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Nonuniform Stability of Difference Equations with Infinite Delay[*]

Luis Barreira, Claudia Valls
Departamento de Matemática
Instituto Superior Técnico, 1049-001 Lisboa, Portugal
e-mail: barreira@math.ist.utl.pt, cvalls@math.ist.utl.pt

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Abstract

In this note we consider the notion of *nonuniform* exponential contraction for delay difference equations with infinite delay (one can argue that the only delay difference equations are those with infinite delay, since otherwise we can always bring them to the standard recurrence form in some higher-dimensional space). We consider the general case of a nonautonomous dynamics. Our main objective is to show that exponential contractions persist under sufficiently small linear and nonlinear perturbations. This includes establishing the continuous dependence with the perturbation of the constants in the notion of contraction. We also characterize the nonuniform exponential contractions in terms of strict Lyapunov sequences, in particular by constructing explicitly a strict Lyapunov sequence for each exponential contraction.

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1 Introduction

In this note we consider the delay equation

$$x(m+1) = L_m x_m, \quad m \in \mathbb{N} \tag{1.1}$$

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for some linear operators L_m in the set of functions $\phi\colon \mathbb{Z}_0^- \to Y$, with values in some Banach space Y. We emphasize that in general the delay may be infinite. Since we are interested in considering exponential contractions, given $\gamma > 0$ we consider the norm

$$\|\phi\|_\gamma = \sup\{|\phi(j)|e^{\gamma j} : j \in \mathbb{Z}_0^-\},$$

where $|\cdot|$ is the norm in Y. Although we can consider other norms (see [7] for a related discussion), $\|\cdot\|_\gamma$ should be considered the natural norm for exponential contractions since it already incorporates the exponential behavior in the constant γ.

Our main objective is to show that the Lyapunov stability of a nonuniform exponential contraction for equation (1.1) persists under sufficiently small linear and nonlinear perturbations. Incidentally, in the case of a dynamics in a finite-dimensional vector space, the existence of a nonuniform exponential contraction is equivalent to have all Lyapunov exponents negative (see [1]), but in general this need not be the case in the present setting. We also establish the continuous dependence with the perturbation of the constants in the notion of contraction. We note that for nonlinear perturbations, the initial conditions may need to be exponentially small in the initial time, although with small exponential speed when compared to the rate of contraction.

In addition, we characterize the nonuniform exponential contractions in terms of strict Lyapunov sequences, in particular by constructing explicitly a strict Lyapunov sequence for each exponential contraction. The use of Lyapunov functions in the study of the stability of trajectories goes back to the seminal work of Lyapunov. According to Coppel in [5], the connection between Lyapunov functions and uniform exponential dichotomies was first considered by Maĭzel' in [6]. We refer to the book by Mitropolsky, Samoilenko and Kulik [8] for a detailed discussion of the relation between Lyapunov functions and uniform exponential dichotomies in the case of continuous time.

2 Nonuniform exponential contractions

Given a Banach space Y, let X_γ be the set of functions $\phi\colon \mathbb{Z}_0^- \to Y$ with $\|\phi\|_\gamma < \infty$. For any function $x\colon (-\infty, m] \cap \mathbb{Z} \to Y$ and $n \le m$, we define $x_n\colon \mathbb{Z}_0^- \to Y$ by $x_n(j) = x(n+j)$ for $j \in \mathbb{Z}_0^-$. Given linear operators $L_m\colon X_\gamma \to Y$ for $m \in \mathbb{N}$, we consider equation (1.1). For each $n \in \mathbb{N}$ and $\phi \in X_\gamma$, we obtain a unique function $x\colon \mathbb{Z} \to Y$, denoted by $x(\cdot, n, \phi)$, such that $x_n = \phi$ and (1.1) holds for all $m \ge n$. For each $m \ge n$ we define the operator $T(m,n)$ in X_γ by

$$T(m,n)\phi = x_m(\cdot, n, \phi), \quad \phi \in X_\gamma.$$

We have

$$\begin{aligned} \|x_{n+1}\|_\gamma &= \sup\bigl\{|x(m)|e^{\gamma(m-n-1)} : m \le n+1\bigr\} \\ &= \max\bigl\{\sup\bigl\{|x(m)|e^{\gamma(m-n-1)} : m \le n\bigr\}, |x(n+1)|\bigr\} \\ &= \max\bigl\{\|x_n\|_\gamma e^{-\gamma}, |x(n+1)|\bigr\}, \end{aligned} \tag{2.1}$$

and hence $T(m,n)X_\gamma \subset X_\gamma$ for each $m \geq n$. Clearly, $T(m,n)$ is linear, $T(m,m) = \mathrm{Id}$, and

$$T(l,m)T(m,n) = T(l,n), \quad l \geq m \geq n.$$

We say that equation (1.1) admits a *nonuniform exponential contraction* if there exist constants $\gamma > 0$, $\varepsilon \geq 0$ and $D \geq 1$ such that for each $\phi \in X_\gamma$, $n \in \mathbb{N}$ and $m \geq n$ we have

$$\|T(m,n)\phi\|_\gamma \leq De^{-\gamma(m-n)+\varepsilon n}\|\phi\|_\gamma. \tag{2.2}$$

