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Elliptic Variational Inequalities with Discontinuous Multi-Valued Lower Order Terms

(Dedicated to Professor Klaus Schmitt)

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Abstract

We consider multi-valued elliptic variational inequalities for operators of the form

$$u \mapsto Au + \partial_2\psi(u,u),$$

where A is a second order elliptic operator of Leray-Lions type, and $u \mapsto \partial_2\psi(u,u)$ is a multi-valued lower order term that may neither be lower nor upper semicontinuous, see e.g., Figure 2. More precisely, the lower order term is generated by some function $(r,s) \to \psi(r,s)$ with $\psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ that is locally Lipschitz with respect to its second variable s for each $r \in \mathbb{R}$, and $\partial_2\psi(r,s)$ denotes Clarke's generalized gradient of $s \mapsto \psi(r,s)$. The novelty of this paper is that the multifunction $r \mapsto \partial_2\psi(r,s)$ may discontinuously depend on r in a certain specified way, which gives rise to the new class of discontinuous multi-valued lower order terms $s \mapsto \partial_2\psi(s,s)$. Though rich in structure, the characteristic features of this new class of multi-valued lower order terms can easily be described and verified. Our main goal is to provide an analytical framework and to prove existence and comparison results for this new class of multi-valued variational inequalities that includes the theory of variational-hemivariational inequalities as special case.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$, $1 < p < +\infty$, denote the usual Sobolev space with its dual space V^*. Let K be a closed, convex subset of V, and let $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function $(r, s) \mapsto \psi(r, s)$ that is supposed to be locally Lipschitz continuous in $s \in \mathbb{R}$ for all $r \in \mathbb{R}$. Denote by $s \mapsto \partial_2\psi(r, s)$ Clarke's generalized gradient of $(r, s) \mapsto \psi(r, s)$ with respect to its second argument s, given by

$$\partial_2\psi(r, s) := \{\zeta \in \mathbb{R} : \psi^o(r, s; t) \geq \zeta t, \ \forall\, t \in \mathbb{R}\}$$

where $\psi^o(r, s; t)$ denotes the generalized directional derivative of $s \mapsto \psi(r, s)$ at s in the direction t depending on r which is defined by

$$\psi^o(r, s; t) = \limsup_{y \to s,\ \alpha \downarrow 0} \frac{\psi(r, y + \alpha\, t) - \psi(r, y)}{\alpha},$$

(cf., e.g., [7, Chap. 2]). Let q denote the Hölder conjugate to p, i.e., $1/p + 1/q = 1$. In this paper we study the following multi-valued variational inequality (MVI for short): Find $u \in K$ and $\eta \in L^q(\Omega)$ such that

$$\begin{cases} \eta(x) \in \partial_2\psi(u(x), u(x)), \text{ for a.e. } x \in \Omega, \\ \langle Au, \varphi - u\rangle + \langle i^*\eta, \varphi - u\rangle \geq 0, \ \forall\, \varphi \in K, \end{cases} \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V^* and V, $i^* : L^q(\Omega) \hookrightarrow V^*$ is the adjoint operator of the embedding $i : V \hookrightarrow L^p(\Omega)$, and A is a second-order quasilinear elliptic differential operator in divergence form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)), \quad \text{ with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right),$$

which, under conditions specified later, gives rise to a continuous, bounded, and monotone operator $A : V \to V^*$ defined by

$$\langle Au, \varphi\rangle := \int_\Omega \sum_{i=1}^{N} a_i(x, \nabla u) \frac{\partial\varphi}{\partial x_i}\, dx, \quad \forall\, \varphi \in V.$$

In view of $\langle i^*\eta, \varphi\rangle = \int_\Omega \eta\,\varphi\, dx$, problem (1.1) may be rewritten in the form

$$\begin{cases} \eta(x) \in \partial_2\psi(u(x), u(x)), \text{ for a.e. } x \in \Omega, \\ \langle Au, \varphi - u\rangle + \displaystyle\int_\Omega \eta\,(\varphi - u)\, dx \geq 0, \ \forall\, \varphi \in K. \end{cases} \tag{1.2}$$

Regarding the dependence of $(r, s) \mapsto \psi(r, s)$ with respect to the first argument, we basically assume only that $r \mapsto \psi^o(r, s; 1)$ is decreasing and $r \mapsto \psi^o(r, s; -1)$ is increasing. These two main features of ψ already generate a a new class of multi-valued mappings of the form $s \mapsto \partial_2\psi(s, s)$ that is rich in structure to cover a wide spectrum of multifunctions, by which a number of constitutive laws in the applied science, most notably in mechanical engineering, can be modeled.

The novelty of this paper is to provide an analytical frame work to deal with this new class of multi-valued lower order terms which not only include multi-valued terms of Clarke's gradient as special case, but which include also upper semicontinuous multi-valued functions that are different from Clarke's gradient (see Figure 1), and even multi-valued functions that may neither be lower nor upper semicontinuous, see Figure 2. Before we give two examples of this new class of multifunctions $\partial_2\psi$, let us recall the notion of semicontinuous multifunctions in a general setting.

Definition 1.1 (Semicontinuous Multifunctions) *Let X, Y be Banach spaces, and $T : X \to 2^Y$ be a multifunction.*

(i) T is called upper semicontinuous at x_0, if for every open subset $O \subset Y$ with $T(x_0) \subset O$, there exists a neighborhood $U(x_0)$ such that $T(U(x_0)) \subset O$. If T is upper semicontinuous at every $x_0 \in X$, we call T upper semicontinuous in X.

(ii) T is called lower semicontinuous at x_0 if for every neighborhood $O(y)$ of every $y \in T(x_0)$, there exists a neighborhood $U(x_0)$ such that

$$T(u) \cap O(y) \neq \emptyset \quad \text{for all} \;\; u \in U(x_0).$$

If T is lower semicontinuous at every $x_0 \in X$, we call T lower semicontinuous in X.

(iii) T is called continuous at x_0 if T is both upper and lower semicontinuous at x_0. If T is continuous at every $x_0 \in X$, we call T continuous in X.

Example 1.1. Define $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follows: $\psi(r, s) = g(r)s + j(s)$ where $j(s) = |s|$, and $g : \mathbb{R} \to \mathbb{R}$ is the following discontinuous function:

$$g(r) = \begin{cases} \frac{1}{2} & \text{if} \quad r < 0, \\ 0 & \text{if} \quad r = 0, \\ -\frac{1}{2} & \text{if} \quad r > 0. \end{cases}$$

The generalized directional derivative $\psi^o(r, s; -1)$ is given by $\psi^o(r, s; -1) = -g(r) + j^o(s; -1)$, which is increasing with respect to r, since $r \mapsto g(r)$ is decreasing. Similarly, we see that $\psi^o(r, s; 1) = g(r) + j^o(s; 1)$, which is decreasing with respect to r. By the calculus of Clarke's generalized gradient we readily obtain: $\partial_2\psi(s, s) = g(s) + \partial j(s)$ where Clarke's gradient $s \mapsto \partial j(s)$ of the Lipschitz function $s \mapsto j(s) = |s|$ is given by

$$\partial j(s) = \begin{cases} -1 & \text{if} \quad s < 0, \\ [-1, 1] & \text{if} \quad s = 0, \\ 1 & \text{if} \quad s > 0. \end{cases}$$

A sketch of the graph of $s \mapsto \partial_2\psi(s, s)$ is displayed in Figure 1. Even though this multifunction is upper semicontinuous, it cannot be represented by Clarke's generalized gradient of some locally Lipschitz function $s \mapsto \hat{j}(s)$. (Note, any Clarke's gradient of a locally Lipschitz function $s \mapsto \tilde{j}(s)$ is an upper semicontinuous multifunction, but not vice versa.) The graph shown in Figure 1 has been used in [1] to model certain friction laws, see also [9].

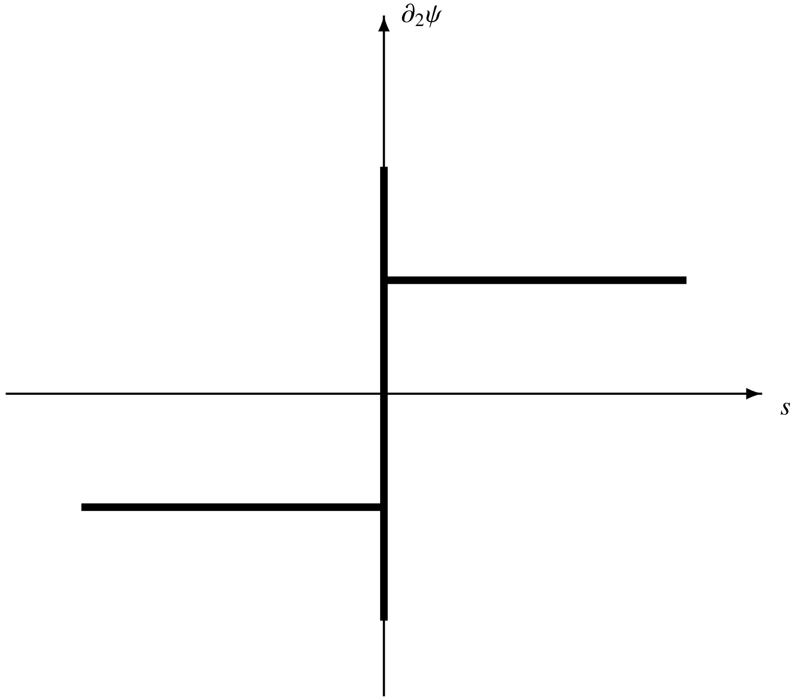

Figure 1

Example 1.2. Let $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by: $\psi(r, s) = g(r)j(s)$, where $j(s) = s^+ := \max\{s, 0\}$, and $g : \mathbb{R} \to \mathbb{R}$ is the following discontinuous function:

$$g(r) = \begin{cases} 1 & \text{if} \quad r \le 0, \\ -1 & \text{if} \quad r > 0. \end{cases}$$

Clarke's generalized gradient $s \mapsto \partial s^+$ of the Lipschitz function $s \mapsto s^+$ is readily seen as follows:

$$\partial s^+ = \begin{cases} 0 & \text{if} \quad s < 0, \\ [0, 1] & \text{if} \quad s = 0, \\ 1 & \text{if} \quad s > 0. \end{cases}$$

Thus a sketch of the graph of $s \mapsto \partial_2\psi(s, s) = g(s)\partial s^+$ is displayed in Figure 2. Since $\partial j(s) = \partial s^+$ is nonnegative ($\partial s^+ \subseteq [0, 1]$), we readily observe that $r \mapsto \psi^o(r, s; -1)$ is increasing with respect to r, and $r \mapsto \psi^o(r, s; 1)$ is decreasing with respect to r. Moreover, one can easily check that $\partial_2\psi(s, s) = g(s)\partial s^+$ is neither a lower nor a upper semicontinuous multifunction at $s = 0$ in the sense of Definition 1.1.

