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Smoothness of Asymptotic Phase Revisited

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Abstract

It is well-known that solutions on the stable manifold of a hyperbolic periodic solution of an autonomous system of ordinary differential equations have an asymptotic phase which has the same order of smoothness as the vector field. In this paper we show if the system depends on a parameter that, in general, the asymptotic phase loses one order of smoothness in the parameter.

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1 Introduction

Consider a system

$$\dot{z} = F(z, \varepsilon), \tag{1.1}$$

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where $\varepsilon \in \mathbb{R}^m$ is a small parameter and F is a C^r function, $r \geq 1$. Suppose when $\varepsilon = 0$, the system

$$\dot{z} = F(z, 0)$$

has a hyperbolic periodic solution $u_0(t)$ with minimal period $T > 0$. Then it is well-known that $u_0(t)$ can be continued to a $T(\varepsilon)$–periodic solution $u(t, \varepsilon)$ of

$$\dot{z} = F(z, \varepsilon)$$

such that $u(t, 0) = u_0(t)$, $T(0) = T$ and both $u(t, \varepsilon)$ and $T(\varepsilon)$ are C^r functions.

Now, according to Chicone [1], Coppel [3], Hale [4], Hartman [5], Hsu [6], Perko [9], (see also Chicone and Liu [2]), each solution $z(t)$ in the local stable manifold of $u_0(t, \varepsilon)$ has an asymptotic phase τ, that is,

$$|z(t) - u(t + \tau, \varepsilon)| \to 0 \quad \text{as} \quad t \to \infty.$$

Note that τ is unique up to an integer multiple of $T(\varepsilon)$. It is known (see, for example, [7]) that for a fixed value of the parameter, the asymptotic phase τ is a C^r function on the stable manifold and when the vector field is analytic, Zinchenko [10] has shown that the asymptotic phase is analytic. However, as far as we know, nobody has discussed the smooth dependence of τ on the parameter.

In Section 2 we show that τ is a C^{r-1} function of the parameter ε but in Section 3 we give an example showing in general it may not be a C^r function. The proofs needed in Section 2 are given in Sections 4 and 5.

2 C^{r-1}–smoothness of the asymptotic phase

Our assumption is that $u_0(t)$ is a hyperbolic periodic solution of

$$\dot{z} = F(z, 0). \tag{2.1}$$

Then it follows that (1.1) has a $T(\varepsilon)$–periodic solution $u(t, \varepsilon)$ (note: $T(\varepsilon)$ is the minimal period), where $T(\varepsilon)$ and $u(t, \varepsilon)$ are both C^r functions. Now we know that for small ε, $u(t, \varepsilon)$ is a hyperbolic periodic solution of (1.1) and every solution $z(t)$ on the stable manifold has an asymptotic phase, that is, a number τ such that

$$|z(t) - u(t + \tau, \varepsilon)| \to 0 \quad \text{as} \quad t \to \infty.$$

This number τ is unique up to a multiple of $T(\varepsilon)$.

In this section our object is to show that if $F(z, \varepsilon)$ is C^r, then the asymptotic phase $\tau(x, \varepsilon)$ is a C^{r-1}–function, where x is in the local stable manifold of $u(t, \varepsilon)$.

The stable manifold of the periodic orbit is foliated by leaves, each leaf corresponding to an asymptotic phase. In the proposition below, we construct these leaves locally near a given point $u(0, \varepsilon)$ on the periodic orbit, which clearly can be chosen arbitrarily. In fact, we just construct the leaf corresponding to asymptotic phase zero as the others arise simply by following the flow.

Proposition 2.1 *Let the function $F(z, \varepsilon)$ be C^r ($r \geq 1$) and suppose* (2.1) *has a hyperbolic T-periodic solution $u_0(t)$. Then*

(i) *the system*

$$\dot{z} = F_z(u_0(t), 0)z \tag{2.2}$$

has a trichotomy with projections P_0, P_+ *and* P_- *and (positive) constants* K, α; *that is,*

$$\begin{array}{lll} |X(t)P_0X^{-1}(s)| & \leq K & \text{for all} \quad t \\ |X(t)P_+X^{-1}(s)| & \leq Ke^{-\alpha(t-s)} & \text{for} \quad t \geq s \\ |X(t)P_-X^{-1}(s)| & \leq Ke^{-\alpha(s-t)} & \text{for} \quad t \leq s, \end{array}$$

where $X(t)$ *is the fundamental matrix of* (2.2) *with* $X(0) = I$ *and the range of* P_0 *is spanned by* $\dot{u}_0(0)$;

(ii) *the perturbed system* (1.1) *has a hyperbolic* $T(\varepsilon)$*-periodic solution* $u(t, \varepsilon)$ *where* $u(t, \varepsilon)$ *and* $T(\varepsilon)$ *are* C^r *functions with* $u(t, 0) = u_0(t)$ *and* $T(0) = T$.

(iii) *suppose* $0 < \gamma < \alpha$. *For* $\Delta > 0$ *sufficiently small, there are positive numbers* ε_0, ξ_0 *depending on* Δ *such that if* $|\varepsilon| < \varepsilon_0$ *and* $\xi \in \mathcal{R}P_+$ *satisfies* $|\xi| < \xi_0$, *there exists a unique solution* $z(t) = z^+(t, \xi, \varepsilon)$ *of* (1.1) *such that*

$$|z(t) - u(t, \varepsilon)|e^{\gamma t} \leq \Delta \ (t \geq 0), \quad P_+[z(0) - u(0, \varepsilon)] = \xi.$$

Moreover,

$$z^+(t, 0, \varepsilon) = u(t, \varepsilon)$$

and $z^+(t, \xi, \varepsilon)$ *is* C^{r-1} *in its arguments. Furthermore the* $(r-1)$*th order derivatives of* $z^+(t, \xi, \varepsilon)$ *are differentiable with respect to* (t, ξ) *and the derivatives are continuous in* (t, ξ, ε). *In particular, for* $\tilde{\xi} \in \mathcal{R}P_+$, $z^+_\xi(t, \xi, \varepsilon)\tilde{\xi}$ *is the unique solution of*

$$\dot{z} = F_z(z^+(t, \xi, \varepsilon), \varepsilon)z$$

such that $\sup_{t\geq 0} e^{\gamma t}|z(t)| < \infty$ *and* $P_+z(0) = \tilde{\xi}$.

We prove Proposition 1 in Sections 4 and 5. Now we use it to derive the smoothness of the asymptotic phase.

Corollary 2.1 *Suppose the conditions of Proposition 1 hold. Then the asymptotic phase* $\tau(x, \varepsilon)$ *is a* C^{r-1}*–function of* (x, ε), *where* x *belongs to the local stable manifold of* $u(t, \varepsilon)$.