We note that if

$$\|T(m,n)\phi\|_\gamma \leq De^{-a(m-n)+\varepsilon n}\|\phi\|_\gamma$$

for some $a > 0$, then $a \leq \gamma$. Indeed, writing $x_m = T(m,n)\phi$, for each $m \geq n$ we have

$$\begin{aligned}
\|x_m\|_\gamma &\leq De^{-a(m-n)+\varepsilon n}\|x_n\|_\gamma \\
&= De^{-a(m-n)+\varepsilon n} \sup\left\{|x(n+j)|e^{\gamma j} : j \in \mathbb{Z}_0^-\right\} \\
&\leq De^{-a(m-n)+\varepsilon n} \sup\left\{|x(n+j)|e^{\gamma(j-m+n)} : j \leq m-n\right\} e^{\gamma(m-n)} \\
&= De^{(-a+\gamma)(m-n)+\varepsilon n}\|x_m\|_\gamma,
\end{aligned}$$

and hence, $a \leq \gamma$. To clarify the meaning of the notion of exponential contraction with infinite delay, we first note that proceeding in a similar manner to that in (2.1) we obtain

$$\begin{aligned}
\|x_m\|_\gamma &= \max\left\{\|x_{m-1}\|_\gamma e^{-\gamma}, |x(m)|\right\} \\
&= \max\left\{\|x_{m-2}\|_\gamma e^{-2\gamma}, |x(m-1)|e^{-\gamma}, |x(m)|\right\} \\
&\cdots \\
&= \max\left\{\|x_n\|_\gamma e^{-\gamma(m-n)}, |x(m-j)|e^{-\gamma j} : j = 0, \ldots, m-n-1\right\}.
\end{aligned}$$

Therefore, (2.2) is equivalent to

$$|x(n+k)|e^{-\gamma(m-n-k)} \leq De^{-\gamma(m-n)+\varepsilon n}\|x_n\|_\gamma,$$

or also to

$$|x(n+k)| \leq De^{-\gamma k+\varepsilon n}\|x_n\|_\gamma$$

for $k = 1, \ldots, m-n$. For example, when $\varepsilon = 0$ (uniform case) we obtain

$$|x(n+k)| \leq De^{-\gamma k}\|x_n\|_\gamma, \quad k = 1, \ldots, m-n.$$

3 Robustness of nonuniform exponential contractions

We establish in this section the persistence of the stability under sufficiently small linear perturbations. More precisely, we consider the equation

$$y(m+1) = (L_m + N_m)y_m, \tag{3.1}$$

for some linear operators $L_m, N_m \colon X_\gamma \to Y$ for $m \in \mathbb{N}$.

Theorem 3.1 *If equation* (1.1) *admits a nonuniform exponential contraction, and there exists* $\delta > 0$ *such that* $\|N_m\|_\gamma \le \delta e^{-\varepsilon|m+1|}$ *for* $m \in \mathbb{N}$, *with* $\delta D < 1 - e^{-\gamma}$, *then equation* (3.1) *admits a nonuniform exponential contraction, with the constant* γ *replaced by* $\gamma - \log(1 + \delta D e^\gamma)$.

Proof. We follow the proof of Theorem 8 in [3]. By Lemma 5.1, if $\hat{T}(m,n)$ is the operator defined in X_γ by

$$\hat{T}(m,n)\phi = y_m = y_m(\cdot, n, \phi),$$

then

$$\hat{T}(m,n) = T(m,n) + \sum_{l=n}^{m-1} T(m, l+1)(\Gamma N_l \hat{T}(l,n)),$$

since $y_m = \hat{T}(m,n)y_n$. Setting $\alpha_m = \|y_m\|_\gamma$, we obtain

$$\begin{aligned}\alpha_m &\le \|T(m,n)y_n\|_\gamma + \sum_{l=n}^{m-1} \|T(m,l+1)(\Gamma N_l y_l)\|_\gamma \\ &\le De^{-\gamma(m-n)+\varepsilon n}\alpha_n + \delta D \sum_{l=n}^{m-1} e^{-\gamma(m-l-1)}\alpha_l.\end{aligned}$$

