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Bifurcation Analysis at a Singular Soliton State in Fiber Couplers

Boris Buffoni

Section de mathématiques (MATHAA)
École Polytechnique Fédérale, Lausanne, CH-1015
e-mail: boris.buffoni@epfl.ch

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Abstract

The system

$$\begin{cases} -u'' + u = u^3 + \lambda v, & u(x) = u(-x) \in \mathbb{R}, \quad x \in \mathbb{R}, \\ -v'' + v = v^3 + \lambda u, & v(x) = v(-x) \in \mathbb{R}, \quad x \in \mathbb{R}, \end{cases}$$

describes pulses in nonlinear fiber couplers. It has the family $(U_{1+\lambda}, -U_{1+\lambda})$, $-1 < \lambda < \infty$, of soliton states (that is, homoclinic solutions to the origin), where

$$U_{1+\lambda}(x) = \frac{\sqrt{2(1+\lambda)}}{\cosh(\sqrt{1+\lambda}\, x)}\,.$$

For $\lambda \geq 1$, the equilibrium $(0,0)$ is not hyperbolic and therefore the soliton state $(U_{1+\lambda}, -U_{1+\lambda})$ can be qualified as "singular". In N. Akhmediev and A. Ankiewicz [1], it is observed numerically that a branch of homoclinic solutions bifurcates subcritically at $\lambda = 1$ from the family $(U_{1+\lambda}, -U_{1+\lambda})$. The aim of the present paper is to give a rigorous proof of the existence of this bifurcation, as desired in A. Ambrosetti and D. Arcoya [3]. A particular feature of the present problem is that the linearized system at $(U_2, -U_2)$ has a non-constant bounded solution that does not vanish at infinity. Hence the bifurcating homoclinic solutions have a transient "spatial" region where they are well described with the help of this bounded function. Moreover the decay to 0 is governed by two different scales, the larger one originating from the singular aspect of $(U_2, -U_2)$. The existence proof developed here relies on the "broken geodesic" technique to match the inside transient region with the outside region.

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1 Introduction

The aim of this work is to prove the existence of a family of homoclinic orbits that has been observed numerically in the following system of nonlinear Schrödinger equations

$$\begin{cases} -u'' + u = u^3 + \lambda v, & x \in \mathbb{R}, \\ -v'' + v = v^3 + \lambda u, & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

under the conditions that u and v are even functions [1, 3]. There are two families of nontrivial solutions:

$$(U_{1-\lambda}, U_{1-\lambda}), \quad -\infty < \lambda < 1 \ \text{(symmetric states)},$$

$$(U_{1+\lambda}, -U_{1+\lambda}), \quad -1 < \lambda < \infty \ \text{(anti-symmetric states)},$$

where

$$U_{1+\lambda}(x) = \frac{\sqrt{2(1+\lambda)}}{\cosh(\sqrt{1+\lambda}\, x)} = \sqrt{2(1+\lambda)}\,\operatorname{sech}(\sqrt{1+\lambda}\, x) \ \ (\lambda > -1)$$

satisfies the equation $-u'' + (1+\lambda)u = u^3$.

It is known that there is a branch of solutions branching off from the solution $(U_1, 0)$ (corresponding to $\lambda = 0$) and another branch from the solution $(0, U_1)$. At $\lambda = 3/5$, there is an additional branch of solutions branching off from the family of symmetric states $(U_{1-\lambda}, U_{1-\lambda})$. See [1, 2, 3, 4],

An additional branch is known to exist that looks like

$$\Big(U_1(\cdot + \xi_\lambda) + U_1(\cdot - \xi_\lambda), -U_1\Big), \ \ \xi_\lambda \text{ near } \ln(1/\lambda), \ \ \lambda > 0 \text{ small};$$

see [4].

In this paper, we wish to prove that a bifurcation branches off at $\lambda = 1$ from the family of anti-symmetric states $(U_{1+\lambda}, -U_{1+\lambda})$. Such a branch is observed numerically in [1] and in Remark 13.3.6 in [3].

2 Formulation of the problem

By setting $\phi = u + v$ and $\psi = u - v$, we are lead to studying the following system:

$$\begin{cases} -\phi'' + \phi - \frac{1}{4}\phi^3 - \frac{3}{4}\phi\psi^2 = \lambda\phi \\ \\ -\psi'' + \psi - \frac{1}{4}\psi^3 - \frac{3}{4}\psi\phi^2 = -\lambda\psi \end{cases}$$

with $\phi, \psi \in H^2(\mathbb{R})$, where the parameter $\lambda < 1$ is near 1. We follow the notations in [1, 2, 3, 4]. Observe that $(0, 2U_{1+\lambda})$ is a solution. We wish to establish that a bifurcation branches off at $\lambda = 1$ from the family $(0, 2U_{1+\lambda})$. We restrict ourselves to even solutions: $(\phi(-x), \psi(-x)) = (\phi(x), \psi(x))$.

For $\lambda < 1$ near 1, we set $\epsilon = \sqrt{1-\lambda}$ and $x_\epsilon = \epsilon^{-1/4}$, and we divide the real line into two regions: $\{x \in \mathbb{R} : |x| \geq x_\epsilon\}$ and $\{x \in \mathbb{R} : |x| \leq x_\epsilon\}$. Instead of ϕ, we shall work with the two functions $\phi_0 : [0, \infty[\to \mathbb{R}$ and $\phi_1 : [-x_\epsilon, x_\epsilon] \to \mathbb{R}$ defined by

$$\phi(x) = \epsilon\phi_0\left(\epsilon|x| - \epsilon^{3/4}\right) \ \text{ if } \ |x| \geq x_\epsilon \ \text{ and } \ \phi|_{[-x_\epsilon, x_\epsilon]} = \epsilon\phi_1 \ .$$

