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# Nonlinear Schrödinger Equations with Sign-Changing Potential

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**Abstract**

In this paper, we show that the following system

$$-\Delta u + \lambda V(x)u = g(x, v), \quad -\Delta v + \lambda V(x)v = f(x, u), \quad x \in \mathbb{R}^N \tag{0.1}$$

possesses at least one non-trivial solution pair $(u, v)$ for $\lambda > 0$ large enough, where $f(x, t)$, $g(x, t)$ are continuous functions on $\mathbb{R}^N \times \mathbb{R}$ and super-linear at $t = 0$ as well as at $t = +\infty$, $V(x)$ is allowed to be sign-changing. We mention that we do not assume that $f$ or $g$ satisfies the Ambrosetti-Rabinowitz condition.

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*Key words*. Schrödinger equations, Without the $(AR)$ condition, Sign-changing potential, Super-linear, Linking structure

# 1 Introduction

In this paper, we study the existence of non-trivial solutions for the following problem

$$-\Delta u + \lambda V(x)u = g(x, v), \quad -\Delta v + \lambda V(x)v = f(x, u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

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where $f(x,t)$ and $g(x,t)$ are continuous functions on $\mathbb{R}^N \times \mathbb{R}$, and super-linear at $t = 0$ as well as at $t = +\infty$, and the potential $V(x)$ may change sign.

When $\lambda = 1$, the problem is related to Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

which has been extensively studied recently, see for instance [4], [5], [6], [11], [12] and [15]. Particularly, it was considered in [7] the case that $V(x)$ is changing-sign and $f(x,t)$ is either asymptotically linear or superlinear (but subcritical). Solutions of (1.2) were found as critical points of the related functional in

$$E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V^+(x)u^2\, dx < +\infty\},$$

where $V^+(x) = \max\{V(x), 0\}$, $V^-(x) = \max\{-V(x), 0\}$. A decomposition of $E = E^+ \oplus E^-$ so that the quadratic form $a(u,v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\, dx$ is negative semi-definite on $E^-$ and positive definite on $E^+$ plays an important role in applying linking theorems.

A natural way to study the existence of non-trivial solutions to (1.1) is to use the variational method. In a special case $V \equiv 1$, (1.1) becomes

$$-\Delta u + u = g(x,v)\,, -\Delta v + v = f(x,u), \quad x \in \Omega; \quad u = v = 0,\ x \in \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f, g \in C(\Omega \times \mathbb{R}, \mathbb{R})$. Such a strongly indefinite problem was considered in [2], [3], [8], [9] etc for $f(x,t)$, $g(x,t)$ subcritical and superlinear. With $f(x,t)$, $g(x,t)$ being asymptotically linear, [14] showed problem (1.3) possesses a positive solution in $\mathbb{R}^N$, where the lack of compactness of the Sobolev embeddings in $\mathbb{R}^N$ was involved. In [13], using a linking theorem and concentration-compactness principle, they got an existence result for problem (1.3), where $f(x,t)$, $g(x,t)$ is superlinear at $t = 0$, subcritical near $t = +\infty$. However, they did not assume that $f(x,t)$, $g(x,t)$ satisfy the $(AR)$ condition. Recently, if the potential $V(x)$ is positive, M. Ramos [18] studied the existence, multiplicity and shape of positive solutions of the system (1.1) as the parameter is large enough.

For problem (1.1), due to $V(x)$ sign-changing, the potential $V(x)$ affects the decomposition of the space $\mathcal{E}_\lambda := E_\lambda \times E_\lambda$ in a way that the quadratic part

$$a_\lambda(u,v) = \int_{\mathbb{R}^N} \nabla u \nabla v + \lambda V(x)uv\, dx$$

of the associated functional

$$I_\lambda(u,v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V(x)uv)\, dx - \int_{\mathbb{R}^N} F(x,u) + G(x,v)\, dx \tag{1.4}$$

need to be taken into consideration. The definition of $E_\lambda$ will be founded in Section 2. A key ingredient in use of the linking theorem is to find a proper decomposition of $E_\lambda \times E_\lambda$ into a direct sum of two subspaces so that $a_\lambda(u,v)$ is definite in each subspace. Let

$$F(x,t) = \int_0^t f(x,s)\, ds,\ G(x,t) = \int_0^t g(x,s)\, ds,$$

$$\widetilde{F}(x,t) = \frac{1}{2} f(x,t)t - F(x,t), \quad \widetilde{G}(x,t) = \frac{1}{2} g(x,t)t - G(x,t).$$

In this paper we suppose that:

$(V_1)$ $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x)$ is bounded from below;

$(V_2)$ There exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is non-empty and has finite measure;

$(B_1)$ $f,\ g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $F(x,t) \geq 0$, $G(x,t) \geq 0$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and

$$\lim_{t\to 0} \frac{f(x,t)}{t} = \lim_{t\to 0} \frac{g(x,t)}{t} = 0$$

uniformly with respect to $x \in \mathbb{R}^N$;

$(B_2)$ $\lim\limits_{|t|\to+\infty} \frac{F(x,t)}{t^2} = \lim\limits_{|t|\to+\infty} \frac{G(x,t)}{t^2} = +\infty$ uniformly in $x \in \mathbb{R}^N$;

$(B_3)$ $\widetilde{F}(x,t) > 0$, $\widetilde{G}(x,t) > 0$ whenever $t \neq 0$;

$(B_4)$ $|f(x,t)|^\tau \leq a_1\widetilde{F}(x,t)|t|^\tau$, $|g(x,t)|^\tau \leq b_1\widetilde{G}(x,t)|t|^\tau$ for some $a_1, b_1 > 0$, $\tau > \max\{1, \frac{N}{2}\}$ and all $(x,t)$ with $|t|$ large enough.

Clearly $(B_1)$ implies that problem (1.1) has a trivial solution $(u,v) = (0,0)$. Solutions $(u,v) \neq (0,0)$ are called nontrivial.

If $(B_1)$ and $(B_4)$ hold, then

$$|f(x,t)|^\tau \leq \frac{1}{2}a_1|f(x,t)||t|^{\tau+1},\ |g(x,t)|^\tau \leq \frac{1}{2}b_1|g(x,t)||t|^{\tau+1}$$

for large $t$, hence $f$ or $g$ satisfies the growth restriction

$$|f(x,t)| \leq a_2(|t| + |t|^{p-1}), \quad |g(x,t)| \leq b_2(|t| + |t|^{p-1}), \tag{1.5}$$

where $p = \frac{2\tau}{\tau-1} \in (2, 2^*)(2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1$ or 2). On the other hand, if $f$ or $g$ satisfies (1.5) with $p \in (2, 2^*)$ and the Ambrosetti-Rabinowitz superlinearity condition

$$0 < \mu F(x,t) \leq f(x,t)t,\ 0 < \mu G(x,t) \leq g(x,t)t \tag{1.6}$$

for some $\mu > 2$ and all $(x,t)$ with $t \neq 0$, then it is easy to see that $(B_2)$ and $(B_3)$ hold, and it will be shown in Lemma 2.3 that so does $(B_4)$. We shall also show in this lemma that $(B_2) - (B_4)$ imply

$$\widetilde{F}(x,t) \to \infty, \quad \widetilde{G}(x,t) \to \infty \; as \; |t| \to \infty.$$

An example of $f$ or $g$ satisfying $(B_1) - (B_4)$ but not (1.6 ) is $f(x,t) = \bar{f}(x)t\ln(1+|t|)$, $0 < \inf \bar{f} \leq \sup \bar{f} < \infty$.

Now we are ready to state our main result.

**Theorem 1.1** *Suppose* $(V_1) - (V_2)$ *and* $(B_1) - (B_4)$ *are satisfied. Then there exists* $\lambda_0$ *such that problem (1.1) has at least one non-trivial solution whenever* $\lambda > \lambda_0$.

Next, we investigate the asymptotic shape of the solutions as $\lambda \to \infty$.

