescribed scalar curvature problem on manifolds without boundary, in particular on spheres, there are few known results on (1.1). In dimension four, M. Ben Ayed and M. Hammami [11] have studied problem (1.1); they characterized the critical points at infinity for the associated variational problem and using an Euler-Poincaré argument, they gave an existence result. See also [16] for the dimension four. For higher dimensions, E. Hebey [15] studied the case where Ω is a ball and $K = K(|x|)$. We observe that when K is constant, the problem (1.1) is called the Yamabe problem on manifolds with boundary. It has also been studied through the works [5], [10] and some references therein.

In order to state our main results, we need to introduce some notations, and to pose the assumptions used in this paper. We denote by G the Green function and by H its regular part, that is for each $x \in \Omega$,

$$\begin{cases} G(x,y) & = |x-y|^{-(n-2)} - H(x,y) \ \text{ in } \Omega \\ \Delta H(x,\,.\,) & = 0 \quad \text{in } \Omega \\ G(x,\,.\,) & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Throughout this paper, we assume that K has only non-degenerate critical points $y_0, y_1, \ldots, y_h$, ordered in the following way:

$$K(y_0) \geqslant K(y_1) \geqslant \ldots \geqslant K(y_h) \tag{1.3}$$

with

$$\begin{cases} -\dfrac{\Delta K(y_i)}{3\,K(y_i)}+8H(y_i,y_i)\neq 0 & \text{for } i=0,1,\dots,h, \text{ if } n=4\\[2ex] \Delta K(y_i)\neq 0 & \text{for } i=0,1,\dots,h, \text{ if } n\geqslant 5.\end{cases} \tag{1.4}$$

Let F^+ be the subset of the critical points of K satisfying

$$\begin{cases} -\dfrac{\Delta K(y_i)}{3\,K(y_i)}+8H(y_i,y_i)>0 & \text{if } n=4\\[2ex] -\Delta K(y_i)>0 & \text{if } n\geqslant 5.\end{cases} \tag{1.5}$$

($\mathbf{A_1}$) Assume that for each $x\in\partial\Omega$, we have $\dfrac{\partial K(x)}{\partial\nu}<0$, where ν is the outward normal vector on $\partial\Omega$.

In the first part of this work, we focus on the case $n\geq 5$ and characterize the critical points at infinity of the associated variational problem (see definition 2.1), which will be used to prove some existence results. A similar problem has been studied in [5], [2] and [3] but on a compact Riemannian manifold without boundary. For $a\in\Omega$ and $\lambda>0$, let

$$\delta_{a,\lambda}(x)=c_n\Big(\frac{\lambda}{1+\lambda^2|x-a|^2}\Big)^{\frac{n-2}{2}} \tag{1.6}$$

where c_n is a positive constant chosen such that $\delta_{a,\lambda}$ is the family of solutions of the following problem

$$-\Delta u=|u|^{\frac{4}{n-2}}u,\ u>0 \text{ in } \mathbb{R}^n. \tag{1.7}$$

Let P be the projection from $H^1(\Omega)$ on $H^1_0(\Omega)$; that is, $u=Pf$ is the unique solution of

$$\Delta u=\Delta f \qquad \text{in } \Omega\,, \qquad\qquad u=0 \qquad \text{on } \partial\Omega. \tag{1.8}$$

Our first main result is the following:

Theorem 1.1 *Let $n\geq 5$. Assume that* (1.1) *has no solutions. Under the assumption* ($\mathbf{A_1}$)*, the critical points at infinity of the associated variational problem (see definition* 2.1 *below) correspond to*

$$(y_{i_1},\dots,y_{i_p})_\infty:=\sum_{j=1}^{p}\frac{1}{K(y_{i_j})^{\frac{n-2}{n}}}P\delta_{(y_{i_j},\infty)}$$

*where $p\in\mathbb{N}^*$ and the points y_{i_j} are the critical points of K satisfying $-\Delta K(y_{i_j})>0$, for any $i=1,\dots,p$ and any $y_{i_j}\neq y_{i_k}$ if $j\neq k$. Such a critical point at infinity has a Morse index equal to $i(y_{i_1},\dots,y_{i_p})_\infty:=p-1+\sum_{j=1}^{p}n-k_{i_j}$, where $k_{i_j}:=$*ind*(K,y_{i_j}).*

The proof of this theorem is quite delicate and extremely technical. It relies on the construction of a suitable pseudo-gradient Z at infinity, as in [5], [8], which is based on very delicate expansion of the associated Euler-Lagrange functional J and its gradient ∂J in the neighborhood of its potential $critical\,point\,at\,infinity$. With respect to the closed case, new difficulties arise here. For example, looking at the expansion of J near infinity, one notices that the regular part of the Green function goes to infinity, and therefore dominates ΔK (see the expansion of J below). To overcome such a difficulty, we need to control the flow lines near the boundary. In [1], T. Aubin showed that any concentration point on $\partial\Omega$ of a sequence of subcritical solutions has to satisfy $\frac{\partial K}{\partial \nu} < 0$ (see Proposition 6.44 of [1]). Using the assumption $(\mathbf{A_1})$, we prove in our case that the boundary does not have any role in the existence of a critical point at infinity. To analyze the effect of the boundary, we define a pseudo-gradient vector field for J, which allows us to control the minimal distance to the boundary, in fact along the flow lines generated by this vector field, the minimal distance to the boundary is an increasing function whenever it is small enough. Furthermore, if all the points are in a compact set of Ω, the regular part of the Green function becomes bounded and we can extend the construction of [5] to our situation. We hope that this theorem and its related delicate construction will be useful elsewhere. For example it should be useful for the study of the existence of solutions to (1.1). At this point, we will illustrate its usefulness through the following result.Set $\mathcal{C}^\infty$ the set of the critical points at infinity of the associated variational problem:

$$\begin{aligned}\mathcal{C}^\infty \quad = \quad & \Big\{\tau_p^\infty := (y_{i_1},\dots,y_{i_p})_\infty;\ p\in\mathbb{N}^*,\ y_{i_k}\in F^+\ \forall\ k=1,\dots,p \\ & \text{and } y_{i_k}\neq y_{i_j}\ if\ k\neq j\Big\}.\end{aligned}$$

Let $l_\sharp := \sup\{i(\tau_p^\infty),\ \tau_p^\infty\in\mathcal{C}^\infty\}$. For $\ell = 0,\dots,\ell_\sharp$ we define the following set

$$X_\ell^\infty = \bigcup_{\tau_p^\infty\in\mathcal{C}^\infty,\, i(\tau_p^\infty)\le\ell} W_u^\infty(\tau_p^\infty)$$

where $W_u^\infty(\tau_p^\infty)$ is the unstable manifold associated to the critical point at infinity τ_p^∞. X_ℓ^∞ is a stratified set of top dimension $\le \ell$. Without loss of generality, we may assume it equals to ℓ. Moreover X_ℓ^∞ is contractible in Σ^+, since Σ^+ is a contractible set. Let $\theta(X_\ell^\infty)$ be a contraction of X_ℓ^∞ in Σ^+, $\theta(X_\ell^\infty)$ is a stratified set of dimension $\ell+1$. Now for each $\tau_p^\infty\in\mathcal{C}^\infty$ such that $i(\tau_p^\infty)=\ell+1$, we define the intersection number modulo 2, between the suspension of the complex at infinity of order $\ell+1$, $\theta(X_\ell^\infty)$ and the stable manifold of the critical point at infinity τ_p^∞ by

$$\mu(\tau_p^\infty) = \theta(X_\ell^\infty)\cdot W_s^\infty(\tau_p^\infty).$$

Observe that this intersection number is well defined since we may assume by transversality that

$$\partial(\theta(X_\ell^\infty))\cap W_s^\infty(\tau_p^\infty)=\emptyset.$$

Indeed, $\dim\partial(\theta(X_\ell^\infty))=\ell$ while $\mathrm{codim} W_s^\infty(\tau_p^\infty)=\ell+1$.
We are now ready to state the following existence result

Theorem 1.2 *Let* $\Omega \subset \mathbb{R}^n; n \geq 5$ *and* $0 < K \in C^2(\Omega)$ *meeting the assumption* $(\mathbf{A_1})$. *If there exists* $\ell \in \mathbb{N}$ *such that*

(i) $\sum_{\tau_p^\infty \in \mathcal{C}^\infty,\; i(\tau_p^\infty) \leq \ell} (-1)^{i(\tau_p^\infty)} \neq 1,$

(ii) $\mu(\tau_p^\infty) = 0 \ \ (mod\, 2)$, *for any* $\tau_p^\infty \in \mathcal{C}^\infty$ *such that* $i(\tau_p^\infty) = \ell + 1,$

then there exists a solution w *of* (1.1) *such that:*

$$\text{Morse}(w) \leq \ell + 1,$$

where Morse(w) *is the* Morse index *of* w.

Now observe that under our assumption, for $n \geq 5$ the above sum, when taking into account all the critical points at infinity is always equal to 1. Hence there is no hope to obtain an existence result by using the total sum. However, it might happen that some critical points at infinity induce some difference of topology between the level sets of J. A natural question arises therefore: under which condition can we use this local information to deduce an existence result? Regarding this question, Theorem 1.2 gives a sufficient condition under which the fact that some partial sum is different from one implies that our problem has at least one solution. An example of functions satisfying the conditions of Theorem 1.2 is the family of functions given by A. Bahri (see page 324 of [2]). Indeed, assume the following:
$(\mathbf{H_1})$ $K(y_0) \geq K(y_1) \geq \ldots \geq K(y_h)$, with $F^+ = \{y_0, y_1\}$,
$(\mathbf{H_2})$ Ω is a contractible set.

Corollary 1.1 *On* $\Omega \subset \mathbb{R}^n; n \geq 5$, *let* $0 < K \in C^2(\Omega)$ *satisfying the assumptions* $(\mathbf{A_1})$, $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$. *There exists a universal constant* $\varepsilon(n) > 0$ *such that if*
$(\mathbf{H_3})$ $\|K - 1\|_{L^\infty(\Omega)} \leq \varepsilon(n)$,
then problem (1.1) *has a solution.*

Our aim in the second part of this work is to extend the above argument to dimension four, in order to give some existence and multiplicity results generalizing the previous existence result obtained by Ben Ayed and Hammami [11]. For $s \in \mathbb{N}^*$ and for any s-tuple $\tau_s = (y_{i_1}, ..., y_{i_s}) \in (F^+)^s$ such that $y_{i_p} \neq y_{i_q}$ if $p \neq q$, we define a matrix $M(\tau_s) = (M_{pq})_{1 \leq p,q \leq s}$ by

$$\begin{cases} M_{pp} & = -\dfrac{\Delta K(y_{i_p})}{3\, K(y_{i_p})^2} + \dfrac{8H(y_{i_p},\, y_{i_p})}{K(y_{i_p})} \\ \\ M_{pq} & = -8\dfrac{G(y_{i_p}, y_{i_q})}{\big(K(y_{i_p})K(y_{i_q})\big)^{\frac{1}{2}}} \qquad\qquad \text{for } p \neq q. \end{cases} \tag{1.9}$$

We denote by $\rho(\tau_s)$ the least eigenvalue of $M(\tau_s)$. It was first pointed out by A. Bahri [4], that when the self interactions of the functions failing the Palais-Smale condition and the mutual interactions between two different such functions are of the same size, the function ρ plays a fundamental role in the existence of solutions to problems like (1.1). Regarding problem (1.1), such a phenomenon appears for $n = 4$, see [11].

$(\mathbf{A_2})$ Assume that for any $s \in \mathbb{N}^*$, $\rho(\tau_s) \neq 0$.
We set

$$F_\infty := \{\tau_p = (y_1, \dots, y_p) \in F^+ \ \ (see\, 1.5),\, p \in \mathbb{N}^*,\, such\, that \ \ \rho(\tau_p) > 0\}$$

and we define an index $i : F_\infty \to \mathbb{N}$, defined by $i(\tau_p) = p - 1 + \sum_{i=1}^{p} 4 - \text{ind}(K, y_i)$ where $\text{ind}(K, y_i)$ denotes the Morse index of K at y_i.
Now we state our main result for dimension four.

Theorem 1.3 *Let* $\Omega \subset \mathbb{R}^4$ *and* $0 < K \in C^2(\Omega)$ *satisfying the assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$.
If there exists $\ell \in \mathbb{N}$ *such that:*
(i) $\sum_{\tau_p \in F_\infty,\, i(\tau_p) \leq \ell} (-1)^{i(\tau_p)} \neq 1,$

(ii) $\forall\ \tau_p \in F_\infty,\ \ i(\tau_p^\infty) \neq \ell + 1.$
Then, there exists a solution w *to the problem* (1.1) *of Morse index* $\leq \ell + 1$.
Moreover for generic K, *we have*

$$N_\ell \geq \left| 1 - \sum_{\tau_p \in F_\infty,\, i(\tau_p) \leq \ell} (-1)^{i(\tau_p)} \right|$$

where N_ℓ *denotes the set of solutions of* (1.1) *having their Morse indices less or equal to* $\ell + 1$.

We point out here that, taking in the above statement, ℓ to be $\ell_\sharp$, where $\ell_\sharp$ is the maximal index over all elements of F_∞, the second assumption of theorem 1.3 is trivially satisfied. Thus, as consequence of the above theorem we have the following corollary, which recovers previous existence result of [11].

Corollary 1.2 *On* $\Omega \subset \mathbb{R}^4$, *let* $0 < K \in C^2(\Omega)$ *satisfying the assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$. *If,*

$$\sum_{\tau_p \in F_\infty} (-1)^{i(\tau_p)} \neq 1,$$

then there exists at least one solution of (1.1). *Moreover for generic* K, *we have*

$$N \geq \left| 1 - \sum_{\tau_p \in F_\infty} (-1)^{i(\tau_p)} \right|$$

where N *denotes the set of solutions of* (1.1).

The rest of the paper is organized as follows. In the second section, we set up the variational structure and we recall some well known facts. In the third section, we perform an accurate expansion of J near potential critical points at infinity and we prove a Morse Lemma at infinity. In section four we construct a pseudo-gradient W, that will be useful in the proof of Theorem 1.1; and in the last section, we give the proofs of our existence and multiplicity results.

