Advanced Nonlinear Studies **9** (2009), 825–836

On Some Quasilinear Critical Problems

Sébastien de Valeriola

Département de Mathématique,
Université catholique de Louvain, *1348 Louvain-le-Neuve, Belgique*
e-mail: sebastien.devaleriola@uclouvain.be

Michel Willem

Département de Mathématique,
Université catholique de Louvain, *1348 Louvain-le-Neuve, Belgique*
e-mail: michel.willem@uclouvain.be

To Vieri Benci, with esteem and friendship.

Communicated by Donato Fortunato

Abstract

In this paper we prove the almost everywhere convergence of the gradient of Palais-Smale sequences, allowing us to apply the Brezis-Lieb lemma. This leads us to show that infima are attained, and thus to prove the existence of optimal solutions for some critical problems. Our method does not use the concentration-compactness principle.

2000 Mathematics Subject Classification. 35J20, 35J25, 35J60, 35J65.
Key words. Critical exponents, p-Laplacian, Quasilinear elliptic problems.

Introduction

In the framework of the classical paper by Brezis and Nirenberg ([8]), the analysis of a critical minimization problem involves

1. a strict inequality,

2. a convergence theorem.

The first step is always difficlut and technical. On the other hand, the second step uses only the Hilbert space structure of $W_0^{1,2}(\Omega)$ and the Brezis-Lieb lemma ([7]).

This method is applicable to many other problems, like the existence of optimal functions for the critical Poincaré-Sobolev inequality in $W^{1,2}(\Omega)$ (see [15]).

For quasilinear critical problems, the spaces $W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$ are replaced by $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$. Since there is no Hilbert space structure, various sophisticated tools were used.

1. The concentration-compactness principle ([17]) for the critical Dirichlet problem ([14]), for the critical Neumann problem ([12]), for the critical Poincaré-Sobolev inequality ([12]) and for the critical trace inequality ([6]).

2. Approximation by subcritical problems and regularity theorems, for the critical Dirichlet problem ([16] and [10]) and for the critical trace inequality ([5] and [11]).

Our aim is to deduce the convergence theorems for quasilinear critical problems from an elementary result. We prove the almost everywhere convergence of the gradients of Palais-Smale sequences. It is then possible to use the Brezis-Lieb lemma for the sequence of the gradients instead of the Hilbert space structure.

1 The convergence theorem

Let us recall an elementary inequality due to J. Simon ([18]).

Lemma 1.1 *Let $x, y \in \mathbb{R}^N$ and $\langle \cdot, \cdot \rangle$ the standard scalar product in $\mathbb{R}^N$. Then there exists $c > 0$ such that*

$$\langle |x|^{p-2} x - |y|^{p-2} y, x - y \rangle \geqslant \begin{cases} c\,|x-y|^p & \text{if } p \geqslant 2, \\ c\frac{|x-y|^2}{(|x|+|y|)^{p-2}} & \text{if } 1 < p < 2. \end{cases}$$

Our main result on the almost everywhere convergence of the gradients is the following.

We define

$$T(s) = \begin{cases} s & \text{if } |s| \leqslant 1, \\ \frac{s}{|s|} & \text{if } |s| > 1, \end{cases}.$$

Theorem 1.1 *Let $p > 1$, let Ω be an open bounded subset of $\mathbb{R}^N$ and let $(u_n) \subset W^{1,p}(\Omega)$ be such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and*

$$\int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla T(u_n - u) dx \;\to\; 0, \quad n \to \infty. \tag{1.1}$$

Then

1. *there exists a subsequence (u_{n_k}) such that*

$$\nabla u_{n_k} \to \nabla u \quad \text{a.e. on } \Omega,$$

2.

$$\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^p\,dx - \int_\Omega |\nabla(u_n-u)|^p\,dx\right) = \int_\Omega |\nabla u|^p\,dx,$$

3. for any $1 \leqslant q < p$, $u_n \to u$ *in* $W^{1,q}(\Omega)$.