**Theorem 1.2** *Suppose* $(V_1)-(V_2)$ *and (1.5) with* $p \in (2, 2^*)$ *are satisfied, and let* $z_m = (u_m, v_m) \in E$ *be a solution of problem (1.1) with* $\lambda = \lambda_m$. *If* $\lambda_m \to \infty$ *and* $\|z_m\|_{\mathcal{E}_\lambda} = \|u_m\|_\lambda + \|v_m\|_\lambda \le C$ *for some* $C > 0$ *and all m. Then passing to a subsequence,* $(u_m, v_m) \to (\bar{u}, \bar{v})$ *in* $L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$ *for all* $q \in (2, 2^*)$, $(\bar{u}, \bar{v})$ *is a weak solution of the equation*

$$\begin{cases} -\Delta u = g(x, v), x \in \Omega, \\ -\Delta v = f(x, u), x \in \Omega \end{cases} \tag{1.7}$$

*and* $(\bar{u}, \bar{v}) = (0, 0)$ *a.e. in* $\mathbb{R}^N \setminus V^{-1}(0)$, *where* $\Omega$ *is the interior of* $V^{-1}(0)$. *In addition if* $V(x) \ge 0$ *and* $(B_1)$ *is satisfied, then* $(u_m, v_m) \to (\bar{u}, \bar{v})$ *in* $E \times E$.

In section 2, we decompose $\mathcal{E}_\lambda := E_\lambda \times E_\lambda = \mathcal{E}^+ \oplus \mathcal{E}^-$ such that $a_\lambda(u, v)$ is negative definite on $\mathcal{E}_\lambda^-$ and positive definite on $\mathcal{E}_\lambda^+$, where both $\mathcal{E}_\lambda^+$ and $\mathcal{E}_\lambda^-$ are infinite-dimensional. Thus we cannot use the similar linking theorem as in [1], [6], which requires *dim* $\mathcal{E}_\lambda^- < \infty$. On the other hand, we have to treat the loss of compactness caused by $\mathbb{R}^N$. In section 3, we prove Theorem 1.1 by using a linking theorem in [12] without $(C)_c$- condition. Finally, we will give the proof of Theorem 1.2.

## 2 Preliminaries

Solutions of (1.1) will be found as critical points of $I$ by linking theorem in [12]. We recall the linking theorem as follows.

Let $E^-$ be a closed subspace of a separable Hilbert space $E$ with the norm $\|.\|_E$ and $E^+ := (E^-)^\perp$. We define a new norm

$$\|u\|_\tau := \max\{\|u^+\|_E : \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle u^-, e_k\rangle|\},$$

where $\{e_k\}$ is a total orthonormal sequence in $E^-$ and $u = u^+ + u^-$, $u^\pm \in E^\pm$. The topology induced by $\|.\|_\tau$ will be called the $\tau$-topology. A homotopy $h = I - g : A \times [0, 1] \to E (A \subset E)$ is said to be *admissible* if :

$(i)$ $h$ is $\tau$-continuous, i.e., $h(u_n, s_n) \to h(u, s)$ in $\tau$-topology as $n \to \infty$ whenever $u_n \to u$ in $\tau$-topology and $s_n \to s$ as $n \to \infty$.
$(ii)$ $g$ is $\tau$-locally finite-dimensional, i.e., for each $(u, s) \in A \times [0, 1]$ there is a neighborhood $U$ of $(u, s)$ in the product topology of $(E, \tau)$ and $[0, 1]$ such that $g(U \cap (A \times [0, 1]))$ is contained in a finite-dimensional subspace of $E$.

Admissible maps are defined in a similar way and we mention that admissible maps and homotopies are continuous in the strong topology, and on bounded set $(E, \tau)$ coincides with the product topology $E^-_{weak}$ and $E^+_{strong}$.

Recall that $(z_n)$ is called a Cerami sequence for $I$ if $I(z_n)$ is bounded and $(1+\|z_n\|)I'(z_n) \to 0$ in $E^*$ as $n \to \infty$, and $I$ satisfies the Cerami condition if each such sequence has a convergent subsequence. A Cerami sequence with $I(z_n) \to c$ will be called a $(C)_c$-sequence, and and we shall say that $I$ satisfies the $(C)_c$-condition if each $(C)_c$-sequence has a convergent subsequence.

For $z_0 \in E^+ \setminus \{0\}$ and $R > r > 0$, let

$$\mathcal{M} = \{z = z^- + tz_0 : z^- \in E^-, \|z\| = R, t \ge 0\}, \qquad \mathcal{N} := \{z \in E^+ : \|z\| = r\},$$

and
$$\Gamma = \{h \in C(\mathcal{M} \times [0,1], E): \quad h \text{ is admissible, } h(u,0) = u \text{ and } I(h(u,s)) \le \max\{I(u), -1\} \text{ for all } s \in [0,1]\}.$$

**Proposition 2.1** *[12] Let $E = E^+ \oplus E^-$ be a separable Hilbert space with $E^-$ orthogonal to $E^+$. Suppose*
*(i) $I(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \Phi(z)$, where $\Phi \in C^1(E, \mathbb{R})$ is bounded below, weakly sequentially lower semi-continuous and $\Phi'$ is weakly sequentially continuous.*
*(ii) There exist $z_0 \in E^+ \setminus \{0\}$, $\alpha > 0$ and $R > r > 0$ such that $I|_N \ge \alpha$ and $I|_{\partial\mathcal{M}} \le 0$.*
*Then there exists a $(C)_c$-sequence for I, where*

$$c := \inf_{h\in\Gamma} \sup_{u\in\mathcal{M}} I(h(u,1)).$$

*Moreover $c \ge \alpha$.*

Proposition 2.1 was proved in [12] which is a generalization of Theorem 3.4 of [11].

In order to study problem (1.1), we shall consider the problem

$$-\Delta u + V(x)u = g(x,v), \quad -\Delta v + V(x)v = f(x,u), \quad x \in \mathbb{R}^N, \tag{2.1}$$

with $V(x)$ and $f, g$ satisfying $(V_1) - (V_2)$ and $(B_1) - (B_4)$. Let

$$E = \{u \in H^1(\mathbb{R}^N): \int_{\mathbb{R}^N} V^+(x)u^2\, dx < +\infty\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V^+(x)uv)\, dx, \|u\| := \langle u, u\rangle^{\frac{1}{2}}.$$

In this paper, we shall need the following inner product

$$\langle u, v\rangle_\lambda := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + \lambda V^+(x)uv)\, dx, \ \lambda > 0.$$

The corresponding norm will be denoted by $\|\cdot\|_\lambda$ and we set $E_\lambda := (E, \|\cdot\|_\lambda)$. Clearly, $\|u\| \le \|u\|_\lambda$ if $\lambda > 1$. It follows from $(V_1)$, $(V_2)$ and the Poincaré inequality that the embedding $E \hookrightarrow H^1(\mathbb{R}^N)$ is continuous. Indeed, set

$$\omega = \{x \in \mathbb{R}^N |\, V^+(x) < b\}, \ \omega_\delta = \{x \in \mathbb{R}^N |\, dist(x, \omega) < \delta\},$$

$\varphi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $\varphi = 1$ in $\omega$ and $\varphi = 0$ in $\mathbb{R}^N \setminus \omega_\delta$. Since $V^+(x) \ge b$ in $\mathbb{R}^N \setminus \omega$, we have

$$\begin{aligned}
\int_\omega (|\nabla u|^2 + u^2)\, dx &\le \int_{\omega_\delta} (|\nabla(u\varphi)|^2 + (u\varphi)^2)\, dx \\
&\le 2\int_{\omega_\delta} |\nabla(u\varphi)|^2\, dx \le C\int_{\omega_\delta} (\varphi^2|\nabla u|^2 + |\nabla\varphi|^2u^2) dx \\
&\le C\int_{\mathbb{R}^N} |\nabla u|^2\, dx + C\int_{\omega_\delta\setminus\omega} |\nabla\varphi|^2u^2\, dx \\
&\le C\int_{\mathbb{R}^N} (|\nabla u|^2 + V^+(x)u^2)\, dx,
\end{aligned}$$

which implies that

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)\,dx &= \int_{\omega}(|\nabla u|^2 + u^2)\,dx + \int_{\mathbb{R}^N\backslash\omega}(|\nabla u|^2 + u^2)\,dx\\ &\leq C\int_{\mathbb{R}^N}(|\nabla u|^2 + V^+(x)u^2)\,dx.\end{aligned}$$

In order to get the main result, we will obtain the direct sum decompositions of the Space $E$.

**Lemma 2.1** *If $V(x)$ satisfies $(V_1) - (V_2)$, then*

$$E = E^- \oplus E^+ \oplus F,$$

*and dim $E^- < \infty$, where $F = \{u \in E : \ supp\ u \in V^{-1}([0,\infty))\}$,*

$$E^- = span\{\varphi_j : \ \mu_j < 1\} \text{ and } E^+ = span\{\varphi_j : \ \mu_j > 1\}.$$

*$\varphi_j$ is the eigenfunction of $\mu_j$ of the following problem*

$$-\Delta u + V^+(x)u = \mu V^-(x)u.$$

*Proof.* The result was proved in [7]. We can also find a more general result in [19].