2 Variational structure and preliminary results.

Our problem (1.1) has a variational structure. Indeed, solutions of (1.1) correspond to positive critical points of the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{n-2}{2n}\int_\Omega K|u|^{\frac{2n}{n-2}} \tag{2.1}$$

defined on $H_0^1(\Omega)$. Let $\Sigma = \{\, u \in H_0^1(\Omega)\,,\ \text{s.t. } \|u\|^2 = \int_\Omega |\nabla u|^2 = 1 \,\}$, and $\Sigma^+ = \{\, u \in \Sigma\,,\ u > 0 \,\}$. Instead of working with the functional I defined above, it is more convenient here to work with the functional

$$J(u) = \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega K|u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}} \tag{2.2}$$

defined on Σ. One can easily verify that if u is a critical point of J on Σ^+, then $J(u)^{\frac{n}{4}} u$ is a solution of (1.1). Notice that the functional J does not satisfy the Palais-Smale condition (P.S for short). That is that there exist sequences along which J is bounded, its gradient goes to zero and which is not convergent. The analysis of the sequences failing (P.S) condition can be performed following the ideas introduced in [12] and [18]. Let for $\varepsilon > 0$ $p \in \mathbb{N}^*$ and w either a solution of (1.1) or zero,

$$\begin{aligned} V(p,\varepsilon,w) = \Big\{ & u \in \Sigma^+ \text{ s.t. } \exists\ a_i \in \Omega,\ \lambda_i > \tfrac{1}{\varepsilon},\ \alpha_i > 0 \text{ for } 1 \leqslant i \leqslant p, \\ & \text{and} \ \exists\ \alpha_0 > 0,\ \text{with} \ \|u - \alpha_0 w - \textstyle\sum_{i=1}^p \alpha_i \mathrm{P}\delta_{a_i,\lambda_i}\| < \varepsilon,\ \varepsilon_{ij} < \varepsilon\ \forall\, i \neq j, \\ \lambda_i d_i > \frac{1}{\varepsilon},\ & \Big| \frac{\alpha_i^{\frac{4}{n-2}} K(a_i)}{\alpha_j^{\frac{4}{n-2}} K(a_j)} - 1 \Big| < \varepsilon\ \ \forall\, i,j = 1,\dots,p, \text{ and } |\alpha_0^{\frac{4}{n-2}} J(u)^{\frac{n}{n-2}} - 1| < \varepsilon \ \Big\} \end{aligned}$$

where $d_i = d(a_i,\, \partial\Omega)$ and $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{-\frac{(n-2)}{2}}$. If u is a function in $V(p,\varepsilon,w)$, one can find an optimal representation of u following the ideas introduced in [6] and [5]. Namely we have

Proposition 2.1 *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon < \varepsilon_p$ and $u \in V(p,\varepsilon,w)$, then the following minimization problem*

$$\min\{\ \|u - \textstyle\sum_{i=1}^p \alpha_i \mathrm{P}\delta_{a_i,\lambda_i} - \alpha_0(w+h)\|\ ,\ \alpha_i > 0\,,\ \lambda_i > 0\,,\ a_i \in \Omega\,,\ h \in T_w W_u(w)\ \} \tag{2.3}$$

has a unique solution $(\bar\alpha,\, \bar a,\, \bar\lambda,\, \bar h)$ (up to permutation) . Thus we can uniquely write u as follows (we drop the bar)

$$u = \textstyle\sum_{i=1}^p \alpha_i \mathrm{P}\delta_{a_i,\lambda_i} + \alpha_0(w+h) + v \tag{2.4}$$

where v belongs to $H_0^1(\Omega) \cap T_w W_s(w)$, and satisfies

$$\begin{cases} \langle v,\, \phi_i\rangle & = 0 \quad \textit{for } \ i = 1,\dots,p \ \textit{and}\ \phi_i = \mathrm{P}\delta_i,\ \dfrac{\partial \mathrm{P}\delta_i}{\partial\lambda_i},\ \dfrac{\partial \mathrm{P}\delta_i}{\partial a_i} \\ \langle v,\, w\rangle & = 0 \\ \langle v,\, h\rangle & = 0 \quad \textit{for all } h \in T_w W_u(w). \end{cases} \tag{2.5}$$

Here, $\mathrm{P}\delta_i = \mathrm{P}\delta_{a_i,\lambda_i}$ and $\langle\ ,\ \rangle$ denotes the scalar product defined on $H_0^1(\Omega)$ by

$$\langle u,\, v\rangle = \int_\Omega \nabla u \nabla v.$$

Notice that Proposition 2.1 is also true if we take $w = 0$, therefore $h = 0$ and $u \in V(p,\varepsilon) := V(p,\varepsilon,0)$. The failure of the (P.S) condition can be described following the ideas developed in [12], [18] and [21]. Such a description is by now standard and reads as follows. Let ∂J be the gradient of J.

Proposition 2.2 *Let $(u_j)_j \subset \Sigma^+$ be a sequence such that ∂J tends to zero and $J(u_j)$ is bounded. Then, there exists an integer $p \in \mathbb{N}^*$, a sequence $\varepsilon_j > 0$, $\varepsilon_j \to 0$, and an extracted subsequence of u_j's, again denoted by u_j, such that $u_j \in V(p,\varepsilon_j,w)$, where w is zero or a solution of* (1.1).

Following A. Bahri [4], we introduce the following definition:

Definition 2.1 A *critical point at infinity* of J in Σ^+ is a limit of a flow line $u(s)$ of the equation

$$\begin{cases} \dfrac{\partial u}{\partial s} & = -\partial J(u) \\ \\ u(0) & = u_0 \end{cases}$$

such that $u(s)$ remains in $V(p,\varepsilon(s),w)$, for $s \geqslant s_0$. .

Here, w is either zero or a solution of (1.1), and $\varepsilon(s)$ is some function tending to zero when $s \to +\infty$. Using Proposition 2.1, $u(s)$ can be written as:

$$u(s) = \sum_{i=1}^{p} \alpha_i(s)\mathrm{P}\delta_{a_i(s),\lambda_i(s)} + \alpha_0(s)\big(w + h(s)\big) + v(s).$$

Denoting by $a_i = \lim a_i(s)$ and $\alpha_i = \lim \alpha_i(s)$, we denote by

$$(a_1, \dots,\, a_p,\, w)_\infty \text{ or } \sum_{i=1}^{p} \alpha_i \mathrm{P}\delta_{a_i,\,\infty} + \alpha_0 w$$

such a critical point at infinity. If $w \neq 0$, it is called of *w-type.*

3 Expansion of the functional and its gradient.

In this section we will give the expansion of the functional in the potential sets $V(p,\varepsilon,w)$ where w is critical point of J or zero, $(\partial J(u),\, \frac{1}{\lambda_i}\frac{\partial \mathrm{P}\delta_i}{\partial a_i})$ and $(\partial J(u),\, \lambda_i \frac{\partial \mathrm{P}\delta_i}{\partial \lambda_i})$ in $V(p,\, \varepsilon)$, where δ_i denotes $\delta_{(a_i,\,\lambda_i)}$.

Proposition 3.1 *For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^{p} \alpha_i \mathrm{P}\delta_{a_i,\lambda_i} + \alpha_0(w+h) + v$ in $V(p,\varepsilon,w)$, we have the following expansion*

$$\begin{aligned}
J(u) \quad &= \quad \frac{S_n \sum_{i=1}^{p} \alpha_i^2 + \alpha_0^2 \parallel w \parallel^2}{\left(S_n \sum_{i=1}^{p} \alpha_i^{\frac{2n}{n-2}} K(a_i) + \alpha_0^{\frac{2n}{n-2}} \parallel w \parallel^2\right)^{\frac{n-2}{n}}} \Bigg[1 - \frac{2c_2\alpha_0}{\gamma_1} \sum_{i=1}^{p} \frac{\alpha_i w(a_i)}{\lambda_i^{\frac{n-2}{2}}} \\
&- \quad \frac{1}{\beta_1} \frac{n-2}{n} c_3 \sum_{i=1}^{p} \frac{\alpha_i^{\frac{2n}{n-2}} \Delta K(a_i)}{\lambda_i^2} + \frac{1}{\gamma_1} c_2 \sum_{i=1}^{p} \frac{\alpha_i^2 H(a_i,\, a_i)}{\lambda_i^{n-2}} - \frac{1}{\gamma_1} c_2 \sum_{i \neq j} \alpha_i \alpha_j \\
&\quad \times \Big(\varepsilon_{ij} - \frac{H(a_i,\, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \Big) + f_1(v) + Q_1(v) + f_2(h) + Q_2(h) \\
&+ \quad o\Big(\sum_k \frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k\, d_k)^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} + \|v\|^2 + \|h\|^2 \Big) \Bigg]
\end{aligned}$$

where,

$$S_n = \int_{\mathbb{R}^n} \delta_{(O,\,1)}^{\frac{2n}{n-2}}(x)dx, \quad c_2 = c_n^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^{\frac{n+2}{2}}}, \quad c_3 = c_n^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} dy$$

$$\beta_1 = S_n \sum_{i=1}^{p} \alpha_i^{\frac{2n}{n-2}} K(a_i) + \alpha_0^{\frac{2n}{n-2}} \|w\|^2, \ \gamma_1 = S_n \sum_{i=1}^{p} \alpha_i^2 + \alpha_0^2 \parallel w \parallel^2$$

$$Q_1(v) = \frac{1}{\gamma_1} \|v\|^2 - \frac{1}{\beta_1} \frac{n+2}{n-2} \int K \sum_{i=1}^{p} (\alpha_i \mathrm{P}\delta_i)^{\frac{8}{n-4}} v^2 - \frac{1}{\beta_1} \frac{n+2}{n-2} \alpha_0^{\frac{4}{n-2}} \int K w^{\frac{4}{n-2}} v^2$$

$$Q_2(h) = \frac{1}{\gamma_1} \|h\|^2 - \frac{1}{\beta_1} \frac{n+2}{n-2} \alpha_0^{\frac{4}{n-2}} \int K w^{\frac{4}{n-2}} h^2$$

$$f_1(v) = -\frac{2}{\beta_1} \int K \sum_{i=1}^{p} (\alpha_i \mathrm{P}\delta_i)^{\frac{n+2}{n-2}} v, \text{ and}$$

$$f_2(h) = \frac{2\alpha_0}{\gamma_1} \sum_{i=1}^{p} \alpha_i \langle \mathrm{P}\delta_i,\, h \rangle - \frac{2\alpha_0}{\beta_1} \int K \big(\sum_{i=1}^{p} \alpha_i \mathrm{P}\delta_i + \alpha_0 w \big)^{\frac{n+2}{n-2}} h$$

Proof. Recall that

$$J(u) = \frac{\|u\|^2}{\left(\int_\Omega K|u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}.$$

We need to estimate $N = \|u\|^2$ and $D^{\frac{n}{n-2}} = \int_\Omega K|u|^{\frac{2n}{n-2}}$. Expanding N we get

$$\sum_{i=1}^{p} \alpha_i \|\mathrm{P}\delta_i\|^2 + 2\alpha_i\alpha_0 \langle \mathrm{P}\delta_i,\, w+h \rangle + \alpha_0^2 \big(\|h\|^2 + \|w\|^2 \big) + \|v\|^2 + \sum_{i \neq j} \alpha_i \alpha_j \langle \mathrm{P}\delta_i,\, \mathrm{P}\delta_j \rangle.$$

Using a computation similar to the one performed in [4], we have

$$\|\mathrm{P}\delta_i\|^2 = S_n - c_2\frac{H(a_i,\, a_i)}{\lambda_i^{n-2}} + O\Big(\frac{log\lambda_i d_i}{(\lambda_i\lambda_j)^n}\Big).$$

$$\langle \mathrm{P}\delta_i,\, \mathrm{P}\delta_j\rangle = c_2\Big(\varepsilon_{ij} - \frac{H(a_i,\, a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\Big) + O\Big(\varepsilon_{ij}^{\frac{n}{n-2}} log(\varepsilon_{ij}^{-1}) + \sum_{k=i,\, j}\frac{log\lambda_k d_k}{(\lambda_k d_k)^n}\Big), \text{ For } i \neq j$$

$$\langle \mathrm{P}\delta_i,\, w\rangle = \int_\Omega \delta_i^{\frac{n+2}{n-2}} w = c_2\frac{w(a_i)}{\lambda_i^{\frac{n-2}{2}}} + O\Big(\frac{log\lambda_i d_i}{\lambda_i^{\frac{n-2}{2}}(\lambda_i d_i)^2}\Big).$$

Thus

$$\begin{aligned} N &= \gamma_1 + 2c_2\alpha_0\sum_{i=1}^p \alpha_i\frac{w(a_i)}{\lambda_i^{\frac{n-2}{2}}} + 2\alpha_0\sum_{i=1}^p \alpha_i\langle \mathrm{P}\delta_i,\, h\rangle + c_2\sum_{i\neq j}\alpha_i\alpha_j(\varepsilon_{ij} - \frac{H(a_i,\, a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}) \\ &- c_2\sum_{i=1}^p \alpha_i^2\frac{H(a_i,\, a_i)}{\lambda_i^{n-2}} + \alpha_0^2\|h\|^2 + \|v\|^2 + o\Big(\sum_i\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \frac{1}{(\lambda_i\, d_i)^{n-1}} + \sum_{i\neq j}\varepsilon_{ij}\Big). \end{aligned} \tag{3.1}$$

$$\begin{aligned} D^{\frac{n}{n-2}} &= \int_\Omega K|u|^{\frac{2n}{n-2}} \\ &= \int K\big(\sum_{i=1}^p\alpha_i\mathrm{P}\delta_i\big)^{\frac{2n}{n-2}} + \alpha_0^{\frac{2n}{n-2}}\int Kw^{\frac{2n}{n-2}} + \tfrac{2n}{n-2}\alpha_0\int K\big(\sum_{i=1}^p\alpha_i\mathrm{P}\delta_i\big)^{\frac{n+2}{n-2}}w \\ &+ \tfrac{2n}{n-2}\alpha_0^{\frac{n+2}{n-2}}\int K\big(\sum_{i=1}^p\alpha_i\mathrm{P}\delta_i\big)w^{\frac{n+2}{n-2}} + \tfrac{2n}{n-2}\int K\big(\sum_{i=1}^p\alpha_i\mathrm{P}\delta_i + \alpha_0 w\big)^{\frac{n+2}{n-2}} \times \\ &\quad (\alpha_0 h + v) + \tfrac{n(n+2)}{(n-2)^2}\int K\big(\sum_{i=1}^p\alpha_i\mathrm{P}\delta_i + \alpha_0 w\big)^{\frac{4}{n-2}}(\alpha_0^2h^2 + v^2 + 2\alpha_0 hv) + \\ &+ O\Big(\sum_i\int w^{\frac{4}{n-2}}\mathrm{P}\delta_i^2 + w^2\mathrm{P}\delta_i^{\frac{4}{n-2}}\Big) + o\big(\|v\|^2 + \|h\|^2\big). \end{aligned} \tag{3.2}$$