Proof. By Rellich Theorem, we can assume, up to a subsequence, that $u_n \to u$ a.e. on Ω. Let

$$E_n = \{x \in \Omega : |u_n(x) - u(x)| \leqslant 1\},$$
$$e_n = \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla\left(u_n - u\right).$$

It follows from assumption (1.1) that

$$\int_{E_n} e_n \mathrm{d}x = \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla T(u_n-u)\mathrm{d}x \;\to\; 0, \quad n\to\infty.$$

Extracting a subsequence, we can assume that $e_n\chi_{E_n} \to 0$ a.e. on Ω. Since $\chi_{E_n} \to 1$ a.e. on Ω, it follows that $e_n \to 0$ a.e. on Ω. Lemma 1.1 implies that $|\nabla u_n - \nabla u| \to 0$ a.e. on Ω.

Finally $\nabla u_n \to \nabla u$ a.e. on Ω and (∇u_n) is bounded in $L^p(\Omega;\mathbb{R}^N)$, so that, for any $1 \leqslant q < p$, $\nabla u_n \to \nabla u$ in $L^q(\Omega;\mathbb{R}^N)$ by Proposition 20.6 in [20]. Statement 2 follows from the Brezis-Lieb Lemma ([7]). Since this is true for an arbitrary subsequence, 2. and 3. hold for all the sequence.

Remark 1.1 Truncated functions are used as test functions in [4] and [3].

2 Dirichlet problem

Let Ω be a bounded smooth subset of $\mathbb{R}^N$ and $1 < p < N$. Let us define the first eigenvalue of the p-Laplacian operator:

$$\lambda_1^p = \min_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p\,\mathrm{d}x}{\int_\Omega |u|^p\,\mathrm{d}x}.$$

The critical exponent is defined as

$$p^* = \frac{Np}{N-p}.$$

Let $0 < \lambda < \lambda_1^p$. We define also

$$c_1 = c_1(p,\Omega,\lambda) = \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p\,\mathrm{d}x - \lambda\int_\Omega |u|^p\,\mathrm{d}x}{\left(\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}}.$$

By the critical Sobolev inequality,

$$S = S(p,\Omega) = \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\left(\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}} > 0.$$

Hence, by assumption $0 < c_1 \leqslant S$ (let us recall that $S(p,\Omega) = S(p,\mathbb{R}^N)$).

Theorem 2.1 *Let p, Ω and λ be such that*

$$c_1(p,\Omega,\lambda) < S(p,\mathbb{R}^N). \tag{2.2}$$

Then there exists $u \in W_0^{1,p}(\Omega)$ optimal for $c_1(p,\Omega,\lambda)$.

Proof. Define

$$\phi_1 : W_0^{1,p}(\Omega) \to \mathbb{R} : u \mapsto \phi_1(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \lambda\,|u|^p\right)\mathrm{d}x,$$
$$\psi_1 : W_0^{1,p}(\Omega) \to \mathbb{R} : u \mapsto \psi_1(u) = \frac{1}{p^*}\int_\Omega |u|^{p^*}\,\mathrm{d}x.$$

Using the Ekeland variational principle (see [13] or [19]), we obtain a sequence $(u_n) \subset W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \phi_1(u_n) &\to \frac{c_1}{p}, \\ \psi_1(u_n) &= \frac{1}{p^*} \quad \forall n, \\ f_{1,n} = \phi_1'(u_n) - c_1\psi_1'(u_n) &\to 0 \quad \text{in } \left(W_0^{1,p}(\Omega)\right)'. \end{aligned} \tag{2.3}$$

Up to a subsequence we can now assume the existence of $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \nabla u_n &\rightharpoonup \nabla u \quad \text{in } L^p(\Omega,\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L^p(\Omega), \\ u_n &\to u \quad \text{a.e. on } \Omega. \end{aligned} \tag{2.4}$$