Now we consider the sequence Φ_m defined recursively by

$$\Phi_m = De^{-\gamma(m-n)+\varepsilon n}\alpha_n + \delta D \sum_{l=n}^{m-1} e^{-\gamma(m-l-1)}\Phi_l. \tag{3.2}$$

Setting $\Gamma_m = e^{\gamma(m-n)}\Phi_m$, we can rewrite (3.2) in the form

$$\Gamma_m = De^{\varepsilon n}\alpha_n + \delta D e^\gamma \sum_{l=n}^{m-1} \Gamma_j.$$

We can easily verify that

$$\Gamma_{m+1} = (1 + \delta D e^\gamma)\Gamma_m,$$

and by (3.2), $\Gamma_n = \Phi_n = De^{\varepsilon n}\alpha_n$. Hence, for each $m \ge n$ we have

$$\begin{aligned}\Gamma_m &= (1+\delta De^\gamma)^{m-n}\Gamma_n = (1+\delta De^\gamma)^{m-n}De^{\varepsilon n}\alpha_n \\ &= De^{\log(1+\delta De^\gamma)(m-n)+\varepsilon n}\alpha_n.\end{aligned}$$

On the other hand, it follows easily from the construction that $\alpha_m \le \Phi_m$ for each $m \ge n$. Therefore,

$$\alpha_m \le \Phi_m = e^{-\gamma(m-n)}\Gamma_m \le De^{-(\gamma - \log(1+\delta De^\gamma))(m-n)+\varepsilon n}\alpha_n,$$

and we obtain the desired result. □

4 Lyapunov functions and nonuniform exponential contractions

We consider continuous functions $V_m\colon X_\gamma \to \mathbb{R}_0^-$ for $m \in \mathbb{N}$, and we assume that there exist $C > 0$ and $\delta \geq 0$ such that

$$\|x\|_\gamma \leq |V_m(x)| \leq Ce^{\delta m}\|x\|_\gamma \tag{4.1}$$

for every $m \in \mathbb{N}$ and $x \in X_\gamma$. We say that $(V_m)_{m\in\mathbb{N}}$ is a *strict Lyapunov sequence* for equation (1.1) if there exists $\kappa \in (0,1)$ such that

$$V_{m+1}(L_m x) - V_m(x) \geq \kappa|V_m(x)| \tag{4.2}$$

for every $m \in \mathbb{N}$ and $x \in X_\gamma$.

The following is a characterization of nonuniform exponential contractions in terms of strict Lyapunov functions.

Theorem 4.1 *The following properties are equivalent.*

1. *equation* (1.1) *admits a nonuniform exponential contraction;*
2. *there is a strict Lyapunov sequence for equation* (1.1).

Proof. We follow partly arguments in [4]. We first assume that there is a strict Lyapunov sequence for equation (1.1). For each $m \geq n$ set

$$\kappa_{m,n} = \sup\left\{\frac{|V_m(T(m,n)x)|}{|V_n(x)|} : x \in X_\gamma \setminus \{0\}\right\}. \tag{4.3}$$

For every $m \geq l \geq n$ we have

$$\begin{aligned}\kappa_{m,n} &= \sup\left\{\frac{|V_m(T(m,n)x)|}{|V_l(T(l,n)x)|}\cdot\frac{|V_l(T(l,n)x)|}{|V_n(x)|} : x \neq 0\right\}\\ &\leq \sup\left\{\frac{|V_m(T(m,l)y)|}{|V_l(y)|} : y \neq 0\right\}\kappa_{l,n} = \kappa_{m,l}\kappa_{l,n}.\end{aligned} \tag{4.4}$$

It follows from (4.2) that

$$\frac{|V_{l+1}(L_l x)|}{|V_l(x)|} \leq 1-\kappa \quad \text{for every} \quad x \in X_\gamma \setminus \{0\}.$$

Therefore, $\kappa_{l+1,l} \leq 1-\kappa \in (0,1)$ for each $l \in \mathbb{N}$, and it follows from (4.4) that for every $m \geq n$,

$$\kappa_{m,n} \leq \prod_{l=n}^{m-1}\kappa_{l+1,l} \leq (1-\kappa)^{m-n}. \tag{4.5}$$