Instead of ψ, we shall work with ψ_0 defined by

$$\psi(x) = 2U_2(x) + \epsilon^2\psi_0(x).$$

Finally we introduce two additional variables $b, c \in \mathbb{R}$ related to ϕ_0 and ϕ_1 by

$$b = \phi_1(-x_\epsilon) = \phi_1(x_\epsilon) \text{ and } c = \phi_0'(0).$$

We get the following equivalent set of equations:

$$-\phi_1'' + \epsilon^2\phi_1 - \frac{1}{4}\epsilon^2\phi_1^3 - 3\phi_1(U_2 + \epsilon^2\frac{\psi_0}{2})^2 = 0,\ \phi_1(x_\epsilon) = b,$$
$$\phi_1 \text{ even},\ \phi_1 \in H^2(-x_\epsilon, x_\epsilon), \quad (2.2)$$

$$-\phi_0'' + \phi_0 - \frac{1}{4}\phi_0^3 - \frac{3}{\epsilon^2}\phi_0\left(\frac{\epsilon^2}{2}\psi_0(\frac{x+\epsilon^{3/4}}{\epsilon}) + U_2(\frac{x+\epsilon^{3/4}}{\epsilon})\right)^2 = 0,\ \phi_0'(0) = c$$
$$\phi_0 \in H^2(0,\infty), \quad (2.3)$$

$$-\psi_0'' + 2\psi_0 - \frac{2}{\epsilon^2}\left(\frac{\epsilon^2}{2}\psi_0 + U_2\right)^3 + \frac{2}{\epsilon^2}U_2^3 - (2U_2 + \epsilon^2\psi_0) - \frac{3}{2}(\frac{\epsilon^2}{2}\psi_0 + U_2)(\phi/\epsilon)^2 = 0,$$
$$\psi_0 \text{ even}, \psi_0 \in H^2(\mathbb{R}), \quad (2.4)$$

$$b = \phi_0(0),\ c = \epsilon^{-1}\phi_1'(x_\epsilon). \quad (2.5)$$

Equation (2.5) ensures that the corresponding function ϕ and its derivative are continuous at $\pm x_\epsilon$.

In the limit $\epsilon \to 0$, we want c to converge to 0, so that $\phi_0'(0) \to 0$ and $\phi_1'(x_\epsilon) = o(\epsilon)$. We shall study the limit system

$$-\widehat{\phi}_1'' - 3\widehat{\phi}_1U_2^2 = 0,\ \lim_{x\to\infty}\widehat{\phi}_1(x) = \widehat{b},\ \widehat{\phi}_1 \text{ even}, \widehat{\phi}_1 \in H^2_{loc}(\mathbb{R}),$$

$$-\widehat{\phi}_0'' + \widehat{\phi}_0 - \frac{1}{4}\widehat{\phi}_0^3 = 0,\ \widehat{\phi}_0'(0) = 0,\ \widehat{\phi}_0 \in H^2(0,\infty), \quad (2.6)$$

$$-\widehat{\psi}_0'' + 2\widehat{\psi}_0 - 3U_2^2\widehat{\psi}_0 - 2U_2 - \frac{3}{2}U_2\widehat{\phi}_1^2 = 0,\ \widehat{\psi}_0 \text{ even},\ \widehat{\psi}_0 \in H^2(\mathbb{R}), \quad (2.7)$$

$$\widehat{b} = \widehat{\phi}_0(0).$$

As we shall see, a solution to (2.2)-(2.5) with small ϵ will be found near a solution to the limit system, which thus deserves its name. Equation (2.6) has three solutions: $\widehat{\phi}_0 = 0$, $\widehat{\phi}_0 = 2U_1$ and $\widehat{\phi}_0 = -2U_1$. We choose the second one: $\widehat{\phi}_0 = 2U_1$. [1] Then $\widehat{b} = 2U_1(0)$ and $\widehat{\phi}_1$ is given by $\widehat{\phi}_1 = 2U_1(0)(1 - \frac{3}{2}\cosh^{-2}(\sqrt{2}\,x))$. Indeed the function

$$f(x) = 1 - \frac{3}{2}\cosh^{-2}(\sqrt{2}\,x)$$

[1] If (ϕ, ψ) is a solution, so is $(-\phi, \psi)$. The choice $\widehat{\phi}_0 = -2U_1$ would lead to "mirror" bifurcating solutions.

satisfies $-f'' - 3U_2^2 f = 0$. Moreover, as $\widehat{\phi}_1$ is now given, equation (2.7) allows one to determine $\widehat{\psi}_0$ in a unique way (see (5.19) below).

In what follows, we shall assume that ϕ_0 is near $\widehat{\phi}_0$, ϕ_1 is near $\widehat{\phi}_1$, ψ_0 is near $\widehat{\psi}_0$, b is near $2U_1(0) = \widehat{b}$ and c is near 0, which means that

$$\|\phi_0 - \widehat{\phi}_0\|_{H^2(0,\infty)} \le \delta_1, \ \ \|\phi_1 - \widehat{\phi}_1\|_{L^\infty(-x_\epsilon,x_\epsilon)} + \|\phi_1' - \widehat{\phi}_1'\|_{H^1(-x_\epsilon,x_\epsilon)} \le \delta_2,$$
$$\|\psi_0 - \widehat{\psi}_0\|_{H^2(\mathbb{R})} \le \delta_3, \ \ |b - 2U_1(0)| \le \delta_4, \ \ |c| \le \delta_5, \quad (2.8)$$

uniformly as $\epsilon \to 0$, where $\delta_2 = \delta_3 = 1$ and $\delta_1, \delta_4, \delta_5 > 0$ can (and will) be chosen as small as needed.

We are lead to four problems ($\epsilon > 0$ being fixed as small as needed):

Problem I for ϕ_1. Given b near $2U_1(0)$ and ψ_0 even near $\widehat{\psi}_0$ in $H^2(\mathbb{R})$, solve (2.2) on $[-x_\epsilon, x_\epsilon]$ near $\widehat{\phi}_1|_{[-x_\epsilon,x_\epsilon]}$. Check that $\phi_1'(x_\epsilon) = o(\epsilon)$ as $\epsilon \to 0$ (uniformly in b and ψ_0).