Using Lemmas 6.1, 6.2, 6.3, 6.4 and 6.15, we derive

$$\begin{aligned} \int K\big(\sum_{i=1}^p\alpha_i\mathrm{P}\delta_i\big)^{\frac{2n}{n-2}} &= \sum_{i=1}^p\alpha_i^{\frac{2n}{n-2}}\Big(K(a_i)S_n + c_3\frac{\Delta K(a_i)}{\lambda_i^2} - \frac{2n}{n-2}c_2K(a_i)\times \\ &\quad \frac{H(a_i,a_i)}{\lambda_i^{n-2}}\Big) + \frac{2n}{n-2}c_2\sum_{i\neq j}\alpha_i^{\frac{n+2}{n-2}}\alpha_jK(a_i)\varepsilon_{ij} \\ &- \tfrac{2n}{n-2}c_2\sum_{i\neq j}\alpha_i^{\frac{n+2}{n-2}}\alpha_jK(a_i)\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}} \\ &+ o\Big(\sum_i\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i\, d_i)^{n-1}} + \sum_{i\neq j}\varepsilon_{ij}\Big). \end{aligned} \tag{3.3}$$

$$\int Kw^{\frac{2n}{n-2}} = \|w\|^2. \tag{3.4}$$

$$\int Kw^{\frac{n+2}{n-2}}\mathrm{P}\delta_i = c_2\frac{w(a_i)}{\lambda_i^{\frac{n-2}{2}}} + o\Big(\frac{1}{\lambda_i^{\frac{n-2}{2}}}\Big). \tag{3.5}$$

$$\int K\big(\sum_{i=1}^{p}\alpha_i\mathrm{P}\delta_i\big)^{\frac{n+2}{n-2}}w = c_2\sum_{i=1}^{p}\alpha_i^{\frac{n+2}{n-2}}K(a_i)\frac{w(a_i)}{\lambda_i^{\frac{n-2}{2}}} + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\frac{n-2}{2}}}\Big). \tag{3.6}$$

$$\int w^{\frac{4}{n-2}}\mathrm{P}\delta_i^2 + w^2\mathrm{P}\delta_i^{\frac{4}{n-2}} = O\Big(\frac{1}{\lambda_i^{n-2}}\Big). \tag{3.7}$$

$$\int K\big(\sum_{i=1}^{p}\alpha_i\mathrm{P}\delta_i + \alpha_0 w\big)^{\frac{4}{n-2}}vh = o\Big(\|v\|^2 + \|h\|^2 + \sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2}}\Big). \tag{3.8}$$

Concerning the linear form in v, since $v \in T_w\big(W_s(w)\big)$, it can be written as

$$\begin{aligned}
&\int K\big(\sum_{i=1}^{p}\alpha_i\mathrm{P}\delta_i + \alpha_0 w\big)^{\frac{n+2}{n-2}}v\\
=\ &\int K\big(\sum_{i=1}^{p}\alpha_i\mathrm{P}\delta_i\big)^{\frac{n+2}{n-2}}v + O\Big(\sum_i\int\Big(w^{\frac{4}{n-2}}\mathrm{P}\delta_i + w\mathrm{P}\delta_i^{\frac{4}{n-2}}\Big)|v|\Big)\\
=\ &f_1(v) + o\Big(\sum_i\frac{\|v\|}{\lambda_i^{\frac{n-2}{2}}}\Big).
\end{aligned}$$

Finally we have

$$\int K\big(\sum_{i=1}^{p}\alpha_i\mathrm{P}\delta_i+\alpha_0 w\big)^{\frac{4}{n-2}}v^2 = \sum_{i=1}^{p}\int K\big(\alpha_i\mathrm{P}\delta_i\big)^{\frac{4}{n-2}}v^2+\alpha_0^{\frac{4}{n-2}}\int Kw^{\frac{4}{n-2}}v^2+o(\|v\|^2) \tag{3.9}$$

$$\int K\big(\sum_{i=1}^{p}\alpha_i\mathrm{P}\delta_i + \alpha_0 w\big)^{\frac{4}{n-2}}h^2 = \alpha_0^{\frac{4}{n-2}}\int Kw^{\frac{4}{n-2}}h^2 + o(\|h\|^2). \tag{3.10}$$

Combining (3.1) to (3.10), and since $\frac{\alpha_i^{\frac{4}{n-2}}K(a_i)}{\alpha_j^{\frac{4}{n-2}}K(a_j)} = 1 + o(1)$, our proposition follows.

Proposition 3.2 *We have*
(**i**) $Q_1(v)$ *is a definite positive quadratic form on*

$$E_v = \Big\{\ v \in H_0^1(\Omega) \cap T_w\big(W_s(w)\big) \quad \text{s.t } v \text{ satisfies } (2.5)\Big\}.$$

(**ii**) $Q_2(h)$ *is a definite negative quadratic form on* $T_w\big(W_u(w)\big)$.

Proof. The proof is similar up to minor modifications to the corresponding statement in [5](see the proof of Lemma 6 of [5]). As a Corollary of Proposition 3.2, we have the following result.

Proposition 3.3 *Let* $u=\sum_{i=1}^p\alpha_i \mathrm{P}\delta_{a_i,\lambda_i}+\alpha_0(w+h)+v$ *in* $V(p,\varepsilon,w)$. *There is an optimal* $(\bar v,\,\bar h)$ *and a change of variables,* $v-\bar v\longmapsto V$ *and* $h-\bar h\longmapsto H$, *such that* J *reads as*

$$J(u)=J\Big(\sum_{i=1}^p\alpha_i\mathrm{P}\delta_{a_i,\lambda_i}+\alpha_0 w+\bar h+\bar v\Big)+\|V\|^2-\|H\|^2.$$

Furthermore, we have the following estimates

$$\|\bar h\|=O\Big(\sum_{i=1}^p\frac{1}{\lambda_i^{\frac{n-2}{2}}}\Big)$$

and

$$\begin{aligned}\|\bar v\| &= O\Big(\sum_{i=1}^p\frac{|DK(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big)\\ &+\begin{cases} O\Big(\sum_{i\neq j}\varepsilon_{ij}\big(\log\varepsilon_{ij}^{-1}\big)^{\frac{n-2}{n}}+\sum_i\dfrac{1}{(\lambda_i\,d_i)^{n-2}}\Big) & \textit{if } n<6\\ O\Big(\sum_{i\neq j}\varepsilon_{ij}^{\frac{n+2}{2(n-2)}}\big(\log\varepsilon_{ij}^{-1}\big)^{\frac{n+2}{2n}}+\sum_i\dfrac{(\log\lambda_i\,d_i)^{\frac{n+2}{2n}}}{(\lambda_i\,d_i)^{\frac{n+2}{2}}}\Big) & \textit{if } n\geqslant 6.\end{cases}\end{aligned}$$

Hence

$$\begin{aligned}J(u) &= \frac{S_n\sum_{i=1}^p\alpha_i^2+\alpha_0^2\parallel w\parallel^2}{\big(S_n\sum_{i=1}^p\alpha_i^{\frac{2n}{n-2}}K(a_i)+\alpha_0^{\frac{2n}{n-2}}\parallel w\parallel^2\big)^{\frac{n-2}{n}}}\Bigg[1-\frac{2c_2\alpha_0}{\gamma_1}\sum_{i=1}^p\frac{\alpha_i w(a_i)}{\lambda_i^{\frac{n-2}{2}}}\\ &- \frac{1}{\beta_1}\frac{n-2}{n}c_3\sum_{i=1}^p\frac{\alpha_i^{\frac{2n}{n-2}}\Delta K(a_i)}{\lambda_i^2}+\frac{1}{\gamma_1}c_2\sum_{i=1}^p\frac{H(a_i,\,a_i)}{K(a_i)^{\frac{n-2}{2}}\lambda_i^{n-2}}\\ &- \frac{1}{\gamma_1}c_2\sum_{i\neq j}\alpha_i\alpha_j\Big(\varepsilon_{ij}-\frac{H(a_i,\,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\Big)+o\Big(\sum_i\frac{1}{\lambda_i^2}+\frac{1}{(\lambda_i\,d_i)^{n-2}}+\sum_{i\neq j}\varepsilon_{ij}\Big)\Bigg]\\ &+ \|V\|^2-\|H\|^2.\end{aligned}$$

Proof. The expansion of J with respect to h (respectively to v) is very close up to a multiplicative constant to $Q_2(h)+f_2(h)$ (respectively to $Q_1(v)+f_1(v)$). Since Q_2 is negative definite (respectively Q_1 is positive definite), there is a unique maximum $\bar h$ in the space of h (respectively a unique minimum $\bar v$ in the space of v). Furthermore, it is easy to derive that $\|\bar h\|\leqslant c\|f_2\|$ and $\|\bar v\|\leqslant c\|f_1\|$. The estimate of $\bar v$ follows from Proposition 5.4 of [4]. For the estimate of $\bar h$ we use the fact that for each $h\in T_wW_u(w)$ which is a finite dimensional space, we have $\|h\|_\infty\leqslant c\|h\|$. Therefore, we derive that $\|f_2\|=O\big(\sum_{i=1}^p\frac{1}{\lambda_i^{\frac{n-2}{2}}}\big)$. Then our result follows.

Proposition 3.4 *Let $n \geq 5$. For ε small enough and $u = \sum_{i=1}^{p}\alpha_i P\delta_i \in V(p,\varepsilon)$, we have the following expansion:*

$$\begin{aligned}(\partial J(u),\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i})_{H_0^1} &= 2J(u)\Big[\alpha_i\widetilde{c_1}(\frac{n-2}{n})\frac{\Delta K(a_i)}{K(a_i)\lambda_i^2} - \frac{n-2}{2}\alpha_i\widetilde{c_2}\frac{H(a_i,a_i)}{\lambda_i^{n-2}}\\ &- \widetilde{c_2}\sum_{j\neq i}\alpha_j(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+\frac{n-2}{2}\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}})\Big]\\ &+ o\Big(\sum_{i\neq j}\varepsilon_{ij}+\frac{1}{\lambda_i^2}+\sum_{k=1}^{p}\frac{1}{(\lambda_k d_k)^{n-1}}\Big)+O\Big(\sum_{k\neq j}\varepsilon_{kj}^{\frac{n}{n-2}}log(\varepsilon_{kj}^{-1})\Big),\end{aligned}$$

where $\widetilde{c_1}$ and $\widetilde{c_2}$ are two positif constants.

We have

$$\partial J(u) = 2J(u)\Big[u + J(u)^{\frac{n}{n-2}}\Delta^{-1}\Big(Ku^{\frac{n+2}{n-2}}\Big)\Big].$$

Thus

$$(\partial J(u),\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i})_{H_0^1} = 2J(u)\Big[\sum_j\alpha_j(P\delta_j,\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}) - J(u)^{\frac{n}{n-2}}\int Ku^{\frac{n+2}{n-2}}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\Big]. \tag{3.11}$$

Observe that

$$\begin{aligned}\Big(\sum_j^p\alpha_j P\delta_j\Big)^{\frac{n+2}{n-2}} &= \sum_j^p(\alpha_j P\delta_j)^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}(\alpha_i P\delta_i)^{\frac{4}{n-2}}\sum_{j\neq i}\alpha_j P\delta_j\\ &+ O\Big((\alpha_i P\delta_i)^{\frac{2}{n-2}}\sum_{j\neq i}(\alpha_j P\delta_j)^{\frac{n}{n-2}}\Big)\\ &+ O\Big(\sum_{j\neq k,\ k;j\neq i}(\alpha_j P\delta_j)^{\frac{2}{n-2}}(\alpha_k P\delta_k)^{\frac{2}{n-2}}\Big).\end{aligned}$$

Combining (3.12), (3.13), Lemmas 6.5, …, 6.9 and the fact that $\mid \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\mid \leq c\delta_i, P\delta_k \leq \delta_k$ and $J(u)^{\frac{n}{n-2}}\alpha_j^{\frac{4}{n-2}}K(a_j) = 1 + o(1)\ \forall\, j = 1,\ldots,p$. Proposition 3.4 follows.

Proposition 3.5 *Let $n \geq 5$. For ε small enough and $u = \sum_{i=1}^{p}\alpha_i P\delta_i \in V(p,\varepsilon)$, we have*

$$\begin{aligned}(\partial J(u),\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i})_{H_0^1} &= J(u)\Big[-\alpha_i^{\frac{n+2}{n-2}}\widetilde{c_4}J^{\frac{n}{n-2}}\frac{\nabla K(a_i)}{\lambda_i}(1+o(1))\\ &+ \frac{\alpha_i}{\lambda_i^{n-1}}\widetilde{c_3}\frac{\partial H(a_i,a_i)}{\partial a_i}(1+o(1))\\ &- \widetilde{c_2}\sum_{j\neq i}\alpha_j(\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} - \frac{1}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}\lambda_i}\frac{\partial H(a_i,a_j)}{\partial a_i})(1+o(1))\Big]\\ &+ O\Big(\sum_{k\neq j}\varepsilon_{kj}^{\frac{n}{n-2}}log(\varepsilon_{kj}^{-1}) + \frac{1}{\lambda_i^2} + \sum_{k=1}^{p}\frac{log(\lambda_k d_k)}{(\lambda_k d_k)^n}\\ &+ \lambda_j|a_i-a_j|\varepsilon_{ij}^{\frac{n+1}{n-2}}\Big),\end{aligned}$$

where $\widetilde{c_3}$ and $\widetilde{c_4}$ are two positif constants

Proof. As in the proof of Proposition 3.4, we get (3.12) but with $\lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}$ changed by $\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}$. Thus, using lemmas $6.10,\dots,6.14$ the proposition follows.