Observe that

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla T\,(u_n - u)\,\mathrm{d}x \\ &\quad = \langle f_{1,n}\ ,\ T(u_n-u)\rangle + c_1\int_\Omega |u_n|^{p^*-2}\,u_n\ T(u_n-u)\mathrm{d}x \\ &\qquad + \lambda\int_\Omega |u_n|^{p-2}\,u_n\ T(u_n-u)\mathrm{d}x - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla T(u_n-u) \end{aligned}$$

goes to 0 because of (2.3), (2.4) and the fact that, for all $r \geqslant 1$,

$$\int_\Omega |T(u_n-u)|^r\,\mathrm{d}x \to 0 \ \text{ as } \ n\to\infty. \tag{2.5}$$

By Theorem 1.1,

$$\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^p\,\mathrm{d}x - \int_\Omega |\nabla(u_n-u)|^p\,\mathrm{d}x\right) = \int_\Omega |\nabla u|^p\,\mathrm{d}x. \tag{2.6}$$

We have also by the Brezis-Lieb Lemma

$$1 - \lim_{n\to\infty}\int_\Omega |u_n-u|^{p^*}\,\mathrm{d}x = \int_\Omega |u|^{p^*}\,\mathrm{d}x. \tag{2.7}$$

By the definition of c_1 and by (2.6) we have

$$\begin{aligned} c_1 &= \int_\Omega |\nabla u|^p\,\mathrm{d}x - \lambda\int_\Omega |u|^p\,\mathrm{d}x + \lim_{n\to\infty}\int_\Omega |\nabla(u_n-u)|^p\,\mathrm{d}x \\ &\geqslant c_1\left(\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} + S\lim_{n\to\infty}\left(\int_\Omega |u_n-u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} \\ &= c_1\left(\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} + S\left(1-\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} \end{aligned}$$

using the Sobolev inequality and (2.7) successively. Now if $\int_\Omega |u|^{p^*}\,\mathrm{d}x \neq 1$, the strict inequality (2.2) leads to a contradiction with the concavity of the function $h(t) = t^{p/p^*}$. Hence c_1 is attained by u. The proof of Theorem 2.1 is complete.

Garcia and Peral proved in [14] the following result. See also Guedda and Veron ([16]).

Theorem 2.2 *Let $1 < p^2 < N$ and $0 < \lambda < \lambda_1^p$. Then we have*

$$c_1(p,\Omega,\lambda) < S(p,\mathbb{R}^N).$$

Combining theorems 2.1 and 2.2, we obtain the following corollary.

Corollary 2.1 *Under the assumptions of the previous theorem, the infimum $c_1(p,\Omega,\lambda)$ is achieved. Moreover, there exists $u \in W_0^{1,p}(\Omega)$ positive solution of the problem*

$$\begin{cases} -div\left(|\nabla u|^{p-2}\nabla u\right) = \lambda|u|^{p-2}u + |u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

3 Neumann problem

Let Ω be a bounded smooth subset of $\mathbb{R}^N$ and $1 < p < N$. Define $S = S(p,\mathbb{R}^N)$. It is easily seen that

$$\inf_{u\in W^{1,p}(\mathbb{R}^N_+)\setminus\{0\}} \frac{\int_{\mathbb{R}^N_+}|\nabla u|^p\,\mathrm{d}x}{\left(\int_{\mathbb{R}^N_+}|u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}} = \frac{S}{2^{\frac{p}{N}}}.$$

Let us recall a result of Cherrier (see [9] or [1]).

Lemma 3.1 *For all $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that, for every $u \in W^{1,p}(\Omega)$,*

$$\left(\frac{S}{2^{\frac{p}{N}}} - \epsilon\right)\left(\int_\Omega |u|^{p^*}\,dx\right)^{\frac{p}{p^*}} \leqslant \int_\Omega |\nabla u|^p\,dx + C_\epsilon \int_\Omega |u|^p\,dx. \tag{3.8}$$

Define

$$c_2 = c_2(p,\Omega) = \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega |u|^p\,\mathrm{d}x}{\left(\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}}.$$

We will prove the following.

