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Global Bifurcations of Critical Orbits via Equivariant Conley Index

Sławomir Rybicki*

Faculty of Mathematics and Computer Science
Nicolaus Copernicus University of Toruń
87-100 Toruń ul. Chopina 12/18, Poland
e-mail: rybicki@mat.umk.pl

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Abstract

Let $\mathbb{V}$ be a finite-dimensional representation of a compact connected Lie group G and let $\varphi : \mathbb{V} \times [\lambda_1, \lambda_2] \to \mathbb{R}$ be a family of G-invariant potentials of the class C^2 such that $\nabla_v \varphi(0, \lambda) = 0$ for every $\lambda \in [\lambda_1, \lambda_2]$, where $\nabla_v \varphi$ is the gradient of φ with respect to the first coordinate. Denote by $\mathbb{V}_i'$ a direct sum of eigenspaces of the isomorphism $\nabla_v^2 \varphi(0, \lambda_i)$ corresponding to positive eigenvalues, $i = 1, 2$. We have proved that if $\dim \mathbb{V}_1' \geqq \dim \mathbb{V}_2'$ and the representation $\mathbb{V}_1'$ is not equivalent (as a linear representation of G) to the representation $\mathbb{V}_2' \oplus \mathbb{R}[2k], k \in \{0\} \cup \mathbb{N}$ (where $\mathbb{R}[2k]$ is a $2k$-dimensional trivial representation of G), then there is a global bifurcation of G-orbits of non-zero solutions of problem $\nabla_v \varphi(v, \lambda) = 0$ from the interval $\{0\} \times [\lambda_1, \lambda_2]$. We have proved that a global bifurcation and symmetry-breaking of solutions phenomenon occurs simultaneously for elliptic differential equations considered by Smoller and Wasserman in [25], which improves results of [25].

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1 Introduction

Let $\varphi \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}))$ be such that $\nabla_x \varphi(0, \lambda) = 0$ for every $\lambda \in \mathbb{R}$, where $\nabla_x \varphi$ is the gradient of φ with respect to the first coordinate. We study bifurcations of non-zero critical points of the potential φ from the set $\mathcal{T} = \{0\} \times \mathbb{R}$ i.e. we study a bifurcation problem

$$\nabla_x \varphi(x, \lambda) = 0. \tag{1.1}$$

We proceed in a natural way i.e topological invariants are computed and compared at different levels of the trivial family $\mathcal{T}$. A change of the Conley index (the Brouwer degree) implies a local (global) bifurcation of non-zero solutions of equation (1.1). For a formal definition of a local (global) bifurcation point and a branching point of non-zero solutions of equation (1.1) see Definition 2.1. Fix $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $0 \in \mathbb{R}^n$ is an isolated critical point of the potential $\varphi(\cdot, \lambda_i), i = 1, 2$. Let $C\mathcal{I}_{\varphi(\cdot,\lambda_i)}(0)$ denote the Conley index of the isolated invariant set $\{0\} \subset \mathbb{R}^n$ given by the flow induced by the equation $\dot{x}(t) = \nabla_x \varphi(x(t), \lambda_i), i = 1, 2$, see [7] for the definition and properties of the Conley index. Moreover, let $\deg_B(\nabla_x \varphi(\cdot, \lambda_i), B^n_\alpha, 0) \in \mathbb{Z}$ denote the Brouwer degree for sufficiently small $\alpha > 0$ and $i = 1, 2$, see [16] for the definition of the Brouwer degree. If $C\mathcal{I}_{\varphi(\cdot,\lambda_1)}(0) \neq C\mathcal{I}_{\varphi(\cdot,\lambda_2)}(0)$ then there is a local bifurcation of non-zero solutions of equation (1.1) in the interval $\{0\} \times [\lambda_1, \lambda_2]$, see [6, 24]. On the other hand it is known fact that, generally speaking, a change of the Conley index does not imply a global bifurcation of non-zero solutions of equation (1.1), see [1, 2, 14, 18, 26].

Suppose that

$$\varphi(x_1, x_2, \lambda) = \frac{1 - \lambda}{2}(x_1^2 + x_2^2) + \frac{1}{4}(x_1^2 + x_2^2)^2 + \mathcal{B}(x_1, x_2),$$

where $\mathcal{B} \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is the map introduced by Böhme in [2]. Since $\nabla^2_x \varphi(0, 0, \lambda) = (1 - \lambda)Id$, we have $C\mathcal{I}_{\varphi(\cdot,0)}(0) = [S^2] \neq [S^0] = C\mathcal{I}_{\varphi(\cdot,2)}(0)$ and consequently the point $((0, 0), 1) \in \mathbb{R}^2 \times \mathbb{R}$ is a local bifurcation point of non-zero solutions of equation (1.1). On the other hand, $\deg_B(\nabla_x \varphi(\cdot, 0), B^n_\alpha, 0) = \deg_B(\nabla_x \varphi(\cdot, 2), B^n_\alpha, 0)$. Moreover, $((0, 0), 1) \in \mathbb{R}^2 \times \mathbb{R}$ is not a global bifurcation point of non-zero solutions of (1.1), see [1].

Suppose now that $\varphi(x_1, x_2, \lambda) = \frac{\lambda}{2}(x_1^2 + x_2^2) + h.o.t.$ and note that $C\mathcal{I}_{\varphi(\cdot,\epsilon)}(0) = [S^2] \neq [S^0] = C\mathcal{I}_{\varphi(\cdot,-\epsilon)}(0)$ for every $\epsilon > 0$. Ambrosetti has studied in [1] the branching of non-zero solutions of equation (1.1). In our language he has proved that, under some additional assumptions, a change of the Conley index implies the existence of a branching point of non-zero solutions of equation (1.1).

Our goal is to investigate an equivariant version of problem (1.1). Assume that G is a compact Lie group and denote by $\mathbb{V}$ an orthogonal representation of G, see [4] for definition. For abbreviation we write G-representation. Fix a G-invariant potential $\varphi : \mathbb{V} \times \mathbb{R} \to \mathbb{R}$ of the class C^2 i.e. a potential satisfying the following condition $\varphi(gv, \lambda) = \varphi(v, \lambda)$ for every $v \in \mathbb{V}, \lambda \in \mathbb{R}$ and $g \in G$. In the presence of symmetries of a compact Lie group G it is natural to relate the equivariant Conley index, see [10, 25], to the degree for equivariant gradient maps, see [3, 11, 23].