Problem II for ϕ_0. Given c near 0 and ψ_0 even near $\widehat{\psi}_0$, solve (2.3) on $[0, \infty[$ near $\widehat{\phi}_0$.

Problem III for ψ_0. Given b near $2U_1(0)$ and c near 0, find ψ_0 even near $\widehat{\psi}_0$, such that for ϕ_0 near $\widehat{\phi}_0$ and ϕ_1 near $\widehat{\phi}_1|_{[-x_\epsilon,x_\epsilon]}$ given by the previous steps, ψ_0 solve (2.4) on $\mathbb{R}$.

Problem IV for b and c. Find b near $2U_1(0)$ and c near 0 such that for ϕ_0 near $\widehat{\phi}_0$, ϕ_1 near $\widehat{\phi}_1|_{[-x_\epsilon,x_\epsilon]}$ and ψ_0 even near $\widehat{\psi}_0$ given by the previous steps, $\phi_0(0) = b$ and $\epsilon^{-1}\phi_1'(x_\epsilon) = c$.

This subdivision into four problems is inspired by the technique of broken geodesics (see [7] and, for a previous application to homoclinic orbits, [6]). Let us comment a little more on (2.8).

To deal with problem I, we first set $\delta_2 = \delta_3 = 1$ and assume that δ_4, ϵ are small enough. We shall get a solution ϕ_1 with a small bound on $\|\phi_1 - bf\|_{H^2(-x_\epsilon,x_\epsilon)}$. Moreover $\phi_1'(x_\epsilon) = o(\epsilon)$ uniformly in b and ψ_0.

To deal with problem II, we suppose $\delta_1, \delta_5, \epsilon$ small enough and we shall get a solution ϕ_0.

To deal with problem III, we shall assume that $\delta_4, \delta_5, \epsilon$ are small enough and then get a solution ψ_0 with a small bound on $\|\psi_0 - \widehat{\psi}_0\|_{H^2(\mathbb{R})}$.

In problem IV, we further diminish the sizes of $\delta_4, \delta_5, \epsilon$ and, in addition, assume that δ_5/δ_4 is small enough too.

In the next sections, all the Sobolev spaces we work with on $[-x_\epsilon, x_\epsilon]$ and on $\mathbb{R}$ are made of even functions only. Moreover the Sobolev norms we use are the usual ones, with the following exception: we often endow $H^2(-x_\epsilon, x_\epsilon)$ (consisting of even functions) with the equivalent norm

$$\|\phi_1\|_{L^\infty(-x_\epsilon,x_\epsilon)} + \|\phi_1'\|_{H^1(-x_\epsilon,x_\epsilon)}$$

(but not uniformly equivalent in ϵ). We then use the notation $H^2_\infty(-x_\epsilon, x_\epsilon)$ for this Banach space and $\|\cdot\|_{H^2_\infty(-x_\epsilon,x_\epsilon)}$ for this norm. The notation $\|\cdot\|_{H^2(-x_\epsilon,x_\epsilon)}$ is for the usual norm

$$\left(\|\phi_1\|^2_{L^2(-x_\epsilon,x_\epsilon)} + \|\phi_1'\|^2_{H^1(-x_\epsilon,x_\epsilon)}\right)^{1/2}.$$

3 Problem I for ϕ_1

Let us recall the problem: given b near $2U_1(0)$ and ψ_0 even near $\widehat{\psi}_0$, solve (2.2) on $[-x_\epsilon, x_\epsilon]$ near $\widehat{\phi}_1|_{[-x_\epsilon,x_\epsilon]}$ under the conditions $\phi_1(x_\epsilon) = b$ and ϕ_1 even.

Consider the linear problem

$$-\widetilde{\phi}_1'' - 3\widetilde{\phi}_1 U_2^2 = 0,\ \widetilde{\phi}_1 \in H^2_\infty(-x_\epsilon, x_\epsilon),$$

under the conditions $\widetilde{\phi}_1(-x_\epsilon) = \widetilde{\phi}_1(x_\epsilon) = \widetilde{b}$ (and $\widetilde{\phi}_1$ even). A solution is given by

$$\widetilde{\phi}_1(x) = \widetilde{b}\frac{f(x)}{f(x_\epsilon)}$$

if x_ϵ is chosen so large that $f(x_\epsilon) \neq 0$. The function

$$h(x) = \sqrt{2}\,x\left(\frac{2}{3} - \cosh^{-2}(\sqrt{2}\,x)\right) - \tanh(\sqrt{2}\,x)$$

is an odd linearly independent solution. We assume that x_ϵ is large enough, so that $h(x_\epsilon) \neq 0$.

The nonhomogeneous linear problem

$$-\widetilde{\phi}_1'' - 3\widetilde{\phi}_1 U_2^2 = \gamma \in L^2(-x_\epsilon, x_\epsilon),\ \widetilde{\phi}_1(-x_\epsilon) = \widetilde{\phi}_1(x_\epsilon) = 0,\ \widetilde{\phi}_1 \in H^2_\infty(-x_\epsilon, x_\epsilon),$$

(with γ even) can be solved explicitly as follows. Set

$$h_\pm(x) = h(x) \mp h(x_\epsilon)\frac{f(x)}{f(x_\epsilon)}$$

and $\Delta_\epsilon = -h_+'(x)h_-(x) + h_-'(x)h_+(x) \neq 0$, which is in fact independent of x. Then $h_\pm$ is a solution to the homogeneous linear equation such that $h_\pm(\pm x_\epsilon) = 0$, and

$$\widetilde{\phi}_1(x) = \Delta_\epsilon^{-1}\int_{-x_\epsilon}^{x} h_+(x)h_-(y)\gamma(y)dy + \Delta_\epsilon^{-1}\int_{x}^{x_\epsilon} h_-(x)h_+(y)\gamma(y)dy. \tag{3.9}$$

Observe that $-h_+(-x) = h_-(x)$. Hence Problem I for ϕ_1 can be written in integral form

$$\begin{cases} \phi_1(x) = b\frac{f(x)}{f(x_\epsilon)} + \Delta_\epsilon^{-1}\int_{-x_\epsilon}^{x} h_+(x)h_-(y)\gamma(y)dy + \Delta_\epsilon^{-1}\int_{x}^{x_\epsilon} h_-(x)h_+(y)\gamma(y)dy \\ \gamma = -\epsilon^2\phi_1 + \frac{1}{4}\epsilon^2\phi_1^3 + 3\phi_1(\epsilon^2\frac{\psi_0}{2})^2 + 6\phi_1 U_2\epsilon^2\frac{\psi_0}{2} \end{cases} \tag{3.10}$$

where b is given near $2U_1(0)$ and $\psi_0 \in H^2(\mathbb{R})$ (even) near $\widehat{\psi}_0$ in the $H^2(\mathbb{R})$-distance.