Theorem 3.1 *Let p and Ω be such that*

$$c_2(p,\Omega) < \frac{S(p,\mathbb{R}^N)}{2^{\frac{p}{N}}}. \tag{3.9}$$

Then there exists $u \in W^{1,p}(\Omega)$ optimal for $c_2(p,\Omega)$.

Proof. Define

$$\phi_2 : W^{1,p}(\Omega) \to \mathbb{R} : u \mapsto \phi_2(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right)\mathrm{d}x$$
$$\psi_2 : W^{1,p}(\Omega) \to \mathbb{R} : u \mapsto \psi_2(u) = \frac{1}{p^*}\int_\Omega |u|^{p^*}\,\mathrm{d}x$$

Using the Ekeland variational principle, we obtain a sequence $(u_n) \subset W^{1,p}(\Omega)$ such that

$$\begin{aligned} \phi_2(u_n) &\to \frac{c_2}{p}, \\ \psi_2(u_n) &= \frac{1}{p^*} \quad \forall n, \\ f_{2,n} = \phi_2'(u_n) - c_2\psi_2'(u_n) &\to 0 \quad \text{in } \left(W^{1,p}(\Omega)\right)'. \end{aligned} \tag{3.10}$$

Up to a subsequence we can now assume the existence of $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \nabla u_n &\rightharpoonup \nabla u \quad \text{in } L^p(\Omega,\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L^p(\Omega), \\ u_n &\to u \quad \text{a.e. on } \Omega. \end{aligned} \tag{3.11}$$

Observe that

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla T\,(u_n - u)\,\mathrm{d}x \\ &\quad = \langle f_{2,n}\ ,\ T(u_n - u)\rangle + c_2\int_\Omega |u_n|^{p^*-2}\,u_n\,T(u_n-u)\mathrm{d}x \\ &\qquad - \int_\Omega |u_n|^{p-2}\,u_n\,T(u_n-u)\mathrm{d}x - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla T(u_n-u) \end{aligned}$$

goes to 0 because of (3.10), (3.11) and (2.5). By Theorem 1.1,

$$\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^p\,\mathrm{d}x - \int_\Omega |\nabla(u_n-u)|^p\,\mathrm{d}x\right) = \int_\Omega |\nabla u|^p\,\mathrm{d}x. \tag{3.12}$$

We have also

$$1 - \lim_{n\to\infty}\int_\Omega |u_n-u|^{p^*}\,\mathrm{d}x = \int_\Omega |u|^{p^*}\,\mathrm{d}x. \tag{3.13}$$

Fix $0<\epsilon<\frac{S}{2^{\frac{p}{N}}}-c_2$. By the definition of c_2 and by (3.12) we have

$$\begin{aligned} c_2 &= \int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega |u|^p\,\mathrm{d}x + \lim_{n\to\infty}\int_\Omega |\nabla(u_n-u)|^p\,\mathrm{d}x \\ &\geqslant c_2\left(\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} + \left(\frac{S}{2^{\frac{p}{N}}}-\epsilon\right)\lim_{n\to\infty}\left(\int_\Omega |u_n-u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} \\ &= c_2\left(\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} + \left(\frac{S}{2^{\frac{p}{N}}}-\epsilon\right)\left(1-\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} \end{aligned}$$

using the Cherrier inequality (3.8) and (3.13) successively. Now if $\int_\Omega |u|^{p^*}\,\mathrm{d}x \neq 1$, the strict inequality (3.9) leads to a contradiction. Hence c_2 is attained by u. The proof of Theorem 3.1 is complete.

Demyanov and Nazarov proved in [12] the following result.

Theorem 3.2 *Let $1<p<(N+1)/2$. Then we have*

$$c_2(p,\Omega) < \frac{S(p,\mathbb{R}^N)}{2^{\frac{p}{N}}}.$$

Combining theorems 3.1 and 3.2, we obtain the following corollary.