Let $C\mathcal{I}_{G,\varphi(\cdot,\lambda_i)}(0)$ denote the G-equivariant Conley index of the isolated invariant set $\{0\} \subset \mathbb{R}^n$ for $i = 1, 2$. We will prove that, under an additional assumption, the condition $C\mathcal{I}_{G,\varphi(\cdot,\lambda_1)}(0) \neq C\mathcal{I}_{G,\varphi(\cdot,\lambda_2)}(0)$ implies a global bifurcation of G-orbits of non-zero solutions of equation

$$\nabla_v \varphi(v, \lambda) = 0 \tag{1.2}$$

from the interval $\{0\} \times [\lambda_1, \lambda_2]$.

We hope that our viewpoint sheds some new light on the equivariant bifurcation theory.

After introduction, this article is organized as follows. In Section 2 we have summarized without proofs the relevant material on the bifurcation theory. We have reminded the reader of the definition of a local (global) bifurcation and branching of non-zero solutions of equation (1.2), see Definition 2.1, and have formulated sufficient conditions for the existence of a local and global bifurcation of G-orbits of non-zero solutions of equation (1.2) from the interval $\{0\} \times [\lambda_1, \lambda_2]$, see Theorem 2.1. At the end of this section we have discussed the notion of a nice group introduced by Smoller and Wasserman [25] and its consequences for the equivariant bifurcation theory.

In Section 3 our main results are stated and proved. Namely, Corollary 3.1 and Remark 3.1 say that if the group G is connected then the comparison of the G-equivariant Conley indices is equivalent to the comparison of the degrees for G-equivariant gradient maps. Additionally, it is equivalent to the comparison of orthogonal G-representations $\mathbb{V}_1', \mathbb{V}_2'$ of G. Moreover, it is enough to conduct these comparisons after the restriction to symmetries of a maximal torus $T \subset G$. Theorems 3.3 and 3.4 say that, under some conditions, a change of the equivariant Conley indices implies a global bifurcation of G-orbits of non-zero solutions of equation (1.2).

In Section 4 we proceed with the study of a global symmetry-breaking phenomenon of non-radial solutions of elliptic differential equations. The basic idea is to apply Theorems 3.3 and 3.4 to problems considered by Smoller and Wasserman in [25]. These theorems ensures the simultaneously existence of a global and symmetry-breaking bifurcation of non-radial solutions of elliptic differential equations considered in [25]. In other words we have strengthened the main results proved in [25].

In Section 5 we have formulated some remarks concerning the relation of the G-equivariant Conley index to the degree for G-equivariant gradient maps.

2 Preliminaries

Throughout this article G stands for a compact Lie group and $T \subset G$ for a maximal torus. Let $\mathcal{F}_\star(G)$ be the set of finite pointed G-CW-complexes, $\mathcal{F}_\star[G]$ be the set of G-homotopy classes of finite pointed G-CW-complexes and let $U(G)$ be the Euler ring of G, see [8, 9] for more details. By $[\mathbb{X}]_G \in \mathcal{F}_\star[G]$ we denote the G-homotopy class of $\mathbb{X} \in \mathcal{F}_\star(G)$.

Two finite-dimensional orthogonal G-representations $\mathbb{V}, \mathbb{W}$ are said to be equivalent if there is a G-equivariant linear isomorphism $L : \mathbb{V} \to \mathbb{W}$. We denote this relation briefly $\mathbb{V} \approx_G \mathbb{W}$. Let $S_\alpha(\mathbb{V}), D_\alpha(\mathbb{V}), B_\alpha(\mathbb{V}) \subset \mathbb{V}$ be the sphere, closed and open unit discs of radius $\alpha > 0$, respectively. Since the G-representation $\mathbb{V}$ is orthogonal, the sets $S_\alpha(\mathbb{V}), D_\alpha(\mathbb{V}), B_\alpha(\mathbb{V})$ are G-invariant. For simplicity we write $S(\mathbb{V}), D(\mathbb{V}), B(\mathbb{V})$ instead of $S_1(\mathbb{V}), D_1(\mathbb{V}), B_1(\mathbb{V})$, respectively. Finally put $S^{\mathbb{V}} = D(\mathbb{V})/S(\mathbb{V})$. It is known that $S^{\mathbb{V}} \in \mathcal{F}_\star(G)$, see [19].

Let $\Omega \subset \mathbb{V}$ be an open, bounded and G-invariant subset such that $0 \in \Omega$ and let $C^k_G(\Omega, \mathbb{R}), k \in \mathbb{N} \cup \{0\}$, denote the space of G-invariant potentials of the class C^k i.e. $C^k_G(\Omega, \mathbb{R})$ consists of maps $\phi : \Omega \to \mathbb{R}$ of the class C^k such that $\phi(gv) = \phi(v)$ for every $v \in \Omega$ and $g \in G$. Moreover, let $C^k_G(\Omega, \mathbb{V}), k \in \mathbb{N} \cup \{0\}$, denote the space of G-equivariant maps of the class C^k i.e. the set $C^k_G(\Omega, \mathbb{V})$ consists of maps $\psi : \Omega \to \mathbb{V}$ of the class C^k such that $\psi(gv) = g\psi(v)$ for every $v \in \Omega$ and $g \in G$. It is clear that if $\phi \in C^2_G(\Omega, \mathbb{R})$ then $\nabla\phi \in C^1_G(\Omega, \mathbb{V})$, where $\nabla\phi$ is the gradient of ϕ. If $v_0 \in \mathbb{V}$ then $G(v_0) = \{gv_0 : g \in G\}$ is called the G-orbit of v_0. It is easy to verify that if $\nabla\phi(v_0) = 0$ then $G(v_0) \subset (\nabla\phi)^{-1}(0)$.

Fix $\phi \in C^2_G(\Omega, \mathbb{R})$ such that $\nabla\phi(0) = 0$ and that $\nabla^2\phi(0)$ is nondegenerate i.e. $\nabla^2\phi(0) : \mathbb{V} \to \mathbb{V}$ is

a linear G-equivariant isomorphism. Then there is $\alpha > 0$ such that the following holds true

1. $D_\alpha(\mathbb{V}) \subset \Omega$,
2. $(\nabla\phi)^{-1}(0) \cap D_\alpha(\mathbb{V}) = \{0\}$,
3. $\nabla_G\text{-deg}(\nabla\phi, B_\alpha(\mathbb{V})) = \nabla_G\text{-deg}(\nabla^2\phi(0), B(\mathbb{V})) \in U(G)$,

where ∇_G-deg denotes the degree for G-equivariant gradient maps, see [3, 11, 23] for a definition and properties.