Here are some useful estimates (uniformly in $x \in [-x_\epsilon, x_\epsilon]$ as $\epsilon \to 0$): $h_+(x) = O(|x - x_\epsilon|)$, $h_-(x) = O(|x + x_\epsilon|)$,

$$\Delta_\epsilon = -h_+'(x_\epsilon)h_-(x_\epsilon) = h_+'(x_\epsilon)h_+(-x_\epsilon) = -\frac{4\sqrt{2}}{3}\left(\frac{2\sqrt{2}}{3}x_\epsilon - 1\right) + O(e^{-x_\epsilon}),$$

$$\widetilde{\phi}_1(x) \overset{(3.9)}{=} \frac{3}{4\sqrt{2}}\left(\frac{2\sqrt{2}}{3}x_\epsilon - 1.5\right)^{-1}\|\gamma\|_{L^2(-x_\epsilon,x_\epsilon)}\Big(O(|x - x_\epsilon||x + x_\epsilon|^{3/2}) + O(|x + x_\epsilon|\,|x - x_\epsilon|^{3/2}\Big) = O(x_\epsilon^{3/2})\|\gamma\|_{L^2} = O(\epsilon^{-3/8})\|\gamma\|_{L^2},$$

$$\widetilde{\phi}_1''(x) = O(|\widetilde{\phi}_1(x)|) + O(|\gamma(x)|) = O(\epsilon^{-3/8})\|\gamma\|_{L^2} + O(|\gamma(x)|) \text{ a.e.},$$
$$\left\|\phi_1 - b\frac{f}{f(x_\epsilon)}\right\|_{H^2(-x_\epsilon,x_\epsilon)} = O(\epsilon^{-3/8}x_\epsilon^{1/2})O(\|\gamma\|_{L^2}) = O(\epsilon^{-1/2})O(\|\gamma\|_{L^2}), \tag{3.11}$$

$$\|\gamma\|_{L^2(-x_\epsilon,x_\epsilon)} \le \epsilon^2\|\phi_1\|_{L^2(-x_\epsilon,x_\epsilon)} + \frac{1}{4}\epsilon^2\|\phi_1\|_{L^2(-x_\epsilon,x_\epsilon)}\|\phi_1\|^2_{L^\infty(-x_\epsilon,x_\epsilon)}$$
$$+ \frac{3}{4}\epsilon^4\|\phi_1\|_{L^2(-x_\epsilon,x_\epsilon)}\|\psi_0\|^2_{L^\infty(-x_\epsilon,x_\epsilon)} + 3\epsilon^2\|\phi_1\|_{L^2(-x_\epsilon,x_\epsilon)}\|U_2\psi_0\|_{L^\infty(-x_\epsilon,x_\epsilon)}$$
$$\overset{(2.8)}{=} O(\epsilon^2)\|\phi_1\|_{L^2(-x_\epsilon,x_\epsilon)}. \tag{3.12}$$

By substituting the expression for γ in (3.10) into the r.h.s. of the first equation in (3.10), the l.h.s. defines a smooth map

$$(\phi_1, b, \psi_0) \to \mathcal{R}_\epsilon(\phi_1, b, \psi_0)$$

from $H^2_\infty(-x_\epsilon, x_\epsilon) \times \mathbb{R} \times H^2(\mathbb{R})$ to $H^2_\infty(-x_\epsilon, x_\epsilon)$. Namely

$$\mathcal{R}_\epsilon(\phi_1, b, \psi_0) = b\frac{f(x)}{f(x_\epsilon)} +$$
$$\Delta_\epsilon^{-1}\int_{-x_\epsilon}^{x} h_+(x)h_-(y)\left\{-\epsilon^2\phi_1 + \frac{1}{4}\epsilon^2\phi_1^3 + 3\phi_1(\epsilon^2\frac{\psi_0}{2})^2 + 6\phi_1U_2\epsilon^2\frac{\psi_0}{2}\right\}dy$$
$$+ \Delta_\epsilon^{-1}\int_{x}^{x_\epsilon} h_-(x)h_+(y)\left\{-\epsilon^2\phi_1 + \frac{1}{4}\epsilon^2\phi_1^3 + 3\phi_1(\epsilon^2\frac{\psi_0}{2})^2 + 6\phi_1U_2\epsilon^2\frac{\psi_0}{2}\right\}dy$$

We get

$$\lim_{|b-2U_1(0)|+\epsilon\to 0}\|\mathcal{R}_\epsilon(\widehat{\phi}_1, b, \psi_0) - \widehat{\phi}_1\|_{H^2_\infty(-x_\epsilon,x_\epsilon)} \overset{(3.11),(3.12)}{=} 0,$$

uniformly in ψ_0 in the closed ball $\overline{B}_{H^2(\mathbb{R})}(\widehat{\psi}_0, \delta_3)$ in $H^2(\mathbb{R})$ (where $\delta_3 = 1$).