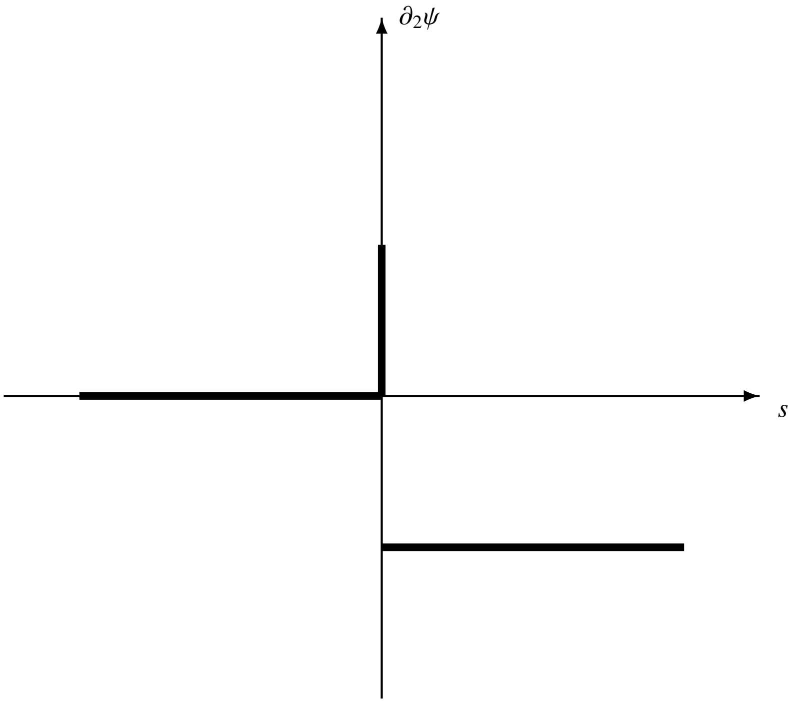

Figure 2

The MVI (1.2) includes as special case lower order multi-valued terms given by Clarke's generalized gradient, which is the case when the function $(r, s) \mapsto \psi(r, s)$ is independent on r. These kind of MVI with Clarke's gradient have extensively been studied in [2, Chap.3] and [3], and have been shown in [4] to be basically equivalent with an associated variational-hemivariational inequality. The theory of variational-hemivariational inequalities, initiated with the pioneering work of P. D. Panagiotopoulos, has attracted much attention over the last two decades mainly due to its many applications in mechanical engineering, cf., e.g. [5, 11, 12, 13, 14]. Elliptic inclusions with discontinuous multifunctions of very restricted and special structure have been treated in [2, Chap.4], which are included as special cases of the MVI (1.2) when $K = V_0 := W_0^{1,p}(\Omega)$. We note that the topic of this paper is closely related to a recent paper by Le [9]. In [9], MVI for a class of multi-valued lower order terms $f : \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ have been studied with $f(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ being closed-convex-valued and upper semicontinuous in the sense of Definition 1.1. While the case of Figure 1 is also covered by [9], the case of Figure 2 cannot be treated by means of the theory developed in [9], because the multi-valued function in Figure 2 is neither upper nor lower semicontinuous. The MVI (1.2) allows for a number of generalizations, e.g., the operator A may be extended to a more general Leray-Lions operator in the form

$$\mathcal{A}u(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)) + a_0(x, u, \nabla u(x)),$$

and the function ψ may, in addition, depend on the space variable $x \in \Omega$, and also a term $\langle D, v - u\rangle$ where $D \in V^*$ may be included in (1.2). To even enlarge the class of considered problems, a multi-valued functions $(r, s) \mapsto \partial_2\omega(r, s)$ acting on the boundary $\partial\Omega$ may be taken into account, which leads to the following more general MVI: Find $u \in K$, $\eta \in L^q(\Omega)$, and $\xi \in L^q(\partial\Omega)$ such that

$$\begin{cases} \eta(x) \in \partial_2\psi(x, u(x), u(x)), \text{ a.e. } x \in \Omega, \;\; \xi(x) \in \partial_2\omega(x, \gamma u(x), \gamma u(x)), \text{ a.e. } x \in \partial\Omega, \\ \langle \mathcal{A}u - D, \varphi - u\rangle + \int_\Omega \eta\,(\varphi - u)\,dx + \int_{\partial\Omega} \xi\,(\gamma\varphi - \gamma u)\,d\sigma \geq 0, \;\forall\, \varphi \in K, \end{cases} \tag{1.3}$$

where $\gamma : V \to L^p(\partial\Omega)$ denotes the trace operator. Only for the sake of simplifying our presentation and in order to emphasize the key ideas we have confined our consideration to problem (1.2).

Our goal is to provide existence and comparison results for the new class of MVI (1.2) that are based on an appropriate notion of sub-supersolution for (1.2) in case that the multi-function $\partial_2\psi$ satisfies only a local $L^q(\Omega)$-boundedness condition within the interval of sub-supersolution (noncoercive case). We will show that the coercive case, in which $\partial_2\psi$ satisfies an appropriate growth condition, may be treated by using the before mentioned sub-supersolution method. Finally we remark that by specifying the closed convex set K as well as the function ψ, the MVI (1.2) is seen to include a wide range of elliptic boundary value problems and variational inequalities, such as, e.g., if K is the linear manifold $K = g \oplus V_0$ with $g \in V$, then we get the inhomogeneous Dirichlet boundary value problem with boundary condition $u = g$ on $\partial\Omega$ for the operator $u \mapsto Au + \partial_2\psi(u, u)$, or if $K = V$, then (1.2) reduces to the Neumann boundary value problem. Mixed boundary value problems and obstacle problems are likewise included and much more.

The paper continues as follows. In Section 2, we provide basic definitions, hypotheses, examples, and some preliminary results. In Section 3, we present our main result on the existence of extremal solutions in the noncoercive situation via sub-supersolution, and in Section 4 the coercive case is treated.

2 Hypotheses, definitions and preliminary results

Let us assume the following hypotheses of Leray–Lions type on the coefficient functions a_i, $i = 1, ..., N$, of the operator A:

(A1) Each $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., $a_i(x, \zeta)$ is measurable in $x \in \Omega$ for all $\zeta \in \mathbb{R}^N$, and continuous in ζ for a.a. $x \in \Omega$. There exist a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ such that

$$|a_i(x, \zeta)| \leq k_0(x) + c_0\,|\zeta|^{p-1},$$

for a.a. $x \in \Omega$ and for all $\zeta \in \mathbb{R}^N$.

(A2) For a.a. $x \in \Omega$, and for all $\zeta, \zeta' \in \mathbb{R}^N$ with $\zeta \neq \zeta'$ the following monotonicity holds:

$$\sum_{i=1}^{N}(a_i(x,\zeta) - a_i(x,\zeta'))(\zeta_i - \zeta_i') > 0.$$

(A3) There is some constant $\nu > 0$ such that for a.a. $x \in \Omega$ and for all $\zeta \in \mathbb{R}^N$ the inequality

$$\sum_{i=1}^{N} a_i(x,\zeta)\zeta_i \geq \nu|\zeta|^p - k_1(x)$$

is satisfied for some function $k_1 \in L^1(\Omega)$.

We note that the negative p-Laplacian, i.e., $Au = -\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$, is a particular case of an operator A satisfying (A1)–(A3) which is given by the following coefficients:

$$a_i(x,\zeta) = |\zeta|^{p-2}\zeta_i, \quad i = 1,\ldots,N, \ \zeta = (\zeta_1,\ldots,\zeta_N).$$

In view of (A1), (A2) the operator A defined by

$$\langle Au, \varphi\rangle := \int_\Omega \sum_{i=1}^{N} a_i(x,\nabla u)\frac{\partial\varphi}{\partial x_i}\,dx, \qquad \forall\, \varphi \in V$$

is known to provide a continuous, bounded, and monotone mapping from V into V^*.

Let us equip the space V with the natural partial ordering, i.e., $u \leq v$ iff $v - u \in L^p_+(\Omega)$, where $L^p_+(\Omega)$ is the positive cone of $L^p(\Omega)$. It is well known that V possess a lattice structure with respect to the natural partial ordering. For functions $w,\ z$ and sets W and Z of functions defined on Ω (resp. on $\partial\Omega$) we use the notations: $w \wedge z = \min\{w,z\}$, $w \vee z = \max\{w,z\}$, $W \wedge Z = \{w \wedge z : w \in W,\ z \in Z\}$, $W \vee Z = \{w \vee z : w \in W,\ z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$. In particular, we denote $w^+ = w \vee 0$, and $w^- = (-w) \vee 0$.

Throughout this paper the closed, convex subset $K \subseteq V$ is assumed to satisfy the following lattice property

$$K \wedge K \subseteq K, \quad K \vee K \subseteq K. \tag{2.1}$$

As for a number of relevant examples for K satisfying (2.1) we refer to [5, p.216].