Proof. Note that τ is the asymptotic phase of the solution of (1.1) starting at $z^+(\tau, \xi, \varepsilon)$ and that $(\tau, \xi) \mapsto z^+(\tau, \xi, \varepsilon)$ gives a parametrization of the local stable manifold of the periodic orbit $u(t, \varepsilon)$ in the neighbourhood of $u(0, \varepsilon)$. In particular

$$z^+(0, 0, \varepsilon) = u(0, \varepsilon).$$

We now compute the derivatives $z^+_\tau(\tau, 0, 0)$ and $z^+_\xi(\tau, 0, 0)$. From Proposition 1 we know that $z^+(\tau, 0, \varepsilon) = u(\tau, \varepsilon)$. Hence:

$$z^+_\tau(0, 0, 0) = \dot{u}_0(0).$$

Next we know that for $\tilde{\xi} \in \mathcal{R}P_+$, $z_{\xi}(t,0,0)\tilde{\xi}$ is the unique solution such that $\sup_{t\geq 0}|z(t)|e^{\gamma t} < \infty$ of the equation

$$\begin{cases} \dot{z} = F_z(u(t,0),0)z \\ P_+z(0) = \tilde{\xi} \end{cases}$$

so that $z^+_{\xi}(t,0,0)\tilde{\xi} = X(t)\tilde{\xi}$. As a consequence

$$z^+_{\xi}(0,0,0)\tilde{\xi} = \tilde{\xi} \quad \text{for all} \quad \tilde{\xi} \in \mathcal{R}P_+.$$

Now we know the local stable manifold of $u(t,\varepsilon)$ near $u(0,\varepsilon)$ can be represented as the graph of a C^r smooth function $w(\eta,\varepsilon)$, where $w(0,0) = u_0(0)$ and $\eta \in \mathcal{R}[P_0 + P_+]$ is small, $\mathcal{R}[P_0 + P_+]$ being the tangent space to the stable manifold of $u_0(t)$ at $u_0(0)$. This means that

$$\eta = (P_0 + P_+)(w(\eta,\varepsilon) - u_0(0)).$$

Our object now is to find the asymptotic phase of the point on the local stable manifold of $u(t,\varepsilon)$ corresponding to the coordinate η. Thus we need to derive the relation between the coordinates η and (τ,ξ). That is, given $\eta \in \mathcal{R}[P_0 + P_+]$ and ε, we need to solve $z^+(\tau,\xi,\varepsilon) = w(\eta,\varepsilon)$ for τ and ξ. So we need to solve the equation

$$g(\tau,\xi,\eta,\varepsilon) = \eta - (P_0 + P_+)(z^+(\tau,\xi,\varepsilon) - u_0(0)) = 0 \tag{2.3}$$

for (τ,ξ), where g is a C^{r-1}–function; note also when $r = 1$ the derivative of $g(\tau,\xi,\eta,\varepsilon)$ with respect to (τ,ξ) exists and is continuous in $(\tau,\xi,\eta,\varepsilon)$. Next note that

$$\begin{aligned} g(0,0,0,0) &= 0 - (P_0 + P_+)(z^+(0,0,0) - u_0(0)) = 0, \\ g_\tau(0,0,0,0) &= -(P_0 + P_+)\dot{u}_0(0) = -\dot{u}_0(0) \end{aligned}$$

and if $\tilde{\xi} \in \mathcal{R}P_+$

$$g_\xi(0,0,0,0)\tilde{\xi} = -(P_0 + P_+)\tilde{\xi} = -\tilde{\xi}.$$

Then if $\tilde{\tau}$ is real and $\tilde{\xi}$ is in $\mathcal{R}P_+$,

$$0 = g_\tau(0,0,0,0)\tilde{\tau} + g_\xi(0,0,0,0)\tilde{\xi} = -\dot{u}_0(0)\tilde{\tau} - \tilde{\xi}$$

implies that $\tilde{\tau} = 0$ and $\tilde{\xi} = 0$ so that the derivative of g with respect to (τ,ξ) at $(0,0,0,0)$ is an invertible linear map from $\mathbb{R} \times \mathcal{R}P_+$ onto $\mathcal{R}[P_0 + P_+]$. Then, by the implicit function theorem, given $\eta \in \mathcal{R}[P_0 + P_+]$ and ε sufficiently small, equation (2.3) has a unique C^{r-1} solution $\tau = \tau(\eta,\varepsilon)$, $\xi = \xi(\eta,\varepsilon)$. In particular, this means the asymptotic phase has C^{r-1} dependence on the parameter ε and so the proof of the Corollary is complete. ■

Remark 2.1 *In the proof of the corollary the rth derivative of g with respect to* (τ,ξ,η) *exists and is continuous in* $(\tau,\xi,\eta,\varepsilon)$. *So the rth derivative of* $\tau(\eta,\varepsilon)$ *with respect to* η *exists and is a continuous function of* (η,ε). *Hence we recover the result that, for fixed* ε, *the asymptotic phase depends in a* C^r *way on the point on the stable manifold.*

3 Example

In this section we construct a C^1 system with an asymptotically stable periodic orbit for which the asymptotic phase depends continuously on the parameter but is not C^1.

Consider the planar system in polar coordinates

$$\dot{r} = 1 - r, \quad \dot{\theta} = 1 + \varepsilon f(r, \varepsilon),$$

where f is C^1 with $f(1, \varepsilon) = 0$. In rectangular coordinates, this equation is

$$\begin{aligned} \dot{x} &= \frac{x}{\sqrt{x^2 + y^2}} - x - y - \varepsilon y f(\sqrt{x^2 + y^2}, \varepsilon) \\ \dot{y} &= \frac{y}{\sqrt{x^2 + y^2}} + x - y + \varepsilon x f(\sqrt{x^2 + y^2}, \varepsilon). \end{aligned}$$

The system has the asymptotically stable periodic orbit, $x = \cos t$, $y = \sin t$, that is,

$$r = 1, \quad \theta = t$$

for all ε. Note that if $r(t) = 1 + Ce^{-t}$ is a solution of the r−equation, then for any τ

$$\theta(t) = \tau + t - \int_t^\infty \varepsilon f(1 + Ce^{-u}, \varepsilon) du$$

is a solution of the θ−equation such that

$$\theta(t) - t - \tau \to 0$$

as $t \to \infty$. So τ is the asymptotic phase of the solution with initial value $1 + C$ for r and

$$\theta(0) = \tau - \int_0^\infty \varepsilon f(1 + Ce^{-u}, \varepsilon) du$$

for θ. So the asymptotic phase of the solution with initial value $(r, \theta) = (1 + C, 0)$ is

$$\tau(C, \varepsilon) = \int_0^\infty \varepsilon f(1 + Ce^{-u}, \varepsilon) du.$$

By change of variable $v = 1 + Ce^{-u}$, we can write

$$\tau(C, \varepsilon) = \int_1^{1+C} \varepsilon \frac{f(v, \varepsilon)}{v - 1} dv.$$

Note that

$$g(C, \varepsilon) = \frac{\tau(C, \varepsilon) - \tau(C, 0)}{\varepsilon} = \int_1^{1+C} \frac{f(v, \varepsilon)}{v - 1} dv = \int_0^C \frac{f(1 + w, \varepsilon)}{w} dw.$$