Moreover the map $\phi_1 \to \mathcal{R}_\epsilon(\phi_1, b, \psi_0)$ is a 1/2-contraction on $\overline{B}_{H^2_\infty(-x_\epsilon,x_\epsilon)}(\widehat{\phi}_1, 1)$ for all small enough ϵ. If $b \in [2U_1(0)-\delta_4, 2U_1(0)+\delta_4]$ with $\delta_4 > 0$ chosen small enough, a standard application of the Banach fixed point theorem [9] applied in the closed ball $\overline{B}_{H^2_\infty(-x_\epsilon,x_\epsilon)}(\widehat{\phi}_1, 1)$ leads to the existence and uniqueness as $\epsilon \to 0$ of a solution $\phi_1 = \Phi_{1,\epsilon}(b, \psi_0)$ to $\phi_1 = \mathcal{R}_\epsilon(\phi_1, b, \psi_0)$

Remember that, in the present section, all the Sobolev spaces we work with on $[-x_\epsilon, x_\epsilon]$ and on $\mathbb{R}$ are made of even functions only. Hence the fixed point is an even function. Moreover, thanks to the implicit function theorem [10], the dependence of the fixed point $\Phi_{1,\epsilon}(b, \psi_0) \in H^2_\infty(-x_\epsilon, x_\epsilon)$ on b and ψ_0 is smooth with respect to the $\mathbb{R} \times H^2(\mathbb{R})$-distance,

$$\lim_{\epsilon\to 0}\left\|\frac{\partial\Phi_{1,\epsilon}}{\partial\psi_0}\right\| = 0 \tag{3.13}$$

uniformly in b and ψ_0, where the norm is the standard one for bounded operators from $H^2(\mathbb{R})$ to $H^2_\infty(-x_\epsilon, x_\epsilon)$, and

$$\left\|\Phi_{1,\epsilon}(b, \psi_0) - bf\right\|_{H^2(-x_\epsilon,x_\epsilon)} \overset{(3.11),(3.12)}{=} O(\epsilon^{3/2}) \tag{3.14}$$

as $\epsilon \to 0$, uniformly in b and ψ_0. As a consequence

$$\lim_{(\epsilon,b)\to(0,2U_1(0))}\|\Phi_{1,\epsilon}(b, \psi_0) - \widehat{\phi}_1\|_{H^2_\infty(-x_\epsilon,x_\epsilon)} = 0$$

uniformly in ψ_0.

It also follows from (3.14) that $\frac{d}{dx}\Phi_{1,\epsilon}(b,\psi_0)|_{x=x_\epsilon} = o(\epsilon)$ uniformly in b and ψ_0.

4 Problem II for ϕ_0

Given c near 0 and ψ_0 even near $\widehat{\psi}_0$, the problem is to solve (2.3) on $[0,\infty[$ near $\widehat{\phi}_0$ under the condition $\phi_0'(0) = c$. Equation (2.3) can be written

$$-(\phi_0'' - 2U_1)'' + (\phi_0 - 2U_1) - 3U_1^2(\phi_0 - 2U_1) = \frac{1}{4}(\phi_0 - 2U_1)^3 + \frac{3}{2}(\phi_0 - 2U_1)^2 U_1 + \frac{3}{\epsilon^2}\phi_0\left(\frac{\epsilon^2}{2}\psi_0(\frac{x+\epsilon^{3/4}}{\epsilon}) + U_2(\frac{x+\epsilon^{3/4}}{\epsilon})\right)^2.$$

Consider the linear problem

$$-\widetilde{\phi}_0'' + \widetilde{\phi}_0 - 3U_1^2\widetilde{\phi}_0 = 0,\ \widetilde{\phi}_0 \in H^2(0,\infty),$$

under the conditions $\widetilde{\phi}_0'(0) = \widetilde{c}$. If $\widetilde{c} = U_1''(0)$, the function $\widetilde{\phi}_0 = U_1'$ is the unique solution. It is also known that the nonhomogeneous linear problem

$$-\widetilde{\phi}_0'' + \widetilde{\phi}_0 - 3U_1^2\widetilde{\phi}_0 = \gamma \in L^2(0,\infty),\ \widetilde{\phi}_0'(0) = 0,\ \widetilde{\phi}_0 \in H^2(0,\infty),$$

has a unique solution and the map $\gamma \to \widetilde{\phi}_0$ is continuous from $L^2(0,\infty)$ to $H^2(0,\infty)$ (this can be proved like Lemmas 13.3.2 and 13.3.3 in [3]). Let us write $\gamma = A_0\widetilde{\phi}_0$ and $\widetilde{\phi}_0 = A_0^{-1}\gamma$ (A_0 and A_0^{-1} being linear operators).

Hence Problem II for ϕ_0 can be written in the form

$$\begin{cases} \phi_0 = c\frac{U_1'}{U_1''(0)} + 2U_1 + A_0^{-1}\gamma \\ \gamma = \frac{1}{4}(\phi_0 - 2U_1)^3 + \frac{3}{2}(\phi_0 - 2U_1)^2U_1 + \frac{3}{\epsilon^2}\phi_0\left(\frac{\epsilon^2}{2}\psi_0(\frac{x+\epsilon^{3/4}}{\epsilon}) + U_2(\frac{x+\epsilon^{3/4}}{\epsilon})\right)^2 \end{cases}$$

where c is given near 0 and $\psi_0 \in H^2(\mathbb{R})$ (even) near $\widehat{\psi}_0$ in the $H^2(\mathbb{R})$-distance.

As above, by substituting the expression for γ into the r.h.s. of the first equation, the l.h.s. defines a smooth map

$$(\phi_0, c, \psi_0) \to \mathcal{S}_\epsilon(\phi_0, c, \psi_0)$$

from $H^2(0,\infty)\times\mathbb{R}\times H^2(\mathbb{R})$ to $H^2(0,\infty)$:

$$\mathcal{S}_\epsilon(\phi_0, c, \psi_0) = c\frac{U_1'}{U_1''(0)} + 2U_1 + A_0^{-1}\Big\{\frac{1}{4}(\phi_0 - 2U_1)^3 + \frac{3}{2}(\phi_0 - 2U_1)^2U_1 + \frac{3}{\epsilon^2}\phi_0\left(\frac{\epsilon^2}{2}\psi_0(\frac{x+\epsilon^{3/4}}{\epsilon}) + U_2(\frac{x+\epsilon^{3/4}}{\epsilon})\right)^2\Big\}. \quad (4.15)$$