4 Construction of a pseudo-gradient.

Observe that for $u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i\lambda_i} + v \in V(p,\varepsilon)$, using Proposition 3.3 after a change of variables we can write

$$J(u) = J(\sum_{i=1}^{p} \alpha_i P\delta_{a_i\lambda_i} + \overline{v}) + |V|^2.$$

In the V variable we define a pseudo-gradient by setting

$$\frac{\partial V}{\partial s} = -\mu V,$$

where μ is a very large constant. Then at $s = 1$ $V(s) = e^{-\mu s}V(0)$ will be very small as we wish. This shows that in order to define our deformation, we can work as if V was zero. The deformation will extend immediately with the same properties to a neighborhood of zero in the V variable. Therefore we need to define a vector field in $\{\sum_{i=1}^{p} \alpha_i P\delta_{a_i\lambda_i} + \overline{v} \in V(p,\varepsilon)\}$.

Theorem 4.1 *Let $n \geq 5$. There exists a pseudo-gradient W so that the following holds. There is a constant $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i\lambda_i} \in V(p,\varepsilon)$ so that*

$$\textbf{(i)}\Big(\partial J(u), W(u)\Big) \leq -c\Big(\sum_{i}^{p}(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{n-1}}) + \sum_{i\neq j} \varepsilon_{ij}^{\frac{n-1}{n-2}}\Big).$$

$$\begin{aligned}\textbf{(ii)}\Big(\partial J(u+\overline{v}), W(u) + \frac{\partial \overline{v}}{\partial(\alpha,a,\lambda)}(W)\Big) &\leq -c\Big(\sum_{i}^{p}(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\\ &+ \frac{1}{(\lambda_i d_i)^{n-1}}) + \sum_{i\neq j} \varepsilon_{ij}^{\frac{n-1}{n-2}}\Big).\end{aligned}$$

(iii) *The minimal distance to the boundary $d_i(t) = d(a_i(t)\partial\Omega)$, only increases if it is small enough.*

(iv) *$|W|$ is bounded. Furthermore, the only case where the maximum of the λ_i's is not bounded is when each point a_i is close to a critical point y_{j_i} of K in F^+ (see* 1.5*) with $y_{j_i} \neq y_{j_k}$ for each $i \neq k$.*

(v)*If each a_i belongs to a neighborhood of $y_{j_i} \in F^+$ with $y_{j_i} \neq y_{j_k}$ for $i \neq k$ there exists a change of variables*

$$(a_i, \lambda_i) \longrightarrow (a_i^{'}, \lambda_i^{'})$$

such that

$$J(\sum_{i=1}^{p}\alpha_i P\delta_i + v) = \frac{S_n^{\frac{2}{n}}\ \sum_{i=1}^{p}\alpha_i^2}{\left(\sum_{i=1}^{p}\alpha_i^{\frac{2n}{n-2}}K(a_i^{'})\right)^{\frac{n-2}{n}}}\{1-\eta\sum_{i=1}^{p}\frac{\Delta K(y_{j_i})}{\lambda_i^{'2}}\} + |V|^2$$

where η is a positive constant.

Before giving the proof of theorem 4.1, we need to state two results which deal with two specific cases of theorem 4.1. The proof of these results will be given later. Let $d_0 > 0$ be a constant small enough such that

$\frac{\partial K}{\partial \nu}(x) < -c_0,\ \forall x \in \Omega_{d_0} := \{x \in \Omega, d(x, \partial\Omega) \leq 2d_0\}$, where, $c_0 > 0$, a fixed constant.

Then we have the following propositions

Proposition 4.1 *In $\widetilde{V}(p,\varepsilon) := \{u = \sum_{i=1}^{p}\alpha_i P\delta_i \in V(p,\varepsilon),\ d(a_i, \partial\Omega) \geq d_0,\ \forall\ 1 \leq i \leq p\}$, there exists a pseudo-gradient W_1 so that the following holds: There is a constant $c > 0$ independent of $u \in \widetilde{V}(p,\varepsilon)$ so that*

$$(\partial J(u), W_1(u)) \leq -c\Big(\sum_{i}^{p}[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}] + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proposition 4.2 *In $V_b(p,\varepsilon) := \{u = \sum_{i=1}^{p}\alpha_i P\delta_i \in V(p,\varepsilon),\ d(a_i, \partial\Omega) \leq 2d_0,\ \forall\ 1 \leq i \leq p\}$, there exists a pseudo-gradient W_2 so that the following holds: There is a constant $c > 0$ independent of $u \in V_2(p,\varepsilon)$ so that*

$$(\partial J(u), W_2(u)) \leq -c\Big(\sum_{i}^{p}[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{(\lambda_i d_i)^{n-1}}] + \sum_{i\neq j}\varepsilon_{ij}^{\frac{n-1}{n-2}}\Big).$$

Proof of Theorem 4.1 We divide the set $\{1, ..., p\}$ into two sets.
The first contains the indices of the points near the $\partial\Omega$ and the second contains the indices of the points far away from $\partial\Omega$. Let us define
$B := \{1 \leq i \leq p \quad s.t \quad d_i \geq 2d_0\}$.

$$B_1 : = B \cup \{i \notin B \quad s.t \quad \exists (i_1, ..., i_r)\, with\, i_1 = i,\ i_r \in B, \quad and \quad |a_{i_{k-1}} - a_{i_k}| < \frac{d_0}{p} \quad \forall \quad k \leq r\}.$$

$B_2 := \{1, ..., p\}\backslash B_1$.
Observe that

(O_1) $d_i := d(a_i, \partial\Omega) \leq 2d_0 \quad \forall\, i \in B_2$.

(O_2) The advantage of B_1 is that if $i \in B_1$ and $j \notin B_1$, then $|a_i - a_j| \geq \frac{d_0}{p}$.

Now we write u as

$$u := u_1 + u_2, \quad u_k := \sum_{i\in B_k}\alpha_i P\delta_i \quad (1 \leq k \leq 2).$$

Observe that $u_1 \in \widetilde{V}(card(B_1), \varepsilon)$. Then we use the previous construction as in Propositions 4.1 to u_1, which means we apply the previous construction to the sub-pack of functions $u = \sum_{i=1}^{card(B_1)} \alpha_i P\delta_i$ forgetting the indices $i \notin B_1$. Let $W_1(u_1)$ be the vector field thus defined. The same argument can be repeated for u_2 which is in $V_b(card(B_2), \varepsilon)$ and we will denote by $W_2(u_2)$ the vector field thus defined. Define W as $W(u) = W_1(u_1) + W_2(u_2)$. Thus we have

$$\begin{aligned}\Big(\nabla J(u), W(u)\Big) &= \Big(\nabla J(u_1), W_k(u_1)\Big) + \Big(\nabla J(u_2), W_k(u_2)\Big) \\ &+ o\Big(\sum_k^p [\frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k d_k)^{n-1}}] + \sum_{i\in B_1,\, j\notin B_1} \varepsilon_{ij}\Big).\end{aligned}$$

From the observation (O_1) we get $\varepsilon_{ij} = o(\sum_{k=i,\,j} \frac{1}{(\lambda_k)^{\frac{n-2}{2}}}) = o(\sum_{k=i,\,j} \frac{1}{(\lambda_k)^2})$, since $n \geq 5$. Thus claim **(i)**of Theorem 4.1 follows. The proof of claim **(ii)** is similar to the proof of Lemma 3.3 of [8] and Appendix 2 of [5]. The conditions **(iii)** and **(iv)** are satisfied by the definition of the vector field W and the proof of **(v)** follows from the expansion of the functional J given by Proposition 3.3 taking $w = 0$ and $v = 0$ so that $H = 0$, $V = 0$ and $O\Big(\sum_{i\neq j} \varepsilon_{ij} + \sum_{i,j} \frac{H(a_i,\, a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\Big) = o\Big(\sum_{i=1}^p \frac{1}{\lambda_i^2}\Big)$.

Proof of Proposition 4.1. Let $\eta > 0$ a fixed constant small enough with $|y_i - y_j| > \eta \;\forall\, i \neq j$. We divide the set $\widetilde{V}(p, \varepsilon)$ into four sets

$$\begin{aligned}V_1(p,\varepsilon) \;&:=\; \{u = \sum_{i=1}^p \alpha_i P\delta_i \in \widetilde{V}(p,\varepsilon),\, a_i \in B(y_{j_i}, \eta)\, with\, -\Delta K(y_{j_i}) > 0, \\ &\forall\, i = 1,\dots,p,\ \text{and}\, y_{j_i} \neq y_{j_k}\, \forall\, i \neq k\}.\end{aligned}$$

$$\begin{aligned}V_2(p,\varepsilon) \;&:=\; \{u = \sum_{i=1}^p \alpha_i P\delta_i \in \widetilde{V}(p,\varepsilon),\ a_i \in B(y_{j_i}, \eta),\ \ \forall i \neq k,\, \forall\, i = 1,\dots,p, \\ &y_{j_i} \neq y_{j_k}\ \text{and}\ \exists\, i_1,\dots,i_q\ s.t\, -\Delta K(y_{i_k}) < 0,\, \forall\, k = 1,\dots,q\}.\end{aligned}$$

$$\begin{aligned}V_3(p,\varepsilon) \;&:=\; \{u = \sum_{i=1}^p \alpha_i P\delta_i \in \widetilde{V}(p,\varepsilon),\, a_i \in B(y_{j_i}, \eta),\, \forall\, i = 1,\dots,p,\ \text{and}\, \exists\ i \neq k \\ &\text{s.t}\ y_{j_i} = y_{j_k}\}.\end{aligned}$$

$$V_4(p,\varepsilon) \;:= \{\quad u = \sum_{i=1}^p \alpha_i P\delta_i \in \widetilde{V}(p,\varepsilon),\ \text{s.t}\ \exists\ a_i \notin \cup_{\nabla K(y)=0} B(y,\eta)\}.$$

We will define the pseudo-gradient depending on the sets $V_i(p,\varepsilon)$, $i = 1, ..., 4$ to which u belongs.

Lemma 4.1 *In $V_1(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W_1}$ so that the following holds: There is a constant $c>0$ independent of $u\in V_1(p,\varepsilon)$ so that*

$$(\partial J(u),\widetilde{W_1}(u))\leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big]+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof. In this region, we define Z_1, by $Z_1=\sum_{i=1}^{p}\alpha_i\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}$. From Proposition 3.4 we obtain

$$\begin{aligned}(\partial J(u),Z_1) &= 2J(u)\left[\sum_{i=1}^{p}\alpha_i^{\frac{2n}{n-2}}c_2J(u)^{\frac{n}{n-2}}(\frac{n-2}{n})\frac{\Delta K(a_i)}{K(a_i)\lambda_i^2}-c_2\sum_{j\neq i}\alpha_j\alpha_i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\right]\\ &+ o\Big(\sum_{i\neq j}\varepsilon_{ij}+\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\Big).\end{aligned}$$

We have $-\Delta K(a_i)$ is close to $-\Delta K(y_{j_i})>0$, $\forall\ i=1,\dots,p$. Moreover $|a_i-a_j|\geq\eta\,\forall\ i\neq j$, so $\lambda_i\dfrac{\partial\varepsilon_{ij}}{\partial\lambda_i}=O(\varepsilon_{ij})=O\Big(\dfrac{1}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\Big)=o\Big(\sum_{i=1}^{p}\dfrac{1}{\lambda_i^2}\Big)$, since $n-2>2$. Thus

$$(\partial J(u),Z_1)\leq -c\Big(\sum_{i\neq j}\varepsilon_{ij}+\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\Big).$$

We let now: $Z_a=\sum_{i=1}^{p}\phi(\lambda_i|\nabla K(a_i)|)\dfrac{1}{\lambda_i}\dfrac{\nabla K(a_i)}{|\nabla K(a_i)|}\dfrac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}$

$$\text{where}\qquad \phi:\mathbb{R}\longrightarrow\mathbb{R},\ t\mapsto\phi(t)=\begin{cases}0 & \text{if}\,|t|\leq\frac12\\ 1 & \text{if}\,|t|\geq 1.\end{cases}$$

From Proposition 3.5 we obtain

$$(\partial J(u),Z_a)\leq -c\sum_{i=1}^{p}\phi(\lambda_i|\nabla K(a_i)|)\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+O(\frac{1}{\lambda_i^2})+O(\sum_{i\neq j}\varepsilon_{ij})\Big].$$

For $M>0$ a fixed constant large enough we derive

$$(\partial J(u),M.Z_1+Z_a)\leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big]+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

We let $\widetilde{W_1}=M.Z_1+Z_a$. Then $\widetilde{W_1}$ satisfies Lemma 4.1.