Definition 2.1 *A function $\underline{u} \in V$ is called a* **subsolution** *of (1.2) if there is an $\underline{\eta} \in L^q(\Omega)$ such that*

$$(i)\ \underline{u} \vee K \subseteq K,$$
$$(ii)\ \underline{\eta}(x) \in \partial_2\psi(\underline{u}(x),\underline{u}(x)),\ \textit{a.e. } x \in \Omega,$$
$$(iii)\ \langle A\underline{u}, \varphi - \underline{u}\rangle + \int_\Omega \underline{\eta}\,(\varphi - \underline{u})\,dx \geq 0,\ \textit{for all } \ \varphi \in \underline{u} \wedge K.$$

Definition 2.2 *A function $\overline{u} \in V$ is called a* **supersolution** *of (1.2) if there is an $\overline{\eta} \in L^q(\Omega)$ such that*

$$(i)\ \overline{u} \wedge K \subseteq K,$$
$$(ii)\ \overline{\eta}(x) \in \partial_2\psi(\overline{u}(x),\overline{u}(x)),\ \textit{a.e. } x \in \Omega,$$
$$(iii)\ \langle A\overline{u}, \varphi - \overline{u}\rangle + \int_\Omega \overline{\eta}\,(\varphi - \overline{u})\,dx \geq 0,\ \textit{for all } \varphi \in \overline{u} \vee K.$$

Remark 2.1 (i) Note that the notions for sub- and supersolution defined in Definition 2.1 and Definition 2.2 have a symmetric structure. One obtains the definition for the supersolution $\overline{u}$ from the definition of the subsolution by replacing $\underline{u}$ in Definition 2.1 by $\overline{u}$, and interchanging $\vee$ by $\wedge$. Furthermore, the lattice condition (2.1) readily implies that any solution of the MVI (1.2) is both a subsolution and a supersolution for (1.2).

(ii) Definition 2.1 and Definition 2.2 reduce to the usual definition of sub-supersolution for elliptic boundary value problems and MVI whose lower order multi-valued term is given in terms of Clarke's gradient of some locally Lipschitz function $s \mapsto j(s)$. This has been demonstrated by a number of examples in [3, 4] or [2, Chap.3].

Let $\underline{u}$ and $\overline{u}$ be an ordered pair of sub-supersolutions of (1.2) such that $\underline{u} \le \overline{u}$, and let $[\underline{u}, \overline{u}]$ denote the order interval. We assume the following hypotheses for the function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

(H1) Let ψ be superpositionally measurable, i.e., if $x \mapsto v(x)$ and $x \mapsto u(x)$ are measurable in Ω, then $x \mapsto \psi(v(x), u(x))$ is measurable in Ω.

(H2) The function $s \mapsto \psi(r, s)$ is locally Lipschitz for all $r \in \mathbb{R}$, and $r \mapsto \psi^o(r, s; 1)$ is decreasing for all $s \in \mathbb{R}$, and $r \mapsto \psi^o(r, s; -1)$ is increasing for all $s \in \mathbb{R}$.

(H3) Let $s \mapsto \partial_2\psi(r, s)$ denote Clarke's generalized gradient of ψ with respect to its second variable s. We assume the following local $L^q(\Omega)$-boundedness of $\partial_2\psi$: There exists a function $k \in L^q_+(\Omega)$ such that for a.e. $x \in \Omega$ and for all $r, s \in [\underline{u}(x), \overline{u}(x)]$ the growth condition

$$|\eta| \le k(x), \quad \forall\, \eta \in \partial_2\psi(r, s)$$

is fulfilled.

Remark 2.2 We remark that the $L^q(\Omega)$-boundedness of $\partial_2\psi$ with respect to the ordered interval $[\underline{u}, \overline{u}]$ is fulfilled provided that $(r, s) \mapsto \partial_2\psi(r, s)$ satisfies the following global growth:

$$|\eta| \le c(1 + |r|^{p-1} + |s|^{p-1}), \ \ \forall\, r, s \in \mathbb{R}.$$

Let us provide next a few important special cases of function ψ that satisfy (H1)–(H2).

Example 2.1 Let $g : \mathbb{R} \to \mathbb{R}$ be a decreasing (not necessarily continuous) function, and $j : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. Define $\psi(r, s) = g(r)s + j(s)$. We claim $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ fulfills (H1)–(H2). First, since $r \mapsto \psi(r, s)$ is monotone for fixed s, it follows that for each measurable function $x \mapsto v(x)$ we have that $x \mapsto \psi(v(x), s)$ is measurable for fixed s. Thus $(x, s) \mapsto \psi(v(x), s)$ is a Carathéodory function, which implies that $x \mapsto \psi(v(x), u(x))$ is measurable for any measurable $x \mapsto u(x)$, which is (H1). Applying Clarke's calculus (see [7, Chap.2.3]), we readily obtain $\partial_2\psi(r, s) = g(r) + \partial j(s)$ and

$$\psi^o(r, s; 1) = g(r) + j^o(s; 1), \quad \psi^o(r, s; -1) = -g(r) + j^o(s; -1),$$

which implies (H2), since $g : \mathbb{R} \to \mathbb{R}$ is supposed to be decreasing. In particular, Example 1.1 fulfills (H1)–(H2), and also (H3) for any pair of sub-supersolutions, because $|\partial_2\psi(r, s)| \le c$ for all $r, s \in \mathbb{R}$.

Example 2.2 Let $g : \mathbb{R} \to \mathbb{R}$ be a decreasing function, and $j : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that Clarke's generalized gradient of $s \mapsto j(s)$ fulfills $\partial j(s) \subset [0, \infty)$. Then $\psi(r, s) = g(r) j(s)$ satisfies (H1)–(H2). To show that (H1) holds true, we follow basically the arguments as in Example 2.1. By Clarke's calculus we see that $\partial_2\psi(r, s) = g(r)\partial j(s)$. To show (H2) we apply the following relation

$$\psi^o(r, s; 1) = \max\{\zeta : \zeta \in \partial_2\psi(r, s)\}, \quad \psi^o(r, s; -1) = \max\{-\zeta : \zeta \in \partial_2\psi(r, s)\},$$

which in view of $\partial_2\psi(r, s) = g(r)\partial j(s)$ in conjunction with the monotonicity of g and the nonnegativity of $\partial j(s)$ implies (H2). In particular, Example 1.2 fulfills (H1)–(H2), and also (H3) for any pair of sub-supersolutions, because $|\partial_2\psi(r, s)| \le c$ for all $r, s \in \mathbb{R}$.

The proof of our main result, which will be given in Section 3, makes use of existence and comparison results for multi-valued variational inequalities with Clarke's generalized gradient, and an abstract fixed point theorem for monotone mappings in partially ordered sets that we recall next.

Let us consider the following MVI with Clarke's generalized gradient $s \mapsto \partial j(x, s)$ of some Carathéodory function function $j : \Omega \times \mathbb{R} \to \mathbb{R}$, that is locally Lipschitz in s: Find $u \in K$ and $\eta \in L^q(\Omega)$ such that

$$\begin{cases} \eta(x) \in \partial j(x, u(x)), \text{ for a.e. } x \in \Omega, \\ \langle Au, \varphi - u\rangle + \int_\Omega \eta\,(\varphi - u)\, dx \ge 0, \ \forall\, \varphi \in K. \end{cases} \tag{2.2}$$

For the proof of the following existence of extremal solutions of the MVI (2.2) we refer to [2, Theorem 3.30], [3, Theorem 29] and also to [4, Theorem 3.2].

Theorem 2.1 *Let (A1)–(A3), and the lattice condition (2.1) be satisfied. Assume the existence of sub- and supersolutions $\underline{u}$ and $\overline{u}$, respectively, of the MVI (2.2) with $\underline{u} \le \overline{u}$ such that ∂j satisfies the following local $L^q(\Omega)$-boundedness: There exists a function $k_\Omega \in L^q_+(\Omega)$ such that for a.e. $x \in \Omega$ and for all $s \in [\underline{u}(x), \overline{u}(x)]$ we have*

$$|\eta| \le k_\Omega(x), \quad \forall\, \eta \in \partial j(x, s). \tag{2.3}$$

Then there exist the greatest solution u^* and the smallest solution u_* of the MVI (2.2) within the ordered interval $[\underline{u}, \overline{u}]$ in the sense that if u is any solution of (2.2) within $[\underline{u}, \overline{u}]$, then $u_* \le u \le u^*$ holds true.

The following abstract fixed point theorem is a consequence of [2, Proposition 2.39], or [6, Theorem 1.1.1].

Theorem 2.2 *Let $\mathcal{P}$ be a subset of an ordered normed space X, and let $G : \mathcal{P} \to \mathcal{P}$ be an increasing mapping, that is, if $x, y \in \mathcal{P}$ and $x \le y$, then $Gx \le Gy$. Then the following holds true:*

(a) If the image $G(\mathcal{P})$ has a lower bound in $\mathcal{P}$ and increasing sequences of $G(\mathcal{P})$ converge weakly in $\mathcal{P}$, then G has the smallest fixed point x_* given by $x_* = \min\{x : Gx \le x\}$.

(b) If the image $G(\mathcal{P})$ has an upper bound in $\mathcal{P}$ and decreasing sequences of $G(\mathcal{P})$ converge weakly in $\mathcal{P}$, then G has the greatest fixed point x^* given by $x^* = \max\{x : x \le Gx\}$.

Theorem 2.1 and Theorem 2.2 will be the key tools to prove our main result.

3 Main result

Our main result reads as follows.

Theorem 3.1 *Let (A1)–(A3), and the lattice condition (2.1) be satisfied. Assume the existence of sub- and supersolutions* $\underline{u}$ *and* $\overline{u}$*, respectively, of the MVI (1.2) with* $\underline{u} \le \overline{u}$ *such that* ψ *satisfies (H1)-(H3). Then there exist the greatest solution* u^* *and the smallest solution* u_* *of the MVI (1.2) within the ordered interval* $[\underline{u}, \overline{u}]$.