We define

$$f(r, \varepsilon) = \begin{cases} \frac{1}{\ln(r-1+|\varepsilon|)} - \frac{1}{\ln |\varepsilon|} & r \geq 1,\ 0 < |\varepsilon| < 1 \\ \frac{1}{\ln(r-1)} & r > 1,\ \varepsilon = 0 \\ 0 & r = 1,\ \varepsilon = 0 \\ -f(1/r, |\varepsilon|) & 0 < r < 1,\ 0 \leq |\varepsilon| < 1 \end{cases}$$

It is clear that $f(1,\varepsilon) = 0$ for any ε. We verify below that $f(r,\varepsilon)$ is C^0 and $\varepsilon f(r,\varepsilon)$ is a C^1 function for $r > 0$, $|\varepsilon| < 1$. Now if $\tau(C,\varepsilon)$ is a C^1 function in a neighbourhood of $(0,0)$, then, if $\alpha > 0$:

$$g(\varepsilon^\alpha,\varepsilon) = \frac{\tau(\varepsilon^\alpha,\varepsilon) - \tau(\varepsilon^\alpha,0)}{\varepsilon} = \int_0^1 \tau_\varepsilon(\varepsilon^\alpha,\theta\varepsilon)d\theta \to \tau_\varepsilon(0,0)$$

as $\varepsilon \to 0^+$. In particular, this would imply that

$$g(\varepsilon^\gamma,\varepsilon) - g(\varepsilon^\beta,\varepsilon) \to 0$$

as $\varepsilon \to 0^+$, where $0 < \gamma < \beta < 1$. Now

$$\begin{aligned}
|g(\varepsilon^\gamma,\varepsilon) - g(\varepsilon^\beta,\varepsilon)| &= \left|\int_{\varepsilon^\beta}^{\varepsilon^\gamma} \frac{f(1+w,\varepsilon)}{w}dw\right| \\
&= \int_{\varepsilon^\beta}^{\varepsilon^\gamma} \frac{1}{w}\left[\frac{1}{\ln\varepsilon} - \frac{1}{\ln(w+\varepsilon)}\right]dw \\
&= \gamma - \beta + \int_{\varepsilon^\beta}^{\varepsilon^\gamma} \frac{1}{w}\frac{1}{|\ln(w+\varepsilon)|}dw \\
&\geq \gamma - \beta + \int_{\varepsilon^\beta}^{\varepsilon^\gamma} \frac{1}{w}\frac{1}{|\ln(\varepsilon^\beta+\varepsilon)|}dw \\
&= (\gamma-\beta)\left(1 - \tfrac{\ln\varepsilon}{\ln(\varepsilon^\beta+\varepsilon)}\right) \\
&\to (\gamma-\beta)\left(1 - \tfrac{1}{\beta}\right) > 0, \text{ as } \varepsilon \to 0^+.
\end{aligned}$$

The conclusion is that $\tau(C,\varepsilon)$ is not C^1 in any neighbourhood of $(0,0)$.

Now we verify that $f(r,\varepsilon)$ is C^0 and $\varepsilon f(r,\varepsilon)$ is a C^1 function for $r > 0$, $|\varepsilon| < 1$.

A. *$f(r,\varepsilon)$ is continuous.* Of course $f(r,\varepsilon)$ is continuous in $\Omega = \{(r,\varepsilon) \mid 0 < r \neq 1,\ 0 < |\varepsilon| < 1\}$. Then $f(r,\varepsilon)$ is also continuous in $U = \{(r,\varepsilon) \mid r \geq 1,\ \varepsilon \geq 0\}$ since for $r^* > 1$ and $0 < \varepsilon^* < 1$ we have:

$$\lim_{\substack{r\to r^*\\ \varepsilon\to 0^+}} f(r,\varepsilon) = \lim_{\substack{r\to r^*\\ \varepsilon\to 0^+}} \frac{1}{\ln(r-1+\varepsilon)} - \frac{1}{\ln\varepsilon} = \frac{1}{\ln(r^*-1)} = f(r^*,0),$$

$$\lim_{\substack{r\to 1^+\\ \varepsilon\to \varepsilon^*}} f(r,\varepsilon) = \lim_{\substack{r\to 1^+\\ \varepsilon\to \varepsilon^*}} \left[\frac{1}{\ln(r-1+\varepsilon)} - \frac{1}{\ln\varepsilon}\right] = \frac{1}{\ln\varepsilon^*} - \frac{1}{\ln\varepsilon^*} = 0 = f(1,\varepsilon^*),$$

$$\lim_{\substack{r\to 1^+\\ \varepsilon\to 0^+}} f(r,\varepsilon) = \lim_{\substack{r\to 1^+\\ \varepsilon\to 0^+}} \left[\frac{1}{\ln(r-1+\varepsilon)} - \frac{1}{\ln\varepsilon}\right] = 0 = f(1,0).$$

Then it is easy to see that $f(r,\varepsilon)$ is continuous in $\{(r,\varepsilon) \mid r > 0,\ |\varepsilon| < 1\}$. In fact we have, for $\varepsilon^* > 0$

and $0 < r^* < 1$:

$$\lim_{\substack{r\to 1^-\\ \varepsilon\to\varepsilon^*}} f(r,\varepsilon) = \lim_{\substack{\rho\to 1^+\\ \varepsilon\to\varepsilon^*}} -f(\rho,\varepsilon) = 0 = f(1,\varepsilon^*),$$
$$\lim_{\substack{r\to r^*\\ \varepsilon\to 0^+}} f(r,\varepsilon) = \lim_{\substack{\rho\to \frac{1}{r^*}\\ \varepsilon\to 0^+}} -f(\rho,\varepsilon) = -f\left(\frac{1}{r^*},0\right) = f(r^*,0),$$
$$\lim_{\substack{r\to 1^-\\ \varepsilon\to 0^+}} f(r,\varepsilon) = \lim_{\substack{\rho\to 1^+\\ \varepsilon\to 0^+}} -f(\rho,\varepsilon) = 0 = f(1,0),$$

and, since $f(r,\varepsilon) = f(r,-\varepsilon)$, the same conclusions holds even when $r \to r^* > 0$ and $\varepsilon \to -\varepsilon^* < 0$ or $\varepsilon \to 0^-$.