We get

$$\lim_{|c|+\epsilon\to 0}\|\mathcal{S}_\epsilon(\widehat{\phi}_0, c, \psi_0) - \widehat{\phi}_0\|_{H^2(0,\infty)} = 0,$$

uniformly in $\psi_0 \in \overline{B}_{H^2(\mathbb{R})}(\widehat{\psi}_0, \delta_3)$ (where $\delta_3 = 1$). In addition the map $\phi_0 \to \mathcal{S}_\epsilon(\phi_0, c, \psi_0)$ is a 1/2-contraction on $\overline{B}_{H^2(0,\infty)}(\widehat{\phi}_0, \delta_1)$ for all small enough ϵ and small enough $\delta_1 < 1$. Banach's fixed point theorem can be applied in the closed ball $\overline{B}_{H^2(0,\infty)}(\widehat{\phi}_0, \delta_1)$ and it gives the existence and uniqueness as $|c| + \epsilon \to 0$ of a solution $\phi_0 = \Phi_{0,\epsilon}(c, \psi_0)$ to $\phi_0 = \mathcal{S}_\epsilon(\phi_0, c, \psi_0)$. In what follows, we choose therefore $\delta_5, \epsilon > 0$ small enough.

Moreover the dependence of the fixed point $\Phi_{0,\epsilon}(c, \psi_0) \in H^2(0, \infty)$ on c and ψ_0 is smooth,

$$\lim_{\epsilon \to 0} \left\| \frac{\partial \Phi_{0,\epsilon}}{\partial \psi_0} \right\| = 0 \tag{4.16}$$

and

$$\limsup_{\epsilon \to 0} \|\Phi_{0,\epsilon}(c, \psi_0) - 2U_1\|_{H^2(0,\infty)} \leq \text{Const}\,|c|$$

uniformly in c and ψ_0 if δ_1 is chosen small enough. We thus get

$$\limsup_{\epsilon \to 0} \left| \Phi_{0,\epsilon}(c, \psi_0)|_{x=0} - 2U_1(0) \right| \leq \text{Const}\,|c| \tag{4.17}$$

uniformly in c and ψ_0.

5 Problem III for ψ_0

Given b near $2U_1(0)$ and c near 0, we wish to find ψ_0 even near $\widehat{\psi}_0$, such that for $\phi_0 = \Phi_{0,\epsilon}(c, \psi_0)$ near $\widehat{\phi}_0$ and $\phi_1 = \Phi_{1,\epsilon}(b, \psi_0)$ near $\widehat{\phi}_1|_{[-x_\epsilon, x_\epsilon]}$ given by the previous steps, ψ_0 solve (2.4) on $\mathbb{R}$.

Let $\widehat{\phi}$ be defined from $\widehat{\phi}_0$ and $\widehat{\phi}_1$ by

$$\widehat{\phi}(x) = \epsilon \widehat{\phi}_0 \left(\epsilon |x| - \epsilon^{3/4} \right) \text{ if } |x| \geq x_\epsilon \text{ and } \widehat{\phi}|_{[-x_\epsilon, x_\epsilon]} = \epsilon \widehat{\phi}_1 \,. \tag{5.18}$$

Note that $\widehat{\phi}$ is not continuous at x_ϵ .

For $\gamma \in L^2(\mathbb{R})$ even, consider the nonhomogeneous linear problem

$$-\widetilde{\psi}_0'' + 2\widetilde{\psi}_0 - 3U_2^2 \widetilde{\psi}_0 = \gamma \in L^2(\mathbb{R}),\ \widetilde{\psi}_0 \in H^2(\mathbb{R}).$$

It is known that there is a unique solution and the map $\gamma \to \widetilde{\psi}_0$ is continuous from $L^2(\mathbb{R})$ to $H^2(\mathbb{R})$ (only even functions are kept in these spaces). Let us write $\gamma = B_0 \widetilde{\psi}_0$ and $\widetilde{\psi}_0 = B_0^{-1} \gamma$. In particular

$$\widehat{\psi}_0 = B_0^{-1} \left(2U_2 + \frac{3}{2} U_2 \widehat{\phi}_1^2 \right) \tag{5.19}$$

is well defined. In fact it is straightforward to relate B_0 to the operator A_0 above via a rescaling.

Hence equation (2.4) for ψ_0 can be written in the form

$$\begin{cases} \psi_0(x) = \widehat{\psi}_0 + B_0^{-1} \gamma \\ \gamma = \frac{\epsilon^4}{4} \psi_0^3 + \frac{3\epsilon^2}{2} \psi_0^2 U_2 + \epsilon^2 \psi_0 + \frac{3\epsilon^2}{4} \psi_0 (\phi/\epsilon)^2 + \frac{3}{2} U_2 \{ (\phi/\epsilon)^2 - \widehat{\phi}_1^2 \} \end{cases} \tag{5.20}$$

where $\phi = \Phi_\epsilon(b, c, \psi_0)$ is defined from $\phi_0 = \Phi_{0,\epsilon}(c, \psi_0)$ and $\phi_1 = \Phi_{1,\epsilon}(b, \psi_0)$ as in (5.18). The function ϕ/ϵ is even, near $\widehat{\phi}/\epsilon$ in the $L^\infty(\mathbb{R})$-distance and

$$\|\phi/\epsilon\|_{L^\infty(\mathbb{R})} = O(\|\widehat{\phi}_0\|_{L^\infty} + \|\widehat{\phi}_1\|_{L^\infty} + \delta_1 + \delta_2). \tag{5.21}$$