Before we prove Theorem 3.1, we provide first several preparatory results. For $v \in [\underline{u}, \overline{u}]$ fixed, we consider the the following auxiliary MVI: Find $u \in K$ and $\xi \in L^q(\Omega)$ such that

$$\begin{cases} \xi(x) \in \partial_2 \psi(v(x), u(x)), \text{ for a.e. } x \in \Omega, \\ \langle Au, \varphi - u \rangle + \displaystyle\int_\Omega \xi\,(\varphi - u)\,dx \ge 0, \ \forall\, \varphi \in K. \end{cases} \tag{3.1}$$

Lemma 3.1 *Let the hypotheses of Theorem 3.1 be satisfied, and let* $v \in [\underline{u}, \overline{u}]$ *be any fixed supersolution of the MVI (1.2). Then the MVI (3.1) has the greatest solution* v^* *and the smallest solution* v_* *within* $[\underline{u}, v]$*. Analogously, if* $w \in [\underline{u}, \overline{u}]$ *is any fixed subsolution of the MVI (1.2), then problem (3.1) with* v *replaced by* w *has the greatest solution* w^* *and the smallest solution* w_* *within* $[w, \overline{u}]$.

Proof. To prove the first part of the lemma, let $v \in [\underline{u}, \overline{u}]$ be any fixed supersolution of the MVI (1.2). Set $j(x, s) := \psi(v(x), s)$. Then (3.1) can be rewritten in the form: Find $u \in K$ and $\xi \in L^q(\Omega)$ such that

$$\begin{cases} \xi(x) \in \partial j(x, u(x)), \text{ for a.e. } x \in \Omega, \\ \langle Au, \varphi - u \rangle + \displaystyle\int_\Omega \xi\,(\varphi - u)\,dx \ge 0, \ \forall\, \varphi \in K, \end{cases} \tag{3.2}$$

where $s \mapsto \partial j(x, s)$ denotes Clarke's generalized gradient of the locally Lipschitz function $s \mapsto j(x, s)$. Taking into account Theorem 2.1, the proof of the first part is complete provided we are able to show that the subsolution $\underline{u}$ and supersolution v of the original MVI (1.2) are also sub-and supersolution of the auxiliary MVI (3.2). According to Definition 2.2, $v \in V$ is a supersolution of the original MVI (1.2) if there is a $\overline{\xi} \in L^q(\Omega)$ satisfying

$$(i)\ v \wedge K \subseteq K,$$
$$(ii)\ \overline{\xi}(x) \in \partial_2 \psi(v(x), v(x)), \text{ a.e. } x \in \Omega,$$
$$(iii)\ \langle Av - h, \varphi - v \rangle + \int_\Omega \overline{\xi}\,(\varphi - v)\,dx \ge 0, \text{ for all } \varphi \in v \vee K.$$

Since $\partial_2 \psi(v(x), v(x)) = \partial j(x, v(x)$, it is obvious that v is also a supersolution of the MVI (3.2). Next, let us show that the subsolution $\underline{u}$ of (1.2), is also a subsolution of (3.2). We recall Definition 2.1 according to which $\underline{u}$ is a subsolution of (1.2) if there is an $\underline{\eta} \in L^q(\Omega)$ satisfying

$$(i')\ \underline{u} \vee K \subseteq K,$$
$$(ii')\ \underline{\eta}(x) \in \partial_2 \psi(\underline{u}(x), \underline{u}(x)), \text{ a.e. } x \in \Omega,$$
$$(iii')\ \langle A\underline{u}, \varphi - \underline{u} \rangle + \int_\Omega \underline{\eta}\,(\varphi - \underline{u})\,dx \ge 0, \text{ for all } \varphi \in \underline{u} \wedge K.$$

For $\underline{u}$ being a subsolution of (3.2) we need to show that there is a $\underline{\xi} \in L^q(\Omega)$ satisfying

$$(a)\ \underline{u} \vee K \subseteq K,$$
$$(b)\ \underline{\xi}(x) \in \partial_2\psi(v(x), \underline{u}(x)),\ \text{ a.e. } x \in \Omega,$$
$$(c)\ \langle A\underline{u}, \varphi - \underline{u}\rangle + \int_\Omega \underline{\xi}\,(\varphi - \underline{u})\,dx \geq 0,\ \text{ for all } \varphi \in \underline{u} \wedge K.$$

Therefore, we only need to verify (b) and (c). Note, $v \in [\underline{u}, \overline{u}]$, and any $\varphi \in \underline{u} \wedge K$ is of the form $\varphi = \underline{u} - (\underline{u} - z)^+$ with $z \in K$. From (ii') and (iii') we get

$$\psi^o(\underline{u}(x), \underline{u}(x); t) \geq \underline{\eta}(x)t, \quad \forall\, t \in \mathbb{R},$$

and thus, in particular,

$$\psi^o(\underline{u}(x), \underline{u}(x); \varphi(x) - \underline{u}(x)) \geq \underline{\eta}(x)(\varphi(x) - \underline{u}(x)), \quad \text{a.e. } x \in \Omega, \tag{3.3}$$

which due to the monotonicity hypothesis (H2) and taking into account $\underline{u} \leq v$ yields

$$\begin{aligned}&\psi^o(\underline{u}(x), \underline{u}(x); \varphi(x) - \underline{u}(x)) = \psi^o(\underline{u}(x), \underline{u}(x); -1)(\underline{u}(x) - z(x))^+\\ &\leq \psi^o(v(x), \underline{u}(x); -1)(\underline{u}(x) - z(x))^+.\end{aligned} \tag{3.4}$$

By Clarke's calculus we have for a.e. $x \in \Omega$

$$\psi^o(v(x), \underline{u}(x); -1) = \max\{\zeta(-1) : \zeta \in \partial_2\psi(v(x), \underline{u}(x))\}. \tag{3.5}$$

Defining $\underline{\xi}(x) := \min\{\zeta : \zeta \in \partial_2\psi(v(x), \underline{u}(x))\}$ and observing that the left-hand side of (3.5) is measurable, we see that $\underline{\xi}(x) \in \partial_2\psi(v(x), \underline{u}(x))$, and from (H3) that $\underline{\xi} \in L^q(\Omega)$, which is (b). Further, from (3.5) it follows that

$$\psi^o(v(x), \underline{u}(x); -1) = -\underline{\xi}(x), \tag{3.6}$$

and thus, from (3.4) and noting that $\varphi = \underline{u} - (\underline{u} - z)^+$ we obtain

$$\begin{aligned}&\psi^o(\underline{u}(x), \underline{u}(x); \varphi(x) - \underline{u}(x)) \leq \psi^o(v(x), \underline{u}(x); -1)(\underline{u}(x) - z(x))^+\\ &= -\underline{\xi}(x)(\underline{u}(x) - z(x))^+ = \underline{\xi}(x)(\varphi(x) - \underline{u}(x)).\end{aligned} \tag{3.7}$$

Finally from (3.3)and (3.7) we get

$$\underline{\eta}(\varphi - \underline{u}) \leq \underline{\xi}(\varphi - \underline{u}), \quad \forall\, \varphi \in \underline{u} \wedge K,$$

which is (c). The function $(x, s) \mapsto j(x, s) = \psi(v(x), s)$ is easily seen to fulfill the conditions of Theorem 2.1 with respect to the interval $[\underline{u}, v] \subseteq [\underline{u}, \overline{u}]$, which allows us to apply Theorem 2.1 to the MVI (3.2) to ensure the existence of the greatest solution v^* and the smallest solution v_* of (3.2), i.e. of (3.1), within $[\underline{u}, v]$. This proves the first part. By obvious analogous reasoning the second part of the lemma can be proved, which completes the proof of the lemma. □

The next lemma shows a certain monotone dependence of the extremal solutions guaranteed by Lemma 3.1.

Lemma 3.2 *Let the hypotheses of Theorem 3.1 be satisfied. Let $v_k \in [\underline{u}, \overline{u}]$, $k = 1, 2$, be supersolutions of (1.2), and let v_k^* be the corresponding greatest solutions of (3.1) with v replaced by v_k within $[\underline{u}, v_k]$, i.e., of*

$$\begin{cases} \xi_k(x) \in \partial_2\psi(v_k(x), u(x)), \ \textit{for a.e.}\ x \in \Omega, \\ \langle Au, \varphi - u\rangle + \int_\Omega \xi_k\,(\varphi - u)\,dx \ge 0, \ \forall\, \varphi \in K. \end{cases} \tag{3.8}$$

Then the following monotone property holds true: If $v_1 \le v_2$, then $v_1^* \le v_2^*$.

Proof. By definition $v_1^* \in [\underline{u}, v_1]$ is the greatest solution of

$$\begin{cases} \xi_1(x) \in \partial_2\psi(v_1(x), u(x)), \ \text{for a.e.}\ x \in \Omega, \\ \langle Au, \varphi - u\rangle + \int_\Omega \xi_1\,(\varphi - u)\,dx \ge 0, \ \forall\, \varphi \in K. \end{cases} \tag{3.9}$$

We are going to show that v_1^* is a subsolution for the MVI (3.8) with $k = 2$, i.e, of

$$\begin{cases} \xi_2(x) \in \partial_2\psi(v_2(x), u(x)), \ \text{for a.e.}\ x \in \Omega, \\ \langle Au, \varphi - u\rangle + \int_\Omega \xi_2\,(\varphi - u)\,dx \ge 0, \ \forall\, \varphi \in K, \end{cases} \tag{3.10}$$

which means that we need to verify the following properties: There is a $\xi_1^* \in L^q(\Omega)$ such that

$$\begin{aligned} &(a)\ v_1^* \vee K \subseteq K, \\ &(b)\ \xi_1^*(x) \in \partial_2\psi(v_2(x), v_1^*), \ \text{a.e.}\ x \in \Omega, \\ &(c)\ \langle Av_1^*, \varphi - v_1^*\rangle + \int_\Omega \xi_1^*\,(\varphi - v_1^*)\,dx \ge 0, \ \text{for all}\ \ \varphi \in v_1^* \wedge K. \end{aligned}$$

Since $v_1^* \in K$, condition (a) is true due to the lattice condition on K. Let us verify (b) and (c). The point of departure is (3.9), which is satisfied by v_1^*, i.e.,

$$\begin{cases} \xi_1(x) \in \partial_2\psi(v_1(x), v_1^*(x)), \ \text{for a.e.}\ x \in \Omega, \\ \langle Au, \varphi - v_1^*\rangle + \int_\Omega \xi_1\,(\varphi - v_1^*)\,dx \ge 0, \ \forall\, \varphi \in K. \end{cases} \tag{3.11}$$