Lemma 4.2 *In $V_2(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W_2}$ so that the following holds: There is a constant $c>0$ independent of $u\in V_2(p,\varepsilon)$ so that*

$$(\partial J(u),\widetilde{W_2})\leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big]+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof. We can assume that $a_j \in B(y_{i_j},\eta)$, with $-\Delta K(y_{i_j}) < 0$, $\forall\ j = 1,\dots,q$

Let $Z_2 = \sum_{i=1}^{q} -\alpha_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}$. From Proposition 3.4 we obtain

$$\begin{aligned}(\partial J(u), Z_2) &= 2J(u)\Big[\sum_{i=1}^{q} -\alpha_i^{\frac{2n}{n-2}} c_2 J(u)^{\frac{n}{n-2}} \frac{n-2}{n}\frac{\Delta K(a_i)}{K(a_i)\lambda_i^2} \\ &+ c_2 \sum_{j\neq i, 1\le i\le q} \alpha_j\alpha_i\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}\Big] + o\Big(\sum_{i\neq j,} \varepsilon_{ij} + \sum_{i=1}^{q}\frac{1}{\lambda_i^2}\Big).\end{aligned}$$

In this region we have $|a_i - a_j| \geq \eta$, $\forall\ i \neq j$, then $\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq -c\varepsilon_{ij}$. Furthermore we have $-\Delta K(a_i) < -\gamma$ $(\gamma > 0)\ \forall i = 1,\dots,q$. Thus we derive

$$(\partial J(u), Z_2) \leq -c\Big(\sum_{i\neq j, 1\le i\le q} \varepsilon_{ij} + \sum_{i=1}^{q}\frac{1}{\lambda_i^2}\Big) + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\Big).$$

Let now

$$I := \Big\{1 \leq i \leq p \quad s.t \quad \lambda_i < \frac{1}{10}\mathrm{Min}\{\mathrm{\lambda_j,\ j = 1,\dots,q}\}\Big\}.$$

We have

$$(\partial J(u), Z_2) \leq -c\Big(\sum_{i\neq j, i\notin I} \varepsilon_{ij} + \sum_{i\notin I}\frac{1}{\lambda_i^2}\Big) + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\Big).$$

If $I = \emptyset$, then we get all the indices in the last upper bound and we obtain Lemma 4.2 in this case. If $I \neq \emptyset$, we let $\widetilde{u} = \sum_{i\in I}\alpha_i P\delta_i$. Observe that $\widetilde{u} \in V_1(\ell,\varepsilon)$, where $\ell = \mathrm{Card(I)}$. Define $Z_2^{'}(\widetilde{u}) = Z_1(\widetilde{u})$, where Z_1 is the pseudo-gradient defined in the proof of Lemma 4.1. We have

$$(\partial J(u), Z_2^{'}(\widetilde{u})) \leq -c\Big(\sum_{i\neq j,\, i,j\in I} \varepsilon_{ij} + \sum_{i\in I}\frac{1}{\lambda_i^2}\Big) + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\Big) + O\Big(\sum_{i\in I, j\notin I}\varepsilon_{ij}\Big).$$

For $m > 0$ a positive constant small enough

$$(\partial J(u), M.Z_1 + Z_2^{'} + mZ_a) \leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Big] + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

We let

$$\widetilde{W_2} = M.Z_1 + Z_2^{'} + mZ_a.$$

Thus Lemma 4.2 follows.

Lemma 4.3 *In $V_3(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W_3}$ so that the following holds: There is a constant $c > 0$ independent of $u \in V_3(p,\varepsilon)$ so that*

$$(\partial J(u), \widetilde{W_3}(u)) \leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Big] + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof. We order the concentrations λ_i's in such a way: $\lambda_1 \leq \ldots \leq \lambda_p$. For each critical point y_k of K, we set $B_k := \{j,\ a_j \in B(y_k, \eta)\}$. Without loss of generality, we can assume $y_1, \ldots, y_q$ are the critical points such that $\mathrm{Card}(B_k) \geq 2$, $\forall\, k = 1, \ldots, q$. Let c_2 and $M_1 > 0$ two fixed constant large enough. We set

$$I_1 := \{2 \leq i \leq p \quad \text{s.t} \quad \lambda_i |\nabla K(a_i)| \geq c_2\}.$$

$$I_2 := \{1\} \cup \{2 \leq i \leq p \quad \text{s.t} \quad \lambda_i \leq M_1 \lambda_j \ \forall\ j \leq i\}.$$

We define

$$Z_3 = \sum_{i \in I_1} \frac{1}{\lambda_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|} \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}, \text{ and}$$

$$Z_3^{'} = -M_1 \sum_{i \notin I_2} 2^i \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} - m_1 \sum_{i \in I_2} \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i},$$

where, M_1 is a positif constant large enough and m_1 a positif constant small enough. From Proposition 3.5 we obtain

$$(\partial J(u), Z_3(u)) \leq -c \sum_{i \in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} + O\Big(\sum_{i \in I_1} \frac{1}{\lambda_i^2} + \sum_{i \neq j,\ i \in I_1} \frac{1}{\lambda_i} |\frac{\partial \varepsilon_{ij}}{\partial a_i}|\Big).$$

Observe that

$$\frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} = o(\varepsilon_{ij}) \qquad \forall\ 1 \leq i \leq p,\ j \in I_2.$$

Then

$$\begin{aligned}(\partial J(u), Z_3(u)) \quad &\leq \quad -c \sum_{i \in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} + O\Big(\sum_{i \in I_1} \frac{1}{\lambda_i^2} + \sum_{i \neq j,\, i \in I_1,\, j \notin I_2} \varepsilon_{ij}\Big) \\ &\quad + \quad o\Big(\sum_{i \neq j,\, i \in I_1,\, j \in I_2} \varepsilon_{ij}\Big).\end{aligned}$$

From the definition of I_1 we can make appear the quantity $c_2 \sum_{i \in I_1} \frac{1}{\lambda_i^2}$ in the last upper bound. Taking c_2 large enough we get

$$\begin{aligned}(\partial J(u), Z_3(u)) \quad &\leq \quad -c\Big(\sum_{i \in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \in I_1} \frac{1}{\lambda_i^2}\Big) + O\Big(\sum_{i \neq j,\, i \in I_1,\, j \notin I_2} \varepsilon_{ij}\Big) \\ &\quad + \quad o\Big(\sum_{i \neq j,\, i \in I_1,\, j \in I_2} \varepsilon_{ij}\Big).\end{aligned}$$

Now from Propositions 3.4, we have

$$\begin{aligned}(\partial J(u), Z_3^{'}(u)) \quad &\leq \quad -cM_1\Big(\sum_{i \notin I_2} O(\frac{1}{\lambda_i^2}) - \sum_{i \notin I_2,\, i \neq j} 2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}\Big) - m_1\Big(\sum_{i \in I_2} O(\frac{1}{\lambda_i^2}) \\ &\quad - \quad \sum_{i \in I_2,\, i \neq j} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}\Big) + \Big(\sum_{i = 1^p} \frac{1}{\lambda_i^2} + \sum_{k \neq r} \varepsilon_{kr}\Big).\end{aligned}$$

Observe that $2^i\lambda_i\dfrac{\partial\varepsilon_{ij}}{\partial\lambda_i}+2^j\lambda_j\dfrac{\partial\varepsilon_{ij}}{\partial\lambda_j}\leq -c\varepsilon_{ij}$, $\forall\ i\neq j$ and for $i\neq j\in I_2$ we have $\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\leq -c\varepsilon_{ij}$ since $\lambda_i\sim\lambda_j$. Thus for $m_1\ll M_1$ we have

$$(\partial J(u),Z_3^{'}(u))\leq -c\Big(M_1\sum_{i\in I_2,\,i\neq j}\varepsilon_{ij}+m_1\sum_{i,j\in I_2,\,i\neq j}\varepsilon_{ij}\Big)+O(\sum_{i\notin I_2}\frac{M_1}{\lambda_i^2})+\sum_{i\in I_2}\frac{m_1}{\lambda_i^2}\Big).$$

Then we derive

$$(\partial J(u),Z_3(u)+Z_3^{'}(u))\leq -c\Big(\sum_{i\in I_1}[\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}]+\sum_{i\neq j}\varepsilon_{ij}\Big)+O\Big(\sum_{i\notin I_2}\frac{M_1}{\lambda_i^2}+\sum_{i\in I_2}\frac{m_1}{\lambda_i^2}\Big).$$

We distinguish two cases:

case 1: $I_1\cap I_2\neq\emptyset$. In this case, we can make appear $\dfrac{1}{\lambda_1^2}$ in the last upper bound and so all the quantity $\displaystyle\sum_{i=1}^{p}\frac{1}{\lambda_i^2}$.

For m_1 small enough, the quantity $O\Big(\displaystyle\sum_{i\in I_2}\frac{m_1}{\lambda_i^2}\Big)$ is then absorbed. For M_1 large enough, we have

$$O\Big(\sum_{i\notin I_2}\frac{M_1}{\lambda_i^2}\Big)=o(\frac{1}{\lambda_1^2}).$$

We obtain :

$$(\partial J(u),Z_3(u)+Z_3^{'}(u))\leq -c\Big(\sum_{i\in I_1}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

We define

$$X_3=\sum_{i\in I_2}\frac{1}{\lambda_i}\frac{\nabla K(a_i)}{|\nabla K(a_i)|}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}.$$

From Proposition 3.5 we obtain

$$(\partial J(u),X_3(u))\leq -c\sum_{i\in I_2}\frac{|\nabla K(a_i)|}{\lambda_i}+O\Big(\sum_{i\in I_2}\frac{1}{\lambda_i^2}+\sum_{i\neq j}\varepsilon_{ij}\Big).$$

For $c_3>0$ a positive constant large enough, the vector field $\widetilde{W}_3:=c_3X_3+Z_3+Z_3^{'}$ satisfies Lemma 4.3.

case 2: $I_1\cap I_2=\emptyset$. In this case, for each $i\in I_2$ the point a_i is close to a critical point y_{k_i} of K. If we suppose that there exist $i,j\in I_2$ such that a_i and a_j in $B(y,\eta)$ for η small enough and y a critical point of K, then $|\nabla K(a_k)|\geq c|y-a_k|$ for $k=i,j$, since y is non-degenerate critical point of K. Therefore, $\lambda_i|a_i-a_j|\leq c$ (we assume that $\lambda_i\leq\lambda_j$). This implies that $\varepsilon_{ij}\geq c\Big(\dfrac{\lambda_i}{\lambda_j}\Big)^{\frac{n-2}{2}}$ which is a contradiction with the fact that λ_i and λ_j are of the same order. Thus our assumption is false. And for $\widetilde{u} = \sum_{i\in I_2}\alpha_i P\delta_i$, we have then $a_i \in B(y_{j_i},\eta)$ with $y_{j_i} \neq y_{j_k}$, $\forall\, i \neq k \in I_2$ and therefore $\widetilde{u} \in V_i(\ell,\varepsilon)$ where $i = 1 \,\mathrm{or}\, 2$ and $\ell = \mathrm{Card}(I_2)$. Let then $Z_3^{''}$ the corresponding vector field in $V_i(\ell,\varepsilon)$ $(i = 1 \,\mathrm{or}\, 2)$. We have

$$(\partial J(u), Z_3^{'}(u)) \leq -c\Big(\sum_{i\neq j,\, i,j\in I_2}\varepsilon_{ij} + \sum_{i\in I_2}\frac{1}{\lambda_i^2}\Big) + O\Big(\sum_{i\in I_2, j\notin I_2}\varepsilon_{ij}\Big).$$

Observe that $\dfrac{1}{\lambda_1^2}$ appear in this upper bound, so we can make appear $\displaystyle\sum_{i=1}^{p}\frac{1}{\lambda_i^2}$. For $M_2 > 0$ a fixed constant large enough, the vector field $\widetilde{W_3} = M_2(Z_3 + Z_3{}') + Z_3{}''$ satisfies the next upper bound:

$$(\partial J(u), \widetilde{W_3}(u)) \leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Big] + \sum_{i\neq j}\varepsilon_{ij}\Big),$$

and so Lemma 4.3 follows in this case.

Lemma 4.4 *In $V_4(p,\varepsilon)$, there exists a pseudo-gradient $\widetilde{W_4}$ so that the following holds: There is a constant $c > 0$ independent of $u \in V_4(p,\varepsilon)$ so that*

$$(\partial J(u), \widetilde{W_4}(u)) \leq -c\Big(\sum_{i=1}^{p}\Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Big] + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

Proof. Without loss of generality, we suppose $\lambda_1 \leq \lambda_1 \leq \ldots \leq \lambda_p$.
We denote by i_1 the index satisfying $a_{i_1} \notin \cup_{\nabla K(y)=0} B(y,\eta)$ and $a_i \in B(y_{j_i},\eta)$, $\forall\, i < i_1$. Let

$$\widetilde{u} = \sum_{i<i_1}\alpha_i P\delta_i.$$

Observe that $\widetilde{u} \in V_i(i_1 - 1,\varepsilon)$, $i = 1 \,\mathrm{or}\, 2 \,\mathrm{or}\, 3$. Then we define $Z_4{}'(\widetilde{u})$ the corresponding vector field and we have

$$(\partial J(u), Z_4{}'(\widetilde{u})) \leq -c\Big(\sum_{i<i_1}\Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Big] + \sum_{i\neq j,\, i,j<i_1}\varepsilon_{ij}\Big) + O\Big(\sum_{i<i_1,\, j\geq i_1}\varepsilon_{ij} + \sum_{i\geq i_1}\frac{1}{\lambda_i^2}\Big).$$

Let now

$$Z_4 = \frac{1}{\lambda_{i_1}}\frac{\nabla K(a_{i_1})}{|\nabla K(a_{i_1})|}\frac{\partial P\delta_{a_{i_1},\lambda_{i_1}}}{\partial a_{i_1}} - M_3\sum_{i\geq i_1}2^i\lambda_i\frac{\partial P\delta_{a_i,\lambda_i}}{\partial\lambda_i},$$

where $M_3 > 0$ a fixed constant large enough. From Propositions 3.4 and 3.5 we obtain

$$\begin{aligned}(\partial J(u), Z_4(u)) \;&\leq\; \frac{-c}{\lambda_{i_1}} - c\sum_{i,j\geq i_1}\varepsilon_{ij} - M_3\sum_{i>i_1, j<i_1}2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\\ &+\; O\Big(\sum_{i\geq i_1}\frac{M_3}{\lambda_i^2}\Big) + O\Big(\frac{1}{\lambda_{i_1}^2}\sum_{j\neq i_1}\varepsilon_{i_1 j}\Big).\end{aligned}$$

Observe that if $i \geq i_1$ and $j < i_1$, we have $\lambda_i \geq \lambda_j$, Then $\lambda_i \dfrac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq -c\varepsilon_{ij}$. Thus

$$(\partial J(u), Z_4(u)) \leq \frac{-c}{\lambda_{i_1}} - M_3 \sum_{i \geq i_1,\, i \neq j} \varepsilon_{ij} + O\Big(\sum_{i \geq i_1} \frac{M_3}{\lambda_i^2} + \sum_{j \neq i_1} \varepsilon_{i_1 j}\Big).$$

We choose $M_3 \gg 1$ so that $\Big(\sum_{j \neq i_1} \varepsilon_{i_1 j}\Big)$ is absorbed by $M_3 \sum_{i \geq i_1,\, i \neq j} \varepsilon_{ij}$. On the other hand we have $\lambda_i \geq \lambda_{i_1}$ for $i \geq i_1$, then $\dfrac{1}{\lambda_i^2} = o(\dfrac{1}{\lambda_{i_1}})$ which makes $O\Big(\sum\limits_{i \geq i_1} \dfrac{M_3}{\lambda_i^2}\Big)$ absorbed by $\dfrac{1}{\lambda_{i_1}}$. We deduce

$$(\partial J(u), Z_4(u)) \leq \frac{-c_2}{\lambda_{i_1}} - M_3 \sum_{i \geq i_1,\, i \neq j} \varepsilon_{ij}.$$

Also $\dfrac{1}{\lambda_{i_1}}$ make $\sum\limits_{i \geq i_1} \dfrac{1}{\lambda_i}$ appear. We let

$$\widetilde{W_4}(u) = MZ_4 + Z_4^{'} + mZ_a,$$

and thus derive

$$(\partial J(u), \widetilde{W_4}(u)) \leq -c\Big(\sum_{i=1}^{p} \Big[\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Big] + \sum_{i \neq j} \varepsilon_{ij}\Big).$$

The claim of Lemma 4.4 follows. The vector field W_1 required in Proposition 4.1 will be defined by a convex combination of the vector fields $\widetilde{W_1}(u)$, $\widetilde{W_2}(u)$, $\widetilde{W_3}(u)$ and $\widetilde{W_4}(u)$.