Corollary 3.1 *Under the assumption of the previous theorem, the infimum $c_2(p,\Omega)$ is achieved. Moreover, there exists $u\in W^{1,p}(\Omega)$ positive solution of*

$$\begin{cases} -div\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = |u|^{p^*-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

4 Poincare-Sobolev inequality

Let Ω be a bounded smooth subset of $\mathbb{R}^N$ and $1<p<N$. For $u\in W^{1,p}(\Omega)$ we introduce the mean of u on Ω:

$$\overline{u} = \frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}x.$$

Define

$$c_3 = c_3(p,\Omega) = \inf_{u\in W^{1,p}(\Omega),\ u\neq\overline{u}} \frac{\int_\Omega |\nabla u|^p\,\mathrm{d}x}{\left(\int_\Omega |u-\overline{u}|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}} > 0.$$

We will prove the following.

Theorem 4.1 *Let p and Ω be such that*

$$c_3(p,\Omega) < \frac{S(p,\mathbb{R}^N)}{2^{\frac{p}{N}}}. \tag{4.14}$$

Then there exists $u \in W^{1,p}(\Omega)$ optimal for $c_3(p,\Omega)$.

Proof. Define

$$X = \left\{u \in W^{1,p}(\Omega) : \overline{u} = 0\right\}$$

and

$$\phi_3 : X \to \mathbb{R} : u \mapsto \phi_3(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x,$$
$$\psi_3 : X \to \mathbb{R} : u \mapsto \psi_3(u) = \frac{1}{p^*}\int_\Omega |u|^{p^*} \,\mathrm{d}x.$$

Using the Ekeland variational principle, we obtain a sequence $(u_n) \subset X$ such that

$$\begin{aligned} \phi_3(u_n) &\to \frac{c_3}{p}, \\ \psi_3(u_n) &= \frac{1}{p^*} \quad \forall n, \\ f_{3,n} = \phi_3'(u_n) - c_3\psi_2'(u_n) &\to 0 \quad \text{in } X'. \end{aligned} \tag{4.15}$$

Up to a subsequence we can now assume the existence of $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \nabla u_n &\rightharpoonup \nabla u \quad \text{in } L^p(\Omega,\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L^p(\Omega), \\ u_n &\to u \quad \text{a.e. on } \Omega. \end{aligned} \tag{4.16}$$

It is clear that

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot \nabla T\,(u_n - u)\,\mathrm{d}x \\ &\quad = \langle f_{3,n}\ ,\ T(u_n-u)\rangle + c_3\int_\Omega |u_n|^{p^*-2}\,u_n\,T(u_n-u)\mathrm{d}x \\ &\qquad - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla T(u_n-u). \end{aligned}$$

This goes to 0 because of (4.16), (2.5) and $\langle f_{3,n}\ ,\ T(u_n-u)\rangle \to 0$. Indeed, one can see that, defining $d_n = \overline{T(u_n-u)} \to 0$ as $n\to\infty$,

$$\begin{aligned} \langle f_{3,n}\ ,\ T(u_n-u)\rangle &= \langle f_{3,n}\ ,\ T(u_n-u)-d_n\rangle + \langle f_{3,n}\ ,\ d_n\rangle \\ &= \langle f_{3,n}\ ,\ T(u_n-u)-d_n\rangle + \langle \phi_3'(u_n)\ ,\ d_n\rangle - c_3\langle \psi_3'(u_n)\ ,\ d_n\rangle \\ &= \langle f_{3,n}\ ,\ T(u_n-u)-d_n\rangle - c_3\int_\Omega |u_n|^{p^*-2}\,u_n\,d_n\,\mathrm{d}x \end{aligned}$$

goes to 0 as $n \to \infty$ because of (4.15) and (4.16). By Theorem 1.1,

$$\lim_{n\to\infty} \left(\int_\Omega |\nabla u_n|^p \, \mathrm{d}x - \int_\Omega |\nabla (u_n - u)|^p \, \mathrm{d}x \right) = \int_\Omega |\nabla u|^p \, \mathrm{d}x. \tag{4.17}$$