The lattice condition on K and $v_1^* \in K$ imply that the variational-inequality (3.11) is satisfied, in particular, for all test functions $\varphi \in v_1^* \wedge K$, which can be represented in the form $\varphi = v_1^* \wedge z = v_1^* - (v_1^* - z)^+$ with $z \in K$. By (3.11) we have $\xi_1(x) \in \partial_2\psi(v_1(x), v_1^*(x))$ which implies

$$\psi^o(v_1, v_1^*; -1)(v_1^* - z)^+ = \psi^o(v_1, v_1^*; \varphi - v_1^*) \ge \xi_1(\varphi - v_1^*),$$

and thus by the monotonicity assumption (H2) of $r \mapsto \psi^o(r, s; -1)$ and applying $v_1 \le v_2$ we deduce the inequality

$$\begin{aligned} \psi^o(v_2, v_1^*; \varphi - v_1^*) &= \psi^o(v_2, v_1^*; -1)(v_1^* - z)^+ \\ &\ge \psi^o(v_1, v_1^*; -1)(v_1^* - z)^+ \ge \xi_1(\varphi - v_1^*) \end{aligned} \tag{3.12}$$

The function $x \mapsto \psi^o(v_2(x), v_1^*(x); -1)$ is measurable, and, by applying Clarke's gradient calculus, we have

$$\psi^o(v_2(x), v_1^*(x); -1) = \max\{\zeta(-1) : \zeta \in \partial_2\psi(v_2(x), v_1^*(x))\}. \tag{3.13}$$

Defining $\xi_1^*(x) := \min\{\zeta : \zeta \in \partial_2\psi(v_2(x), v_1^*(x))\}$, we deduce from (H3) that $\xi_1^* \in L^q(\Omega)$ and $\xi_1^*(x) \in \partial_2\psi(v_2(x), v_1^*(x))$, which is (b), and from (3.12) and (3.13) we get

$$\xi_1^*(\varphi - v_1^*) = -\xi_1^*(v_1^* - z)^+ \geq \xi_1(\varphi - v_1^*), \ \ \forall\, \varphi = v_1^* \wedge K,$$

which in view of (3.11) results in (c). Since v_2 is a supersolution of the original problem (1.2), it is, in particular, also a supersolution of the MVI (3.10). Moreover, in view of $v_1^* \leq v_1 \leq v_2$, we see that v_1^* and v_2 is pair of sub-supersolution of (3.10), which, by applying Theorem 2.1, yields the existence of solutions of (3.10) within the interval $[v_1^*, v_2]$. However, v_2^* is the greatest solution of (3.10) within the bigger interval $[\underline{u}, v_2]$, therefore we conclude $v_1^* \leq v_2^*$. □

In a similar way one proves the following lemma.

Lemma 3.3 *Let the hypotheses of Theorem 3.1 be satisfied. Let* $w_k \in [\underline{u}, \overline{u}]$, $k = 1, 2$, *be subsolutions of (1.2), and let* w_{k*} *be the corresponding smallest solutions of (3.1) with* v *replaced by* w_k *within* $[w_k, \overline{u}]$, *i.e., of*

$$\begin{cases} \eta_k(x) \in \partial_2\psi(w_k(x), u(x)), \ \textit{for a.e. } x \in \Omega, \\ \langle Au, \varphi - u\rangle + \displaystyle\int_\Omega \eta_k\,(\varphi - u)\,dx \geq 0, \ \forall\, \varphi \in K. \end{cases} \tag{3.14}$$

Then the following monotone property holds true: If $w_1 \leq w_2$, *then* $w_{1*} \leq w_{2*}$.

Now we again consider the auxiliary MVI (3.1): Find $u \in K$ and $\xi \in L^q(\Omega)$ such that

$$\begin{cases} \xi(x) \in \partial_2\psi(v(x), u(x)), \ \text{for a.e. } x \in \Omega, \\ \langle Au, \varphi - u\rangle + \displaystyle\int_\Omega \xi\,(\varphi - u)\,dx \geq 0, \ \forall\, \varphi \in K, \end{cases}$$

where $v \in [\underline{u}, \overline{u}]$ is fixed.

Lemma 3.4 *Let the hypotheses of Theorem 3.1 be satisfied, and let* $v \in [\underline{u}, \overline{u}]$ *be any fixed supersolution of the original MVI (1.2). Then the MVI (3.1) has the greatest solution* v^* *within* $[\underline{u}, v]$, *and* v^* *is again a supersolution of (1.2). Analogously, if* $w \in [\underline{u}, \overline{u}]$ *is any fixed subsolution of the MVI (1.2). Then problem (3.1) with* v *replaced by* w *has the smallest solution* w_* *within* $[w, \overline{u}]$, *and* w_* *is again a subsolution of (1.2).*

Proof. Let us focus on the first part of the lemma. The existence of the greatest solution v^* of (3.1) within the interval $[\underline{u}, v]$ is ensured by Lemma 3.1, and by definition v^* satisfies the following MVI: $v^* \in K$ and there exists a $\xi^* \in L^q(\Omega)$ such that

$$\begin{cases} \xi^*(x) \in \partial_2\psi(v(x), v^*(x)), \ \text{for a.e. } x \in \Omega, \\ \langle Av^*, \varphi - v^*\rangle + \displaystyle\int_\Omega \xi^*\,(\varphi - v^*)\,dx \geq 0, \ \forall\, \varphi \in K. \end{cases} \tag{3.15}$$

We are going to show that v^* is a supersolution of the original MVI (1.2), which means, that we need to show the following: There is a $\eta^* \in L^q(\Omega)$ such that

$$\begin{aligned}
&(a)\ v^* \wedge K \subseteq K,\\
&(b)\ \eta^*(x) \in \partial_2\psi(v^*(x), v^*),\ \text{ a.e. } x \in \Omega,\\
&(c)\ \langle Av^*, \varphi - v^*\rangle + \int_\Omega \eta^*\,(\varphi - v^*)\,dx \ge 0,\ \text{ for all }\ \varphi \in v^* \vee K.
\end{aligned}$$

Since $v^* \in K$, condition (a) is true due to the lattice condition on K. Let us verify (b) and (c). The point of departure is (3.15). Due to the lattice condition on K, the variational inequality (3.15) holds true, in particular, for all $\varphi \in v^* \vee K$, i.e., for φ of the form $\varphi = v^* \vee z = v^* + (z - v^*)^+$, where z may be any element of K. By (3.15) we have $\xi^*(x) \in \partial_2\psi(v(x), v^*(x))$ which implies

$$\psi^o(v, v^*; 1)(z - v^*)^+ = \psi^o(v, v^*; \varphi - v^*) \ge \xi^*(\varphi - v^*),\ \ \forall\, \varphi \in v^* \vee K,$$

and thus by the monotonicity assumption (H2) of $r \mapsto \psi^o(r, s; 1)$ (is decreasing), and applying $v^* \le v$ we deduce the inequality

$$\begin{aligned}
\psi^o(v^*, v^*; \varphi - v^*) &= \psi^o(v^*, v^*; 1)(z - v^*)^+\\
&\ge \psi^o(v, v^*; 1)(z - v^*)^+ \ge \xi^*(\varphi - v^*).
\end{aligned} \tag{3.16}$$

The function $x \mapsto \psi^o(v^*(x), v^*(x); 1)$ is measurable, and, by applying Clarke's gradient calculus, we have

$$\psi^o(v^*(x), v^*(x); 1) = \max\{\zeta : \zeta \in \partial_2\psi(v^*(x), v^*(x))\}. \tag{3.17}$$

Defining $\eta^*(x) := \max\{\zeta : \zeta \in \partial_2\psi(v^*(x), v^*(x))\}$, we deduce from (H3) that $\eta^* \in L^q(\Omega)$ and $\eta^*(x) \in \partial_2\psi(v^*(x), v^*(x))$, which is (b), and from (3.16) and (3.17) it follows

$$\eta^*(\varphi - v^*) = \eta^*(z - v^*)^+ \ge \xi^*(\varphi - v^*),\ \ \forall\, \varphi \in v^* \vee K. \tag{3.18}$$

Since the variational inequality (3.15) holds true, in particular, for all $\varphi \in v^* \vee K$, from (3.18) we get (c), which proves the first part. The second part can be shown by obvious dual reasoning. □

We define subsets $\mathcal{V}$ and $\mathcal{W}$ as follows:

$$\mathcal{V} := \{v \in V : v \in [\underline{u}, \overline{u}]\ \text{ and } v \text{ is a supersolution of (1.2)}\}, \tag{3.19}$$

$$\mathcal{W} := \{w \in V : w \in [\underline{u}, \overline{u}]\ \text{ and } w \text{ is a subsolution of (1.2)}\}. \tag{3.20}$$

Let us introduce operators G and S on $\mathcal{V}$ and $\mathcal{W}$, respectively, that are defined in the following way: For $v \in \mathcal{V}$, let $Gv := v^*$ denote the greatest solution of (3.1) within $[\underline{u}, v]$, and for $w \in \mathcal{W}$, let $Sw := w_*$ denote the smallest solution within $[w, \overline{u}]$ of (3.1) with v replaced by w. By Lemma 3.4 the operators $G : \mathcal{V} \to \mathcal{V}$ and $S : \mathcal{W} \to \mathcal{W}$ are well defined, and the following lemma holds true.

Lemma 3.5 *The operator $G : \mathcal{V} \to \mathcal{V}$ is increasing, and any fixed point of G is a solution of (1.2) within $[\underline{u}, \overline{u}]$, and vice versa.*

The operator $S : \mathcal{W} \to \mathcal{W}$ is increasing, and any fixed point of S is a solution of (1.2) within $[\underline{u}, \overline{u}]$, and vice versa.