In a similar way, we also have

**Lemma 2.2** *If $V(x)$ satisfies $(V_1) - (V_2)$, then*

$$E_\lambda = E_\lambda^- \oplus E_\lambda^+ \oplus F,$$

*and dim $E_\lambda^- < \infty$, where $F = \{u \in E : \ supp\ u \in V^{-1}([0,\infty))\}$,*

$$E_\lambda^- = span\{\varphi_j : \ \mu_j < 1\} \text{ and } E_\lambda^+ = span\{\varphi_j : \ \mu_j > 1\}.$$

*$\varphi_j$ is the eigenfunction of $\mu_j$ of the following problem*

$$-\Delta u + \lambda V^+(x)u = \mu\lambda V^-(x)u.$$

Let $E = E^1 \oplus E^2$, and

$$P : \ E \to E^1, \quad Q : \ E \to E^2$$

be the orthogonal projections, where $E^1 = E^+ \oplus F$, $E^2 = E^-$.

Let $\mathcal{E} := E \times E$. The inner product on $\mathcal{E}$ is given by

$$\langle (u,v), (\varphi, \psi)\rangle_E = \int_{\mathbb{R}^N}(\nabla u\nabla\varphi + V^+(x)u\varphi)\,dx + \int_{\mathbb{R}^N}(\nabla v\nabla\psi + V^+(x)v\psi)\,dx,$$

where $z = (u,v) \in E \times E$. Denote

$$E_{11} = \{(u,\ u)|\ u \in E^1\},\ E_{12} = \{(u,\ -u)|\ u \in E^1\},$$

$$E_{21} = \{(u,\ u)|\ u \in E^2\},\ E_{22} = \{(u,\ -u)|\ u \in E^2\}.$$

Hence, $\mathcal{E} = E_{11} \oplus E_{12} \oplus E_{21} \oplus E_{22}$. We may write for any $(u, v) \in \mathcal{E}$ that

$$(u, v) = (u_{11}, u_{11}) + (u_{12}, -u_{12}) + (u_{21}, u_{21}) + (u_{22}, -u_{22}),$$

where

$$u_{11} = Q(\frac{u+v}{2}) \in E^1, \quad u_{12} = Q(\frac{u-v}{2}) \in E^1,$$
$$u_{21} = P(\frac{u+v}{2}) \in E^2, \quad u_{22} = P(\frac{u-v}{2}) \in E^2.$$

We may verify that $a(u, v) = \int_{\mathbb{R}^N} \nabla u \nabla v + V(x)uv\, dx$ is positive definite in $E_{11} \oplus E_{22}$ and negative definite in $E_{12} \oplus E_{21}$. Set $\mathcal{E}^+ = E_{11} \oplus E_{22}$ and $\mathcal{E}^- = E_{12} \oplus E_{21}$ and in $\mathcal{E}$, we define a new inner product

$$\langle z, \eta \rangle = B[z^+ - z^-, \eta], \quad \forall\, z, \eta \in \mathcal{E},$$

which induces a norm

$$\|z\| = \langle z, z \rangle^{\frac{1}{2}}, \; z \in \mathcal{E},$$

where

$$B[(u, v), (\varphi, \psi)] = \int_{\mathbb{R}^N} (\nabla u \nabla \psi + V(x)u\psi)\, dx + \int_{\mathbb{R}^N} (\nabla v \nabla \varphi + V(x)v\varphi)\, dx.$$

The subspaces $\mathcal{E}^+, \mathcal{E}^-$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$. Furthermore,

$$\|z\|^2 = \|z^+\|^2 + \|z^-\|^2 = B[z^+ - z^-, z^+ + z^-],$$

and the norms $\| \cdot \|$ and $\| \cdot \|_{\mathcal{E}}$ are equivalent.

Similarly, we have $\mathcal{E}_\lambda := E_\lambda \times E_\lambda = \mathcal{E}_\lambda^+ \oplus \mathcal{E}_\lambda^-$, where $a_\lambda(u, v)$ is positive definite in $\mathcal{E}_\lambda^+$ and negative definite in $\mathcal{E}_\lambda^-$. We will also define a new inner product

$$\langle z, \eta \rangle_\lambda = B_\lambda[z^+ - z^-, \eta], \quad \forall\, z, \eta \in \mathcal{E}_\lambda,$$

which induces a norm

$$\|z\|_\lambda = \langle z, z \rangle^{\frac{1}{2}}, \; z \in \mathcal{E}_\lambda,$$

where

$$B_\lambda[(u, v), (\varphi, \psi)] = \int_{\mathbb{R}^N} (\nabla u \nabla \psi + \lambda V(x)u\psi)\, dx + \int_{\mathbb{R}^N} (\nabla v \nabla \varphi + \lambda V(x)v\varphi)\, dx.$$

The subspaces $\mathcal{E}_\lambda^+, \mathcal{E}_\lambda^-$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$. Furthermore,

$$\|z\|_\lambda^2 = \|z^+\|_\lambda^2 + \|z^-\|_\lambda^2 = B_\lambda[z^+ - z^-, z^+ + z^-],$$

and the norms $\| \cdot \|_\lambda$ and $\| \cdot \|_{\mathcal{E}_\lambda}$ are equivalent.

The corresponding function of (1.1) is

$$I_\lambda(z) = I_\lambda(u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V(x)uv)\, dx - \int_{\mathbb{R}^N} F(x, u) + G(x, v)\, dx \tag{2.2}$$

where $z = (u, v) \in \mathcal{E}_\lambda$, $z = z^+ + z^-$ with $z_\lambda^+ = (Q(\frac{u+v}{2}) + P(\frac{u-v}{2}), Q(\frac{u+v}{2}) - P(\frac{u-v}{2}))$, $z_\lambda^- = (Q(\frac{u-v}{2}) + P(\frac{u+v}{2}), -Q(\frac{u-v}{2}) + P(\frac{u+v}{2}))$. Furthermore, we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V(x)uv)\, dx = \frac{1}{2} B_\lambda[z, z] = \frac{1}{2} B_\lambda[z^+ + z^-, z^+ + z^-] = \frac{1}{2}(\|z^+\|_\lambda^2 - \|z^-\|_\lambda^2).$$

It is well known that $I_\lambda \in C^1(\mathcal{E}, \mathbb{R})$ (see [17]) and the Fréchet derivative $I'_\lambda$ satisfies

$$\begin{aligned}\langle I'_\lambda(u,v),(\varphi,\psi)\rangle &= \int_{\mathbb{R}^N}(\nabla u\nabla\psi + \lambda V(x)u\psi)\,dx + \int_{\mathbb{R}^N}(\nabla v\nabla\varphi + \lambda V(x)v\varphi)\,dx \\ &\quad - \int_{\mathbb{R}^N}(f(x,u)\varphi + g(x,v)\psi)\,dx\end{aligned}$$

for $(\varphi,\psi) \in E$. So solutions of (1.1) correspond to critical points of $I_\lambda$ in $\mathcal{E}$.

We denote that

$$I_1 = I(u,v) = \int_{\mathbb{R}^N}(\nabla u\nabla v + V(x)uv)\,dx - \int_{\mathbb{R}^N} F(x,u) + G(x,v)\,dx,$$

which is the functional of the problem (2.1).

**Lemma 2.3** *(i) If $f$ or $g$ satisfies* (1.5) *and* (1.6) *for some $a_2, b_2 > 0$, $p \in (2, 2^*)$ and $\mu > 2$, then $(B_4)$ holds with $\tau \in (\frac{N}{2}, \frac{p}{p-2}), \tau > 1$.*

*(ii) If $(B_1) - (B_4)$ are satisfied, then $\widetilde{F}(x,t) \to \infty$, $\widetilde{G}(x,t) \to \infty$ uniformly in $x$ as $|t| \to \infty$.*

*Proof.* This result can be found in [7]. However, for the reader's convenience, we give a proof here. Here we only show that $f$ satisfies this lemma. The proof for $g$ is similar.