Proof of Proposition 4.2. We will introduce some technical lemmas for the proof of Proposition 4.2. Without loss of generality, we suppose $\lambda_1 d_1 \leq ... \leq \lambda_p d_p$. Let $c_1 > 0$ a fixed constant small enough. We define

$$I_2 := \{1\} \cup \{1 \leq i \leq p, \text{ s.t } c_1 \lambda_k d_k \leq \lambda_{k-1} d_{k-1} \leq \lambda_k d_k,\ \forall\, k \leq i\}.$$

In I_2, we order the λ_i's : $\lambda_{i_1} \leq ... \leq \lambda_{i_s}$. For $c_2 > 0$ a fixed constant small enough, we define

$$I_{\lambda_{i_s}} := \{i_s\} \cup \{1 \leq k \leq s,\ \text{ s.t } c_2 \lambda_{i_{j+1}} \leq \lambda_{i_j} \leq \lambda_{i_{j+1}},\, \forall\, j \geq k\}.$$

For $u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V_b(p, \varepsilon)$, we introduce the following condition: for $i \in \{1, \ldots, p\}$

$$\frac{1}{2^{p+1}} \sum_{k \neq i} \varepsilon_{ki} \leq \sum_{j=1}^{p} \frac{H_{ij}}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \tag{4.1}$$

We divide the set $\{1,...,p\}$ into $T_1 \cup T_2$, where

$$T_1 = \{1 \leq i \leq p,\ \text{ s.t } \quad i\ satisfies\,(4.1)\},\ \text{and}$$

$$T_2 = \{1,...,p\}\backslash T_1.$$

Lemma 4.5 *For $j \notin I_{\lambda_{i_s}}$, $k \in I_{\lambda_{i_s}}$, and $j \in T_1$ we have*

$$\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{n}{n-2}} = o\biggl(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}\biggr).$$

Proof. We distinguish two cases
case 1: $j \in I_2$. In this case, we have $\lambda_j \leq c_2\lambda_k$. Observe that

$$\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{n}{n-2}} = O\Bigl((\frac{\lambda_j}{\lambda_k})^{\frac{1}{2}}\varepsilon_{kj}^{\frac{n-1}{n-2}}\Bigr).$$

On the other hand we have

$$\varepsilon_{kj}^{\frac{n}{n-2}} \leq c\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{n-1}{2}}(\lambda_j d_j)^{\frac{n-1}{2}}},\ \ \text{since } j \in T_1.$$

We obtain

$$\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{n}{n-2}} \leq cc_2^{\frac{1}{2}}\Bigl(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}\Bigr) = o\Bigl(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}\Bigr)$$

case 2: $j \notin I_2$. Let $c_3 > 0$ a fixed constant small enough. If $d_k \leq c_3 d_j$, using the fact that $j \in T_1$, we get

$$\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{n}{n-2}} \leq c(\frac{\lambda_j}{\lambda_k})^{\frac{1}{2}}\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{n-1}{2}}(\lambda_j d_j)^{\frac{n-1}{2}}}.$$

This gives the lower bound

$$\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{n}{n-2}} \leq cc_3^{\frac{1}{2}}\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{n-1}{2}}(\lambda_j d_j)^{\frac{n-2}{2}}(\lambda_k d_k)^{\frac{1}{2}}}.$$

We derive

$$\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{n}{n-2}} = o\Bigl(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}\Bigr).$$

If $c_3 d_j \leq d_k \leq \frac{1}{c_3}d_j$ we get

$$\begin{aligned}\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{n}{n-2}} &\leq \frac{c}{c_3^{\frac{1}{2}}}\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{n-1}{2}}(\lambda_j d_j)^{\frac{n-2}{2}}(\lambda_k d_k)^{\frac{1}{2}}}\\ &\leq \frac{cc_1^{\frac{n-2}{2}}}{c_3^{\frac{1}{2}}}\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{n-1}{2}}(\lambda_k d_k)^{\frac{n-1}{2}}},\end{aligned}$$

since $j \notin I_2$ and $k \in I_2$. Taking $\frac{c_1^{\frac{n-2}{2}}}{c_3^{\frac{1}{2}}}$ small enough, thus we derive

$$\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k-a_j|\varepsilon_{kj}^{\frac{n}{n-2}}=o\Big(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}\Big).$$

If $d_j \leq c_3 d_k$, observe that in this case $|a_k-a_j| \geq c d_k$, so we have $d_j \leq \frac{c_3}{c}|a_k-a_j|$. We derive that

$$\begin{aligned}\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k-a_j|\varepsilon_{kj}^{\frac{n}{n-2}} &\leq c\frac{1}{(\lambda_j|a_k-a_j|)^{\frac{n-2}{2}}(\lambda_k|a_k-a_j|)^{\frac{n}{2}}}\\ &\leq cc_3^{\frac{n-2}{2}}\frac{1}{(\lambda_j d_j)^{\frac{n-2}{2}}(\lambda_k d_k)^{\frac{n}{2}}}\\ &\leq cc_3^{\frac{n-2}{2}}\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}\\ &= o\Big(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}\Big).\end{aligned}$$

Lemma 4.6 *There exist a vector field X_1 such that*

$$\Big(\partial J(u),X_1\Big)\leq -c\Big(\frac{1}{\lambda_{i_s}}+\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}+\sum_{k\in T_2,1\leq j\leq p}\varepsilon_{kj}\Big)+O\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\Big).$$

Proof. We define the vector field Y_1^2 by $Y_1^2 := \frac{1}{\lambda_{i_s}}\sum_{i\in I_{\lambda_{i_s}}}\frac{\partial P\delta_i}{\partial a_i}(-\alpha_i\nu_i)$. From Proposition 3.5, we obtain

$$\begin{aligned}&(\partial J(u),Y_1^2)\\ &\leq\\ &-\quad c\Big(\frac{1}{\lambda_{i_s}}+\frac{1}{(\lambda_{i_s}d_{i_s})^{n-1}}\Big)+\frac{1}{\lambda_{i_s}}O\Big(\sum_{k,j\in I_{\lambda_{i_s}}}\lambda_k\lambda_j|a_k-a_j||\nu_k-\nu_j|\varepsilon_{kj}^{\frac{n}{n-2}}\Big)\\ &+\quad\frac{1}{\lambda_{i_s}}O\Big(\sum_{k\in I_{\lambda_{i_s}},j\notin I_{\lambda_{i_s}}}\lambda_k\lambda_j|a_k-a_j|\varepsilon_{kj}^{\frac{n}{n-2}}\Big)+o\Big(\sum_{k\neq j}\varepsilon_{kj}^{\frac{n-1}{n-2}}\Big)\\ &+\quad o\Big(\sum_{k=1}^{p}\frac{1}{(\lambda_k d_k)^{n-1}}\Big)+O\Big(\sum_{i\in I_{\lambda_{i_s}}}\frac{1}{\lambda_i^2}\Big).\end{aligned}$$

We can make the term $(\lambda_1 d_1)^{1-n}$ appear in the last upper bound and so all the $(\lambda_i d_i)^{1-n}$. Observe that for $k,j\in I_{\lambda_{i_s}}$, $|\nu_k-\nu_j|=O(|a_k-a_j|)$. So we get

$$\frac{1}{\lambda_{i_s}}\lambda_k\lambda_j|a_k-a_j||\nu_k-\nu_j|\varepsilon_{kj}^{\frac{n}{n-2}}=\frac{1}{\lambda_{i_s}}O(\varepsilon_{kj})=o(\frac{1}{\lambda_{i_s}}).$$

For $j \in T_1$ and $k \neq j$ we have

$$\varepsilon_{kj}^{\frac{n-1}{n-2}} \leq c \sum_{i=1}^{p} (\lambda_i d_i)^{1-n}.$$

We are left with the estimation of $\frac{1}{\lambda_{i_s}} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{kj}^{\frac{n}{n-2}}$ with $j \notin I_{\lambda_{i_s}}$ and $k \in I_{\lambda_{i_s}}$. If $k \in T_2$ or $j \in T_2$, we get:

$$\frac{1}{\lambda_{i_s}} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{kj}^{\frac{n}{n-2}} = O(\lambda_j |a_k - a_j| \varepsilon_{kj}^{\frac{n}{n-2}}) = O\Big((\frac{\lambda_j}{\lambda_k})^{\frac{1}{2}} \varepsilon_{kj}^{\frac{n-1}{n-2}} \Big) = O\Big(\varepsilon_{kj} \Big).$$

If $k, j \in T_1$, we use the estimation of Lemma 4.5. As a conclusion of the last discussion, we obtain

$$(\partial J(u), Y_1^2) \leq -c \Big(\frac{1}{\lambda_{i_s}} + \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n-1}} \Big) + O\Big(\sum_{k \in T_2, 1 \leq j \leq p} \varepsilon_{kj} \Big).$$

Let $T_2 = \{i_1, ..., i_r\}$, with $\lambda_{i_1} \leq ... \leq \lambda_{i_r}$. We define

$$Y_2^2 := - \sum_{k=1}^{r} 2^{k-1} \alpha_{i_k} \lambda_{i_k} \frac{\partial P \delta_{i_k}}{\partial \lambda_{i_k}}.$$

We have(see the proof of Lemma 3.4 of [11])

$$(\partial J(u), Y_2^2) \leq -c \sum_{j \in T_2} \Big(\sum_{k \neq j} \varepsilon_{kj} \Big) + O\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} \Big) + o\Big(\sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^{n-1}} \Big).$$

Taking $m > 0$ a fixed constant large enough, the vector field

$$X_1 = Y_1^2 + m Y_2^2,$$

satisfies the claim of Lemma 4.6.

Lemma 4.7 *There exist a vector field X_2 such that*

$$(\partial J(u), X_2) \leq -c \Big(\sum_{i=1}^{p} \frac{1}{\lambda_i} \Big) + O\Big(\sum_{j \in T_2, 1 \leq k \leq p} \varepsilon_{kj} \Big) + O\Big(\sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n-1}} \Big).$$

Proof. Let $\{1, ..., p\} =: \{i_0, ..., i_p\}$, with $\lambda_{i_0} \leq ... \leq \lambda_{i_p}$. We define for M a large constant,

$$I_0^{'} := \{1 \leq i \leq p, \text{ s.t } |a_i - a_{i_0}| \geq \frac{2}{M} d_{i_0}\}.$$

$$I_0^{''} := \{i \notin I_0^{'}, \text{ s.t } \exists (i_1, ..., i_r), \text{ with } i_1 = i,\ i_r \in I_0^{'}, \text{ and } |a_{i_{k-1}} - a_{i_k}| < \frac{d_{i_0}}{pM}, \forall\ k \leq r\}.$$

$$I_0 := \{1, ..., p\} \backslash \{I_0^{'} \cup I_0^{''}\}.$$

Let us define for $c_2 > 0$ a fixed constant small enough

$$I_{\lambda_{i_0}} := \{i_0\} \cup \{1 \le j \le p, \text{ s.t } c_2\lambda_{i_k} \le \lambda_{i_{k-1}} \le \lambda_{i_k}, \forall\, k \le j\}.$$

We set

$$X_2 := \frac{1}{\lambda_{i_0}} \sum_{i \in I_0 \cap I_{\lambda_{i_0}}} \frac{\partial P\delta_i}{\partial a_i}(-\alpha_i \nu_i).$$

Observe that $d_i \sim d_{i_0}$, for $i \in I_0 \cap I_{\lambda_{i_0}}$. From Proposition 3.5

$$\begin{aligned}
(\partial J(u), X_2) \;\; &\le \;\; -c_2^s c\Big(\frac{1}{\lambda_{i_0}}\Big) + \frac{1}{\lambda_{i_0}} O\Big(\sum_{k,j \in I_{\lambda_{i_0}} \cap I_0} \lambda_k \lambda_j |a_k - a_j||\nu_k - \nu_j| \varepsilon_{kj}^{\frac{n}{n-2}}\Big)\\
&+ \;\; \frac{1}{\lambda_{i_0}} O\Big(\sum_{k \in I_{\lambda_{i_0}} \cap I_0, j \notin I_{\lambda_{i_0}} \cap I_0} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{kj}^{\frac{n}{n-2}}\Big) + o\Big(\sum_{k \ne j} \varepsilon_{kj}^{\frac{n-1}{n-2}}\Big)\\
&+ \;\; o\Big(\sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^{n-1}}\Big) + O\Big(\sum_{i \in I_0 \bigcap I_{\lambda_{i_0}}} \frac{1}{\lambda_i^2}\Big)
\end{aligned}$$

Observe that for $k, j \in I_{\lambda_{i_0}} \cap I_0$, $|\nu_k - \nu_j| = O(|a_k - a_j|)$. From this, we deduce

$$\frac{1}{\lambda_{i_0}} \lambda_k \lambda_j |a_k - a_j||\nu_k - \nu_j| \varepsilon_{kj}^{\frac{n}{n-2}} = \frac{1}{\lambda_{i_0}} O(\varepsilon_{kj}) = o(\frac{1}{\lambda_{i_0}}).$$