We have also

$$1 - \lim_{n\to\infty} \int_\Omega |u_n - u|^{p^*} \, \mathrm{d}x = \int_\Omega |u|^{p^*} \, \mathrm{d}x. \tag{4.18}$$

Let $0 < \epsilon < \frac{S}{2^{\frac{p}{N}}} - c_3$. By the definition of c_3 and by (4.17) we have

$$\begin{aligned} c_3 &= \int_\Omega |\nabla u|^p \, \mathrm{d}x + \lim_{n\to\infty} \int_\Omega |\nabla (u_n - u)|^p \, \mathrm{d}x \\ &\geqslant c_3 \left(\int_\Omega |u|^{p^*} \, \mathrm{d}x \right)^{\frac{p}{p^*}} + \left(\frac{S}{2^{\frac{p}{N}}} - \epsilon \right) \lim_{n\to\infty} \left(\int_\Omega |u_n - u|^{p^*} \, \mathrm{d}x \right)^{\frac{p}{p^*}} \\ &= c_3 \left(\int_\Omega |u|^{p^*} \, \mathrm{d}x \right)^{\frac{p}{p^*}} + \left(\frac{S}{2^{\frac{p}{N}}} - \epsilon \right) \left(1 - \int_\Omega |u|^{p^*} \, \mathrm{d}x \right)^{\frac{p}{p^*}} \end{aligned}$$

using the Cherrier inequality (3.8) and (4.18) successively. Now if $\int_\Omega |u|^{p^*} \, \mathrm{d}x \neq 1$, the strict inequality (4.14) leads to a contradiction. Hence c_3 is attained by u. The proof of Theorem 4.1 is complete.

Demyanov and Nazarov proved in [12] the following result.

Theorem 4.2 *Let $1 < p < (N+1)/2$. Then we have*

$$c_3(p, \Omega) < \frac{S(p, \mathbb{R}^N)}{2^{\frac{p}{N}}}.$$

Combining theorems 4.1 and 4.2, we obtain the following corollary.

Corollary 4.1 *Under the assumption of the previous theorem, the infimum $c_3(p, \Omega)$ is achieved.*

5 Trace inequality

Let Ω be a bounded smooth subset of $\mathbb{R}^N$ and $1 < p < N$. The critical exponent for the trace of Sobolev functions is defined as

$$p_* = \frac{(N-1)p}{N-p}.$$

Define

$$c_4 = c_4(p, \Omega) = \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega (|\nabla u|^p + |u|^p) \, \mathrm{d}x}{\left(\int_{\partial\Omega} |u|^{p_*} \, \mathrm{d}\gamma \right)^{\frac{p}{p_*}}}.$$

The critical trace inequality ensures that $c_4 > 0$. Define

$$K = K(p, N) = \inf_{u \in W^{1,p}(\mathbb{R}^N_+) \setminus W_0^{1,p}(\mathbb{R}^N_+)} \frac{\int_{\mathbb{R}^N_+} |\nabla u|^p \, \mathrm{d}x}{\left(\int_{\partial\mathbb{R}^N_+} |u|^{p_*} \, \mathrm{d}\gamma \right)^{\frac{p}{p_*}}}.$$

Let us recall a result of Demengel and Nazaret (see [11] or [3]).

Lemma 5.1 *For all $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that, for every $u \in W^{1,p}(\Omega)$,*

$$(K-\epsilon)\left(\int_{\partial\Omega} |u|^{p_*}\, d\gamma\right)^{\frac{p}{p_*}} \leqslant \int_\Omega |\nabla u|^p\, dx + C_\epsilon \int_\Omega |u|^p\, dx. \tag{5.19}$$

We will need the following lemma.