(i) First we note that $\frac{p}{p-2} > \max\{1, \frac{N}{2}\}$ because $p \in (2, 2^*)$. Fix $\tau \in (\frac{N}{2}, \frac{p}{p-2})$, $\tau > 1$. If $|t| \geq 1$, then $|f(x,t)| \leq a_3|t|^{p-1}$ for some $a_3 > 0$. Choose $r \geq 1$ so large that

$$\frac{1}{\mu} \leq \frac{1}{2} - \frac{a_3^{\tau-1}}{|t|^{p-(p-2)\tau}} \text{ whenever } |t| \geq r.$$

Then, for such $|t|$,

$$F(x,t) \leq \frac{1}{\mu}f(x,t)t \leq (\frac{1}{2} - \frac{a_3^{\tau-1}}{|t|^{p-(p-2)\tau}})f(x,t)t \leq (\frac{1}{2} - \frac{|f(x,t)|^{\tau-1}}{|t|^{\tau-1}t^2})f(x,t)t,$$

and it follows that

$$\frac{|f(x,t)|^\tau}{|t|^\tau} \leq \frac{1}{2}f(x,t)t - F(x,t) = \widetilde{F}(x,t).$$

(ii) Using $(B_2) - (B_4)$, it follows that for $|t|$ large enough,

$$a_1\widetilde{F}(x,t) \geq (\frac{f(x,t)}{t})^\tau \geq (\frac{2F(x,t)}{t^2})^\tau \to \infty$$

uniformly in $x$ as $|t| \to \infty$.

# 3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For a fixed $z_0 \in \mathcal{E}^+ \setminus \{0\}$ and $R > r > 0$, let

$$M_R = \{z = z^- + \rho z_0 : z^- \in \mathcal{E}^-, \|z\| = R, \rho \geq 0\}, \quad N_r = N := \{z \in \mathcal{E}^+ : \|z\| = r\},$$

and

$$\begin{aligned} T = \{h \in C(M_R \times [0,1], \mathcal{E}) : \quad & h \text{ is admissible, } h(u,0) = u \\ & \text{and } I(h(u,s)) \leq \max\{I(u), -1\} \text{ for all } s \in [0,1]\}.\end{aligned}$$

**Lemma 3.1** *There exist $r > 0$ and $\alpha > 0$ such that $I\mid_{N_r} \geq \alpha$.*

*Proof.* For any $z \in N_r$, we may write $z = (u, u),\ u \in E^1$ or $z = (u, -u),\ u \in E^2$. Thus,

$$I(u, v) = \frac{1}{2}\|z^+\|^2 - \int_{\mathbb{R}^N} F(x, u)\, dx - \int_{\mathbb{R}^N} G(x, v)\, dx.$$

By $(B_1) - (B_4)$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x, t)| \leq \epsilon|t|^2 + C_\epsilon|t|^p, \quad |G(x, t)| \leq \epsilon|t|^2 + C_\epsilon|t|^p,$$

where $2 < p < \frac{2N}{N-2}$. The inclusions $E \hookrightarrow H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ then imply

$$I(u, v) \geq (\frac{1}{2} - \epsilon)\|z^+\|^2 - C_\epsilon\|z^+\|^{p+1},$$

the result follows.

**Lemma 3.2** *For any $z_0 = (u_0, u_0) \in \mathcal{E}^+ \setminus \{0\}$ with $\|z_0\| = 1$, there exists $R > r$ so that $I\mid_{\partial M_R} \leq 0$, where*

$$M_R = \{z = z^- + \rho z_0 :\ \|z\| \leq R,\ \rho \geq 0\}.$$

*Proof.* For $z \in \partial M_R$, we can write $z = z^- + \rho z_0$ with either $\|z\| = R,\ \rho > 0$ or $\|z\| < R,\ \rho = 0$. If $\rho = 0$, we have

$$z \in E^-,\ z = (u, u),\ u \in E^2, \ \text{or } z = (u, -u),\ u \in E^1$$

and

$$I(u, v) = -\frac{1}{2}\|z^-\|^2 - \int_{\mathbb{R}^N} (F(x, u) + G(x, v))\, dx \leq 0,$$

since $F(x, t),\ G(x, t) \geq 0$ for $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

Suppose now that $\rho > 0$, we fix $z_0 = (u_0, u_0)$. If the conclusion of the Lemma were not true, we would have a sequence $\{z_n\} \in \partial M_R$, $z_n = \rho_n z_0 + z_n^-$, $z_n^- = (\phi_n, \psi_n)$, $\rho_n > 0$, $\|z_n\| = n$ such that $I(z_n) > 0$. We write $z_n = (u_n,\ v_n) = (\rho_n u_0 + \phi_n,\ \rho_n u_0 + \psi_n)$. Then

$$I(z_n) = \frac{1}{2}(\rho_n^2\|z_0\|^2 - \|z_n^-\|^2) - \int_{\mathbb{R}^N} (F(x, u_n) + G(x, v_n))\, dx > 0,$$

that is

$$\frac{I(z_n)}{\|z_n\|^2} = \frac{1}{2}(\frac{\rho_n^2}{\|z_n\|^2} - \frac{\|z_n^-\|^2}{\|z_n\|^2}) - \int_{\mathbb{R}^N} \frac{F(x, u_n) + G(x, v_n)}{\|z_n\|^2}\, dx > 0. \tag{3.1}$$

Since $F(x, u) \geq 0$, $G(x, u) \geq 0$ and $\|z_0\| = 1$, we have $\rho_n \geq \|z_n^-\|$. On the other hand, $\frac{\rho_n^2 + \|z_n^-\|^2}{\|z_n\|^2} = 1$ implies that $\frac{1}{2} \leq \frac{\rho_n^2}{\|z_n\|^2} \leq 1$. Thus, we may assume that $\frac{\rho_n^2}{\|z_n\|^2} \to \rho^2 > 0$ as $n \to \infty$. It follows that $\rho_n \to +\infty$. We also have $\zeta_n^- = \frac{z_n^-}{\|z_n\|} \rightharpoonup (\zeta_1,\ \zeta_2)$ in $E$, that is, $\frac{\phi_n}{\|z_n\|} \rightharpoonup \zeta_1$, $\frac{\psi_n}{\|z_n\|} \rightharpoonup \zeta_2$ in $E$ as $n \to \infty$, and $\frac{\phi_n}{\|z_n\|} \to \zeta_1$, $\frac{\psi_n}{\|z_n\|} \to \zeta_2$ a.e in $\mathbb{R}^N$ .

If $x \in \mathbb{R}^N$ such that $\rho u_0 + \zeta_1 \neq 0$, then

$$\lim_{n\to\infty} \frac{\rho_n u_0 + \phi_n}{\|z_n\|} = \rho u_0 + \zeta_1 \neq 0 \text{ a.e in } \mathbb{R}^N.$$

So $u_n = \rho_n u_0 + \phi_n \to \infty$ a.e in $\mathbb{R}^N$ as $n \to \infty$. Similarly if $x \in \mathbb{R}^N$ such that $\rho u_0 + \zeta_2 \neq 0$, then $v_n = \rho_n u_0 + \psi_n \to \infty$ a.e in $\mathbb{R}^N$ as $n \to \infty$. As $\frac{I(z_n)}{\|z_n\|^2} > 0$, and $F(x,t)$, $G(x,t) \geq 0$, we get

$$\begin{aligned} 0 \quad &< \tfrac{1}{2}\tfrac{\rho_n^2}{\|z_n\|^2}\|z_0\|^2 - \tfrac{\|z_n^-\|^2}{2\|z_n\|^2} - \textstyle\int_{\mathbb{R}^N}[\tfrac{F(x,u_n)}{u_n^2}(\tfrac{u_n}{\|z_n\|})^2 + \tfrac{G(x,v_n)}{v_n^2}(\tfrac{v_n}{\|z_n\|})^2]\,dx \\ &\leq \tfrac{1}{2}\tfrac{\rho_n^2}{\|z_n\|^2}\|z_0\|^2 - \tfrac{\|z_n^-\|^2}{2\|z_n\|^2} - \textstyle\int_{\{\rho u_0+\zeta_1\neq 0\}} \tfrac{F(x,u_n)}{u_n^2}(\tfrac{u_n}{\|z_n\|})^2\,dx \\ &\quad - \textstyle\int_{\{\rho u_0+\zeta_2\neq 0\}} \tfrac{G(x,v_n)}{v_n^2}(\tfrac{v_n}{\|z_n\|})^2\,dx. \end{aligned} \tag{3.2}$$