B. $\varepsilon f_r(r,\varepsilon)$ *is continuous.* Clearly $f_r(r,\varepsilon)$ is continuous in $\Omega = \{(r,\varepsilon) \mid 0 < r \neq 1,\ \varepsilon > 0\}$. Noting that for $r > 1$, $\varepsilon > 0$ we have:

$$\varepsilon f_r(r,\varepsilon) = -\frac{\varepsilon}{(r-1+\varepsilon)\ln^2(r-1+\varepsilon)},$$

it follows that $\varepsilon f_r(r,\varepsilon)$ is continuous in U as defined above since if $r^* > 1$ and $0 < \varepsilon^* < 1$:

$$\lim_{\substack{r\to r^*\\ \varepsilon\to 0^+}} \varepsilon f_r(r,\varepsilon) = -\lim_{\substack{r\to r^*\\ \varepsilon\to 0^+}} \frac{\varepsilon}{(r-1+\varepsilon)}\frac{1}{\ln^2(r-1+\varepsilon)} = 0 = \frac{\partial}{\partial r}[\varepsilon f(r,\varepsilon)]_{|r=r^*,\varepsilon=0},$$
$$\lim_{\substack{r\to 1^+\\ \varepsilon\to\varepsilon^*}} \varepsilon f_r(r,\varepsilon) = -\frac{1}{\ln^2\varepsilon^*},$$

where the latter equals $\varepsilon^* f_r(1,\varepsilon^*)$ since

$$\lim_{r\to 0^+} \frac{f(r+1,\varepsilon^*) - f(1,\varepsilon^*)}{r} = \lim_{r\to 0^+} \frac{f(r+1,\varepsilon^*)}{r} = \lim_{r\to 0^+} \frac{1}{r}\left[\frac{1}{\ln(r+\varepsilon^*)} - \frac{1}{\ln\varepsilon^*}\right]$$
$$= \frac{d}{dr}\left[\frac{1}{\ln(r+\varepsilon^*)}\right]_{|r=0} = -\frac{1}{\varepsilon^*\ln^2\varepsilon^*} \quad \text{and similarly}$$
$$\lim_{r\to 0^-} \frac{f(r+1,\varepsilon^*) - f(1,\varepsilon^*)}{r} = -\frac{1}{\varepsilon^*\ln^2\varepsilon^*};$$

also

$$\lim_{\substack{r\to 1^+\\ \varepsilon\to 0^+}} \varepsilon f_r(r,\varepsilon) = -\lim_{\substack{r\to 1^+\\ \varepsilon\to 0^+}} \frac{\varepsilon}{(r-1+\varepsilon)\ln^2(r-1+\varepsilon)} = 0$$

since, for $r > 1$,

$$\left|\frac{\varepsilon}{r-1+\varepsilon}\right| \leq 1 \quad \text{and} \quad \lim_{\substack{r\to 1^+\\ \varepsilon\to 0^+}} \frac{1}{\ln^2(r-1+\varepsilon)} = 0.$$

Then $\varepsilon f_r(r,\varepsilon)$ is continuous in $\{(r,\varepsilon) \mid r > 0, |\varepsilon| < 1\}$ since for $0 < r^* < 1$ and $\varepsilon^* > 0$

$$\lim_{\substack{r\to 1^-\\ \varepsilon\to\varepsilon^*}} \varepsilon f_r(r,\varepsilon) = \lim_{\substack{\rho\to 1^+\\ \varepsilon\to\varepsilon^*}} \varepsilon\rho^2 f_r(\rho,\varepsilon) = \varepsilon^* f_r(1,\varepsilon^*)$$
$$\lim_{\substack{r\to r^*\\ \varepsilon\to 0^+}} \varepsilon f_r(r,\varepsilon) = \lim_{\substack{\rho\to \frac{1}{r^*}\\ \varepsilon\to 0}} \varepsilon\rho^2 f_r(\rho,\varepsilon) = 0$$
$$\lim_{\substack{r\to 1^-\\ \varepsilon\to 0^+}} \varepsilon f_r(r,\varepsilon) = \lim_{\substack{\rho\to 1^+\\ \varepsilon\to 0^+}} \varepsilon\rho^2 f_r(\rho,\varepsilon) = 0$$

and since, because $\varepsilon f(r,\varepsilon)$ is an odd function of ε, the same conclusion holds for $r \to r^* > 0$ and $\varepsilon \to -\varepsilon^* < 0$ or $\varepsilon \to 0^-$.

C. $\frac{\partial}{\partial\varepsilon}[\varepsilon f(r,\varepsilon)]$ *is continuous.* We have

$$\frac{\partial}{\partial\varepsilon}[\varepsilon f(r,\varepsilon)] = \begin{cases} f(r,|\varepsilon|) + |\varepsilon| f_\varepsilon(r,|\varepsilon|) & \text{if } r > 1 \text{ and } \varepsilon \neq 0 \\ -f(\frac{1}{r},|\varepsilon|) - |\varepsilon| f_\varepsilon(\frac{1}{r},|\varepsilon|) & \text{if } r < 1 \text{ and } \varepsilon \neq 0 \end{cases}$$

and, for all $r^* > 0$, $\varepsilon^* \neq 0$:

$$\frac{\partial}{\partial\varepsilon}[\varepsilon f(r^*,\varepsilon)]_{|\varepsilon=0} = \lim_{\varepsilon\to 0} f(r^*,\varepsilon) = f(r^*,0)$$

$$\frac{\partial}{\partial\varepsilon}[\varepsilon f(1,\varepsilon)]_{|\varepsilon=\varepsilon^*} = \frac{\partial}{\partial\varepsilon}[0]_{|\varepsilon=\varepsilon^*} = 0.$$

We see that it is enough to prove that $\frac{\partial}{\partial\varepsilon}[\varepsilon f(r,\varepsilon)]$ is continuous in $U = \{(r,\varepsilon) \mid r \geq 1,\ \varepsilon \geq 0\}$. For $r > 1$ and $\varepsilon > 0$ we have

$$\frac{\partial}{\partial\varepsilon}[\varepsilon f(r,\varepsilon)] = f(r,\varepsilon) + \varepsilon f_\varepsilon(r,\varepsilon)$$

where

$$\varepsilon f_\varepsilon(r,\varepsilon) = \frac{1}{\ln^2\varepsilon} - \frac{\varepsilon}{(r-1+\varepsilon)\ln^2(r-1+\varepsilon)} = \frac{1}{\ln^2\varepsilon} + \varepsilon f_r(r,\varepsilon)$$

so that $\frac{\partial}{\partial\varepsilon}[\varepsilon f(r,\varepsilon)]$ is continuous in the interior of U. On the boundary of U,

$$\lim_{\substack{r\to r^* \\ \varepsilon\to\varepsilon^*}} \frac{\partial}{\partial\varepsilon}[\varepsilon f(r,\varepsilon)] = \begin{cases} f(r^*,0) & \text{if } r^* > 1 \text{ and } \varepsilon^* = 0 \\ 0 & \text{if } r^* = 1 \text{ and } \varepsilon^* \geq 0 \end{cases}$$

which equals $\frac{\partial}{\partial\varepsilon}[\varepsilon f(r,\varepsilon)]_{|\substack{r=r^* \\ \varepsilon=\varepsilon^*}}$.

4 A Nemyckii operator

In order to prove Proposition 1, we need to discuss the smoothness of a certain Nemyckii operator. It turns out that, in general, the order of smoothness of this operator is one less than that of the function used to define it. This is the technical reason the asymptotic phase is, in general, less smooth in the parameter than the vectorfield.

Let I be an interval of $\mathbb{R}$. For $\gamma > 0$, $C^0_\gamma(I,\mathbb{R}^n)$ denotes the Banach space of continuous functions $z : I \to \mathbb{R}^n$ such that

$$||z||_\gamma := \sup_{t\in I} |z(t)| e^{\gamma|t|} < \infty$$

with the norm $||z||_\gamma$. We denote by $B(\Delta)$ the open ball in $C^0_\gamma(\mathbb{R}_+,\mathbb{R}^n)$ with centre 0 and radius Δ, where $\mathbb{R}_+ = [0,\infty)$.