Lemma 5.2 *Let γ_0 be the trace operator. We have*

$$\gamma_0(T \circ u) = T \circ (\gamma_0 u)$$

Proof. This is clear for C^∞ functions, and the continuity of the operator $u \mapsto T \circ u$ for the $W^{1,p}(\Omega)$ norm and for the $L^p(\partial\Omega)$ norm ensures that it stands also for $W^{1,p}(\Omega)$ functions.

We will prove the following.

Theorem 5.1 *Let p and Ω be such that*

$$c_4(p,\Omega) < K(p,N). \tag{5.20}$$

Then there exists $u \in W^{1,p}(\Omega)$ optimal for $c_4(p,\Omega)$.

Proof. Define

$$\phi_4 : W^{1,p}(\Omega) \to \mathbb{R} : u \mapsto \phi_4(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right) \mathrm{d}x,$$
$$\psi_4 : W^{1,p}(\Omega) \to \mathbb{R} : u \mapsto \psi_4(u) = \frac{1}{p_*}\int_{\partial\Omega} |u|^{p_*}\, \mathrm{d}\gamma.$$

Using the Ekeland Variational Principle, we obtain a sequence $(u_n) \subset W^{1,p}(\Omega)$ such that

$$\phi_4(u_n) \to \frac{c_4}{p},$$
$$\psi_4(u_n) = \frac{1}{p_*} \quad \forall n, \tag{5.21}$$
$$f_{4,n} = \phi_4'(u_n) - c_4\psi_4'(u_n) \to 0 \quad \text{in } \left(W^{1,p}(\Omega)\right)'. \tag{5.22}$$

Up to a subsequence we can now assume the existence of $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \nabla u_n &\rightharpoonup \nabla u \quad \text{in } L^p(\Omega,\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L^p(\Omega), \\ u_n &\to u \quad \text{a.e. on } \Omega, \\ \gamma_0 u_n &\to \gamma_0 u \quad \text{in } L^p(\partial\Omega), \\ \gamma_0 u_n &\to \gamma_0 u \quad \text{a.e. on } \partial\Omega. \end{aligned} \tag{5.23}$$

It is clear that

$$\begin{aligned}\int_\Omega \Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\Big)\cdot\nabla T\,(u_n-u)\,\mathrm{d}x \\ = \langle f_{4,n}\ ,\ T(u_n-u)\rangle + c_4\int_{\partial\Omega}|u_n|^{p_*-2}\,u_n\ T(u_n-u)\mathrm{d}\gamma \\ - \int_\Omega |u_n|^{p-2}\,u_n\ T(u_n-u)\mathrm{d}x - \int_\Omega |\nabla u|^{p-2}\,\nabla u\cdot\nabla T(u_n-u)\mathrm{d}x.\end{aligned}$$

The first, third and fourth terms go to 0 because of (5.22), (2.5) and (5.23). For the second term, we have

$$\int_{\partial\Omega}|u_n|^{p_*-2}\,u_n\ T(u_n-u)\mathrm{d}\gamma \to 0$$

as $n\to\infty$ by Lemma 5.2. The four terms of the previous equation going to 0, we can use Theorem 1.1:

$$\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^p\,\mathrm{d}x - \int_\Omega |\nabla(u_n-u)|^p\,\mathrm{d}x\right) = \int_\Omega |\nabla u|^p\,\mathrm{d}x. \tag{5.24}$$

We have also

$$1 - \lim_{n\to\infty}\int_{\partial\Omega}|u_n-u|^{p_*}\,\mathrm{d}\gamma = \int_{\partial\Omega}|u|^{p_*}\,\mathrm{d}\gamma. \tag{5.25}$$