Proof. By Lemma 3.2, $G : \mathcal{V} \to \mathcal{V}$ is increasing. The definition of G immediately implies that any fixed point u of G is a solution of (1.2) which belongs to $[\underline{u}, \overline{u}]$. Conversely, if $u \in [\underline{u}, \overline{u}]$ is a solution of (1.2) then u is, in particular, a supersolution, i.e., $u \in \mathcal{V}$, and it is trivially a solution of (3.1) with v replaced by u, and thus u must be the greatest solution of (3.1) within $[\underline{u}, u]$, i.e., $u = Gu$. Similar arguments apply to prove the assertion for the operator S. □

Lemma 3.6 *The range $G(\mathcal{V})$ of G has an upper bound in $\mathcal{V}$, and decreasing sequences of $G(\mathcal{V})$ converge weakly in $\mathcal{V}$.*

The range $S(\mathcal{W})$ of S has a lower bound in $\mathcal{W}$, and increasing sequences of $S(\mathcal{W})$ converge weakly in $\mathcal{W}$.

Proof. The given supersolution $\overline{u} \in \mathcal{V} \subset [\underline{u}, \overline{u}]$ is apparently an upper bound of $G(\mathcal{V})$. Let $(u_n) \subset G(\mathcal{V})$ be a decreasing sequence, i.e., there is a sequence $(v_n) \subset \mathcal{V}$ with $u_n = Gv_n$ which means: $u_n \in K$, and there exist $\eta_n \in L^q(\Omega)$ such that

$$\begin{cases} \eta_n(x) \in \partial_2\psi(v_n(x), u_n(x)), \text{ for a.e. } x \in \Omega, \\ \langle Au_n, \varphi - u_n\rangle + \displaystyle\int_\Omega \eta_n\,(\varphi - u_n)\,dx \geq 0, \ \forall\, \varphi \in K. \end{cases} \tag{3.21}$$

Let $\varphi_0 \in K$ be fixed. Since u_n and v_n belong to $[\underline{u}, \overline{u}]$, both sequences are $L^p(\Omega)$-bounded, and (η_n) is bounded in $L^q(\Omega)$ in view of (H3). Thus from (3.21) we obtain (with c being a generic positive constant)

$$\langle Au_n, u_n\rangle \leq \langle Au_n, \varphi_0\rangle + c. \tag{3.22}$$

Applying (A1) and (A3), and Young's inequality, (3.22) leads to

$$\nu\|\nabla u_n\|^p_{L^p(\Omega)} \leq c + c\int_\Omega |\nabla u_n|^{p-1}|\nabla\varphi_0|\,dx \leq c + \varepsilon\|\nabla u_n\|^p_{L^p(\Omega)} + c(\varepsilon)\|\nabla\varphi_0\|^p_{L^p(\Omega)},$$

for any $\varepsilon > 0$. Taking ε small enough $(0 < \varepsilon < \nu)$, the last inequality shows that $\|\nabla u_n\|_{L^p(\Omega)} \leq c$ for all n, which proves that

$$\|u_n\|_V \leq c, \ \ \forall\, n.$$

The boundedness of (u_n) in V, and $(u_n) \subset K$, along with the monotonicity of the sequence and the compact embedding $V \hookrightarrow\hookrightarrow L^p(\Omega)$ imply:

$$u_n \rightharpoonup u \ \text{ in } V, \quad u_n \to u \ \text{ in } L^p(\Omega), \quad u \in K. \tag{3.23}$$

Further, using the special test function $\varphi = u$ in (3.21), we obtain

$$\langle Au_n, u_n - u\rangle \leq \int_\Omega \eta_n\,(u - u_n)\,dx,$$

which due to the boundedness of (η_n) in $L^q(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ results in

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0.$$

As $A : V \to V^*$ enjoys the (S_+)-property (see, e.g.,[5, Theorem 2.109]), the last inequality in conjunction with the weak convergence of (u_n) (see (3.23)) yields the strong convergence

$$u_n \to u \ \text{ in } \ V.$$

To complete the proof we need to show that the limit $u \in \mathcal{V}$, i.e. that u is a supersolution of (1.2). To this end we recall (3.21) and $u_n = Gv_n$, which means that u_n is the greatest solution of (3.21) with $u_n \in [\underline{u}, v_n]$. Since K satisfies the lattice condition, we may apply the following special test function in the variational inequality (3.21):

$$\varphi = u_n \vee \chi = u_n + (\chi - u_n)^+, \ \ \chi \in K.$$

From $\eta_n(x) \in \partial_2\psi(v_n(x), u_n(x))$ it follows by the definition of Clarke's gradient

$$\psi^o(v_n, u_n; \varphi - u_n) \geq \eta_n\,(\varphi - u_n). \tag{3.24}$$

By (3.21) and (3.24) with the special test function $\varphi = u_n \vee \chi$ we obtain

$$\langle Au_n, (\chi - u_n)^+\rangle + \int_\Omega \psi^o(v_n, u_n; 1)\,(\chi - u_n)^+\,dx \geq 0, \ \forall\, \chi \in K. \tag{3.25}$$

By (H2), $r \mapsto \psi^o(r, s; 1)$ is decreasing, which in view of $u \leq u_n \leq v_n$ results in

$$\psi^o(v_n, u_n; 1) \leq \psi^o(u, u_n; 1),$$

and thus from (3.25) we obtain the following inequality

$$\langle Au_n, (\chi - u_n)^+\rangle + \int_\Omega \psi^o(u, u_n; 1)\,(\chi - u_n)^+\,dx \geq 0, \ \forall\, \chi \in K. \tag{3.26}$$

Since $s \mapsto \psi^o(r, s; 1)$ is upper semicontinous (see [7, Chap.2]), by applying Fatou's lemma and the convergence properties of (u_n) the second term on the left-hand side of (3.26) is seen to satisfy

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega \psi^o(u, u_n; 1)\,(\chi - u_n)^+\,dx &\leq \int_\Omega \limsup_{n\to\infty} \psi^o(u, u_n; 1)\,(\chi - u_n)^+\,dx \\ &\leq \int_\Omega \psi^o(u, u; 1)\,(\chi - u)^+\,dx. \end{aligned} \tag{3.27}$$

Passing in (3.26) to the lim sup as $n \to \infty$ and taking into account (3.27) and the strong convergence of (u_n) in V we get

$$\langle Au, (\chi - u)^+\rangle + \int_\Omega \psi^o(u, u; 1)\,(\chi - u)^+\,dx \geq 0, \ \forall\, \chi \in K. \tag{3.28}$$

Setting $\eta(x) := \psi^o(u(x), u(x); 1)$, in a similar way as has been done already before one easily shows that $\eta \in L^q(\Omega)$ and $\eta(x) \in \partial_2\psi(u(x), u(x))$. Moreover, (3.28) is equivalent to

$$\langle Au, \varphi - u\rangle + \int_\Omega \psi^o(u, u; \varphi - u)\,dx \geq 0, \ \forall\, \varphi \in u \vee K, \tag{3.29}$$

which proves that $u \in \mathcal{V}$. The proof of the second part of the lemma can be omitted, as it follows by similar arguments. □

The proof of our main Theorem 3.1 is now an an immediate consequence of the fixed point Theorem 2.2 and Lemma 3.5–Lemma 3.6.

Proof of Theorem 3.1. First, let us prove the existence of the greatest solution of (1.2) within $[\underline{u}, \overline{u}]$. By Lemma 3.5 any fixed point of $G : \mathcal{V} \to \mathcal{V}$ is a solution of (1.2) in $[\underline{u}, \overline{u}]$ and vice versa. Lemma 3.5 and Lemma 3.6 allow to apply Theorem 2.2 (b), which ensures the existence of the greatest fixed point of G in $\mathcal{V}$, and thus the greatest solution u^* of (1.2) within $[\underline{u}, \overline{u}]$. Similarly, by Lemma 3.5, any fixed point of $S : \mathcal{W} \to \mathcal{W}$ is a solution of (1.2) in $[\underline{u}, \overline{u}]$ and vice versa. Lemma 3.5 and Lemma 3.6 allow to apply Theorem 2.2 (a), which ensures the existence of the smallest fixed point of S in $\mathcal{W}$, and thus the smallest solution u_* of (1.2) within $[\underline{u}, \overline{u}]$. □

4 Coercive MVI

In this section we consider the MVI (1.2), where $(r, s) \mapsto \psi(r, s)$ is given by

$$\psi(r, s) = \frac{\lambda}{p}|s|^p + \beta(r, s), \quad \lambda > 0, \tag{4.1}$$

where $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the following hypotheses:

(B1) Let β be superpositionally measurable. The function $s \mapsto \beta(r, s)$ is locally Lipschitz for all $r \in \mathbb{R}$, and $r \mapsto \beta^o(r, s; 1)$ is decreasing for all $s \in \mathbb{R}$, and $r \mapsto \beta^o(r, s; -1)$ is increasing for all $s \in \mathbb{R}$.