Lemma 4.1 *Let $\Omega(\Delta)$ be the open ball in $\mathbb{R}^n$ with centre 0 and radius Δ. For $r \geq 1$, let $h : \mathbb{R}_+ \times \Omega(\Delta) \times \{|\varepsilon| < \bar{\varepsilon}\} \to \mathbb{R}^n$ be a function such that $h(t,x,\varepsilon)$ and its derivatives up to order r with respect to (x,ε) are bounded and uniformly continuous in (t,x,ε); suppose also that*

$$h(t,0,\varepsilon) = 0.$$

Then the map $\mathcal{H} : B(\Delta) \to C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)$ *given by*

$$\mathcal{H}(x, \varepsilon)(t) := h(t, x(t), \varepsilon)$$

is C^{r-1}. *Moreover its* $(r-1)$*–order derivatives are differentiable with respect to* x *and the derivatives are continuous in* (x, ε).

Proof. For simplicity, we assume ε is scalar. In the whole proof we assume that $x, x_1, x_2 \in \Omega(\Delta)$ and $|\varepsilon|, |\varepsilon_1|, |\varepsilon_2|$ are all $< \bar{\varepsilon}$. Also M denotes an upper bound for the norms of all derivatives of h and $\omega(\cdot)$ denotes an upper bound for the modulus of continuity of a derivative so that, for example,

$$\sup\{|h_x(t, x_2, \varepsilon_2) - h_x(t, x_1, \varepsilon_1)| : t \geq 0\} \leq \omega(|\varepsilon_2 - \varepsilon_1| + |x_2 - x_1|)$$

(here $x_1, x_2 \in \Omega(\Delta)$). Note that $\omega(\rho) \to 0$ as $\rho \to 0$. We claim the following holds:

i) for any nonnegative integers p, q such that either $1 \leq p \leq p + q = r$ or $(p, q) = (0, r - 1)$, the map $(x, \varepsilon) \mapsto L_{p,q}(x, \varepsilon)$ defined by

$$L_{p,q}(x, \varepsilon)(\zeta_1, \ldots, \zeta_p)(t) = h_{x^p\varepsilon^q}(t, x(t), \varepsilon)\zeta_1(t) \ldots \zeta_p(t)$$

is a continuous linear map from $B(\Delta) \times \{|\varepsilon| < \bar{\varepsilon}\}$ into the space of bounded multilinear maps from $[C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)]^p$ into $C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)$ when $p \geq 1$ and a function in $C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)$ when $p = 0$.

First note that $L_{p,q}$ is indeed a bounded multilinear map for $1 \leq p \leq p + q = r$ since

$$||L_{p,q}(x, \varepsilon)(\zeta_1, \ldots, \zeta_p)||_\gamma \leq M||\zeta_1||_\gamma \ldots ||\zeta_p||_\gamma$$

and $L_{0,r-1}$ is in $C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)$ since

$$\begin{aligned}||L_{0,r-1}(x, \varepsilon)||_\gamma &= ||h_{\varepsilon^{r-1}}(\cdot, x(\cdot), \varepsilon) - h_{\varepsilon^{r-1}}(\cdot, 0, \varepsilon)||_\gamma \\ &\leq \sup_{t\geq 0} \int_0^1 |h_{x\varepsilon^{r-1}}(t, \theta x(t), \varepsilon)|d\theta ||x||_\gamma \leq M||x||_\gamma.\end{aligned}$$

Now, we prove the continuity of $L_{p,q}$. To this end we observe first that, for $1 \leq p \leq p + q = r$:

$$|h_{x^p\varepsilon^q}(t, x_2, \varepsilon_2) - h_{x^p\varepsilon^q}(t, x_1, \varepsilon_1)| \leq \omega(|x_2 - x_1| + |\varepsilon_2 - \varepsilon_1|)$$

and hence:

$$\begin{aligned}&||[L_{p,q}(x_2, \varepsilon_2) - L_{p,q}(x_1, \varepsilon_1)](\zeta_1 \ldots \zeta_p)||_\gamma \\ &\leq \omega(||x_2 - x_1||_\gamma + |\varepsilon_2 - \varepsilon_1|)||\zeta_1||_\gamma \ldots ||\zeta_p||_\gamma.\end{aligned} \tag{4.1}$$

Next, if $(p, q) = (0, r - 1)$ we have

$$\begin{aligned}&|h_{\varepsilon^{r-1}}(t, x_2, \varepsilon_2) - h_{\varepsilon^{r-1}}(t, x_1, \varepsilon_1)| \leq |h_{\varepsilon^{r-1}}(t, x_2, \varepsilon_2) - h_{\varepsilon^{r-1}}(t, x_1, \varepsilon_2)| \\ &+|h_{\varepsilon^{r-1}}(t, x_1, \varepsilon_2) - h_{\varepsilon^{r-1}}(t, 0, \varepsilon_2) - [h_{\varepsilon^{r-1}}(t, x_1, \varepsilon_1) - h_{\varepsilon^{r-1}}(t, 0, \varepsilon_1)]| \\ &\leq M|x_2 - x_1| + \int_0^1 |h_{x\varepsilon^{r-1}}(t, \theta x_1, \varepsilon_2) - h_{x\varepsilon^{r-1}}(t, \theta x_1, \varepsilon_1)|d\theta|x_1| \\ &\leq M|x_2 - x_1| + \omega(|\varepsilon_2 - \varepsilon_1|)|x_1|.\end{aligned}$$

As a consequence:

$$\|L_{0,r-1}(x_2,\varepsilon_2) - L_{0,r-1}(x_1,\varepsilon_1)\|_\gamma \le M\|x_2 - x_1\|_\gamma + \omega(|\varepsilon_2 - \varepsilon_1|)\|x_1\|_\gamma. \tag{4.2}$$

From (4.1), (4.2) the continuity of $L_{p,q}$ as stated in i) follows.

If, in the claim i), we take $r = 1$ we see that $[\mathcal{H}(x,\varepsilon)](t) = h(t,x(t),\varepsilon) = L_{0,0}(x,\varepsilon)$ is a continuous map from $B(\Delta)$ to $C^0_\gamma(\mathbb{R}_+,\mathbb{R}^n)$. Now assuming $r = 1$ we prove that the derivative $D_x\mathcal{H}(x,\varepsilon)$ exists and is continuous in (x,ε). First we have

$$\begin{aligned}
&\sup_{t\ge 0} |h(t,x(t)+\zeta(t),\varepsilon) - h(t,x(t),\varepsilon) - h_x(t,x(t),\varepsilon)\zeta(t)|e^{\gamma t}\\
&\le \sup_{t\ge 0}\int_0^1 |h_x(t,x(t)+\theta\zeta(t),\varepsilon) - h_x(t,x(t),\varepsilon)|d\theta|\,|\zeta(t)|e^{\gamma t}\\
&\le \omega(\|\zeta\|_\gamma)\|\zeta\|_\gamma e^{\gamma t}
\end{aligned}$$

so that

$$[D_x\mathcal{H}(x,\varepsilon)\zeta](t) = h_x(t,x(t),\varepsilon)\zeta(t).$$

Then the continuity of $D_x\mathcal{H}(x,\varepsilon)$ follows from i) with $(p,q) = (1,0)$. This completes the proof of the Lemma when $r = 1$.

