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Statistical Functional Equations and p-Harmonious Functions

(Dedicated to Professor Klaus Schmitt, a continuing source of inspiration, a valued mentor and a good friend, on the occasion of his retirement)

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Abstract

Motivated by the mean-value property characterizing harmonic functions and recently established asymptotic statistical formulas characterizing p-harmonic functions, we consider the Dirichlet problem for a functional equation involving a convex combination of the mean and median. We show that this problem has a continuous solution when it has both a subsolution and a supersolution. We demonstrate that solutions of these problems approximate p-harmonic functions and discuss connections with related results of Manfredi, Parviainen and Rossi.

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1 Introduction and notation

Throughout this paper, Ω denotes a bounded, open and connected subset of $\mathbb{R}^N$. A well-known and amazing fact about harmonic functions on such domains is that they can be characterized by the mean-value property: the continuous function u is harmonic in Ω if and only if

$$u(x) = \fint_{\partial B(x,r)} u(s)\,ds = \fint_{B(x,r)} u(y)\,dy \quad \text{for each} \quad x \in \Omega\,, \tag{1.1}$$

where $B(x,r) \subset \Omega$ is a ball with center x and radius $r > 0$, $\partial B(x,r)$ is its boundary, and $\fint_E f$ denotes the average of f over the set E. Using the mean-value property, a harmonic function can be defined without reference to derivatives or to Laplace's equation, and identity (1.1) is the prototypical *statistical* characterization of solutions of a partial differential equation.

Recent work generalizes this result and reveals that p-harmonic functions can be characterized by their statistics. Recall that, for $1 < p < \infty$, a function u is p-harmonic if and only if it is a solution of

$$-\Delta_p u := -\operatorname{div}\left(|Du|^{p-2} Du\right) = 0, \tag{1.2}$$

where the notation $D\varphi$ indicates the gradient of the function φ, and we refer to the operator Δ_p as the p-Laplacian.

Solutions of (1.2) are to be understood in the weak sense or in the viscosity sense, notions that have been shown to be equivalent for this equation [16]. In [32], Manfredi, Parviainen and Rossi proved that a continuous function u is p–harmonic in $\Omega \subset \mathbb{R}^N$ if and only if the functional equation

$$u(x) = \frac{1-q}{2}\left\{\max_{\overline{B(x,\varepsilon)}} u + \min_{\overline{B(x,\varepsilon)}} u\right\} + q \fint_{B(x,\varepsilon)} u(y)\,dy + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0 \tag{1.3}$$

holds in the viscosity sense for all $x \in \Omega$, where

$$q = \frac{2+N}{p+N}\,. \tag{1.4}$$

Note that, when $p = 2$, $q = 1$ and (1.3) reduces to an asymptotic version of the mean-value property; when $p > 2$, $0 < q < 1$ and (1.3) shows that a p-harmonic function can be characterized asymptotically in terms of a convex combination of its mean over a ball and the average of its maximum and minimum values on that ball.

Using a similar approach, we proved in [15] that p-harmonic functions of two variables have additional statistical asymptotic characterizations. Specifically, [15] shows that a continuous function u is p-harmonic in $\Omega \subset \mathbb{R}^2$ if and only if

$$u(x) = \left(\frac{2}{p} - 1\right) \operatorname*{med}_{\partial B(x,\varepsilon)} \{u\} + \left(2 - \frac{2}{p}\right) \fint_{\partial B(x,\varepsilon)} u(s)\,ds + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0 \tag{1.5}$$

holds in the viscosity sense at each $x \in \Omega$, which is equivalent to

$$u(x) = \frac{1}{p} \operatorname*{med}_{\partial B(x,\varepsilon)} \{u\} + \left(\frac{p-1}{2p}\right)\left(\max_{\overline{B_\varepsilon(x)}} \{u\} + \min_{\overline{B_\varepsilon(x)}} \{u\}\right) + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0 \tag{1.6}$$

holding in the viscosity sense at each $x \in \Omega$.