Let $\mathbb{V} = \mathbb{V}' \oplus \mathbb{V}''$, where $\mathbb{V}'(\mathbb{V}'')$ is the direct sum of the eigenspaces of $\nabla^2\phi(0)$ corresponding to positive (negative) eigenvalues of $\nabla^2\phi(0)$. Then, using the homotopy invariance and the product formula of the degree for G-equivariant gradient maps, one can show that

$$\begin{array}{rcl} \nabla_G\text{-deg}(-\nabla\phi, B_\alpha(\mathbb{V})) & = & \nabla_G\text{-deg}(-\nabla^2\phi(0), B(\mathbb{V})) = \\ = \nabla_G\text{-deg}((-Id, +Id), B(\mathbb{V}') \times B(\mathbb{V}'')) & = & \nabla_G\text{-deg}(-Id, B(\mathbb{V}')) = \\ = \chi_G([S^{\mathbb{V}'}]_G) & \in & U(G) \end{array} \tag{1.3}$$

where $\chi_G : \mathcal{F}_\star[G] \to U(G)$ is the G-equivariant Euler characteristic, see [8, 9]. It is worth pointing out that the equality $\nabla_G\text{-deg}(-Id, B(\mathbb{V}')) = \chi_G([S^{\mathbb{V}'}]_G)$ is due to Gęba [11].

Note that if G is trivial then ∇_G-deg $= \deg_{\mathrm{B}}$ and $U(G) = \mathbb{Z}$ i.e. the degree for G-equivariant gradient maps equals the Brouwer degree and the Euler ring equals the ring of integers. Moreover, formula (1.3) has the following form $\deg_{\mathrm{B}}(-\nabla\phi, B_\alpha(\mathbb{V})) = \chi([S^{\dim\mathbb{V}'}]) = (-1)^{\dim\mathbb{V}'} \in \mathbb{Z}$, where χ is the classical Euler characteristic.

Let us consider a G-equivariant dynamical system induced by the following differential equation: $\dot{v}(t) = \nabla\phi(v(t))$. Then the G-equivariant Conley index $C\mathcal{I}_{G,\phi}$ can be defined, see [25]. Since $0 \in \mathbb{V}$ is an isolated critical point of φ, $\{0\} \subset \mathbb{V}$ is an isolated invariant set in the sense of G-equivariant Conley index theory and

$$C\mathcal{I}_{G,\phi}(\{0\}) = [S^{\mathbb{V}'}]_G \in \mathcal{F}_\star[G]. \tag{1.4}$$

In other words the G-equivariant Conley index $C\mathcal{I}_{G,\phi}(\{0\})$ equals the G-homotopy type of a G-CW-complex $S^{\mathbb{V}'}$. If the group G is trivial then $C\mathcal{I}_{G,\phi} = C\mathcal{I}_\phi$ i.e. the G-equivariant Conley index equals the classical one. Moreover, formula (1.4) has the following form $C\mathcal{I}_\phi(0) = [S^{\dim\mathbb{V}'}] \in \mathcal{F}_\star$, where $\mathcal{F}_\star$ is the set of homotopy types of finite pointed CW-complexes. Combining (1.3) with (1.4) we obtain $\nabla_G\text{-deg}(-\nabla\phi, B_\alpha(\mathbb{V})) = \chi_G(C\mathcal{I}_{G,\phi}(\{0\}))$.

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_1 < \lambda_2$ and let $C^2_G(\Omega \times [\lambda_1, \lambda_2], \mathbb{R})$ denote the space of families of G-invariant potentials of the class C^2 i.e. $C^2_G(\Omega \times [\lambda_1, \lambda_2], \mathbb{R})$ consists of potentials $\varphi : \Omega \times [\lambda_1, \lambda_2] \to \mathbb{R}$ of the class C^2 such that $\varphi(gv, \lambda) = \varphi(v, \lambda)$ for every $v \in \Omega$, $g \in G$ and $\lambda \in [\lambda_1, \lambda_2]$.

Fix $\varphi \in C^2_G(\Omega \times [\lambda_1, \lambda_2], \mathbb{R})$ such that $\nabla_v\varphi(0, \lambda) = 0$. Our interest is to study non-zero solutions of the following equation

$$\nabla_v\varphi(v, \lambda) = 0. \tag{1.5}$$

Define a set of non-zero solutions of equation (1.5) by $\mathcal{N} = \{(v, \lambda) \in \Omega \times [\lambda_1, \lambda_2] : \nabla_v\varphi(v, \lambda) = 0 \text{ and } v \neq 0\}$, fix $\lambda_0 \in (\lambda_1, \lambda_2)$ and denote by $C(\lambda_0) \subset \mathrm{cl}(\Omega) \times [\lambda_1, \lambda_2]$ a connected component of $\mathrm{cl}(\mathcal{N})$ such that $(0, \lambda_0) \in C(\lambda_0)$. Moreover, put $\mathcal{F} = \{0\} \times [\lambda_1, \lambda_2]$.

Definition 2.1 A point $(0,\lambda_0) \in \mathcal{F}$ is said to be a *local bifurcation point* of non-zero solutions of equation (1.5), if $(0,\lambda_0) \in \mathrm{cl}(\mathcal{N})$. The set of local bifurcation points will be denoted by $\mathcal{BIF}$. A point $(0,\lambda_0) \in \mathcal{F}$ is said to be a *branching point* of non-zero solutions of equation (1.5), if $C(\lambda_0) \neq \{(0,\lambda_0)\}$. The set of branching points will be denoted by $\mathcal{BRANCH}$. A point $(0,\lambda_0) \in \mathcal{F}$ is said to be a *global bifurcation point* of non-zero solutions of equation (1.5), if either $C(\lambda_0) \cap (\{0\} \times [\lambda_1,\lambda_2] \setminus \{(0,\lambda_0)\}) \neq \emptyset$ or $C(\lambda_0) \cap \partial(\Omega \times (\lambda_1,\lambda_2)) \neq \emptyset$. The set of global bifurcation points will be denoted by $\mathcal{GLOB}$.