Now assume that $r \ge 2$. To prove that $\mathcal{H}$ has the stated properties it is enough to prove that

ii) for any nonnegative integers p,q such that either $1 \le p \le p+q = r$ or $(p,q) = (0,r-1)$, the derivative $\mathcal{H}^{(p,q)}(x,\varepsilon)$ exists and satisfies

$$\mathcal{H}^{(p,q)}(x,\varepsilon)(\zeta_1\ldots\zeta_p) = L_{p,q}(x,\varepsilon)(\zeta_1\ldots\zeta_p).$$

Note that we have already proved the statement when $r = 1$. So we assume ii) holds with $r-1 \ge 1$ instead of r and prove it for r. To this end it is then enough to prove that for any pair (p,q) such that either $1 \le p \le p+q = r-1$ or $(p,q) = (0,r-2)$ we have

$$D_x[\mathcal{H}^{(p,q)}(x,\varepsilon)](\zeta_1,\ldots,\zeta_{p+1}) = L_{p+1,q}(x,\varepsilon)(\zeta_1\ldots\zeta_{p+1})$$

and

$$D_\varepsilon[\mathcal{H}^{(p,q)}(x,\varepsilon)](\zeta_1,\ldots,\zeta_p) = L_{p,q+1}(x,\varepsilon)(\zeta_1\ldots\zeta_p).$$

Now, in the first case we have:

$$\begin{aligned}
&\|[\mathcal{H}^{(p,q)}(x+\zeta_{p+1},\varepsilon) - \mathcal{H}^{(p,q)}(x,\varepsilon)](\zeta_1,\ldots,\zeta_p) - L_{p+1,q}(x,\varepsilon)(\zeta_1,\ldots,\zeta_{p+1})\|_\gamma\\
&\le \sup_{t\ge 0}\int_0^1 |h_{x^{p+1}\varepsilon^q}(t,x(t)+\theta\zeta_{p+1}(t),\varepsilon) - h_{x^{p+1}\varepsilon^q}(t,x(t),\varepsilon)|d\theta\cdot\\
&\qquad\cdot|\zeta_1(t)|\ldots|\zeta_{p+1}(t)|e^{\gamma t}\\
&\le \omega(\|\zeta_{p+1}\|_\gamma)\|\zeta_1\|_\gamma\ldots\|\zeta_p\|_\gamma\|\zeta_{p+1}\|_\gamma.
\end{aligned}$$

This proves that $D_x\mathcal{H}^{(p,q)}(x,\varepsilon) = L_{p+1,q}(x,\varepsilon)$. To show the second equality we distinguish the two cases $p = 0$ and $p > 0$. When $p = 0$ we have (using $h(t, 0, \varepsilon) = 0$):

$$\begin{aligned}
&\|\mathcal{H}^{(0,q)}(x,\varepsilon+\tilde{\varepsilon}) - \mathcal{H}^{(0,q)}(x,\varepsilon) - L_{0,q+1}(x,\varepsilon)\tilde{\varepsilon}\|_\gamma \\
&\le \sup_{t\ge 0}\int_0^1 |h_{\varepsilon^{q+1}}(t,x(t),\varepsilon+\theta\tilde{\varepsilon}) - h_{\varepsilon^{q+1}}(t,x(t),\varepsilon)|d\theta\,|\tilde{\varepsilon}|e^{\gamma t} \\
&\le \sup_{t\ge 0}\int_0^1\int_0^1 |h_{x\varepsilon^{q+1}}(t,\mu x(t),\varepsilon+\theta\tilde{\varepsilon}) - h_{x\varepsilon^{q+1}}(t,\mu x(t),\varepsilon)|d\mu d\theta\,|\tilde{\varepsilon}|\|x\|_\gamma \\
&\le \omega(|\tilde{\varepsilon}|)|\tilde{\varepsilon}|\|x\|_\gamma
\end{aligned}$$

from which the claim follows. If, instead, $p > 0$ we have:

$$\begin{aligned}
&\|[\mathcal{H}^{(p,q)}(x,\varepsilon+\tilde{\varepsilon}) - \mathcal{H}^{(p,q)}(x,\varepsilon) - L_{p,q+1}(x,\varepsilon)\tilde{\varepsilon}](\zeta_1,\ldots,\zeta_p)\|_\gamma \\
&\le \sup_{t\ge 0}\int_0^1 |h_{x^p\varepsilon^{q+1}}(t,x(t),\varepsilon+\theta\tilde{\varepsilon}) - h_{x^p\varepsilon^{q+1}}(t,x(t),\varepsilon)|d\theta\,\|\zeta_1\|_\gamma\ldots\|\zeta_p\|_\gamma\,|\tilde{\varepsilon}| \\
&\le \omega(|\tilde{\varepsilon}|)|\tilde{\varepsilon}|\|\zeta_1\|_\gamma\ldots\|\zeta_p\|_\gamma
\end{aligned}$$

from which the claim follows. This completes the proof of Lemma 4.1. ■

Remark 4.1 *i) From the proof of Lemma 4.1 we see that if the rth order derivatives of* $h(t,x,\varepsilon)$ *are Lipschitz-continuous in* (x,ε) *uniformly with respect to t, the* $(r-1)$*th derivative of* $\mathcal{H} : B(\Delta)\times\{|\varepsilon| < \bar{\varepsilon}\} \to C^0_\gamma(\mathbb{R}_+,\mathbb{R}^n)$ *is Lipschitz in* $(x,\varepsilon) \in B(\Delta)\times\{|\varepsilon| < \bar{\varepsilon}\}$.

ii) The proof of Lemma 4.1 goes through under the weaker assumption that the moduli of continuity of $h(t,x,\varepsilon)$ *and its derivatives are well defined. So that all we need is that* $h(t,x,\varepsilon)$ *and its derivatives are continuous in* (t,x,ε) *and uniformly continuous in* (x,ε) *uniformly with respect to t.*

5 Proof of Proposition 1

For the proof of Proposition 1, we need two lemmas.

Lemma 5.1 *Suppose the linear system*

$$\dot{x} = A(t)x \tag{5.1}$$

has a trichotomy on $[0,\infty)$ *with projections* P_0, P_+, P_- *and constants K,* α *and let* $0 < \gamma < \alpha$. *Suppose* $\xi \in \mathcal{R}P_+$ *and* $h(t)$ *is a continuous function with* $\|h\|_\gamma = \sup_{t\ge 0}|h(t)|e^{\gamma t} < \infty$. *Then*

$$x(t) = X(t)\xi + \int_0^t X(t)P_+X^{-1}(s)h(s)ds - \int_t^\infty X(t)(I-P_+)X^{-1}(s)h(s)ds$$

is the unique solution of

$$\dot{x} = A(t)x + h(t) \tag{5.2}$$

such that $\|x\|_\gamma < \infty$ *and* $P_+x(0) = \xi$. *Moreover*

$$\|x\|_\gamma \le K|\xi| + K[(\alpha-\gamma)^{-1} + 2\gamma^{-1}]\|h\|_\gamma.$$

Proof. See the proof of Lemma 3.6 in [8]. ■

From Lemmas 4.1 and 5.1, we deduce another.