(B2) If $s \mapsto \partial_2\beta(r, s)$ denotes Clarke's generalized gradient of β with respect to its second variable s and $0 \le \alpha < p$, then the following growth condition for $\partial_2\beta$ is assumed:

$$|\eta| \le c(1 + |r|^{\alpha-1} + |s|^{\alpha-1}), \;\; \forall\, \eta \in \partial_2\beta(r, s), \;\forall\, r, s \in \mathbb{R}.$$

Throughout this section we suppose hypotheses (A1)–(A3), (B1)–(B2), and the lattice condition (2.1), and consider the MVI (1.2) with ψ specified by (4.1), i.e., we consider the following MVI: Find $u \in K$ and $\eta \in L^q(\Omega)$ such that

$$\begin{cases} \eta(x) \in \partial_2\beta(u(x), u(x)), \text{ for a.e. } x \in \Omega, \\ \langle Au + \lambda|u|^{p-2}u, \varphi - u\rangle + \displaystyle\int_\Omega \eta\,(\varphi - u)\,dx \ge 0, \;\forall\, \varphi \in K, \end{cases} \tag{4.2}$$

where we have taken into account that $\partial_2\psi$ for ψ given by (4.1) results in

$$\partial_2\psi(r, s) = \lambda|s|^{p-2}s + \partial_2\beta(r, s).$$

Only under the above mentioned hypotheses, i.e., (A1)–(A3), (B1)–(B2), and (2.1), we are going to show that the MVI (4.2) possesses extremal solution without assuming sub-supersolutions. To this end we introduce the following auxiliary (single-valued) variational inequalities:

$$u \in K : \langle Au + \lambda|u|^{p-2}u, \varphi - u\rangle - \int_\Omega f(u)\,(\varphi - u)\,dx \ge 0, \;\forall\, \varphi \in K, \tag{4.3}$$

and

$$u \in K : \langle Au + \lambda|u|^{p-2}u, \varphi - u\rangle + \int_\Omega f(u)\,(\varphi - u)\,dx \ge 0, \;\forall\, \varphi \in K, \tag{4.4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined by the right-hand side of the growth condition (B2) for $r = s$, i.e.,

$$f(s) = c(1 + 2|s|^{\alpha-1}), \quad s \in \mathbb{R}. \tag{4.5}$$

The outline of the proof of the extremal solutions for the MVI (4.2) is as follows: We first show that the auxiliary VI (4.3) has a greatest solution v^*, and the VI (4.4) has the smallest solution v_*, and $v_* \le v^*$ holds true. Then it will be shown that v^* is a supersolution of the MVI (4.2), and v_* is a subsolution of the MVI (4.2), and any solution u of the MVI (4.2) belongs to $[v_*, v^*]$, which completes the proof in view of our main result Theorem 3.1.

Lemma 4.1 *The VI (4.3) has the greatest solution v^*.*

Proof. The proof will be done in four steps.

Step 1: Existence.

If G denotes the Nemytskij operator generated by the function $s \mapsto \lambda|s|^{p-2}s - f(s)$, then $G : L^p(\Omega) \to L^q(\Omega)$ is bounded and continuous, and thus $G : V \to V^*$ is continuous and compact due to the compact embedding $V \hookrightarrow\hookrightarrow L^p(\Omega)$. This implies that $A + G : V \to V^*$ is a bounded, continuous, and pseudomonotone operator. Since (4.3) may equivalently be rewritten as

$$u \in K : \langle (A + G)u, \varphi - u\rangle \ge 0, \ \forall\, \varphi \in K,$$

the existence result is proved provided $A + G$ is coercive on K, i.e., for fixed $\varphi_0 \in K$ the following is required to hold:

$$\lim_{\|u\|_V\to\infty} \frac{\langle (A + G)u, u - \varphi_0\rangle}{\|u\|_V} = +\infty. \tag{4.6}$$

Since $\alpha < p$, we get by means of Young's inequality for any $\varepsilon > 0$

$$\langle (A + G)u, u\rangle \ge \nu\, \|\nabla u\|^p_{L^p(\Omega)} + \lambda\|u\|^p_{L^p(\Omega)} - c_1\|u\|_{L^p(\Omega)} - \varepsilon\|u\|^p_{L^p(\Omega)} - c_2(\varepsilon), \tag{4.7}$$

and for any $\delta > 0$ and $\varrho > 0$ we have estimates of the form

$$\begin{cases} |\langle Au, \varphi_0\rangle| \le c(\delta, \varphi_0) + \delta\|\nabla u\|^p_{L^p(\Omega)}, \\ |\langle Gu, \varphi_0\rangle| \le c(\varrho, \varphi_0) + \varrho\|u\|^p_{L^p(\Omega)}. \end{cases} \tag{4.8}$$

Taking ε, δ, and ϱ small enough, the coercivity (4.6) follows from estimates (4.7) and (4.8), which completes the existence proof by applying standard existence results for variational inequalities, i.e., [10, 15].

Step 2: Compactness of the solution set of (4.3).

Let $\mathcal{S}$ denote the set of all solutions of (4.3), and let $(u_n) \subseteq \mathcal{S}$ be any sequence. By the coercivity proof of Step 1, in particular from (4.7) and (4.8), we readily deduce that (u_n) is bounded in V, which implies the existence of a subsequence (u_k) of (u_n) that satisfies

$$u_k \rightharpoonup u \ \text{ in } V, \quad u_k \to u \ \text{ in } L^p(\Omega). \tag{4.9}$$

Since K is closed and convex, it follows by (4.9) that $u \in K$. Moreover, the u_k satisfy

$$u_k \in K : \langle (A + G)u_k, \varphi - u_k\rangle \ge 0, \ \forall\, \varphi \in K, \tag{4.10}$$

which yields with $\varphi = u$ as special test function

$$\langle Au_k, u_k - u\rangle \le \langle Gu_k, u - u_k\rangle,$$

and thus by means of (4.9)

$$\limsup_{k\to\infty}\langle Au_k, u_k - u\rangle \le 0.$$

Taking into account that $A : V \to V^*$ possesses the (S_+)-property, from the last inequality we infer the strong convergence of (u_k) in V, i.e., $u_k \to u$, which allows to pass to the limit in (4.10) showing that $u \in \mathcal{S}$. This completes the compactness proof.

Step 3: The solution set $\mathcal{S}$ of (4.3) is upward directed.

The set $\mathcal{S}$ is called upward directed if for each pair u_1, $u_2 \in \mathcal{S}$ there exists a $u \in \mathcal{S}$ such that $u_k \le u$ for $k = 1, 2$. Let $u_0 = \max\{u_1, u_2\}$. We introduce truncation mappings T_l related to u_l, $l = 0, 1, 2$, respectively, as follows:

$$T_l(u)(x) = \begin{cases} u_l(x) & \text{if} \quad u(x) < u_l(x), \\ u(x) & \text{if} \quad u(x) \ge u_l(x). \end{cases} \tag{4.11}$$

It is well known that $T_l : V \to V$ ($T_l : L^p(\Omega) \to L^p(\Omega)$) are continuous and bounded mappings. This, in particular, implies that the composed mappings

$$F \circ T_l : L^p(\Omega) \to L^q(\Omega), \quad l = 0, 1, 2,$$

are continuous and bounded, where F denotes the Nemytskij operator generated by f, i.e., $F(u)(x) = f(u(x))$. Consider now the following auxiliary variational inequality: Find $u \in K$ such that

$$\begin{aligned} &\langle Au + \lambda|u|^{p-2}u, \varphi - u\rangle + \int_\Omega Bu\,(\varphi - u)\,dx \\ &- \int_\Omega \Big(F \circ T_0(u) + \sum_{i=1}^{2} |F \circ T_0(u) - F \circ T_i(u)|\Big)(\varphi - u)\,dx \ge 0, \end{aligned} \tag{4.12}$$

for all $\varphi \in K$, where B denotes the Nemytskij operator associated to the following cut-off function $b : \Omega \times \mathbb{R} \to \mathbb{R}$

$$b(x, s) = \begin{cases} 0 & \text{if} \quad u_0(x) \le s, \\ -(u_0(x) - s)^{p-1} & \text{if} \quad s < u_0(x). \end{cases} \tag{4.13}$$

The function b is easily seen to be a Carathéodory function satisfying a growth of order $p-1$, which implies that $B : L^p(\Omega) \to L^q(\Omega)$ is continuous and bounded, and thus $\hat{B} = i^* \circ B \circ i : V \to V^*$ is continuous and compact due to the compact embedding $i : V \to L^p(\Omega)$, with i^* denoting its adjoint operator. As for the existence of solutions of (4.12), basically the same arguments as in Step 1 apply. We only need to make sure that the coercivity is not violated due to the new operator $\hat{B}$. To this end we have to control $v \mapsto \langle \hat{B}v, v\rangle$. Since by definition (4.13) the function $s \mapsto b(x, s)$ is increasing and $b(x, u_0(x)) = 0$, by means of Young's inequality we have the following estimate:

$$\langle \hat{B}v, v\rangle = \int_\Omega b(\cdot, v)(v - u_0 + u_0)\,dx \ge \int_\Omega b(\cdot, v)u_0\,dx \ge -\delta\|v\|^p_{L^p(\Omega)} - c(\delta, u_0),$$

for any $\delta > 0$ and a constant $c(\delta, u_0)$ depending on δ and u_0 only. This shows that the coercivity is preserved as δ can be chosen sufficiently small. Let u be any solution of (4.12). We are going to show that $u \geq u_0$, which proves the upward directedness, because then we have $T_l(u) = u$ for $l = 0, 1, 2$, and $Bu = 0$, and thus (4.12) reduces to (4.3), i.e., u is a solution of (4.3) with $u \geq u_k$, $k = 1, 2$. Since u_k is a solution of (4.3), we get by applying the special test function $\varphi = u_k \wedge u = u_k - (u_k - u)^+$ the inequality

$$\langle Au_k + \lambda |u_k|^{p-2} u_k, -(u_k - u)^+ \rangle - \int_\Omega F(u_k)\,(-(u_k - u)^+)\,dx \geq 0. \tag{4.14}$$

Applying the special test function $\varphi = u_k \vee u = u + (u_k - u)^+$ in (4.12) and adding the resulting inequality to (4.14), we obtain by setting $A_\lambda u = Au + \lambda |u|^{p-2} u$ the inequality

$$\begin{aligned}
&\langle A_\lambda u_k - A_\lambda u, (u_k - u)^+ \rangle - \int_\Omega Bu\,(u_k - u)^+\,dx \\
&\leq \int_\Omega \Big(F(u_k) - F \circ T_0(u) - \sum_{i=1}^{2} |F \circ T_0(u) - F \circ T_i(u)| \Big)\,(u_k - u)^+\,dx \\
&\leq \int_{\{u_k > u\}} \Big(F(u_k) - F \circ T_0(u) - |F \circ T_0(u) - F(u_k)| \Big)\,(u_k - u)^+\,dx \\
&\quad - \int_{\{u_k > u\}} \Big(\sum_{i=1, i \neq k}^{2} |F \circ T_0(u) - F \circ T_i(u)| \Big)\,(u_k - u)^+\,dx \leq 0.
\end{aligned} \tag{4.15}$$

Because

$$\langle A_\lambda u_k - A_\lambda u, (u_k - u)^+ \rangle \geq 0,$$

from (4.15) we deduce

$$-\int_\Omega Bu\,(u_k - u)^+\,dx \leq 0,$$

which by definition of B results in

$$\int_\Omega \Big[(u_0 - u)^+ \Big]^{p-1} (u_k - u)^+\,dx \leq 0.$$

Taking into account that $u_k \leq u_0$, the last inequality yields

$$\int_\Omega \Big[(u_k - u)^+ \Big]^{p}\,dx \leq 0,$$

which implies $(u_k - u)^+ = 0$, and thus $u_k \leq u$, $k = 1, 2$, which completes the proof of the upward directedness.