Let now $0<\epsilon<K-c_4$. By the definition of c_4 and by (5.24) and (5.21), we have

$$\begin{aligned}c_4 &= \int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega |u|^p\,\mathrm{d}x + \lim_{n\to\infty}\int_\Omega |\nabla(u_n-u)|^p\,\mathrm{d}x \\ &\geqslant c_4\left(\int_{\partial\Omega}|u|^{p_*}\,\mathrm{d}\gamma\right)^{\frac{p}{p_*}} + (K-\epsilon)\lim_{n\to\infty}\left(\int_{\partial\Omega}|u_n-u|^{p_*}\,\mathrm{d}\gamma\right)^{\frac{p}{p_*}} \\ &= c_4\left(\int_{\partial\Omega}|u|^{p_*}\,\mathrm{d}\gamma\right)^{\frac{p}{p_*}} + (K-\epsilon)\left(1-\int_{\partial\Omega}|u|^{p_*}\,\mathrm{d}\gamma\right)^{\frac{p}{p_*}}\end{aligned}$$

using (5.19) and (5.25) successively. Now if $\int_{\partial\Omega}|u|^{p_*}\,\mathrm{d}\gamma\neq 1$, the strict inequality (5.20) leads to a contradiction. Hence c_4 is attained by u. The proof of Theorem 5.1 is complete.

Bonder and Saintier proved in [6] the following result.

Theorem 5.2 *Let $1<p<(N+1)/2$. Then we have*

$$c_4(p,\Omega) < K(p,N).$$

Combining theorems 5.1 and 5.2, we obtain the following corollary.

Corollary 5.1 *Under the assumption of the previous theorem, the infimum $c_4(p,\Omega)$ is achieved.*

References

[1] T. Aubin, Nonlinear Analysis on Manifolds, Monge-Ampère Equations, Springer, Berlin, 1982.

[2] J.P. Garcia Azorero and I. Peral Alonso, *Existence and nonuniqueness for the p-Laplacian, nonlinear eigenvalue*, Commun. in Partial Diff. Equations **12** (1987), 12, 1389-1430.

[3] R.J. Biezumer, *Best constants in Sobolev trace inequalities*, Nonlinear Analysis **54** (2003), 575–589.

[4] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Analysis **19** (1992), 6, 581–597.

[5] J. F. Bonder and J. Rossi, *On the existence of extremals fot the Sobolev trace embedding theorem with critical exponent*, Bull. London Math. Soc. **37** (2005), 1, 119-125.

[6] J. F. Bonder and N. Saintier, *Estimates for the Sobolev trace constant with critical exponent and applications*, Ann. Mat. Pura Appl. **187** (2008), 4, 683–704.

[7] H. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[8] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 4, 437–477.

[9] P. Cherrier, *Problèmes de Neumann nonlinéaires sur des variétés Riemannienes*, J. Funct. Anal. **57** (1984), 154–207.

[10] F. Demengel and E. Hebey, *On some nonlinear equations involving the p-Laplacian with critical Sobolev growth*, Adv. Differential Equations **3** (1998), 533-574.

[11] F. Demengel and B. Nazaret, *On some nonlinear partial differential equations involving the p-Laplacian and critical Sobolev trace maps*, Asymptotic Analysis **23** (2000), 135-156.

[12] A. V. Demyanov and A. I. Nazarov, *On the existence of extremal functions in Sobolev embedding theorems with critical exponents*, St. Petersburg Math. J. **17** (2006), 5, 773-796.

[13] I. Ekeland, *On the variational principle*, J. Math. Anal Appl. **47** (1974), 324-353.

[14] J. P. Garcia and I. Peral, *Existence and nonuniqueness for the p-Laplacian : nonlinear eigenvalues*, Comm. in Partial Differential Equations **12** (1987), 12, 1389-1430.

[15] P. Girão and T. Weth, *The shape of extremal functions for Poincaré-Sobolev type inequalities in a ball*, J. Funct. Anal. **237** (2006), 1, 194-223.

[16] M. Guedda and L. Veron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Analysis **13** (1989), 8, 879-902.

[17] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I.*, Rev. Mat. Iberoamericana **1** (1985), 1, 145-201.

[18] J. Simon, Régularité de la Solution d'une Équation non Linéaire Dans $\mathbb{R}^N$, Lectures Notes in Math., 665, Springer (1978), 205-227.

[19] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.

[20] M. Willem, Principes d'analyse Fonctionnelle, Cassini, Paris, 2007.