Lemma 5.2 *Suppose the linear system* (5.1) *satisfies the conditions of Lemma 5.1 and let* $0 < \gamma < \alpha$. *Let* $h(t,x,\varepsilon)$ *be a continuous function with period* $T > 0$ *in* t *satisfying*

$$h(t,0,\varepsilon) = 0, \quad |h(t,x_1,\varepsilon) - h(t,x_2,\varepsilon)| \le L|x_1 - x_2|$$

for all $t \ge 0$, $|x_1| \le \Delta$, $|x_2| \le \Delta$, $|\varepsilon| < \varepsilon_0$. *Suppose that*

$$K(\gamma)L = K[(\alpha-\gamma)^{-1} + 2\gamma^{-1}]L < 1.$$

Then if $|\varepsilon| < \varepsilon_0$ *and* $\xi \in \mathcal{R}P_+$ *with*

$$K|\xi| < (1 - K(\gamma)L)\Delta, \tag{5.3}$$

the equation

$$\dot{x} = A(t)x + h(t,x,\varepsilon)$$

has a unique solution $x(t) = x(t,\xi,\varepsilon)$ *such that* $\|x\|_\gamma \le \Delta$ *and* $P_+x(0) = \xi$. *Moreover*

$$x(t,0,\varepsilon) = 0$$

and

$$\|x\|_\gamma \le K(1 - K(\gamma)L)^{-1}|\xi|. \tag{5.4}$$

Also, if $h(t,x,\varepsilon)$ *and its derivatives with respect to* (x,ε) *up to order* r *are bounded and uniformly continuous uniformly with respect to* t, *then* $x(t,\xi,\varepsilon)$ *is* C^{r-1} *in* (ξ,ε) *and is continuous in* (t,ξ,ε) *together with all its derivatives; also the rth derivative with respect to* ξ *exists and is continuous in* (t,ξ,ε). *Moreover, for* $\tilde{\xi} \in \mathcal{R}P_+$, $x_\xi(t,\xi,\varepsilon)\tilde{\xi}$ *is the unique solution of*

$$\dot{z} = [A(t) + h_x(t,x(t,\xi,\varepsilon),\varepsilon)]z \tag{5.5}$$

such that $\|z\|_\gamma < \infty$ *and* $P_+z(0) = \tilde{\xi}$.

Proof. Consider the Banach space $E := C^0_\gamma(\mathbb{R}_+,\mathbb{R}^n)$ of continuous functions $x(t)$ on $[0,\infty)$ with $\|x\|_\gamma < \infty$, with $\|x\|_\gamma$ as norm. Let $B = \bar{B}(\Delta)$ be the closed ball of radius Δ and centre 0. Then B is a complete metric space. Next suppose $|\varepsilon| < \varepsilon_0$ and $\xi \in \mathcal{R}P_+$ satisfies (5.3). Then we define a mapping $\mathcal{T} : B \to B$ as follows: if $x \in B$, we let $\mathcal{T}(x)(t)$ be the unique solution of

$$\dot{x} = A(t)x + h(t,x(t),\varepsilon)$$

such that $P_+\mathcal{T}(x)(0) = \xi$ and $\|\mathcal{T}(x)\|_\gamma < \infty$. This exists by Lemma 5.1 and it also follows from that Lemma that

$$\|\mathcal{T}(x)\|_\gamma \le K|\xi| + K(\gamma)L\|x\|_\gamma \le K|\xi| + K(\gamma)L\Delta \le \Delta. \tag{5.6}$$

Next if x_1 and x_2 are in B, it follows by similar reasoning that

$$\|\mathcal{T}(x_1) - \mathcal{T}(x_2)\|_\gamma \le K(\gamma)L\|x_1 - x_2\|_\gamma. \tag{5.7}$$

Since $K(\gamma)L < 1$, $\mathcal{T}$ is a contraction and so has a unique fixed point $x(t) = x(t, \xi, \varepsilon)$. That $x(t) = 0$ when $\xi = 0$ follows by uniqueness and the inequality (5.4) follows from the first inequality in (5.6) by putting $\mathcal{T}x = x$.

To prove the smoothness, we write $\mathcal{T}(x)$ as $\mathcal{T}(x, \xi, \varepsilon)$ to indicate the dependence on ξ and ε. Note that if $x \in B$

$$\begin{aligned} \mathcal{T}(x, \xi, \varepsilon)(t) &= X(t)\xi + \int_0^t X(t)P_+X^{-1}(s)h(s, x(s), \varepsilon)ds \\ &\quad - \int_t^\infty X(t)(I - P_+)X^{-1}(s)h(s, x(s), \varepsilon)ds. \end{aligned} \tag{5.8}$$

From Lemma 4.1 we know that the map

$$B \times \{|\varepsilon| < \varepsilon_0\} \ni (x, \varepsilon) \mapsto [\mathcal{H}(x, \varepsilon)](t) := h(t, x(t), \varepsilon) \in C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)$$

is C^{r-1}, and that the rth derivative of $\mathcal{H}$ with respect to x exists and is a continuous function of (x, ε). Moreover it is not hard to see that

$$\xi \mapsto X(t)\xi, \quad \zeta(t) \mapsto \int_0^t X(t)P_+X^{-1}(s)\zeta(s)ds$$

and

$$\zeta(t) \mapsto \int_t^\infty X(t)(I - P_+)X^{-1}(s)\zeta(s)ds$$

are linear and bounded maps from $\mathcal{R}P_+$ (resp. $C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)$) to $C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)$ and hence C^∞. Thus:

$$\mathcal{T} : B \times \mathcal{R}P_+ \times \{|\varepsilon| < \varepsilon_0\} \to B$$

is C^{r-1} and its rth derivative with respect to (x, ξ) exists and is a continuous function of (x, ξ, ε).

Since the fixed point of a uniform contraction inherits the same smoothness property as the map, we conclude that $(\xi, \varepsilon) \mapsto x(\cdot, \xi, \varepsilon) \in \bar{B}(\Delta)$ is C^{r-1} and hence the derivatives of $x(\cdot, \xi, \varepsilon)$ with respect to ξ and ε up to the order $r - 1$ are elements of $C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)$; also the rth derivative with respect to ξ exists, is a continuous function of (ξ, ε) and is likewise an element of $C^0_\gamma(\mathbb{R}_+, \mathbb{R}^n)$.