In other words a point $(0,\lambda_0) \in \mathcal{F}$ is a local bifurcation point of non-zero solutions of equation (1.5) provided that it is an accumulation point of non-zero solutions of this equation. A bifurcation point $(0,\lambda_0) \in \mathcal{F}$ is a branching point of non-zero solutions of equation (1.5) if a connected set $C(\lambda_0)$ of non-zero solutions branches from this point. Finally, a branching point $(0,\lambda_0) \in \mathcal{F}$ is a global bifurcation point of non-zero solutions of equation (1.5) provided that a connected set $C(\lambda_0)$ of non-zero solutions of equation (1.5) branching from this point satisfies Rabinowitz type alternative, see [20, 21, 22], i.e. either $C(\lambda_0)$ is not compact in $\Omega \times (\lambda_1,\lambda_2)$ or meets the set $\mathcal{F}$ at least at two points.

Directly form the above definition it follows that $\mathcal{GLOB} \subset \mathcal{BRANCH} \subset \mathcal{BIF}$. Moreover, by the implicit function theorem we obtain that if $(0,\lambda_0) \in \mathcal{BIF}$ then $\det \nabla_v^2\varphi(0,\lambda_0) = 0$.

Assume that $\nabla_v^2\varphi(0,\lambda_i)$ is non-degenerate for $i = 1,2$. Under this assumption we have a decomposition $\mathbb{V} = \mathbb{V}_1' \oplus \mathbb{V}_1''$ ($\mathbb{V} = \mathbb{V}_2' \oplus \mathbb{V}_2''$) with respect to positive and negative part of the spectrum of $\nabla_v^2\varphi(0,\lambda_1)$ ($\nabla_v^2\varphi(0,\lambda_2)$). Since $\nabla_v^2\varphi(0,\lambda_i), i = 1,2$, are nondegenerate, there is $\alpha > 0$ such that the following equalities hold true

$$\nabla_G\text{-deg}(-\nabla_v\varphi(\cdot,\lambda_i), B_\alpha(\mathbb{V})) = \chi_G([S^{\mathbb{V}_i'}]_G) \in U(G), \tag{1.6}$$

and

$$C\mathcal{I}_{G,\varphi(\cdot,\lambda_i)}(\{0\}) = [S^{\mathbb{V}_i'}]_G \in \mathcal{F}_\star[G], \tag{1.7}$$

for $i = 1,2$.

The following theorem is known. Roughly speaking, the first two parts of this theorem say that a change of the G-equivariant Conley index or the classical Conley index implies the existence of a local bifurcation of non-zero solutions of equation (1.5) from the interval $\{0\} \times [\lambda_1,\lambda_2]$, see [24, 25]. The second two parts say that a change of the degree for G-equivariant gradient maps or the Brouwer degree imply a global bifurcation of non-zero solutions of equation (1.5) from the interval $\{0\} \times [\lambda_1,\lambda_2]$.

Theorem 2.1 *Fix $\varphi \in C_G^2(\Omega \times [\lambda_1,\lambda_2],\mathbb{R})$ such that*

1. *$\nabla_v\varphi(0,\lambda) = 0$ for every $\lambda \in [\lambda_1,\lambda_2]$,*
2. *$\nabla_v^2\varphi(0,\lambda_1), \nabla_v^2\varphi(0,\lambda_2)$ are nondegenerate.*

If $\mathbb{V}_i'$ is a direct sum of the eigenspaces of $\nabla_v^2\varphi(0,\lambda_i)$ corresponding to a positive part of spectrum of $\nabla_v^2\varphi(0,\lambda_i), i = 1,2$, then

1. *if $[S^{\mathbb{V}_1'}]_G \neq [S^{\mathbb{V}_2'}]_G$ then $\mathcal{F} \cap \mathcal{BIF} \neq \emptyset$,*
2. *if $\dim \mathbb{V}_1' \neq \dim \mathbb{V}_2'$ then $\mathcal{F} \cap \mathcal{BIF} \neq \emptyset$,*
3. *if $\chi_G([S^{\mathbb{V}_1'}]_G) \neq \chi_G([S^{\mathbb{V}_2'}]_G)$ then $\mathcal{F} \cap \mathcal{GLOB} \neq \emptyset$,*

4. *if* $\dim \mathbb{V}_1' - \dim \mathbb{V}_2'$ *is odd then* $\mathcal{F} \cap \mathcal{GLOB} \neq \emptyset$.

Remark 2.1 Note that a change of the degree implies a "better" bifurcation i.e. we obtain connected set of G-orbits of non-zero solutions of equation (1.5) bifurcating from the interval $\{0\} \times [\lambda_1, \lambda_2]$. On the other hand it can happen that $[S^{\mathbb{V}_1'}]_G \neq [S^{\mathbb{V}_2'}]_G$ and $\chi_G([S^{\mathbb{V}_1'}]_G) = \chi_G([S^{\mathbb{V}_2'}]_G)$. Whereas $\chi_G([S^{\mathbb{V}_1'}]_G) \neq \chi_G([S^{\mathbb{V}_2'}]_G)$ implies $[S^{\mathbb{V}_1'}]_G \neq [S^{\mathbb{V}_2'}]_G$. It is evident that the condition $\mathbb{V}_1' \approx_G \mathbb{V}_2'$ implies $[S^{\mathbb{V}_1'}]_G = [S^{\mathbb{V}_2'}]_G$. On the other hand there are a compact Lie group G and orthogonal G-representations $\mathbb{V}_1', \mathbb{V}_2'$ such that $\mathbb{V}_1' \not\approx_G \mathbb{V}_2'$ and $[S^{\mathbb{V}_1'}]_G = [S^{\mathbb{V}_2'}]_G$, see [5, 25]. Therefore it is interesting to look for compact Lie groups G and G-representations $\mathbb{V}_1', \mathbb{V}_2'$ such that

$$\text{if } \mathbb{V}_1' \not\approx_G \mathbb{V}_2' \text{ then } [S^{\mathbb{V}_1'}]_G \neq [S^{\mathbb{V}_2'}]_G \tag{1.8}$$

or

$$\text{if } \mathbb{V}_1' \not\approx_G \mathbb{V}_2' \text{ then } \chi_G([S^{\mathbb{V}_1'}]_G) \neq \chi_G([S^{\mathbb{V}_2'}]_G). \tag{1.9}$$

Implication (1.9) is very important from the point of view of the equivariant nonlinear analysis. By Theorem 2.1(3) the condition $\chi_G([S^{\mathbb{V}_1'}]_G) \neq \chi_G([S^{\mathbb{V}_2'}]_G)$ implies the existence of global a bifurcation of non-zero solutions of equation (1.5) from the interval $\{0\} \times [\lambda_1, \lambda_2]$. But this assumption is difficult to verify. Condition (1.9) is a reduction of a difficult question of the equivariant algebraic topology to a simpler question of the representation theory of compact Lie groups.