Now, we need to estimate the quantity $\frac{1}{\lambda_{i_0}} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{kj}^{\frac{n}{n-2}}$, for $j \notin I_{\lambda_{i_0}} \cap I_0$, and $k \in I_{\lambda_{i_0}} \cap I_0$. We have two cases

case 1: $j \notin I_0$ In this case, we have $|a_k - a_j| \ge \frac{1}{pM} d_{i_0}$. On the other hand, we observe that $d_k \sim d_{i_0}$. Thus, We deduce

$$\frac{1}{\lambda_{i_0}} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{kj}^{\frac{n}{n-2}} = O\Big(\frac{1}{\lambda_{i_0} d_{i_0}} \varepsilon_{kj}\Big) = O\Big(\frac{1}{(\lambda_{i_0} d_{i_0})^{n-1}}\Big) + o(\varepsilon_{kj}^{\frac{n-1}{n-2}}).$$

case 2: $j \in I_0$ In this case, we observe that $d_j \sim d_k \sim d_{i_0}$ *and* $|a_k - a_j| \le \frac{4}{pM} d_{i_0}$. Using Lemma 6.16 we obtain $H_{jj} \le c\frac{1}{d_j^{n-2}}$, and $H_{ij} \le c\frac{1}{(d_i d_j)^{\frac{n-2}{2}}}$.
If $j \in T_1$, we deduce

$$\begin{aligned}
\frac{1}{\lambda_{i_0}} \lambda_k \lambda_j |a_k - a_j| \varepsilon_{kj}^{\frac{n}{n-2}} \;\; &\le \;\; \frac{1}{c_1^p} \lambda_j |a_k - a_j| \varepsilon_{kj}^{\frac{n}{n-2}}\\
&\le \;\; \frac{c}{c_1^p} \lambda_j |a_k - a_j| \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{\frac{n}{2}} (\lambda_j d_j)^{\frac{n}{2}}}\\
&\le \;\; \frac{c}{c_1^p} \frac{1}{M} \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{\frac{n}{2}} (\lambda_j d_j)^{\frac{n-2}{2}}}.
\end{aligned}$$

We choose $\dfrac{1}{c_1^p}\dfrac{1}{M} = o(1)$, and therefore

$$\frac{1}{\lambda_{i_0}}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{n}{n-2}} = o\Big(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}\Big).$$

If $j \in T_2$, we easily have $\dfrac{1}{\lambda_{i_0}}\lambda_k\lambda_j|a_k - a_j|\varepsilon_{kj}^{\frac{n}{n-2}} = O\Big(\varepsilon_{kj}\Big)$. We conclude that

$$\begin{aligned}(\partial J(u), X_2) \quad &\leq \quad -c_2^s c\Big(\frac{1}{\lambda_{i_0}}\Big) + O\Big(\sum_{j\in T_2, 1\leq k\leq p}\varepsilon_{kj}\Big) + O\Big(\frac{1}{(\lambda_{i_0}d_{i_0})^{n-1}}\Big)\\ &+ \quad o\Big(\sum_{k=1}^{p}\frac{1}{(\lambda_k d_k)^{n-1}}\Big) + O\Big(\sum_{k=1}^{p}\frac{1}{\lambda_k^2}\Big).\end{aligned}$$

Such vector field X_2 satisfies the upper bound of Lemma 4.7. To get the proof of Proposition 4.2, we need to define

$$X_3 := \Big(-\sum_{i=1}^{p}M^i\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\Big)\Big(\sum_{1\leq i,j\leq p,\, i\neq j}\varepsilon_{ij}^{\frac{1}{n-2}}\Big).$$

Observe that $\varepsilon_{kj}^{\frac{1}{n-2}}\dfrac{1}{(\lambda_k d_k)^{\frac{n-2}{2}}} = o\Big(\sum_{k\neq j}\varepsilon_{kj}^{\frac{n-1}{n-2}}\Big) + O\Big(\dfrac{1}{(\lambda_k d_k)^{n-1}}\Big)$. Furthermore, we have $H_{ii} \leq \dfrac{c}{d_i^{n-2}}$ and $H_{ij} \leq \dfrac{c}{(d_i d_j)^{\frac{n-2}{2}}}$, for i, $j = 1, \ldots, p$. From Proposition 3.4 we obtain

$$(\partial J(u), X_3) \leq -c\Big(\sum_{1\leq i,j\leq p}\varepsilon_{ij}^{\frac{n-1}{n-2}}\Big) + O\Big(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-1}}\Big) + o\Big(\sum_{k=1}^{p}\frac{1}{\lambda_k^2}\Big).$$

Thus for $m_1 > 0$ a fixed constant large enough the vector field

$$W_2(u) := X_3 + m_1(X_2 + m_1 X_1)$$

satisfies the claim of Proposition 4.2.

5 Proofs of the theorems

Proof of Theorem 1.1. Using Theorem 4.1 the only region where λ_i's are unbounded is the one where each point a_i is close to a critical point $y_{i_j} \in F^+$ where $y_{i_j} \neq y_{i_k}$ for $j \neq k$. In this region, the normal form of J given by Theorem 4.1 allows us to split the variables λ_i's and a_i's. Then it is easy to see that if $(a_1, \ldots, a_p)$ is equal to $(y_{i_1}, \ldots, y_{i_p})$, only λ_i can move. Since $-\Delta K(y_{i_j}) > 0$, in order to decrease the functional J we have to increase λ_i and therefore we obtain a critical point at infinity only in this region.

In order to compute the Morse index of such a critical point at infinity, we observe that this Morse index corresponds to the Morse index of the critical point of the following function

$$g(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p) = \frac{S_n^{\frac{2}{n}} \sum_{i=1}^p \alpha_i^2}{\left(\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2}} K(a_i)\right)^{\frac{n-2}{n}}}.$$

In the variables α_i's, we have a degenerate critical point $(\overline{\alpha_1}, \dots, \overline{\alpha_p})$ which is a maximum and satisfies

$$\frac{\bar{\alpha}_i^{\frac{4}{n-2}} K(a_i)}{\bar{\alpha}_j^{\frac{4}{n-2}} K(a_j)} = 1.$$

This critical point has an index equal to $p-1$ (since the functional g is homogenous in the variables α_i and the critical point correspond to a maximum). Then we have

$$g(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p) = S_n^{\frac{2}{n}} \Big(\sum_{i=1}^p \frac{1}{K(a_i)^{\frac{n-2}{n}}}\Big)^{\frac{2}{n}} \big(1 - |Y|^2\big).$$

Here $Y \in \mathbb{R}^{p-1}$ is the coordinates $(\alpha_1, \dots, \alpha_p)$. Again using the Morse lemma for g in the variables a_i's, we obtain

$$g(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p) = S_n^{\frac{2}{n}} \Big(\sum_{i=1}^p \frac{1}{K(y_{j_i})^{\frac{n-2}{n}}}\Big)^{\frac{2}{n}} \big(1 - |Y|^2 + \sum_{i=1}^p (|a_i^-|^2 - |a_i^+|^2)\big)$$

where (a_i^+, a_i^-) are the coordinates of a_i near y_{j_i} along the stable and the unstable manifold of K at y_{j_i}. Thus, the Morse index of such a critical point at infinity is equal to

$$p - 1 + \sum_{i=1}^p n - \operatorname{ind}(K, y_{j_i}).$$

Proof of Theorem 1.2. before giving the proof of the Theorem, we will introduce some definitions

Definition 5.1 Let z_∞ and z'_∞ two critical points at infinity. z_∞ is said to be *dominated by* z'_∞ if

$$W_u(z'_\infty) \cap W_s(z_\infty) \neq \emptyset$$

i.e there exists (at least) a flow line of decreasing pseudo-gradient of J descending from z'_∞ *to* z_∞. If we assume that the intersection is transverse, then we obtain

$$\mathrm{index(z'_\infty)} \geq \mathrm{index(z_\infty)} + 1.$$

.

Definition 5.2 Given two sub-manifolds N and M of Σ which have transverse intersection and complementary dimensions, we note by $(N.P)$ their number of intersection points (see [19] for the whole definitions).

Finally, we also introduce the following definition

Definition 5.3 Let N be a sub-manifold of Σ and z_∞ be a critical point at infinity such that $i(z_\infty) = \dim(\mathrm{N})$, where $i(z_\infty)$ is the Morse index of J at z_∞. Following definition 5.2, the number of intersection points between N and $W_s^\infty(z_\infty)$ is well defined. We denote this number by $(N \cdot z_\infty)$.

Arguing by contradiction, we suppose that J has no critical points in Σ^+. It follows from Theorem 1.1 that the critical points at infinity of J are $\tau_p^\infty := (y_{i_1}, \dots, y_{i_p})_\infty$, $p \in \mathbb{N}^*$, $y_{i_j} \in F^+, \forall\, j = 1, \dots, p$ *and* $y_{i_j} \neq y_{i_k}$ *if* $j \neq k$. We denote by $\mathcal{C}^\infty$ the set of the critical points at infinity. Recall that

$$X_\ell^\infty = \bigcup_{\tau_p^\infty \in \mathcal{C}^\infty,\, i(\tau_p^\infty) \leq \ell} W_u^\infty(\tau_p^\infty)$$

X_ℓ^∞ is a stratified set of dimension ℓ, which is contractible in Σ^+. We denote by $\theta(X_\ell^\infty)$ the contraction of X_ℓ^∞ in Σ^+. The dimension of $\theta(X_\ell^\infty)$ is equal to $\ell + 1$. Using the gradient flow of $-J$ to deform $\theta(X_\ell^\infty)$. By transversality arguments, we can assume that deformation avoids all critical points at infinity having their Morse indices greater than $\ell + 1$. It follows then by Proposition 4.21 of [7] (see also the proof of Lemma 7 of [5]) that $\theta(X_\ell^\infty)$ retracts by deformation on the set

$$X_\ell^\infty \cup \bigcup_{\tau_p^\infty \in \mathcal{C}^\infty,\, i(\tau_p^\infty) = \ell+1} (\theta(X_\ell^\infty) \cdot \tau_p^\infty) W_u^\infty(\tau_p^\infty)$$

Using the condition **(ii)** of the Theorem, we may assume that the deformation of $\theta(X_\ell^\infty)$ along any pseudo-gradient flow of $-J$ avoids all critical points at infinity having their Morse indices equal to $\ell + 1$. Thus $\theta(X_\ell^\infty)$ retracts by deformation on

$$X_\ell^\infty = \bigcup_{\tau_p^\infty \in \mathcal{C}^\infty,\, i(\tau_p^\infty) \leq \ell} W_u^\infty(\tau_p^\infty),$$

and hence

$$1 = \mathcal{X}(\theta(X_\ell^\infty)) = \sum_{\tau_p^\infty \in \mathcal{C}^\infty,\, i(\tau_p^\infty) \leq \ell} (-1)^{i(\tau_p^\infty)},$$

since $\theta(X_\ell^\infty)$ is a contractible set. Here $\mathcal{X}$ denotes the Euler-Poincaré characteristic. Such an equality contradicts the assumption (i) of Theorem 1.2.

Proof of corollary 1.1. Let $\ell = n - \operatorname{ind}(K, y_1)$. It follows from the results of Theorem 1.1 that under the assumption $(\mathbf{H_1})$,the only critical points at infinity of the associated variational problem are $(y_0)_\infty$, $(y_1)_\infty$ and $(y_0, y_1)_\infty$ of index respectively 0, ℓ and $\ell + 1$. Thus, the first assumption of Theorem 1.2 follows in this case. Observe that

$$X_\ell^\infty := \bigcup_{\tau_p^\infty \in \mathcal{C}^\infty,\, i(\tau_p^\infty) \leq \ell} W_u^\infty(\tau_p^\infty) = \bigcup_{y \in F^+} W_u^\infty(y)_\infty.$$

On the other hand, the unstable manifold at infinity of such critical points at infinity $W_u^\infty(y)_\infty$, $y \in F^+$ can be described using [5] as the product of $W_s(y)$ (for a pseudo-gradient of K) by $[A, +\infty[$ domain of the variable λ, for some positive number A large enough. Thus X_ℓ^∞ can be parameterized by $X \times [A, +\infty[$, where $X = \bigcup_{y \in F^+} W_s(y)$. Using the assumption $(\mathbf{H_2})$, there exists a contraction

$$h : [0,1] \times X \longrightarrow \Omega$$

h continuous, such that for any $a \in X$, $h(0,a) = a$ and $h(1,a) = a_0$ a point of X. Such a contraction gives rise to the following contraction

$$h_\infty : [0,1] \times X_{k_0}^\infty \longrightarrow \Sigma^+,\ (t, a, \lambda) \mapsto P\delta_{(h(t,a),\lambda)} + \bar{v}.$$

For $t = 0$, $h_\infty(0, a, \lambda) + \bar{v} = P\delta_{(a,\lambda)} + \bar{v} \in X_\ell^\infty$, h_∞ is a continuous and $h_\infty(1, a, \lambda) = P\delta_{(a_0,\lambda)} + \bar{v}$. Setting $\theta(X_\ell^\infty) = h_\infty\Big([0,1] \times X_\ell^\infty\Big) \cdot \theta(X_\ell^\infty)$ is a contraction of X_ℓ^∞ in Σ^+. Now we have the following expansion

$$J\Big(P\delta_{(h(t,a),\lambda)} + \bar{v}\Big) \sim \Big(\frac{S_n}{K((h(t,a))}\Big)^{\frac{n-2}{n}} \Big[1 + O(\frac{1}{A^2})\Big].$$

Therefore under the assumption $(\mathbf{H_3})$ such contraction is performed under the level $S_n^{\frac{n-2}{n}} + \varepsilon$, ε positive small enough. Again by the assumption $(\mathbf{H_3})$ of corollary 1.1, we derive that for each $p \geq 2$, $J(\tau_p^\infty) := \sum_{j=1}^p \dfrac{S_n^{\frac{n-2}{n}}}{K(y_{i_j})^{\frac{n-2}{n}}}$ is above the level $2S_n^{\frac{n-2}{n}} - \varepsilon$. Thus we get

$$\theta(X_\ell^\infty)) \cap W_s(\tau_p^\infty) = \emptyset,\ \forall p \geq 2.$$

In particular, for each critical point at infinity (τ_p^∞) of index $\ell + 1$ (we have only $(y_0,\, y_1)_\infty$ of index $\ell + 1$, in our statement), we have

$$\mu(\tau_p) = \theta(X_\ell^\infty) \cdot W_s(\tau_p^\infty) = 0.$$

Thus the second assumption of Theorem 1.2 is satisfied, which concludes the proof of corollary 1.1.