(see (5.18) for the definition of $\widehat{\phi}$ and remember that $\delta_2 = 1$). If moreover $\phi_0(0) = b$ and $\epsilon^{-1}\phi_1'(x_\epsilon) = c$, then $\phi \in H^2(\mathbb{R})$. Observe that

$$\begin{cases} (\phi/\epsilon)^2 - \widehat{\phi_1^2} = \phi_1^2 - \widehat{\phi_1^2} & \text{on } [-x_\epsilon, x_\epsilon] \\ (\phi/\epsilon)^2 - \widehat{\phi_1^2} = \phi_0(\epsilon|x| - \epsilon^{3/4})^2 - \widehat{\phi_1^2} & \text{elsewhere} \end{cases}$$

and

$$\|U_2\{(\phi/\epsilon)^2 - \widehat{\phi_1^2}\}\|^2_{L^2(\mathbb{R})} \le \|\phi_1^2 - \widehat{\phi_1^2}\|^2_{L^\infty(-x_\epsilon,x_\epsilon)}\|U_2\|^2_{L^2(\mathbb{R})} + 2\left(\|\phi_0\|^2_{L^\infty(0,\infty)} + \|\widehat{\phi_1}\|^2_{L^\infty(0,\infty)}\right)^2 \|U_2\|^2_{L^2(\epsilon^{-1/4},\infty)}$$

Thanks to (3.14),

$$\limsup_{|b-2U_1(0)|+\epsilon\to 0} \|U_2\{(\phi/\epsilon)^2 - \widehat{\phi_1^2}\}\|^2_{L^2(\mathbb{R})} = 0. \tag{5.22}$$

By substituting the expression for γ into the r.h.s of the first equation in (5.20), the l.h.s. defines a smooth map

$$(\psi_0, b, c) \to \mathcal{T}_\epsilon(\psi_0, b, c)$$

from $\overline{B}_{H^2(\mathbb{R})}(\widehat{\psi_0}, 1) \times [2U_1(0) - \delta_4, 2U_1(0) + \delta_4] \times [-\delta_5, \delta_5]$ to $H^2(\mathbb{R})$:

$$\mathcal{T}_\epsilon(\psi_0, b, c) = \widehat{\psi_0} + B_0^{-1}\Big\{\frac{\epsilon^4}{4}\psi_0^3 + \frac{3\epsilon^2}{2}\psi_0^2 U_2 + \epsilon^2\psi_0 + \frac{3\epsilon^2}{4}\psi_0(\phi/\epsilon)^2 + \frac{3}{2}U_2\{(\phi/\epsilon)^2 - \widehat{\phi_1^2}\}\Big\},$$

where $\phi = \Phi_\epsilon(b, c, \psi_0)$.

We get

$$\lim_{|b-2U_1(0)|+\epsilon\to 0} \|\mathcal{T}_\epsilon(\widehat{\psi_0}, b, c) - \widehat{\psi_0}\|_{H^2(\mathbb{R})} \overset{(5.21),(5.22)}{=} 0,$$

uniformly in c. Also the map $\psi_0 \to \mathcal{T}_\epsilon(\psi_0, b, c)$ is a $1/2$-contraction on $\overline{B}_{H^2(\mathbb{R})}(\widehat{\psi_0}, 1)$ for all small enough ϵ, thanks to (3.13) and to (4.16). If δ_4 is small enough and $b \in [2U_1(0) - \delta_4, 2U_1(0) + \delta_4]$, existence and uniqueness as $\epsilon \to 0$ of a solution $\psi_0 = \Psi_{0,\epsilon}(b, c)$ to $\psi_0 = \mathcal{T}_\epsilon(\psi_0, b, c)$ is obtained with the help of the Banach fixed point theorem applied in the closed ball $\overline{B}_{H^2(\mathbb{R})}(\widehat{\psi_0}, 1)$.

As the Sobolev spaces we work with on $\mathbb{R}$ are made of even functions only, the fixed point is an even function. In addition the dependence of the fixed point $\Psi_{0,\epsilon}(b, c) \in H^2(\mathbb{R})$ on b and c is smooth and

$$\|\Psi_{0,\epsilon}(b, c) - \widehat{\psi_0}\|_{H^2(\mathbb{R})} \to 0$$

as $|b - 2U_1(0)| + \epsilon \to 0$, uniformly in c.

In order to get a solution to our problem, it suffices to show the existence of b and c such that

$$\Phi_{0,\epsilon}(c, \Psi_{0,\epsilon}(b, c))|_{x=0} = b \ \text{ and } \ \epsilon^{-1}\frac{d}{dx}\Phi_{1,\epsilon}(b, \Psi_{0,\epsilon}(b, c))|_{x=x_\epsilon} = c.$$

6 Problem IV for b and c

The problem amounts to finding b in some (ϵ-independent) neighborhood of $2U_1(0)$ and c in some neighborhood of 0, such that $\phi_0(0) = b$ and $\epsilon^{-1}\phi_1'(x_\epsilon) = c$, where $\phi_0 = \Phi_{0,\epsilon}(c, \Psi_{0,\epsilon}(b, c))$ near $\widehat{\phi_0}$, $\phi_1 = \Phi_{1,\epsilon}(b, \Psi_{0,\epsilon}(b, c))$ near $\widehat{\phi_1}|_{[-x_\epsilon,x_\epsilon]}$ and $\psi_0 = \Psi_{0,\epsilon}(b, c)$ near $\widehat{\psi_0}$ are given by the previous steps.

The real numbers $\phi_0(0)$ and $\epsilon^{-1}\phi_1'(x_\epsilon)$ depend on b, c and ϵ, and, for every small ϵ, the dependence of $\phi_0(0)$ and $\epsilon^{-1}\phi_1'(x_\epsilon)$ on b and c is smooth. We know that

$$\limsup_{\epsilon\to 0} |\phi_0(0) - 2U_1(0)| \le \text{Const}\,|c| \quad \text{and} \quad \lim_{\epsilon\to 0} \epsilon^{-1}\phi_1'(x_\epsilon) = 0 \tag{6.23}$$

uniformly in b and c (see (4.17) and (3.14)).