Hence, by (4.1), (4.3), and (4.5) we obtain

$$\begin{aligned}\|T(m,n)x\|_\gamma &\leq |V_m(T(m,n)x)|\\ &\leq \kappa_{m,n}|V_n(x)| \leq C(1-\kappa)^{m-n}e^{\delta n}\|x\|_\gamma\end{aligned}$$

for every $m \geq n$ and $x \in X_\gamma$. This shows that equation (1.1) admits a nonuniform exponential contraction with

$$\gamma = -\log(1-\kappa), \quad \varepsilon = \delta, \quad \text{and} \quad D = C.$$

Now we assume that equation (1.1) admits a nonuniform exponential contraction. We construct explicitly a strict Lyapunov sequence for equation (1.1). For each $m \in \mathbb{N}$ and $x \in X_\gamma$ set

$$V_m(x) = -\sup\big\{\|T(k,m)x\|_\gamma e^{\gamma(k-m)} : k \geq m\big\}.$$

It follows from (2.2) that

$$|V_m(x)| \leq De^{\varepsilon m}\|x\|_\gamma.$$

Moreover, by construction $|V_m(x)| \geq \|x\|_\gamma$, and (4.1) holds with $\delta = \varepsilon$ and $C = D$. We also have

$$\begin{aligned}
&V_{m+1}(T(m+1,m)(x)) - V_m(x)\\
&= -\sup\big\{\|T(k,m+1)T(m+1,m)x\|_\gamma e^{\gamma(k-m-1)} : k \geq m+1\big\}\\
&\quad + \sup\big\{\|T(k,m)x\|_\gamma e^{\gamma(k-m)} : k \geq m\big\}\\
&= \sup\big\{\|T(k,m)x\|_\gamma e^{\gamma(k-m)} : k \geq m\big\}\\
&\quad - e^{-\gamma}\sup\big\{\|T(k,m)x\|_\gamma e^{\gamma(k-m)} : k \geq m+1\big\}\\
&\geq (1-e^{-\gamma})\sup\big\{\|T(k,m)x\|_\gamma e^{\gamma(k-m)} : k \geq m\big\}\\
&= (1-e^{-\gamma})|V_m(x)|.
\end{aligned}$$

This shows that (4.2) holds with $\kappa = 1 - e^{-\gamma}$, and $(V_m)_{m\in\mathbb{N}}$ is a strict Lyapunov sequence. □

5 Stability of nonuniform exponential contractions

Now we consider the delay equation

$$x(m+1) = L_m x_m + f_m(x_m) \tag{5.1}$$

for some linear operators L_m as above, and some functions $f_m\colon X_\gamma \to Y$ for $m \in \mathbb{N}$. We assume that $f_m(0) = 0$ for each $m \in \mathbb{N}$, and that there exist constants $\delta, q > 0$ (independent of m) such that

$$|f_m(u) - f_m(v))| \leq \delta\|u-v\|_\gamma(\|u\|_\gamma^q + \|v\|_\gamma^q), \tag{5.2}$$

for every $m \in \mathbb{N}$ and $u, v \in X_\gamma$.

We establish the persistence of the stability under sufficiently small nonlinear perturbations.

Theorem 5.1 *If equation* (1.1) *admits a nonuniform exponential contraction and* $\varepsilon < q\gamma$, *then for every sufficiently small* $\delta > 0$, *the solution of equation* (5.1) *with initial condition* $(n, \phi) \in \mathbb{N} \times X_\gamma$ *and* $\|\phi\|_\gamma \le e^{-\varepsilon(1+2/q)n}$ *satisfies*

$$\|x_m\|_\gamma \le 2De^{-\gamma(m-n)+\varepsilon n}\|\phi\|_\gamma \quad \textit{for every} \quad m \ge n. \tag{5.3}$$

Proof. We have the following result.

Lemma 5.1 *The solution* $x = x(\cdot, n, \phi)$ *of equation* (5.1) *satisfies*

$$x_m = T(m,n)\phi + \sum_{j=n}^{m-1} T(m, j+1)(\Gamma f_j(x_j)), \quad m \ge n, \tag{5.4}$$

where $\Gamma(0) = \mathrm{Id}$ *and* $\Gamma(l) = 0$ *for* $l < 0$.

The symbol $\Gamma f_j(x_j)$ in (5.4) denotes the function $\mathbb{Z}_0^- \ni l \mapsto \Gamma(l) f_j(x_j)$. We refer to [2] for a simple proof of Lemma 5.1 (although the statement in the lemma is well-known, we were not able to to find any other proof).