Proof of Theorem 1.3. We introduce the following lemma.

Lemma 5.1 *Let $n = 4$ and let w be a non-degenerate critical point of J in Σ^+. Then, for each $p \in \mathbb{N}^*$, there is no critical points or critical points at infinity in the set $V(p,\, \varepsilon,\, w)$, that means, we can construct a pseudo-gradient of J so that the Palais-Smale condition is satisfied along the decreasing flow lines.*

Proof. Let $u = \sum_{i=1}^p \alpha_i \mathrm{P}\delta_{a_i,\lambda_i} + \alpha_0(w + h) + v$ in $V(p, \varepsilon, w)$. Using the expansion of $J(u)$ given by Proposition 3.1, we have for $n = 4$,

$$J(u) \quad = \quad \frac{S_4 \sum_{i=1}^p \alpha_i^2 + \alpha_0^2 \parallel w \parallel^2}{\big(S_4 \sum_{i=1}^p \alpha_i^4 K(a_i) + \alpha_0^4 \parallel w \parallel^2\big)^{\frac{1}{2}}} \Big[1 - \frac{2c_2\alpha_0}{\gamma_1} \sum_{i=1}^p \frac{\alpha_i w(a_i)}{\lambda_i}$$

$$- \quad \frac{c_2}{\gamma_1}\sum_{i\neq j}\alpha_i\alpha_j\varepsilon_{ij} + o\Big(\sum_{k=1}^{p}\frac{1}{\lambda_k} + \sum_{i\neq j}\varepsilon_{ij}\Big)\Big] + \|V\|^2 + \|H\|^2.$$

The result follows from the above expansion and the fact that $w > 0, \in \Omega$. Setting

$$\ell_\sharp := \sup\{i(\tau_p);\ \ \tau_p \in F_\infty\},$$

for $k \in \{0, \dots, \ell_\sharp\}$ we define the following sets

$$X_k^\infty := \cup_{\tau_p \in F_\infty,\, i(\tau_p)\leq k}\, \overline{W_u^\infty(\tau_p^\infty)},$$

where $W_u^\infty(\tau_p^\infty)$ is the unstable manifold associated to the critical point at infinity τ_p^∞ and

$$\mathcal{C}(X_k^\infty) := \{tu + (1-t)(y_0)_\infty,\ \ t\in[0,1],\ \ u \in X_k^\infty\},$$

where y_0 is a global maximum of K on $\overline{\Omega}$. By a Theorem of Bahri-Rabinowitz [7], it follows that

$$\overline{W_u^\infty(\tau_p^\infty)} = W_u^\infty(\tau_p^\infty) \cup \bigcup_{x_\infty < \tau_p^\infty} W_u^\infty(x_\infty) \cup \bigcup_{w<\tau_p^\infty} W_u(w),$$

where x_∞ is a critical point at infinity dominated by τ_p^∞ and w is a solution of (1.1) dominated by τ_p^∞. By transversality arguments for we assume that the index of x_∞ and the Morse index of w are no bigger than k. Hence

$$X_k^\infty = \bigcup_{i(\tau_p)\leq k} W_u^\infty(\tau_p^\infty) \cup \bigcup_{w<\tau_p^\infty} W_u(w).$$

It follows that X_k^∞ is a stratified set of top dimension $\leq k$. Without loss of generality we may assume it equals to k, therefore $\mathcal{C}(X_k^\infty)$ is also a stratified set of top dimension $k+1$. Now we use the gradient flow $(-\partial J)$ to deform $\mathcal{C}(X_k^\infty)$. By dimension arguments we can assume that the deformation avoids all critical as well as critical points at infinity having their Morse indices greater than $k+2$. It follows then by a Theorem of Bahri and Rabinowitz [7] that $\mathcal{C}(X_k^\infty)$ retracts by deformation on the set

$$U := X_k^\infty \cup \bigcup_{i(x_\infty)=k+1} W_u^\infty(x_\infty) \cup \bigcup_{w<\tau_p^\infty} W_u(w).$$

Now taking $k = \ell$ and using the assumption of Theorem 1.3, we deduce that there are no critical points at infinity with index $\ell+1$, we derive that $\mathcal{C}(X_\ell^\infty)$ retracts by deformation onto

$$Z_\ell^\infty := X_\ell^\infty \cup \bigcup_{w;\ \nabla J(w)=0;\ w\,\text{dominated by}\,\mathcal{C}(X_\ell^\infty)} W_u(w). \tag{5.1}$$

Now observe that it follows from the above deformation retract that the problem (1.1) has necessary a solution w with Morse$(w) := m(w) \leq \ell+1$. Otherwise it follows from (5.1) that

$$1 = \mathcal{X}(Z_\ell^\infty) = \sum_{\tau_p\in F_\infty,\ i(\tau_p)\leq \ell} (-1)^{i(\tau_p)},$$

where $\mathcal{X}$ denotes the Euler Characteristic. Such an equality contradicts the assumption (i) of the Theorem. Now for generic K, it follows from the Sard-Smale Theorem that all solutions of (1.1) are non-degenerate solutions, in the sense that their associated linearized operator does not admits zero as an eigenvalue, see [22]. We derive now from (5.1), taking the Euler Characteristic of both sides that:

$$1 = \mathcal{X}(Z_\ell^\infty) = \sum_{\tau_p \in F_\infty,\ i(\tau_p) \le \ell} (-1)^{i(\tau_p)} + \sum_{w < \mathcal{C}(X_\ell^\infty),\, \nabla J(w)=0} (-1)^{m(w)}.$$

It follows then that

$$\Big| 1 - \sum_{\tau_p \in F_\infty,\ i(\tau_p) \le \ell} (-1)^{i(\tau_p)} \Big| \le \sum_{w,\, \nabla J(w)=0,\ m(w) \le \ell+1} (-1)^{m(w)} \le N_\ell,$$

where N_ℓ denotes the set of solutions of (1.1) having their Morse indices $\le \ell + 1$.

6 APPENDIX

In this Appendix, we collect the estimates of the different integral quantities which occur in the paper. These estimates were originally introduced by Bahri [4] and Bahri-Coron [6]. For the proofs we refer the interested reader to [4], [6] and [20]. In this Appendix, we suppose that $\lambda_i d_i$ is large enough and ε_{ij} is small enough. We have the following estimates:

Lemma 6.1 $|P\delta|^2 = S_n - c_1 \dfrac{H(a,a)}{\lambda^{n-2}} + O\Big(\dfrac{log(\lambda d)}{(\lambda d)^n}\Big)$
where $c_1 = c_n^{\frac{2n}{n-2}} \displaystyle\int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+2}{2}}}$ *and* c_n *is defined in (1.6).*

Lemma 6.2 $\displaystyle\int_\Omega K P\delta^{\frac{2n}{n-2}} = c_n^{\frac{2n}{n-2}} K(a) S_n + c_1 \frac{\Delta K(a)}{\lambda^2} - \frac{2n}{n-2} c_1 K(a) \frac{H(a,a)}{\lambda^{n-2}} + O\Big(\frac{log(\lambda d)}{(\lambda d)^n}\Big).$

Lemma 6.3 *For* $i \ne j$
$$(P\delta_i, P\delta_j) = c_1\Big(\varepsilon_{ij} - \frac{H(a_i,\, a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\Big) + O\Big(\sum_{k=i,j} \frac{log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}} log(\varepsilon_{ij}^{-1})\Big).$$

Lemma 6.4 *For* $i \ne j$
$$\int_\Omega K P\delta_j^{\frac{n+2}{n-2}} P\delta_i = K(a_j)(P\delta_j, P\delta_i) + O\Big(\sum_{k=i,j} \frac{log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}} log(\varepsilon_{ij}^{-1})\Big) + o\Big(\varepsilon_{ij} + \frac{1}{(\lambda_k d_k)^{n-1}}\Big).$$

Lemma 6.5 $(P\delta, \lambda \dfrac{\partial P\delta}{\partial \lambda}) = \dfrac{n-2}{2} c_1 \dfrac{H(a,a)}{\lambda^{n-2}} + O\Big(\dfrac{log(\lambda d)}{(\lambda d)^n}\Big).$

Lemma 6.6 $\int_\Omega KP\delta^{\frac{n+2}{n-2}}\lambda\frac{\partial P\delta}{\partial\lambda} = -(\frac{n-2}{2})\bar{c}_1\frac{\Delta K(a)}{\lambda^2} + (n-2)\bar{c}_2 K(a)\frac{H(a,a)}{\lambda^{n-2}}$
$+ O(\frac{1}{\lambda^3}) + o\Big(\frac{1}{(\lambda d)^{n-1)}}\Big).$

Lemma 6.7 *For* $i \neq j$
$$(P\delta_j, \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}) = c_1\Big(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{(n-2)}{2}}}\Big) + O\Big(\sum_{k=i,j}\frac{log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}}log(\varepsilon_{ij}^{-1})\Big).$$

Lemma 6.8 *For* $i \neq j$
$$\int_\Omega KP\delta_j^{\frac{n+2}{n-2}}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i} = K(a_j)(P\delta_j, \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}) + O\Big(\sum_{k=i,j}\frac{log(\lambda d)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}}log(\varepsilon_{ij}^{-1})\Big)$$
$$+ o\Big(\frac{1}{(\lambda_k d_k)^{n-1}} + \varepsilon_{ij}\Big).$$

Lemma 6.9 *For* $i \neq j$
$$\int_\Omega KP\delta_j\,\lambda_i\frac{\partial P\delta_i^{\frac{n+2}{n-2}}}{\partial\lambda_i} = K(a_i)(P\delta_j, \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}) + O\Big(\sum_{k=i,j}\frac{log(\lambda d)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}}log(\varepsilon_{ij}^{-1})\Big)$$
$$+ o\Big(\frac{1}{(\lambda_k d_k)^{n-1}} + \varepsilon_{ij}\Big).$$

Lemma 6.10 $(P\delta, \frac{1}{\lambda}\frac{\partial P\delta}{\partial a}) = -\frac{1}{2}\frac{c_1}{\lambda^{n-1}}\frac{\partial H}{\partial a}(a,a) + O\Big(\frac{1}{(\lambda d)^n}\Big).$

Lemma 6.11 $\int_\Omega KP\delta^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial P\delta}{\partial a} = \frac{n-2}{2}c_2\frac{\nabla K(a)}{\lambda}\Big(1 + o(1)\Big) - 2c_2K(a)\frac{c_n}{\lambda^{n-1}}$
$\frac{\partial H}{\partial a}(a,a) + O\Big(\frac{log(\lambda d)}{(\lambda d)^n} + \frac{1}{\lambda^3}\Big).$

Lemma 6.12 *For* $i \neq j$
$$(P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}) = -\frac{c_1}{(\lambda_i\lambda_j)^{\frac{(n-2)}{2}}}\frac{1}{\lambda_i}\frac{\partial H}{\partial a_i}(a_i,a_j) + c_1\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} + O\Big(\sum_{k=i,j}\frac{1)}{(\lambda_k d_k)^n}$$
$$+ \varepsilon_{ij}^{\frac{n+1}{n-2}}\lambda_j|a_i - a_j|\Big).$$

Lemma 6.13 *For* $i \neq j$
$$\int_\Omega KP\delta_j^{\frac{n+2}{n-2}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = K(a_j)(P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}) + O\Big(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}}log(\varepsilon_{ij}^{-1})\Big).$$

Lemma 6.14 *For* $i \neq j$
$$\int_\Omega KP\delta_j\,\frac{1}{\lambda_i}\frac{\partial P\delta_i^{\frac{n+2}{n-2}}}{\partial a_i} = K(a_i)(P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}) + O\Big(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}}log(\varepsilon_{ij}^{-1})\Big).$$

Lemma 6.15 *For* $\theta = \delta_{(a,\lambda)} - P\delta_{(a,\lambda)}$ *and* H *the regular part of the Green's function, we have the following estimates:*

$$\theta(x)=\frac{c_n}{\lambda^{\frac{(n-2)}{2}}}H(a,x)+O\Big(\frac{1}{\lambda^{\frac{(n+2)}{2}}d^n}\Big),\quad |\theta|_{L^{\frac{2n}{n-2}}}=O\Big(\frac{1}{(\lambda d)^{\frac{(n-2)}{2}}}\Big).$$
$$\frac{\partial\theta}{\partial a}(x)=\frac{c_n}{\lambda^{\frac{(n-2)}{2}}}\frac{\partial H}{\partial a}(a,x)+O\left(\frac{1}{\lambda^{\frac{(n+2)}{2}}d^{n+1}}\right),\quad \Big|\frac{1}{\lambda}\frac{\partial\theta}{\partial a}\Big|_{L^{\frac{2n}{n-2}}}=O\Big(\frac{1}{(\lambda d)^{\frac{n}{2}}}\Big).$$
$$\lambda\frac{\partial\theta}{\partial\lambda}(x)=-\frac{n-2}{2}\frac{c_n}{\lambda^{\frac{(n-2)}{2}}}H(a,x)+O\Big(\frac{1}{\lambda^{\frac{(n+2)}{2}}d^n}\Big),\quad \Big|\lambda\frac{\partial\theta}{\partial\lambda}\Big|_{L^{\frac{2n}{n-2}}}=O\Big(\frac{1}{(\lambda d)^{\frac{n-2}{2}}}\Big).$$

Lemma 6.16 *For each $a\in\Omega$, near the boundary of Ω, let $n_a=n$ the outward normal to $\partial\Omega$ at a.*
$H(a,a)=(2d)^{2-n}+o(d^{2-n}),\quad H(x,a)\le cmax(d_x,d_a)^{2-n}$.
$\dfrac{\partial H}{\partial n}(a,a)=\dfrac{n-2}{2^{n-2}d^{n-1}}+o\Big(\dfrac{1}{d^{n-1}}\Big),\quad \Big|\dfrac{\partial H}{\partial x}\Big|(x,a)\le\dfrac{c}{d_x}H(x,a)$.

Lemma 6.17 *Let x_1 and x_2 be two points of Ω such that $d_1\le d_2$ and $c_2d_2\le|x_1-x_2|$ where c_2 is a fixed constant. If d_1 is small enough then* $(\dfrac{\partial G}{\partial n_1})(x_1,x_2)\le 0$.

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