Notice that

$$\frac{u_n}{\|z_n\|} = \frac{\rho_n u_0 + \phi_n}{\|z_n\|} \rightharpoonup \rho u_0 + \zeta_1 \ \text{ and } \quad \frac{v_n}{\|z_n\|} = \frac{\rho_n u_0 + \psi_n}{\|z_n\|} \rightharpoonup \rho u_0 + \zeta_2$$

in $E$ as $n \to \infty$, by Sobolev imbedding theorem, we may assume, up to a subsequence, that

$$\frac{u_n}{\|z_n\|} = \frac{\rho_n u_0 + \phi_n}{\|z_n\|} \to \rho u_0 + \zeta_1 \ \text{ and } \quad \frac{v_n}{\|z_n\|} = \frac{\rho_n u_0 + \psi_n}{\|z_n\|} \to \rho u_0 + \zeta_2$$

a.e in $\mathbb{R}^N$ as $n \to \infty$. Let $z = \rho z_0 + \zeta^-$ with $z_0 = (u_0, u_0)$, $\zeta^- = (\zeta_1, \zeta_2)$. Taking limit in (3.2), we have

$$\begin{aligned} 0 \quad &\leq \tfrac{1}{2}(\rho^2 - \|\zeta^-\|^2) - \textstyle\int_{\{\rho u_0+\zeta_1\neq 0\}} +\infty(\rho u_0 + \zeta_1)^2\,dx \\ &\quad - \textstyle\int_{\{\rho u_0+\zeta_2\neq 0\}} +\infty(\rho u_0 + \zeta_2)^2\,dx \to -\infty. \end{aligned} \tag{3.3}$$

This is a contradiction, the proof is complete.

**Lemma 3.3** *Suppose that $V(x)$ and $f$, $g$ satisfy $(V_1)-(V_2)$ and $(B_1)-(B_4)$. Then any $(C)_c$-sequence for $I$ is bounded.*

*Proof.* We adapt an argument in [6], see also [10].

Let $\{z_n = (u_n, v_n)\} \subset E \times E$ be such that

$$I(u_n, v_n) \to c, \ \textit{and} \quad (1 + \|z_n\|)I'(u_n, v_n) \to 0.$$

Observe that for $n$ large

$$C \geq I(u_n, v_n) - \frac{1}{2}I'(u_n, v_n)(u_n, v_n) = \int_{\mathbb{R}^N} \widetilde{F}(x, u_n)\,dx + \int_{\mathbb{R}^N} \widetilde{G}(x, v_n)\,dx. \tag{3.4}$$

We show that $\{z_n\}$ is bounded. Assume to the contrary that $\|z_n\|_{\mathcal{E}} \to \infty$. Let

$$w_n = \frac{z_n}{\|z_n\|} = (\frac{u_n}{\|z_n\|}, \frac{v_n}{\|z_n\|}) =: (w_n^1, w_n^2),$$

then $\|w_n\|_{\mathcal{E}} = 1$. Observe that, from (3.4) and

$$\begin{aligned} &I'(u_n, v_n)(u_n^+ - u_n^-, v_n^+ - v_n^-) \\ &= \|z_n\|^2 - \textstyle\int_{\mathbb{R}^N} f(x, u_n)(u_n^+ - u_n^-)\,dx - \int_{\mathbb{R}^N} g(x, v_n)(v_n^+ - v_n^-)\,dx \\ &= \|z_n\|^2(1 - \textstyle\int_{\mathbb{R}^N} \tfrac{f(x,u_n)((w_n^1)^+ - (w_n^1)^-)}{\|z_n\|}\,dx - \int_{\mathbb{R}^N} \tfrac{g(x,v_n)((w_n^2)^+ - (w_n^2)^-)}{\|z_n\|}\,dx), \end{aligned}$$

it follows that

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)((w_n^1)^+ - (w_n^1)^-)}{\|z_n\|}\,dx + \int_{\mathbb{R}^N} \frac{g(x,v_n)((w_n^2)^+ - (w_n^2)^-)}{\|z_n\|}\,dx \to 1 \; as\; n \to \infty. \tag{3.5}$$

Set

$$h(r) = \inf\{\widetilde{F}(x,u) : x \in \mathbb{R}^N,\; and\; |u| \geq r\}.$$

By $(B_3)$ and Lemma 2.3, $h(r) > 0$ for all $r > 0$ and $h(r) \to \infty$ as $r \to \infty$. For $0 \leq a < b$, let

$$\Omega_n(a,b) := \{x \in \mathbb{R}^N : a \leq |u_n| < b\}$$

and

$$c_a^b := \inf\{\frac{\widetilde{F}(x,u)}{u^2} : x \in \mathbb{R}^N,\; and\; a \leq |u| \leq b\}.$$

One has $\widetilde{F}(x,u_n) \geq c_a^b u_n^2(x)$ for all $x \in \Omega_n(a,b)$.

It follows from (3.4) that

$$\begin{aligned} C &\geq \int_{\Omega_n(0,a)} \widetilde{F}(x,u_n)\,dx + \int_{\Omega_n(a,b)} \widetilde{F}(x,u_n)\,dx + \int_{\Omega_n(b,\infty)} \widetilde{F}(x,u_n)\,dx \\ &\geq \int_{\Omega_n(0,a)} \widetilde{F}(x,u_n)\,dx + c_a^b \int_{\Omega_n(a,b)} |u_n|^2\,dx + h(b)|\Omega_n(b,\infty)|. \end{aligned} \tag{3.6}$$

Using (3.4),

$$|\Omega_n(b,\infty)| \leq \frac{C}{h(b)} \to 0 \tag{3.7}$$

as $b \to \infty$ uniformly in $n$, and for any fixed $0 \leq a < b$,

$$\int_{\Omega_n(a,b)} |w_n^1|^2\,dx = \frac{1}{\|z_n\|^2} \int_{\Omega_n(a,b)} |u_n|^2\,dx \leq \frac{C}{c_a^b \|z_n\|^2} \to 0 \; as\; n \to \infty. \tag{3.8}$$

It follows from (3.4) and the Hölder inequality that for any $s \in [2, 2^*)$, $p \in (s, 2^*)$ and a suitable constant $C_1$,

$$\begin{aligned} \int_{\Omega_n(b,\infty)} |w_n^1|^s\,dx &\leq (\int_{\Omega_n(b,\infty)} |w_n^1|^p\,dx)^{\frac{s}{p}} |\Omega_n(b,\infty)|^{\frac{p-s}{p}} \\ &\leq \|w_n^1\|^s |\Omega_n(b,\infty)|^{\frac{p-s}{p}} \\ &\leq \|w_n\|^s |\Omega_n(b,\infty)|^{\frac{p-s}{p}} \\ &\leq C_1 |\Omega_n(b,\infty)|^{\frac{p-s}{p}} \to 0 \; uniformly\; in\; n\; as\; b \to \infty. \end{aligned} \tag{3.9}$$

Let $0 < \varepsilon < \frac{1}{6}$, by $(B_1)$, there is $a_\varepsilon > 0$ such that $f(x,u) \leq \frac{\varepsilon}{C_2^2}|u|$ for all $|u| \leq a_\varepsilon$. Consequently

$$\begin{aligned} &\int_{\Omega_n(0,a_\varepsilon)} \frac{f(x,u_n)}{|u_n|} \frac{|u_n|}{\|z_n\|} |(w_n^1)^+ - (w_n^1)^-|\,dx \\ = &\int_{\Omega_n(0,a_\varepsilon)} \frac{f(x,u_n)}{|u_n|} |w_n^1| |(w_n^1)^+ - (w_n^1)^-|\,dx \\ \leq &\int_{\Omega_n(0,a_\varepsilon)} \frac{\varepsilon}{C_2^2} |w_n^1| |(w_n^1)^+ - (w_n^1)^-|\,dx \\ \leq &\frac{\varepsilon}{C_2^2} (\int_{\Omega_n(0,a_\varepsilon)} |w_n^1|^2\,dx)^{\frac{1}{2}} (\int_{\Omega_n(0,a_\varepsilon)} |(w_n^1)^+ - (w_n^1)^-|^2\,dx)^{\frac{1}{2}} \\ \leq &\frac{\varepsilon}{C_2^2} \int_{\Omega_n(0,a_\varepsilon)} |w_n^1|^2\,dx \leq \varepsilon \; for\; all\; n. \end{aligned} \tag{3.10}$$