Now we modify the proof of Theorem 1 in [2]. We consider the operator R defined by

$$(Rx)_m = T(m,n)\phi + \sum_{k=n}^{m-1} T(m, k+1)(\Gamma f_k(x_k)), \quad m \ge n \tag{5.5}$$

in the space

$$\mathcal{C} = \{x\colon (-\infty, n] \to Y : \|x\| \le 2D\|\phi\|_\gamma\},$$

with the norm

$$\|x\| := \sup\{\|x_m\|_\gamma e^{-\alpha(m,n)} : m \ge n\} < \infty, \tag{5.6}$$

and where

$$\alpha(m,n) = -\gamma(m-n) + \varepsilon n.$$

We can easily verify that $\mathcal{C}$ is a complete metric space (since X_γ is a Banach space). By (5.5) we have

$$(Rx)_m = T(m,n)\phi + \sum_{k=n}^{m-1} T(m, k+1)(\Gamma f_k(x_k)). \tag{5.7}$$

Since $\|\Gamma f_k(x_k)\|_\gamma = |f_k(x_k)|$, by (2.2) and (5.2) we obtain

$$\begin{aligned}\|(Rx)_m\|_\gamma &\le \|T(m,n)\phi\|_\gamma + \sum_{k=n}^{m-1} \|T(m,k+1)(\Gamma f_k(x_k))\|_\gamma \\ &\le De^{-\gamma(m-n)+\varepsilon n}\|\phi\|_\gamma + D\delta \sum_{k=n}^{m-1} e^{-\gamma(m-k-1)+\varepsilon(k+1)}\|x_k\|_\gamma^{q+1}.\end{aligned}$$

Moreover, since

$$\|x_k\|_\gamma \le 2De^{\alpha(k,n)}\|\phi\|_\gamma \quad \text{and} \quad \|\phi\|_\gamma \le e^{-\varepsilon(1+2/q)n},$$

we obtain

$$\begin{aligned}\|(Rx)_m\|_\gamma &\le De^{-\gamma(m-n)+\varepsilon n}\|\phi\|_\gamma \\ &\quad + (2D)^{q+1}D\delta\|\phi\|_\gamma \sum_{k=n}^{m-1} e^{-\gamma(m-k-1)+\varepsilon(k+1)-(q+1)\gamma(k-n)-\varepsilon n} \\ &\le De^{\alpha(m,n)}\|\phi\|_\gamma \\ &\quad + (2D)^{q+1}D\delta\|\phi\|_\gamma e^{-\gamma(m-n-1)}e^{\varepsilon}\sum_{k=n}^{m-1} e^{(-q\gamma+\varepsilon)(k-n)} \\ &\le De^{\alpha(m,n)}\|\phi\|_\gamma + (2D)^{q+1}D\delta\|\phi\|_\gamma \frac{1}{1-e^{-q\gamma+\varepsilon}}e^{\gamma+\varepsilon}e^{\alpha(m,n)} \\ &\le De^{\alpha(m,n)}\|\phi\|_\gamma\,(1+\delta\mu)\,,\end{aligned}$$

where

$$\mu = \frac{(2D)^{q+1}e^{\gamma+\varepsilon}}{1-e^{-q\gamma+\varepsilon}}. \tag{5.8}$$

It follows from (5.6) that

$$\|Rx\| \le D(1+\delta\mu)\|\phi\|_\gamma \le 2D\|\phi\|_\gamma,$$

provided that δ is sufficiently small. Therefore, $R(\mathcal{C}) \subset \mathcal{C}$. Now we show that R is a contraction. By (5.7) we have

$$\begin{aligned}\|(Rx)_m-(Ry)_m\|_\gamma &\le D\delta\sum_{k=n}^{m-1} e^{-\gamma(m-k-1)+\varepsilon(k+1)}\|x_k-y_k\|_\gamma(\|x_k\|_\gamma^q+\|y_k\|_\gamma^q) \\ &\le (2D)^{q+1}\delta\|\phi\|_\gamma^q\|x-y\| \\ &\quad \times\sum_{k=n}^{m-1} e^{(q+1)\alpha(k,n)}e^{-\gamma(m-k-1)+\varepsilon(k+1)} \\ &\le (2D)^{q+1}\delta e^{-\gamma(m-n)}e^{\gamma+\varepsilon}\|x-y\|\sum_{k=n}^{m-1} e^{(-q\gamma+\varepsilon)(k-n)} \\ &\le (2D)^{q+1}\delta e^{\alpha(m,n)}\frac{e^{\gamma+\varepsilon}}{1-e^{-q\gamma+\varepsilon}}\|x-y\|.\end{aligned}$$

Therefore,

$$\|Rx-Ry\| \le \delta\mu\|x-y\|,$$

with μ as in (5.8), and R is a contraction in $\mathcal{C}$. Hence, R has a unique fixed point in $\mathcal{C}$, which thus satisfies (5.3). □

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