We introduce the notion of the nice group, following Smoller and Wasserman [25]. A compact Lie group G is said to be nice if for every two orthogonal G-representations $\mathbb{V}_1', \mathbb{V}_2'$ condition (1.8) is fulfilled. It is worth to point out that every connected compact Lie group is nice and the group $O(n)$ is nice, see [15, 25]. Note that for nice groups the verification of the assumption of Theorem 2.1(1) is reduced to a simpler condition $\mathbb{V}_1' \not\approx_G \mathbb{V}_2'$, because for every nice group G we have

$$\mathbb{V}_1' \not\approx_G \mathbb{V}_2' \text{ iff } [S^{\mathbb{V}_1'}]_G \neq [S^{\mathbb{V}_2'}]_G. \tag{1.10}$$

Condition (1.9) has not been considered in paper [25] because the authors have used in the proofs of their results the G-equivariant Conley index and have proved only the existence of a local bifurcation of G-orbits of solutions of equation (1.5). Since a change of the degree for G-equivariant gradient maps implies the existence of a global bifurcation of G-orbits of solutions of equation (1.5) (combine conditions (1.6), (1.9) with Theorem 2.1), in the next section we discuss condition (1.9).

3 Abstract results

In this section we prove the main abstract results of this article. Our aim is to prove that, under some conditions, a change of the equivariant Conley index implies the existence of a global bifurcation of G-orbits of non-zero solutions of equation (1.5). Here G stands for a compact Lie group, $T \subset G$ is a maximal torus and $\mathbb{V}_1', \mathbb{V}_2'$ are as in the previous section.

If $\dim \mathbb{V}_1' > \dim \mathbb{V}_2'$ then condition (1.8) is fulfilled and by Theorems 2.1(1) and 2.1(2) we obtain the existence of a local bifurcation of G-orbits of non-zero solutions of equation (1.5) from the interval $\{0\} \times [\lambda_1, \lambda_2]$. Simultaneously it can happen that the assumptions of Theorems 2.1(3) and 2.1(4) are not satisfied i.e. $\chi_G([S^{\mathbb{V}_1'}]_G) = \chi_G([S^{\mathbb{V}_2'}]_G)$ and $\dim \mathbb{V}_1' - \dim \mathbb{V}_2'$ is even. In fact,

suppose that $\mathbb{R}[s]$ is an s-dimensional trivial G-representation and that for some $k \in \mathbb{N}$ we have $\mathbb{V}_1' \approx_G \mathbb{V}_2' \oplus \mathbb{R}[2k]$ then

$$\chi_G([S^{\mathbb{V}_1'}]_G) = \chi_G([S^{\mathbb{V}_2' \oplus \mathbb{R}[2k]}]_G) = \chi_G([S^{\mathbb{V}_2'}]_G) * \chi_G([S^{\mathbb{R}[2k]}]_G) = \chi_G([S^{\mathbb{V}_2'}]_G),$$

see [17] for details. On the other hand, if $\mathbb{V}_1' \approx_G \mathbb{V}_2' \oplus \mathbb{R}[2k+1]$ for some $k \in \mathbb{N}$ then

$$\chi_G([S^{\mathbb{V}_1'}]_G) = \chi_G([S^{\mathbb{V}_2' \oplus \mathbb{R}[2k+1]}]_G) = \chi_G([S^{\mathbb{V}_2'}]_G) * \chi_G([S^{\mathbb{R}[2k+1]}]_G) = -\chi_G([S^{\mathbb{V}_2'}]_G)),$$

see [17] for details. Since $\chi_G([S^{\mathbb{V}_2'}]_G)$ is nontrivial in $U(G)$, applying Theorem 2.1(3) we obtain of a global bifurcation of G-orbits of non-zero solutions of equation (1.5) from the interval $\{0\} \times [\lambda_1, \lambda_2]$. Note that if $\dim \mathbb{V}_1' - \dim \mathbb{V}_2'$ is odd then by Theorem 2.1(4) we obtain the existence of a global bifurcation of G-orbits of solutions of equation (1.5) from the interval $\{0\} \times [\lambda_1, \lambda_2]$.

Theorem 3.1 *Assume that G is a compact connected Lie group and that* $\dim \mathbb{V}_1' > \dim \mathbb{V}_2'$. *If* $\mathbb{V}_1' \not\approx_G \mathbb{V}_2' \oplus \mathbb{R}[2k]$ *for all* $k \in \mathbb{N}$ *then* $\chi_G([S^{\mathbb{V}_1'}]_G) \neq \chi_G([S^{\mathbb{V}_2'}]_G)$.

Proof. Suppose contrary to our claim that $\chi_G([S^{\mathbb{V}_1'}]_G) = \chi_G([S^{\mathbb{V}_2'}]_G)$. Then by Lemma 3.4 of [17] we obtain $k_0 \in \mathbb{N} \cup \{0\}$ such that $\mathbb{V}_1' \approx_G \mathbb{V}_2' \oplus \mathbb{R}[2k_0]$, a contradiction.
This is the end of the proof.

If $\dim \mathbb{V}_1' = \dim \mathbb{V}_2'$ then the assumptions of Theorems 2.1(2) and 2.1(4) are not satisfied i.e. we can not apply the classical Conley index or the Brouwer degree to prove the existence of a local and global bifurcation of G-orbits of non-zero solutions of equation (1.5) from the interval $\{0\} \times [\lambda_1, \lambda_2]$, respectively. Therefore the G-invariance of problem (1.5) will play a crucial role in the study of bifurcations of G-orbits of non-zero solutions of this problem.

In the theorem below we have proved that if G is connected and $\dim \mathbb{V}_1' = \dim \mathbb{V}_2'$ then implication (1.9) holds true. This theorem will play crucial role in the study of global bifurcations of G-orbits of non-zero solutions of equation (1.5).