We can therefore use the Brouwer degree [5]. Let

$$\Omega = \{(b,c) \in \mathbb{R}^2 : |b - 2U_1(0)| < \delta_4, |c| < \delta_5\}$$

with $\delta_4, \delta_5 > 0$ small enough, let $g : \overline{\Omega} \to \mathbb{R}^2$ be the smooth map defined by

$$\begin{aligned} g(b,c) &= \left(b - \phi_0(0) + 2U_1(0), c - \epsilon^{-1}\phi_1'(x_\epsilon)\right) \\ &= \left(b - \Phi_{0,\epsilon}(c, \Psi_{0,\epsilon}(b,c))|_{x=0} + 2U_1(0), c - \epsilon^{-1}\frac{d}{dx}\Phi_{1,\epsilon}(b, \Psi_{0,\epsilon}(b,c))|_{x=x_\epsilon}\right) \end{aligned}$$

and let $p \in \mathbb{R}^2$ be given by

$$p = (2U_1(0), 0).$$

Let δ_5/δ_4 be so small that we get, for all $(b,c) \in \partial\Omega$ and all small enough ϵ,

$$\begin{aligned} |b - 2U_1(0)| = \delta_4 \Rightarrow |\phi_0(0) - 2U_1(0)| &\overset{(2.8),(6.23)}{=} O(\delta_5) < \delta_4/2 \\ \text{and } |c| = \delta_5 &\overset{(6.23)}{\Rightarrow} |\epsilon^{-1}\phi_1'(x_\epsilon)| < \delta_5/2 \end{aligned} \tag{6.24}$$

Then the degree $\deg(g, \Omega, p)$ satisfies $\deg(g, \Omega, p) = \deg(I_2, \Omega, p)$, where I_2 is the identity map in $\mathbb{R}^2$, as it can be seen by considering the homotopy

$$h(\lambda, x) = (1-\lambda)g(x) + \lambda x, \ \lambda \in [0,1], \ x \in \overline{\Omega}.$$

This homotopy is admissible for small ϵ, in the sense that $h([0,1] \times \partial\Omega)$ does not contain p for all small enough $\epsilon > 0$ (thanks to (6.24)):

$$\begin{aligned} h(\lambda, x) = p \text{ with } x = (b,c) \in \overline{\Omega} \\ \Rightarrow \left(b - (1-\lambda)(\phi_0(0) - 2U_1(0)), \ c - (1-\lambda)\epsilon^{-1}\phi_1'(x_\epsilon)\right) = (2U_1(0), 0) \\ \Rightarrow (b - 2U_1(0), c) = (1-\lambda)\left(\phi_0(0) - 2U_1(0), \ \epsilon^{-1}\phi_1'(x_\epsilon)\right) \overset{(6.24)}{\Rightarrow} x \notin \partial\Omega. \end{aligned}$$

By homotopy invariance, we get that $\deg(g, \Omega, p) = \deg(I_2, \Omega, p)$. By normalization, $\deg(I_2, \Omega, p) = 1$ and thus $g(x) = p$ has at least one solution $x \in \Omega$. Hence $\phi_0(0) = b$ and $\epsilon^{-1}\phi_1'(x_\epsilon) = c$ for some (b,c) near $(2U_1(0), 0)$ (if ϵ is small enough).

7 Conclusion

Hence we have proved the existence of a bifurcation branch of solutions $(\phi_\lambda, \psi_\lambda)$ in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$. Incidentally, the method gives a qualitative description of the branch: it looks like the family of discontinuous maps

$$\left(\widehat{\phi}, 2U_2 + \epsilon^2\widehat{\psi}_0\right), \quad \epsilon = \sqrt{1-\lambda},$$

where $\lambda < 1$ is near 1,

$$\widehat{\phi}(x) = \begin{cases} 2\epsilon U_1(0)(1 - \frac{3}{2}\cosh^{-2}(\sqrt{2}\,x)) & \text{if } |x| \le \epsilon^{-1/4} \\ 2\epsilon U_1(\epsilon x - \epsilon^{3/4}) & \text{if } |x| \ge \epsilon^{-1/4} \end{cases}$$

(see (5.18)) and $\widehat{\psi}_0 \in H^2(\mathbb{R})$ is uniquely determined by

$$-\widehat{\psi}_0'' + 2\widehat{\psi}_0 - 3U_2^2\widehat{\psi}_0 = 2U_2 + 6U_1(0)^2\left(1 - \frac{3}{2}\cosh^{-2}(\sqrt{2}\,x)\right)^2 U_2.$$

We can replace this family by the following family of smooth maps that still describes the bifurcating branch approximatively:

$$\left(2\epsilon U_1(\epsilon x) - 2\epsilon U_1(0)\frac{3}{2}\cosh^{-2}(\sqrt{2}\,x),\ 2U_2 + \epsilon^2\widehat{\psi}_0\right).$$

Indeed the corresponding ϕ_0, ϕ_1 and ψ_0 are given by

$$\phi_0(x) = 2U_1(x + \epsilon^{3/4}) - 3U_1(0)\cosh^{-2}(\epsilon^{-1}\sqrt{2}(x + \epsilon^{3/4})),$$

$$\phi_1(x) = 2U_1(\epsilon x) - 3U_1(0)\cosh^{-2}(\sqrt{2}\,x)$$

and $\psi_0 = \widehat{\psi}_0$. Moreover

$$\|\phi_0 - \widehat{\phi}_0\|_{H^2(0,\infty)} \to 0 \ \text{ and } \ \|\phi_1 - \widehat{\phi}_1\|_{H^2_\infty(-x_\epsilon, x_\epsilon)} \to 0$$

as $\epsilon \to 0$.

Equivalently, this proves the existence of a branch (u_λ, v_λ) in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ of solutions to the system (1.1) that looks like the family

$$\Big(U_2 + \epsilon U_1(\epsilon x) - \epsilon U_1(0)\tfrac{3}{2}\cosh^{-2}(\sqrt{2}\,x) + \tfrac{1}{2}\epsilon^2\widehat{\psi}_0\,,\ - U_2 + \epsilon U_1(\epsilon x) - \epsilon U_1(0)\frac{3}{2}\cosh^{-2}(\sqrt{2}\,x) - \frac{1}{2}\epsilon^2\widehat{\psi}_0\Big)$$

($\epsilon = \sqrt{1-\lambda}$ and $\lambda < 1$ is near 1).

This result is a special instance of bifurcation from the essential spectrum (see e.g.[8]).

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