By $(B_4)$,(3.4)and (3.9) with $s = \frac{2\tau}{\tau-1} \in (2, 2^*)$, we can take $b_\varepsilon$ so large that

$$\begin{aligned}
&\int_{\Omega_n(b_\varepsilon,\infty)} \frac{f(x,u_n)}{|u_n|}|w_n^1||(w_n^1)^+ - (w_n^1)^-|\,dx \\
&\le (\int_{\Omega_n(b_\varepsilon,\infty)} |\frac{f(x,u_n)}{u_n}|^\tau\,dx)^{\frac{1}{\tau}}(\int_{\Omega_n(b_\varepsilon,\infty)} |w_n^1|^s\,dx)^{\frac{1}{s}}(\int_{\Omega_n(b_\varepsilon,\infty)} |(w_n^1)^+ - (w_n^1)^-|^s\,dx)^{\frac{1}{s}} \\
&\le (\int_{\Omega_n(b_\varepsilon,\infty)} a_1\widetilde{F}(x,u_n)\,dx)^{\frac{1}{\tau}}(\int_{\Omega_n(b_\varepsilon,\infty)} |w_n^1|^s\,dx)^{\frac{1}{s}}(\int_{\Omega_n(b_\varepsilon,\infty)} |(w_n^1)^+ - (w_n^1)^-|^s\,dx)^{\frac{1}{s}} \\
&\le \varepsilon\ for\ all\ n.
\end{aligned} \tag{3.11}$$

Note that there is $C_3 = C_3(\varepsilon) > 0$ independent of $n$ such that $|f(x,u_n)| \le C_3|u_n|$ for $x \in \Omega_n(a_\varepsilon, b_\varepsilon)$. By (3.8), there is $n_0$ such that

$$\begin{aligned}
&\int_{\Omega_n(a_\varepsilon,b_\varepsilon)} \frac{f(x,u_n)}{|u_n|}|w_n^1||(w_n^1)^+ - (w_n^1)^-|\,dx \\
&\le C_3 \int_{\Omega_n(a_\varepsilon,b_\varepsilon)} |w_n^1||(w_n^1)^+ - (w_n^1)^-|\,dx \\
&\le C_3(\int_{\Omega_n(a_\varepsilon,b_\varepsilon)} |w_n^1|^2\,dx)^{\frac{1}{2}}(\int_{\Omega_n(a_\varepsilon,b_\varepsilon)} |(w_n^1)^+ - (w_n^1)^-|^2\,dx)^{\frac{1}{2}} \\
&\le \varepsilon\ for\ all\ n \ge n_0.
\end{aligned} \tag{3.12}$$

Now the combination of (3.10)-(3.12) implies that for $n \ge n_0$,

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)((w_n^1)^+ - (w_n^1)^-)}{\|z_n\|}\,dx < 3\varepsilon < \frac{1}{2}.$$

Similarly, we can also prove that

$$\int_{\mathbb{R}^N} \frac{g(x,v_n)((w_n^2)^+ - (w_n^2)^-)}{\|z_n\|}\,dx < 3\varepsilon < \frac{1}{2}\ for\ n \ge n_0.$$

Thus, it contradict (3.5). This completes the proof of the boundedness of $\{z_n\}$.

Similar as [6] and [7], we have the following lemma.

**Lemma 3.4** *Suppose* $(V_1), (V_2)$, *(1.5) are satisfied and let* $(z_n) = (u_n, v_n)$ *be a Palais-Smale sequence for* $I$ *such that* $I(z_n) = I(u_n, v_n) \to c$ *and* $(u_n, v_n) \rightharpoonup (u, v)$. *Then passing to a subsequence, there exists a sequence* $(h_n) = (h_n^1, h_n^2) \to (u, v)$ *such that*

$$I(z_n - h_n) = I(u_n - h_n^1, v_n - h_n^2) \to c - I(u,v), \quad I'(z_n - h_n) = I'(u_n - h_n^1, v_n - h_n^2) \to 0. \tag{3.13}$$

It is clear that the conclusion of this lemma remains valid also for the functional $I_\lambda$ defined in (2.2).

**Lemma 3.5** *Suppose that* $V(x)$ *and* $f$, $g$ *satisfy* $(V_1) - (V_2)$ *and* $(B_1) - (B_4)$. *Then for any* $M > 0$ *there is* $\Lambda = \Lambda(M) > 0$ *such that the functional* $I_\lambda$ *satisfies the Cerami condition for all* $c \le M$, *and* $\lambda > \Lambda$.

*Proof.* Let $\{z_n\} = \{(u_n, v_n)\} \subset E \times E$ be a Cerami sequence. Then by Lemma 3.3, $(z_n)$ is bounded in $E_\lambda \times E_\lambda$, hence passing to a subsequence we may assume that

$$(u_n, v_n) \rightharpoonup (u, v)\ in\ E \times E$$

and $I_\lambda$ satisfies (3.13). Let $w_n = z_n - h_n$. Since $\{V(x) < b\}$ is a set of finite measure and $w_n := (w_n^1, w_n^2) = (u_n - h_n^1, v_n - h_n^2) \rightharpoonup (0,0)$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |w_n^1|^2 + |w_n^2|^2\,dx\\
&= \int_{\{V(x)\geq b\}} |w_n^1|^2 + |w_n^2|^2\,dx + \int_{\{V(x)<b\}} |w_n^1|^2 + |w_n^2|^2\,dx\\
&\leq \tfrac{1}{\lambda b}\int_{\{V(x)\geq b\}} \lambda V(x)(|w_n^1|^2 + |w_n^2|^2)\,dx + o(1)\\
&\leq \tfrac{1}{\lambda b}\int_{\mathbb{R}^N} \lambda V^+(x)(|w_n^1|^2 + |w_n^2|^2)\,dx + o(1)\\
&\leq \tfrac{1}{\lambda b}(\|w_n^1\|_{E_\lambda}^2 + \|w_n^2\|_{E_\lambda}^2) + o(1) = \tfrac{1}{\lambda b}\|w_n\|_{\mathcal{E}_\lambda}^2 + o(1).
\end{aligned}\tag{3.14}$$

Moreover, if $2 < s < p < 2^*$, then by (3.14) and the Hölder inequality and the inclusions $E \hookrightarrow H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, it implies that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |w_n^1|^s + |w_n^2|^s\,dx\\
&\leq \Big(\int_{\mathbb{R}^N} |w_n^1|^2 + |w_n^2|^2\,dx\Big)^{\frac{p-s}{p-2}}\Big(\int_{\mathbb{R}^N} |w_n^1|^p + |w_n^2|^p\,dx\Big)^{\frac{s-2}{p-2}}\\
&\leq d_1\big(\tfrac{1}{\lambda b}\big)^{-\frac{p-s}{p-2}}\|w_n\|_{\mathcal{E}_\lambda}^s + o(1)
\end{aligned}\tag{3.15}$$

where the constant $d_1$ is independent of $w_n$.

By (3.13), we have

$$\begin{aligned}
I_\lambda(w_n^1, w_n^2) - \tfrac{1}{2}I'_\lambda(w_n^1, w_n^2)(w_n^1, w_n^2) &= \int_{\mathbb{R}^N} \widetilde{F}(x, w_n^1)\,dx + \int_{\mathbb{R}^N} \widetilde{G}(x, w_n^2)\,dx\\
&\to c - I_\lambda(u, v),
\end{aligned}\tag{3.16}$$

$$\begin{aligned}
&I'_\lambda(w_n^1, w_n^2)(w_n^2, w_n^1)\\
&= \int_{\mathbb{R}^N} |\nabla w_n^1|^2 + \lambda V(x)|w_n^1|^2\,dx + \int_{\mathbb{R}^N} |\nabla w_n^2|^2 + \lambda V(x)|w_n^2|^2\,dx\\
&\quad - \int_{\mathbb{R}^N} f(x, w_n^1)w_n^2\,dx - \int_{\mathbb{R}^N} g(x, w_n^2)w_n^1\,dx = o(1).
\end{aligned}\tag{3.17}$$