Theorem 3.2 *Assume that G is a nontrivial compact connected Lie group and that* $\dim \mathbb{V}_1' = \dim \mathbb{V}_2'$. *Then,* $\mathbb{V}_1' \approx_G \mathbb{V}_2'$ *iff* $\chi_G([S^{\mathbb{V}_1'}]_G) = \chi_G([S^{\mathbb{V}_2'}]_G)$.

Proof. ($\Rightarrow$) If $\mathbb{V}_1' \approx_G \mathbb{V}_2'$ then $[S^{\mathbb{V}_1'}]_G = [S^{\mathbb{V}_2'}]_G$. Hence $\chi_G([S^{\mathbb{V}_1'}]_G) = \chi_G([S^{\mathbb{V}_2'}]_G)$. ($\Leftarrow$) The natural homomorphism $\varphi : T \to G$ induces a ring homomorphism $\varphi^* : U(G) \to U(T)$, see [8, 9]. It is clear that $\varphi^*(\chi_G([S^{\mathbb{V}_1'}]_G)) = \chi_T([S^{\mathbb{V}_1'}]_T)$ and $\varphi^*(\chi_G([S^{\mathbb{V}_2'}]_G)) = \chi_T([S^{\mathbb{V}_2'}]_T)$. By assumption we obtain $\chi_T([S^{\mathbb{V}_1'}]_T) = \chi_T([S^{\mathbb{V}_2'}]_T)$. Since $\dim \mathbb{V}_1' = \dim \mathbb{V}_2'$ and $\chi_T([S^{\mathbb{V}_1'}]_T) = \chi_T([S^{\mathbb{V}_2'}]_T)$, applying Corollary 3.2 of [17] we obtain $\mathbb{V}_1' \approx_T \mathbb{V}_2'$. Let $\mu^G_{\mathbb{V}_1'}, \mu^G_{\mathbb{V}_2'} : G \to \mathbb{R}$ be characters of G-representations $\mathbb{V}_1', \mathbb{V}_2'$, respectively. Since $\mathbb{V}_1', \mathbb{V}_2'$ can be treated as orthogonal T-representations with natural T-action, we can define characters of T-representations $\mathbb{V}_1', \mathbb{V}_2'$ as follows $\mu^T_{\mathbb{V}_1'} = (\mu^G_{\mathbb{V}_1'})_{|T}, \mu^T_{\mathbb{V}_2'} = (\mu^G_{\mathbb{V}_2'})_{|T} : T \to \mathbb{R}$. Since $\mathbb{V}_1' \approx_T \mathbb{V}_2', \mu^T_{\mathbb{V}_1'} = \mu^T_{\mathbb{V}_2'}$. What is left is to show that $\mu^G_{\mathbb{V}_1'} = \mu^G_{\mathbb{V}_2'}$. Since G is connected, for every $g \in G$ there are $\hat{g} \in G$ and $t \in T$ such that $g = \hat{g} t \hat{g}^{-1}$, see [4], and that $\mu^G_{\mathbb{V}_1'}(g) = \mu^G_{\mathbb{V}_1'}(\hat{g} t \hat{g}^{-1}) = \mu^G_{\mathbb{V}_1'}(t) = \mu^T_{\mathbb{V}_1'}(t) = \mu^T_{\mathbb{V}_2'}(t) = \mu^G_{\mathbb{V}_2'}(t) = \mu^G_{\mathbb{V}_2'}(\hat{g} t \hat{g}^{-1}) = \mu^G_{\mathbb{V}_2'}(g)$, which implies $\mathbb{V}_1' \approx_G \mathbb{V}_2'$, see [13].
This is the end of the proof.

As a consequence of the above theorem we obtain the following corollary.

Corollary 3.1 *If G is a nontrivial connected compact Lie group then taking into account (1.10) and Theorem 3.2 we obtain the following:*

1. $\mathbb{V}_1' \approx_G \mathbb{V}_2'$ *iff* $[S^{\mathbb{V}_1'}]_G = [S^{\mathbb{V}_2'}]_G$ *iff* $\chi_G([S^{\mathbb{V}_1'}]_G) = \chi_G([S^{\mathbb{V}_2'}]_G)$,
2. $\mathbb{V}_1' \approx_T \mathbb{V}_2'$ *iff* $[S^{\mathbb{V}_1'}]_T = [S^{\mathbb{V}_2'}]_T$ *iff* $\chi_T([S^{\mathbb{V}_1'}]_T) = \chi_T([S^{\mathbb{V}_2'}]_T)$.

Moreover, $\chi_G([S^{\mathbb{V}_1'}]) = \chi_G([S^{\mathbb{V}_2'}])$ *iff* $\chi_T([S^{\mathbb{V}_1'}]) = \chi_T([S^{\mathbb{V}_2'}])$, *see Theorem 3.3 of [17].*

Remark 3.1 Let G be a nontrivial connected compact Lie group. Note that we have proved that, under assumptions of Theorem 2.1, a change of the G-equivariant Conley index of $\{0\} \subset \mathbb{V}$ is equivalent to a change of an index of $0 \in \mathbb{V}$ computed in terms of the degree for G-equivariant gradient maps.

Here is another way of stating of Corollary 3.1

3.1.(1)′ $\mathbb{V}_1' \approx_G \mathbb{V}_2'$ iff $\nabla_G\text{-deg}(-\nabla_v\varphi(\cdot,\lambda_i), B_\alpha(\mathbb{V})) = \nabla_G\text{-deg}(-\nabla_v\varphi(\cdot,\lambda_i), B_\alpha(\mathbb{V}))$ iff $C\mathcal{I}_{G,\varphi(\cdot,\lambda_1)}(\{0\}) = C\mathcal{I}_{G,\varphi(\cdot,\lambda_2)}(\{0\})$.

3.1.(2)′ $\mathbb{V}_1' \approx_T \mathbb{V}_2'$ iff $\nabla_T\text{-deg}(-\nabla_v\varphi(\cdot,\lambda_i), B_\alpha(\mathbb{V})) = \nabla_T\text{-deg}(-\nabla_v\varphi(\cdot,\lambda_i), B_\alpha(\mathbb{V}))$ iff $C\mathcal{I}_{T,\varphi(\cdot,\lambda_1)}(\{0\}) = C\mathcal{I}_{T,\varphi(\cdot,\lambda_2)}(\{0\})$.