Since $x(\xi, \varepsilon)$ is a fixed point of $\mathcal{T}$, it follows from (5.8) that

$$\begin{aligned} x(t, \xi, \varepsilon) &= X(t)\xi + \int_0^t X(t)P_+X^{-1}(s)h(s, x(s, \xi, \varepsilon), \varepsilon)ds \\ &\quad - \int_t^\infty X(t)(I - P_+)X^{-1}(s)h(s, x(s, \xi, \varepsilon), \varepsilon)ds. \end{aligned}$$

So if we write $x_\xi(t, \xi, \varepsilon) = Z(t)$, we see that for $\tilde{\xi} \in \mathcal{R}P_+$,

$$\begin{aligned} Z(t)\tilde{\xi} &= X(t)\tilde{\xi} + \int_0^t X(t)P_+X^{-1}(s)h_x(s, x(s, \xi, \varepsilon), \varepsilon)Z(s)\tilde{\xi}ds \\ &\quad - \int_t^\infty X(t)(I - P_+)X^{-1}(s)h_x(s, x(s, \xi, \varepsilon), \varepsilon)Z(s)\tilde{\xi}ds. \end{aligned}$$

This means that for $\tilde{\xi} \in \mathcal{R}P_+$, $z(t) = x_\xi(t, \xi, \varepsilon)\tilde{\xi}$ is a solution of (5.5) such that $||z||_\gamma < \infty$ and $P_+z(0) = \tilde{\xi}$. It is unique because the difference $z(t)$ of any two such solutions would be a solution of (5.5) such that $||z||_\gamma < \infty$ and $P_+z(0) = 0$. Then by Lemma 2, $||z||_\gamma \le K(\gamma)L||z||_\gamma$ so that $z = 0$. ■

Proof of Proposition 1.

(i) is proved as in the proof of Proposition 3.5 in [8] and (ii) follows from standard theorems. In fact our Proposition 1 is essentially a variant of Proposition 3.5 in [8] but here we have sharpened the conclusions about the smoothness.

Now if we change the time scale $\tau = \omega(\varepsilon)t$, where $\omega(\varepsilon) = T/T(\varepsilon)$, then the new system

$$\frac{dz}{d\tau} = \omega(\varepsilon)^{-1}F(z, \varepsilon) \tag{5.9}$$

has the periodic orbit $\hat{u}(\tau, \varepsilon) := u(\omega(\varepsilon)^{-1}\tau, \varepsilon)$ which has period T for all ε. Next, assume we can prove that (5.9) has a solution $\hat{z}(\tau, \xi, \varepsilon)$ satisfying the conclusions of Proposition 2.1 with $\hat{\gamma}$ instead of γ. Then $z(t, \xi, \varepsilon) = \hat{z}(\omega(\varepsilon)t, \varepsilon, \xi, \varepsilon)$ satisfies (1.1) together with the conclusions of Proposition 2.1 with $\hat{\gamma}\omega(\varepsilon)$ instead of $\hat{\gamma}$. Since $\omega(\varepsilon) \to 1$ as $\varepsilon \to 0$ we simply need to take $0 < \gamma = \sup\{\hat{\gamma}\omega(\varepsilon)^{-1} \mid |\varepsilon| < \varepsilon_0\} < \alpha$. So we can assume without loss of generality that in the original system the period of $u(t, \varepsilon)$ is T, independent of ε.

To prove (iii), we look for solutions $z(t)$ of (1.1) such that $|z(t) - u(t, \varepsilon)| \to 0$ at an exponential rate as $t \to \infty$. We write

$$z(t) = u(t, \varepsilon) + x(t).$$

Then

$$\dot{x} = A(t)x + h(t, x, \varepsilon), \tag{5.10}$$

where

$$A(t) = F_z(u_0(t), 0), \quad h(t, x, \varepsilon) = F(u(t, \varepsilon) + x, \varepsilon) - F(u(t, \varepsilon), \varepsilon) - F_z(u_0(t), 0)x.$$

Note that h is a C^r−function, has period T in t, $h(t, 0, \varepsilon) = 0$ and for $|x_1|, |x_2| \le \Delta$,

$$\begin{aligned} &|h(t, x_2, \varepsilon) - h(t, x_1, \varepsilon)| \\ &\le \int_0^1 |F_z(u(t, \varepsilon) + \theta x_2 + (1 - \theta)x_1, \varepsilon) - F_z(u(t, 0), 0)|d\theta|x_2 - x_1| \\ &\le L(\Delta, \varepsilon)|x_2 - x_1| \end{aligned}$$

where

$$L(\Delta, \varepsilon) := \sup\{|F_z(u(t, \varepsilon) + x, \varepsilon) - F_z(u(t, 0), 0)| \mid t \in [0, T], |x| \le \Delta\}$$

is a continuous function nondecreasing in both arguments which $\to 0$ as $(\Delta, |\varepsilon|) \to (0, 0)$.

We fix γ in $0 < \gamma < \alpha$. Given small $\xi \in \mathcal{R}(P_+)$, we look for solutions $x(t)$ of (5.10) such that

$$P_+x(0) = \xi, \quad |x(t)| \le \Delta e^{-\gamma t} \ (t \ge 0),$$

where we assume Δ is chosen so small that

$$K(\gamma)L(\Delta, 0) < 1.$$

Then we take ε_0 as the largest value such that

$$K(\gamma)L(\Delta, \varepsilon) < 1 \quad \text{for } |\varepsilon| < \varepsilon_0$$

and

$$\xi_0 = K^{-1}(1 - K(\gamma)L)\Delta.$$

According to Lemma 5.2, for $|\varepsilon| < \varepsilon_0$ and $|\xi| < \xi_0$, equation (5.10) has a unique solution $x(t) = x(t, \xi, \varepsilon)$ with $\|x\|_\gamma \leq \Delta$ and $P_+x(0) = \xi$, and the map $(\xi, \varepsilon) \mapsto x(\cdot, \xi, \varepsilon) \in B(\Delta)$ is C^{r-1} and the rth derivative with respect to ξ exists and is a continuous function of (ξ, ε). Lemma 5.2 also tells us that for $\tilde{\xi} \in \mathcal{R}P_+$, $z(t) = x_\xi(t, \xi, \varepsilon)\tilde{\xi}$ is the unique solution of

$$\dot{z} = [A(t) + h_x(t, x(t, \xi, \varepsilon), \varepsilon)]z = F_z(u(t, \varepsilon) + x(t, \xi, \varepsilon), \varepsilon)z$$

such that $\sup_{t \geq 0} |z(t)|e^{\gamma t} < \infty$ and $P_+z(0) = \tilde{\xi}$.

Next we note that since here $A(t) = F_z(u_0(t), 0)$ and $h(t, x, \varepsilon)$ are C^r, the corresponding nonautonomous flow has the same property and hence since $x(0, \xi, \varepsilon)$ is a C^{r-1}–function of (ξ, ε), $x(t, \xi, \varepsilon)$ is a C^{r-1}–function of (t, ξ, ε); also since the $(r-1)$th derivatives of $x(0, \xi, \varepsilon)$ are differentiable with respect to ξ and the derivatives are continuous functions of (ξ, ε), the $(r-1)$th derivatives of $x(t, \xi, \varepsilon)$ are differentiable with respect to (t, ξ) and the derivatives are continuous functions of (t, ξ, ε). Finally if we define

$$z^+(t, \xi, \varepsilon) = u(t, \varepsilon) + x(t, \xi, \varepsilon)$$

we see that (iii) and hence Proposition 2.1 has been proved. ■

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