Step 4: Existence of the greatest solution of (4.3).

We need to prove that the solution set $\mathcal{S}$ has the greatest element. For this purpose Zorn's lemma is used to show first that $\mathcal{S}$ has a maximal element (with respect to the partial ordering defined by the cone $L^p_+(\Omega)$). To this end we need to prove that each nonempty chain C in $\mathcal{S}$ has an upper bound. By Step 2, $\mathcal{S}$ is compact, and thus in particular bounded. So C is bounded as well. Therefore, there exists an increasing sequence $(u_n) \subseteq C$, which converges to $u := \sup C$ strongly in $L^p(\Omega)$ and weakly in V, because $L^p_+(\Omega)$ is a fully regular order cone in $L^p(\Omega)$, see e.g. [8, Proposition 1.3.7]. By the compactness of $\mathcal{S}$ we infer that $u = \sup C \in \mathcal{S}$, which shows that C has an upper bound in $\mathcal{S}$. By

Step 3, $\mathcal{S}$ is upward directed, which implies that the maximal element is uniquely defined and must be the greatest element denoted by v^*. This completes the proof of the theorem. □

Making use of the idea of proof of Lemma 4.1, we obtain by obvious dual reasoning the following lemma.

Lemma 4.2 *The VI (4.4) has the smallest solution v_*.*

We continue our study with some characterizations of the extremal solutions v_* and v^* of (4.4) and (4.3), respectively.

Lemma 4.3 *Let v^* and v_* be the greatest and smallest solution of the VI (4.3) and (4.4), respectively. Then $v_* \le v^*$, and v_* is a subsolution and v^* is a supersolution of the MVI (4.2).*

Proof. Let us first verify that $v_* \le v^*$. Setting $A_\lambda u = Au + \lambda |u|^{p-2}u$, and applying in the VI (4.3) the special test function $\varphi = v^* \vee v_* = v^* + (v_* - v^*)^+$, and in the VI (4.4) the special test function $\varphi = v^* \wedge v_* = v_* - (v_* - v^*)^+$, we obtain by adding the two inequalities the following

$$\langle A_\lambda v_* - A_\lambda v^*, (v_* - v^*)^+ \rangle \le - \int_\Omega \Big(f(v_*) + f(v^*) \Big) (v_* - v^*)^+ \, dx \le 0,$$

from which it readily follows $(v_* - v^*)^+ = 0$, i.e., $v_* \le v^*$. Next we are going to show that v_* is a subsolution of the MVI (4.2). We recall that v_* satisfies: $v_* \in K$, and the variational inequality (4.4)

$$\langle A_\lambda v_*, \varphi - v_* \rangle + \int_\Omega f(v_*) \, (\varphi - v_*) \, dx \ge 0, \ \forall \, \varphi \in K.$$

In view of the lattice condition on K, the last inequality, in particular, is satisfied for any $\varphi \in v_* \wedge K$, i.e., for φ of the form $\varphi = v_* \wedge \chi = v_* - (v_* - \chi)^+$, where $\chi \in K$, which yields

$$\langle A_\lambda v_*, -(v_* - \chi)^+ \rangle + \int_\Omega f(v_*) \, (-(v_* - \chi)^+) \, dx \ge 0. \tag{4.16}$$

Let $\eta_* \in L^q(\Omega)$ be any function with $\eta_*(x) \in \partial_2 \beta(v_*(x), v_*(x))$, which exists, e.g., $\eta_*(x) = \beta^o(v_*(x), v_*(x); 1)$. Taking into account that then we have

$$|\eta_*(x)| \le f(v_*(x)), \quad \text{for a.e. } x \in \Omega,$$

from (4.16) it follows

$$\langle A_\lambda v_*, -(v_* - \chi)^+ \rangle + \int_\Omega \eta_* \, (-(v_* - \chi)^+) \, dx \ge 0, \quad \forall \, \chi \in K,$$

which results in

$$\langle A_\lambda v_*, \varphi - v_* \rangle + \int_\Omega \eta_* \, (\varphi - v_*) \, dx \ge 0, \ \forall \, \varphi \in v_* \wedge K,$$

and thus v_* is a subsolution of the MVI (4.2). The proof for v^* follows similar arguments and can be omitted. □

By means of Lemma 4.1–4.3 we are now in the position to prove the main result of this section.

Theorem 4.1 *Let hypotheses (A1)–(A3), (B1)–(B2), and the lattice condition (2.1) be satisfied. Then the MVI (4.2) has the greatest solution u^* and the smallest solution u_*.*

Proof. In view of Lemma 4.3 the smallest solution v_* of (4.4) and the greatest solution v^* of (4.3) represent a pair of sub-supersolution of the MVI (4.2). One readily verifies that the main result of Section 3 (Theorem 3.1) can be applied to the MVI (4.2), which insures the existence of the greatest and smallest solution u^* and u_* of (4.2) within the order interval $[v_*, v^*]$. The proof of the theorem is accomplished provided we are able to show that each solution of the MVI (4.2) belongs to $[v_*, v^*]$. To this end let $\hat{u} \in K$ be any solution of the MVI (4.2). We are going to prove first $\hat{u} \le v^*$. Using $A_\lambda u = Au + \lambda |u|^{p-2} u$, we introduce the following auxiliary (single-valued) variational inequality: Find $u \in K$ such that

$$\langle A_\lambda u, \varphi - u\rangle + \int_\Omega \tilde{b}(\cdot, u)\,(\varphi - u)\,dx - \int_\Omega F \circ \hat{T}(u)\,(\varphi - u)\,dx \ge 0,\ \forall\, \varphi \in K, \tag{4.17}$$

where $\tilde{b} : \Omega \times \mathbb{R} \to \mathbb{R}$ is the following cut-off function, which is defined similarly to (4.13) by

$$\tilde{b}(x, s) = \begin{cases} 0 & \text{if} \quad \hat{u}(x) \le s, \\ -(\hat{u}(x) - s)^{p-1} & \text{if} \quad s < \hat{u}(x), \end{cases} \tag{4.18}$$

and the truncation $\hat{T}$ is defined by

$$\hat{T}(u)(x) = \begin{cases} u(x) & \text{if} \quad u(x) \ge \hat{u}(x), \\ \hat{u}(x) & \text{if} \quad u(x) < \hat{u}(x). \end{cases}$$

The existence of solutions of (4.17) follows basically the arguments as in Step 3 of the proof of Lemma 4.1. Since $\hat{u}$ is a solution of the MVI (4.2), there is an $\hat{\eta} \in L^q(\Omega)$ such that $\hat{\eta}(x) \in \partial_2\beta(\hat{u}(x), \hat{u}(x))$ and

$$\langle A_\lambda \hat{u}, \varphi - \hat{u}\rangle + \int_\Omega \hat{\eta}\,(\varphi - u)\,dx \ge 0,\ \forall\, \varphi \in K. \tag{4.19}$$

Let u be a solution of (4.17). Applying to (4.17) the special test function $\varphi = u \vee \hat{u} = u + (\hat{u} - u)^+$ and to (4.19) the special test function $\varphi = u \wedge \hat{u} = \hat{u} - (\hat{u} - u)^+$, we obtain by adding the resulting inequalities

$$\begin{aligned} &\langle A_\lambda \hat{u} - A_\lambda u, (\hat{u} - u)^+\rangle - \int_\Omega \tilde{b}(\cdot, u)\,(\hat{u} - u)^+\,dx \\ &\le \int_\Omega \Big(- \hat{\eta} - F \circ \hat{T}(u)\Big)\,(\hat{u} - u)^+\,dx \\ &= \int_{\{u<\hat{u}\}} \Big(- \hat{\eta} - f(\hat{u})\Big)\,(\hat{u} - u)\,dx. \end{aligned} \tag{4.20}$$

Since the right-hand side of (4.20) is nonpositive and

$$\langle A_\lambda \hat{u} - A_\lambda u, (\hat{u} - u)^+\rangle \ge 0,$$

from (4.20) we infer

$$- \int_\Omega \tilde{b}(\cdot, u)\,(\hat{u} - u)^+\,dx \le 0,$$

which by using the definition of $\tilde{b}$ yields

$$\int_\Omega \left[(\hat{u} - u)^+\right]^p dx \leq 0,$$

and thus $(\hat{u} - u)^+ = 0$, i.e., $\hat{u} \leq u$. Because of the latter, we have $b(\cdot, u) = 0$, and $\hat{T}u = 0$, and hence it follows that any solution u of the auxiliary variational inequality (4.17) must be a solution of (4.3) exceeding $\hat{u}$. Since v^* is the greatest solution of all the solutions of (4.3), we conclude that $\hat{u} \leq v^*$. The proof $\hat{u} \geq v_*$ is done by obvious dual reasoning and can be omitted. This completes the proof of the theorem. □

Remark 4.1 (i) Hypotheses (B1)–(B2) of Theorem 4.1 are satisfied for β given by $\beta(r, s) = g(r)s + |s|$, where g is as in Example 1.1, or for $\beta(r, s) = g(r)s^+$ with g as in Example 1.2.

(ii) One can show that Theorem 4.1 still remains true if the growth condition (B2) for $\partial_2\beta$ is replaced by

$$|\eta| \leq c_1 + c_2(|r|^{p-1} + |s|^{p-1}), \quad \forall\, \eta \in \partial_2\beta(r, s), \ \forall\, r, s \in \mathbb{R},$$

with $c_1 \geq 0$, $c_2 \geq 0$, and with c_2 sufficiently small, i.e., $0 \leq c_2 < \lambda/2$.

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