The assumption that the group G is nontrivial is important here. Note that if G is trivial then

1. a condition $\mathbb{V}_1' \approx_G \mathbb{V}_2'$ reads $\dim \mathbb{V}_1' = \dim \mathbb{V}_2'$,
2. a condition $\nabla_G\text{-deg}(-\nabla_v\varphi(\cdot,\lambda_i), B_\alpha(\mathbb{V})) = \nabla_G\text{-deg}(-\nabla_v\varphi(\cdot,\lambda_i), B_\alpha(\mathbb{V}))$ is equivalent to $(-1)^{\dim \mathbb{V}_1'} = (-1)^{\dim \mathbb{V}_2'}$,
3. a condition $C\mathcal{I}_{G,\varphi(\cdot,\lambda_1)}(\{0\}) = C\mathcal{I}_{G,\varphi(\cdot,\lambda_2)}(\{0\})$ reads $[S^{\dim \mathbb{V}_1'}] = [S^{\dim \mathbb{V}_2'}]$.

Hence $(-1)^{\dim \mathbb{V}_1'} = (-1)^{\dim \mathbb{V}_2'} \Leftarrow \dim \mathbb{V}_1' = \dim \mathbb{V}_2' \Leftrightarrow [S^{\dim \mathbb{V}_1'}] = [S^{\dim \mathbb{V}_2'}]$; which means that for the trivial group G a change of the Conley index is not equivalent to a change of the Brouwer degree.

In the two theorems below we have proved, under certain conditions, that a change of the G-equivariant Conley index implies a global bifurcation of G-orbits of non-zero solutions of equation (1.5) from the interval $\{0\}\times[\lambda_1, \lambda_2]$. These theorems gain in interest if we realize that there are examples of variational non-equivariant problems for which a change of the Conley index does not imply a global bifurcations of non-zero solutions of variational problems, see [1, 2, 14, 18, 26] for examples and discussion. However, for the trivial group G the condition $C\mathcal{I}_{G,\varphi(\cdot,\lambda_1)}(\{0\}) \neq C\mathcal{I}_{G,\varphi(\cdot,\lambda_2)}(\{0\})$ is equivalent to $\dim \mathbb{V}_1' \neq \dim \mathbb{V}_2'$, which contradicts assumption $\dim \mathbb{V}_1' = \dim \mathbb{V}_2'$ of Theorem 3.4. The point of these theorems is that they allow one to obtain a global bifurcation if $\dim \mathbb{V}_1' - \dim \mathbb{V}_2'$ is even i.e. if we do not obtain a global bifurcation by Theorem 2.1(4). In these theorems we have formulated sufficient conditions for the existence of a global bifurcation of G-orbits of non-zero solutions of equation (1.5). We underline that the problem of the existence of a global bifurcation of non-zero orbits we have reduced to a much simpler problem from representation theory of compact connected Lie groups.

Theorem 3.3 *Let G be a compact connected Lie group,* $\dim \mathbb{V}_1' > \dim \mathbb{V}_2'$ *and* $\mathbb{V}_1' \not\approx_G \mathbb{V}_2' \oplus \mathbb{R}[2k]$ *for all $k \in \mathbb{N}$. Then $\mathcal{F} \cap \mathcal{GLOB} \neq \emptyset$.*

Proof. The proof is a direct consequence of Theorems 3.1 and 2.1(3).
This is the end of the proof.

Theorem 3.4 *Let G be a compact connected Lie group, $\dim \mathbb{V}_1' = \dim \mathbb{V}_2'$ and $\mathbb{V}_1' \not\approx_G \mathbb{V}_2'$. Then $\mathcal{F} \cap \mathcal{GLOB} \neq \emptyset$.*

Proof. First of all note that from the assumptions it follows that the group G is nontrivial. Since G is connected and $\mathbb{V}_1' \not\approx_G \mathbb{V}_2'$, $\chi_G([S^{\mathbb{V}_1'}]_G) \neq \chi_G([S^{\mathbb{V}_2'}]_G)$ by Theorem 3.2. The rest of the proof is a direct consequence of Theorem 2.1(3).
This is the end of the proof.

Corollary 3.2 *Under the hypothesis of Theorem 3.3 or Theorem 3.4, if moreover $\nabla_u^2\varphi(0,\lambda)_{|\mathbb{V}^G} : \mathbb{V}^G \to \mathbb{V}^G$ is an isomorphism for all $\lambda \in [\lambda_1, \lambda_2]$ then there is a global symmetry-breaking bifurcation form the interval $\{0\} \times [\lambda_1, \lambda_2]$ i.e. if $(0,\lambda_0) \in \mathcal{GLOB}$ then there is an open neighbourhood $\mathcal{U} \subset \mathrm{cl}\mathcal{U} \subset \mathbb{V} \times (\lambda_1, \lambda_2)$ of $(0,\lambda_0)$ such that if $(u,\lambda) \in (\mathcal{C}(\lambda_0) \cap \mathcal{U}) \setminus (\{0\} \times \mathbb{R})$ then $G_u \neq G$, where $G_u = \{g \in G : gu = u\}$ is the isotropy group of u.*

Remark 3.2 In order to prove the existence of a global bifurcation of G-orbits it is enough to verify one of conditions of Corollary 3.1(2). We emphasize that usually computations in the Euler ring $U(T)$ are much simpler than those in $U(G)$ because the torus T is commutative, see [17] for some computations in $U(T)$.

4 Global symmetry-breaking

In this section we apply results of the previous one to symmetry breaking bifurcation problems for partial differential equations considered in [25]. Let G be a compact Lie group and $G_0 \subset G$ be the component of the identity. Consider bifurcation problem (1.5) with $\mathbb{V}_1' \not\approx_G \mathbb{V}_2'$. If we assume additionally that

(a1) $\dim \mathbb{V}_1' = \dim \mathbb{V}_2'$ and $\mathbb{V}_1' \not\approx_{G_0} \mathbb{V}_2'$ then applying Theorem 3.4, with G replaced by G_0, we obtain a global bifurcation of G-orbits of solutions of equation (1.5) from the interval $\{0\} \times [\lambda_1, \lambda_2]$,

(a2) $\dim \mathbb{V}_1' > \dim \mathbb{V}_2'$ and $\mathbb{V}_1' \not\approx_{G_0} \mathbb{V}_2' \oplus \mathbb{R}[2k]$ for all $k \in \mathbb{N}$, then applying Theorem 3.3, with G replaced by G_0, we obtain a global bifurcation of G-orbits of solutions of equation (1.5) from the interval $\{0\} \times [\lambda_1, \lambda_2]$.