By $(B_1)$, it is clear that given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x,u)| \leq \varepsilon|u|$, $|g(x,v)| \leq \varepsilon|v|$ for all $x \in \mathbb{R}^N$ with $|u| \leq \delta, |v| \leq \delta$, and $(B_4)$ is satisfied for $|u| \geq \delta, |v| \geq \delta$. It follows from (3.14) that

$$\begin{aligned}
\int_{|w_n^1|\leq\delta} f(x, w_n^1)w_n^2\,dx &\leq \varepsilon\int_{|w_n^1|\leq\delta} w_n^1 w_n^2\,dx\\
&\leq \tfrac{\varepsilon}{2}\Big(\int_{|w_n^1|\leq\delta} |w_n^1|^2\,dx + \int_{|w_n^1|\leq\delta} |w_n^2|^2\,dx\Big)\\
&\leq \tfrac{\varepsilon}{2\lambda b}\|w_n\|_{\mathcal{E}_\lambda}^2 + o(1),
\end{aligned}\tag{3.18}$$

$$\begin{aligned}
\int_{|w_n^2|\leq\delta} g(x, w_n^2)w_n^1\,dx &\leq \varepsilon\int_{|w_n^2|\leq\delta} w_n^1 w_n^2\,dx\\
&\leq \tfrac{\varepsilon}{2}\Big(\int_{|w_n^2|\leq\delta} |w_n^1|^2\,dx + \int_{|w_n^2|\leq\delta} |w_n^2|^2\,dx\Big)\\
&\leq \tfrac{\varepsilon}{2\lambda b}\|w_n\|_{\mathcal{E}_\lambda}^2 + o(1).
\end{aligned}\tag{3.19}$$

Using $(B_4)$, (3.15) with $s = \frac{2\tau}{\tau-1}$ and (3.16), we obtain

$$\begin{aligned}
&\int_{|w_n^1|>\delta} f(x, w_n^1) w_n^2 \, dx \\
&= \int_{|w_n^1|>\delta} \frac{f(x,w_n^1)}{w_n^1} w_n^1 w_n^2 \, dx \\
&\le \Big(\int_{|w_n^1|>\delta} a_1 \widetilde{F}(x, w_n^1) \, dx\Big)^{\frac{1}{\tau}} \Big(\int_{|w_n^1|>\delta} |w_n^1|^s \, dx\Big)^{\frac{1}{s}} \Big(\int_{|w_n^1|>\delta} |w_n^2|^s \, dx\Big)^{\frac{1}{s}} \\
&\le \Big(\frac{1}{4}\Big)^{\frac{1}{s}} \Big(\int_{|w_n^1|>\delta} a_1 \widetilde{F}(x, w_n^1) \, dx\Big)^{\frac{1}{\tau}} \Big(\int_{|w_n^1|>\delta} |w_n^1|^s \, dx + \int_{|w_n^1|>\delta} |w_n^2|^s \, dx\Big)^{\frac{2}{s}} \\
&\le \Big(\frac{1}{4}\Big)^{\frac{1}{s}} d_1^{\frac{2}{s}} (\lambda b)^{-\frac{2(p-s)}{s(p-2)}} \Big(\int_{|w_n^1|>\delta} a_1 \widetilde{F}(x, w_n^1) \, dx\Big)^{\frac{1}{\tau}} \|w_n\|_{\mathcal{E}_\lambda}^2 + o(1),
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\int_{|w_n^2|>\delta} g(x, w_n^2) w_n^1 \, dx \\
&= \int_{|w_n^2|>\delta} \frac{g(x,w_n^2)}{w_n^2} w_n^1 w_n^2 \, dx \\
&\le \Big(\int_{|w_n^2|>\delta} b_1 \widetilde{G}(x, w_n^2) \, dx\Big)^{\frac{1}{\tau}} \Big(\int_{|w_n^2|>\delta} |w_n^1|^s \, dx\Big)^{\frac{1}{s}} \Big(\int_{|w_n^2|>\delta} |w_n^2|^s \, dx\Big)^{\frac{1}{s}} \\
&\le \Big(\frac{1}{4}\Big)^{\frac{1}{s}} \Big(\int_{|w_n^2|>\delta} b_1 \widetilde{G}(x, w_n^2) \, dx\Big)^{\frac{1}{\tau}} \Big(\int_{|w_n^2|>\delta} |w_n^1|^s \, dx + \int_{|w_n^2|>\delta} |w_n^2|^s \, dx\Big)^{\frac{2}{s}} \\
&\le \Big(\frac{1}{4}\Big)^{\frac{1}{s}} d_1^{\frac{2}{s}} (\lambda b)^{-\frac{2(p-s)}{s(p-2)}} \Big(\int_{|w_n^2|>\delta} b_1 \widetilde{G}(x, w_n^2) \, dx\Big)^{\frac{1}{\tau}} \|w_n\|_{\mathcal{E}_\lambda}^2 + o(1).
\end{aligned} \tag{3.21}$$

Thus

$$\begin{aligned}
&\int_{|w_n^1|>\delta} f(x, w_n^1) w_n^2 \, dx + \int_{|w_n^2|>\delta} g(x, w_n^2) w_n^1 \, dx \\
&\le d_2 (\lambda b)^{-\frac{2(p-s)}{s(p-2)}} \Big(\Big(\int_{|w_n^1|>\delta} \widetilde{F}(x, w_n^1) \, dx\Big)^{\frac{1}{\tau}} + \Big(\int_{|w_n^2|>\delta} \widetilde{G}(x, w_n^2) \, dx\Big)^{\frac{1}{\tau}}\Big) \|w_n\|_{\mathcal{E}_\lambda}^2 + o(1) \\
&\le d_3 (\lambda b)^{-\frac{2(p-s)}{s(p-2)}} (c - I_\lambda(u, v))^{\frac{1}{\tau}} \|w_n\|_{\mathcal{E}_\lambda}^2 + o(1) \\
&\le d_4 M^{\frac{1}{\tau}} (\lambda b)^{-\frac{2(p-s)}{s(p-2)}} \|w_n\|_{\mathcal{E}_\lambda}^2 + o(1).
\end{aligned} \tag{3.22}$$

From $I'_\lambda(w_n^1, w_n^2)(w_n^2, w_n^1) \to 0$ and $(w_n) = (w_n^1, w_n^2) \rightharpoonup (0, 0)$, we have

$$\begin{aligned}
o(1) &= I'_\lambda(w_n^1, w_n^2)(w_n^2, w_n^1) \\
&= \int_{\mathbb{R}^N} |\nabla w_n^1|^2 + \lambda V(x) |w_n^1|^2 \, dx + \int_{\mathbb{R}^N} |\nabla w_n^2|^2 + \lambda V(x) |w_n^2|^2 \, dx \\
&\quad - \int_{\mathbb{R}^N} f(x, w_n^1) w_n^2 \, dx - \int_{\mathbb{R}^N} g(x, w_n^2) w_n^1 \, dx \\
&= \int_{\mathbb{R}^N} |\nabla w_n^1|^2 + \lambda V^+(x) |w_n^1|^2 \, dx + \int_{\mathbb{R}^N} |\nabla w_n^2|^2 + \lambda V^+(x) |w_n^2|^2 \, dx \\
&\quad - \int_{\mathbb{R}^N} f(x, w_n^1) w_n^2 \, dx - \int_{\mathbb{R}^N} g(x, w_n^2) w_n^1 \, dx + o(1) \\
&\ge \Big(1 - \frac{\varepsilon}{\lambda b} - d_4 M^{\frac{1}{\tau}} (\lambda b)^{-\frac{2(p-s)}{s(p-2)}}\Big) \|w_n\|_{\mathcal{E}_\lambda}^2 + o(1).
\end{aligned} \tag{3.23}$$

Let $\Lambda = \Lambda(M)$ be so large such that $1 - \frac{\varepsilon}{\lambda b} - d_4 M^{\frac{1}{\tau}} (\lambda b)^{-\frac{2(p-s)}{s(p-2)}} > 0$ when $\lambda > \Lambda$, it implies that $\|w_n\|_{\mathcal{E}_\lambda} \to 0$. Since $w_n = z_n - h_n$ and $h_n = (h_n^1, h_n^2) \to (u, v)$, it follows that also $z_n = (u_n, v_n) \to (u, v)$ in $E \times E$.