The model problem for such an approach has been considered by Smoller and Wasserman [25] with $G = O(n)$ and $G_0 = SO(n)$. Namely, they have studied the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta u &= \lambda^2 f(u) & \text{in } B^n, \\ \alpha u - \lambda\beta \frac{\partial u}{\partial \nu} &= 0 & \text{on } S^{n-1}, \end{cases} \tag{1.11}$$

where $\alpha^2 + \beta^2 = 1$, $B^n \subset \mathbb{R}^n$ is an open disc of radius 1 centred at the origin and ν is the outward normal vector to B^n and f satisfies some technical assumptions. More precisely, they have studied a local symmetry-breaking bifurcations of non-radial solutions of this problem from the families of radial ones.

First of all they have studied an abstract infinite-dimensional variational bifurcation problem

$$M(u, \lambda) = 0,$$

satisfying $M(0, \lambda) = 0$, in the presence of symmetries of the group $O(n)$, see Theorems 3.1, 3.3 of [25]. They have proved the existence of a local bifurcation of critical $O(n)$-orbits of non-zero solutions. To prove these theorems they have reduced the problem to a finite-dimensional, variational, $O(n)$-equivariant problem of the form (1.5) by using the Lyapunov-Schmidt reduction, see proof of Theorem 2.1 of [25] for this reduction.

Summing up, Smoller and Wasserman have considered an abstract $O(n)$-equivariant variational bifurcation problem of the form (1.5) and use the $O(n)$-equivariant Conley index to prove the existence of a local bifurcation of non-zero solutions of this problem.

Next they have studied families of radial solutions (i.e. $O(n)$-invariant solutions) of problem (1.11). And finally, they have applied the abstract results to prove a local bifurcation of non-radial solutions from the families of radial ones, see Theorems 5.1, 5.4.

In the proof of Theorem 5.1 the crucial is condition (5.4) on page 81 of [25] i.e. $P_i \not\approx_{O(n)} P_{i+1}$. Directly from the definitions of $O(n)$-representations P_i, P_{i+1} it follows that $\mathbb{V}_1' = P_{i+1}, \mathbb{V}_2' = P_i$ and $G_0 = SO(n)$ satisfy assumptions **(a2)**. Applying Theorem 3.3 we prove a stronger version of Theorem 5.1 of [25] i.e. there is a global bifurcation from the interval $[q_i, q_{i+1}]$ for $i \geq \overline{N}$. Moreover, the symmetry breaking phenomenon occurs at these global bifurcation points. Similarly in the proof of Theorem 5.4. Putting $\mathbb{V}_1' = P(u(\cdot, q_N^j)), \mathbb{V}_2' = P(u(\cdot, q_{N+1}^j))$, $0 \leqq j \leqq k, N \in \mathbb{N}$ and $G_0 = SO(n)$, one can show that assumptions of **(a2)** are fulfilled (maybe with a change of the role $\mathbb{V}_1'$ and $\mathbb{V}_2'$), see the remarks below the proof of Theorem 5.4 of [25]. Once more applying Theorem 3.3 we prove a stronger version of Theorem 5.4 of [25] i.e. there is a global bifurcation from the interval $[q_N^j, q_{N+1}^j]$ for $0 \leqq j \leqq k, N \in \mathbb{N}$. Additionally, the symmetry breaking phenomenon occurs at these global bifurcation points.

5 Final remarks

Let us consider more general case than in Section 3. As in Section 3 we assume that G is a compact connected Lie group and let $\varphi \in C_G^2(\Omega \times [\lambda_1, \lambda_2], \mathbb{R})$ and $\alpha_0 > 0$ satisfy the following assumptions

1. $\{0\} \times [\lambda_1, \lambda_2] \subset (\nabla_v \varphi)^{-1}(0)$,
2. $0 \in \mathbb{V}$ is isolated in $(\nabla_v \varphi(\cdot, \lambda_i))^{-1}(0)$ for $i = 1, 2$,
3. $C\mathcal{I}_{G,\varphi(\cdot,\lambda_1)}(\{0\}) \neq C\mathcal{I}_{G,\varphi(\cdot,\lambda_2)}(\{0\})$,
4. $(\nabla_v \varphi(\cdot, \lambda_i))^{-1}(0) \cap D_\alpha(\mathbb{V}) = \{0\}$, for all $\alpha_0 > \alpha > 0$ and $i = 1, 2$.

We emphasize that we do not assume that $\det \nabla_v^2 \varphi(0, \lambda_1) \cdot \det \nabla_v^2 \varphi(0, \lambda_2) \neq 0$. It is known that $\chi_G(C\mathcal{I}_{G,\varphi(\cdot,\lambda_i)}(\{0\})) = \nabla_G\text{-deg}(-\nabla_v \varphi(\cdot, \lambda_i), B_\alpha(\mathbb{V})), i = 1, 2$, see [11]. Note that this equality is a generalisation of the equality which we obtain combining (1.6) with (1.7). It is natural to formulate the following question. Does the inequality $C\mathcal{I}_{G,\varphi(\cdot,\lambda_1)}(\{0\}) \neq C\mathcal{I}_{G,\varphi(\cdot,\lambda_2)}(\{0\})$ imply $\nabla_G\text{-deg}(-\nabla_v \varphi(\cdot, \lambda_1), B_\alpha(\mathbb{V})) \neq \nabla_G\text{-deg}(-\nabla_v \varphi(\cdot, \lambda_2), B_\alpha(\mathbb{V}))$? This question at present is far from being solved.

Other interesting questions are the following. Are Theorems 3.1- 3.3 true with a compact connected Lie group G replaced by a 'nice' group that is not connected? In addition, how can we characterize a subclass of the class of compact Lie groups such that the condition $[S^{\mathbb{V}'_1}]_G \neq [S^{\mathbb{V}'_2}]_G$ implies $\chi_G([S^{\mathbb{V}'_1}]_G) \neq \chi_G([S^{\mathbb{V}'_2}]_G)$.

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