*Proof of Theorem 1.1.* We may show as in [17] that the functional $\bar{I}_\lambda(u, v) := \int_{\mathbb{R}^N} F(x, u) + G(x, v) \, dx \ge 0$ is weakly lower semi-continuous and $\bar{I}'_\lambda$ is weakly continuous. Replacing $V$ by $\lambda V$ in Lemma 3.3, and by Lemma 3.5, we see that $I_\lambda$ satisfies the Cerami condition if $\lambda > \Lambda$. Replacing $V$ by $\lambda V$, Lemma 3.1 and Lemma 3.2 are held for $I_\lambda$. Thus, by Proposition 2.1, Theorem 1.1 is proved. □

# 4 Proof of Theorem 1.2

*Proof Theorem 1.2.* We observe that $(z_m)$ is bounded in $\mathcal{E}_\lambda$, and therefore, after passing to a subsequence we have

$$u_m \rightharpoonup \bar{u}, \quad v_m \rightharpoonup \bar{v} \; in \; E_\lambda \times E_\lambda,$$

and

$$u_m \to \bar{u}, \quad v_m \to \bar{v} \; in \; L^q_{loc}(\mathbb{R}^N), \; 2 \le q < 2^*.$$

Since $I'_{\lambda_m}(u_m, v_m)(\varphi, \psi) = 0$, that is

$$\begin{aligned} &\langle I'_{\lambda_m}(u_m, v_m), (\varphi, \psi)\rangle \\ &= \int_{\mathbb{R}^N} (\nabla u_m \nabla \psi + \lambda_m V(x) u_m \psi)\, dx + \int_{\mathbb{R}^N} (\nabla v_m \nabla \varphi + \lambda_m V(x) v_m \varphi)\, dx \\ &\quad - \int_{\mathbb{R}^N} (f(x, u_m)\varphi + g(x, v_m)\psi)\, dx, \end{aligned}$$

it follows that $\int_{\mathbb{R}^N} V(x) u_m \psi \, dx \to 0$ and $\int_{\mathbb{R}^N} V(x) v_m \varphi \, dx \to 0$ as $\lambda_m \to \infty$, and hence $\int_{\mathbb{R}^N} V(x)\bar{u}\psi \, dx = 0$, and $\int_{\mathbb{R}^N} V(x)\bar{v}\varphi \, dx = 0$ for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$.

Next, we show that

$$\bar{u} = 0, \quad \bar{v} = 0 \; a.e. \; in \; \mathbb{R}^N \setminus V^{-1}(0).$$

In fact, consider the sets $A_k = \{x \in \mathbb{R}^N : |x| \le k, V(x) \ge \frac{1}{k}\}, k \in N$. Then

$$\int_{A_k} u_m^2 + v_m^2 \, dx \le k \int_{A_k} V(x)(u_m^2 + v_m^2)\, dx \le \frac{k}{\lambda_m} \|z_n\|_{\mathcal{E}_\lambda} \to 0 \; as \; \lambda_m \to \infty,$$

hence,

$$\int_{A_k} (\bar{u}^2 + \bar{v}^2)\, dx = 0 \; for \; every \; k.$$

This implies $(\bar{u}, \bar{v})|_{A_k} = (0, 0)$ a.e. for every $k \in N$, and therefore $(\bar{u}, \bar{v}) = (0, 0)$ for a.e. $x \in \cup_{k \in N} A_k$, that is $(\bar{u}, \bar{v}) = (0, 0)$ in $\mathbb{R}^N \setminus V^{-1}(0)$. As $f(x, t)$, $g(x, t)$ satisfy (1.5), we obtain that $(\bar{u}, \bar{v})$ is a weak solution of the equation

$$-\Delta u = g(x, v), x \in \Omega, \quad -\Delta v = f(x, u), x \in \Omega. \tag{4.1}$$

Now, we will prove that $u_m \to \bar{u}, v_m \to \bar{v}$ in $L^q(\mathbb{R}^N)$. Assuming the contrary, it follows from Lemma I.1 of [16] that there exists $\delta > 0, \rho > 0$ and a sequence $\{x_m\} \subset \mathbb{R}^N$ such that

$$\int_{B_\rho(x_m)} (|u_m - \bar{u}|^2 + |v_m - \bar{v}|^2)\, dx \ge \delta, \; for \; all \; m \in N.$$

Moreover, $|x_m| \to \infty$ as $m \to \infty$ (for otherwise the left-hand side above tends to 0 as $m \to \infty$). Hence $|B_\rho(x_m) \cap \{x \in \mathbb{R}^N : V(x) < b\}| \to 0$ and $\int_{B_\rho(x_m)} (\bar{u}^2 + \bar{v}^2)\, dx \to 0$ as $m \to \infty$. Consequently,

$$\begin{aligned} \|z_m\|_{\mathcal{E}_\lambda} &\ge \lambda_m b \int_{B_\rho(x_m) \cap \{V(x) \ge b\}} (u_m^2 + v_m^2)\, dx \\ &= \lambda_m b \int_{B_\rho(x_m) \cap \{V(x) \ge b\}} (|u_m - \bar{u}|^2 + |v_m - \bar{v}|^2)\, dx \\ &= \lambda_m b \Big(\int_{B_\rho(x_m)} (|u_m - \bar{u}|^2 + |v_m - \bar{v}|^2)\, dx + o(1)\Big) \to \infty, \end{aligned}$$

as $m \to \infty$. It is a contradiction.

Suppose $V(x) \geq 0$ and $(B_1)$ holds. Since $(u_m, v_m)$ is a weak solution of problem (1.1), we have

$$||z_m||_{\mathcal{E}}^2 \leq ||z_m||_{\mathcal{E}_\lambda} = \int_{\mathbb{R}^N} f(x, u_m)v_m + g(x, v_m)u_m \, dx. \tag{4.2}$$

Similarly, we obtain

$$\langle u_m, \bar{u} \rangle_{E_\lambda} + \langle v_m, \bar{v} \rangle_{E_\lambda} = \int_{\mathbb{R}^N} f(x, u_m)\bar{v} + g(x, v_m)\bar{u} \, dx. \tag{4.3}$$

Let $m \to \infty$ in (4.3) and recalling that $(\bar{u}, \bar{v}) = (0, 0)$ if $V(x) > 0$, we have

$$\langle \bar{u}, \bar{u} \rangle_E + \langle \bar{v}, \bar{v} \rangle_E = \int_{\mathbb{R}^N} f(x, \bar{u})\bar{v} + g(x, \bar{v})\bar{u} \, dx. \tag{4.4}$$

Next we show that

$$\begin{aligned} &\int_{\mathbb{R}^N} f(x, u_m)v_m \, dx \to \int_{\mathbb{R}^N} f(x, \bar{u})\bar{v} \, dx, \\ &\int_{\mathbb{R}^N} g(x, v_m)u_m \, dx \to \int_{\mathbb{R}^N} g(x, \bar{v})\bar{u} \, dx \; as \; m \to \infty. \end{aligned} \tag{4.5}$$

In fact

$$\begin{aligned} &\int_{\mathbb{R}^N} |f(x, u_m)v_m - f(x, \bar{u})\bar{v}| \, dx \\ \leq &\int_{\mathbb{R}^N} |f(x, u_m)||v_m - \bar{v}| \, dx + \int_{\mathbb{R}^N} |f(x, u_m) - f(x, \bar{u})||\bar{v}| \, dx \end{aligned}$$

and $u_m \to \bar{u}$ in $L^q_{loc}(\mathbb{R}^N), 2 \leq q < 2^*$, it is easy to see that the second integral on the right-hand side above tends to 0. Since for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that $|f(x, u)| \leq \varepsilon|u| + C_\varepsilon|u|^{p-1}$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} |f(x, u_m)||v_m - \bar{v}| \, dx \\ &\leq \varepsilon \int_{\mathbb{R}^N} |u_m||v_m - \bar{v}| \, dx + C_\varepsilon \int_{\mathbb{R}^N} |u_m|^{p-1}|v_m - \bar{v}| \, dx \\ &\leq 2\varepsilon\left(\int_{\mathbb{R}^N} |u_m|^2 + |v_m - \bar{v}|^2 \, dx\right) + C_\varepsilon\left(\int_{\mathbb{R}^N} |u_m|^p \, dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N} |v_m - \bar{v}|^p \, dx\right)^{\frac{1}{p}}. \end{aligned}$$

Thus,

$$\int_{\mathbb{R}^N} f(x, u_m)v_m \, dx \to \int_{\mathbb{R}^N} f(x, \bar{u})\bar{v} \, dx$$

follows because $v_m \to \bar{v}$ in $L^q(\mathbb{R}^N), q \in (2, 2^*)$, $(u_m)$ and $(v_m)$ are bounded in $L^2(\mathbb{R}^N)$, and $\varepsilon$ has been chosen arbitrarily. Similarly, we obtain

$$\int_{\mathbb{R}^N} g(x, v_m)u_m \, dx \to \int_{\mathbb{R}^N} g(x, \bar{v})\bar{u} \, dx \; as \; m \to \infty.$$

From (4.2) and (4.4), it follows that

$$\limsup_{m \to \infty} ||u_m||_E + ||v_m||_E \leq ||\bar{u}||_E + ||\bar{v}||_E,$$

hence $u_m \to \bar{u}, v_m \to \bar{v}$ in $E$. □

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