In (1.5) and (1.6), $\operatorname{med}_E \{u\}$ is the median of the function u over the set E. The median of a function does not seem to be commonly used by analysts, but it has been studied at least since the time of Lévy [28] and is also known as the Lévy mean. Medians of random variables, as well as a variational characterization of them, appear in [53]. They also play an important role in the local theory of Banach spaces and concentration of measure (see [27] and [36]). In Section 2 below, we give a definition of the median and explore some of its properties.

The asymptotic characterizations (1.3), (1.5) and (1.6) raise the following interesting question: what happens if we drop the $o(\varepsilon^2)$ error terms? Manfredi, Parviainen and Rossi pursued this question in [33], obtaining the following results. Given $\varepsilon > 0$, define the compact boundary strip Γ_ε by

$$\Gamma_\varepsilon := \left\{ x \in \mathbb{R}^N \setminus \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \leq \varepsilon \right\} ,$$

where $\operatorname{dist}(x, \partial\Omega)$ denotes the distance from $x \in \mathbb{R}^N$ to $\partial\Omega$, and define $0 < q \leq 1$ by (1.4). Given a bounded measurable function $F : \Gamma_\varepsilon \to \mathbb{R}$, there exists a unique function u_ε such that

$$u_\varepsilon(x) = \frac{1-q}{2} \left\{ \sup_{\overline{B(x,\varepsilon)}} \{u_\varepsilon\} + \inf_{\overline{B(x,\varepsilon)}} \{u_\varepsilon\} \right\} + q \fint_{B(x,\varepsilon)} u_\varepsilon(y)\, dy$$

at each $x \in \Omega$ and $u_\varepsilon(x) = F(x)$ at each $x \in \Gamma_\varepsilon$. As $\varepsilon \to 0$, these solutions u_ε converge uniformly to the unique p-harmonic function u on Ω with $u = F$ on $\partial\Omega$; consequently, they extend the terminology in [25] and [26] and refer to the solutions u_ε as p-harmonious functions.

To explain how our results complement those of [33], we must first establish some more notation. Suppose that $h \geq 0$ is given, define $r_h : \overline{\Omega} \to [0, \sqrt{2h}]$ by

$$r_h(x) = \begin{cases} \sqrt{2h} & \text{if} \quad \operatorname{dist}(x, \partial\Omega) \geq \sqrt{2h}\,, \\ \operatorname{dist}(x, \partial\Omega) & \text{otherwise}\,, \end{cases}$$

and let B_x^h denote the ball $B(x, r_h(x))$. We assume until further notice (see Section 4) that h is positive and remains fixed. The function r_h is continuous, constant sufficiently far away from $\partial\Omega$ and vanishes on $\partial\Omega$; the specific function chosen here is for convenience in the convergence arguments of Sections 4 and 5.

Given $q \in (0, 1)$ and $g \in C(\partial\Omega)$, we seek a function $u \in C(\overline{\Omega})$ such that

$$\begin{cases} u = q \operatorname*{mean}_{\partial B_x^h} \{u\} + (1-q) \operatorname*{med}_{\partial B_x^h} \{u\} & \text{in} \quad \Omega\,, \\ u = g & \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.7}$$

We prove in Theorem 3.2 that problem (1.7) has a unique continuous solution when both a subsolution and a supersolution exist; in Theorem 3.3, we show that we can always find a subsolution and a supersolution if Ω is strictly convex. In Section 4, we prove that when $N = 2$, solutions of (1.7) converge as $h \to 0$ to the unique p-harmonic function with boundary values g, where $1 < p \leq 2$ and q is determined by p. By analogy with the work in [33], it therefore seems reasonable to call solutions of (1.7) p-harmonious. In contrast to [33], however, we treat the boundary condition in (1.7) directly, without having to work on the boundary strip Γ_ε. Moreover, we show in Section 5 that our elementary proof of Theorem 3.2 adapts easily to the problem

$$\begin{cases} u = q \operatorname*{mean}_{B_x^h} \{u\} + \dfrac{1-q}{2} \left\{ \max_{\overline{B_x^h}} u + \min_{\overline{B_x^h}} u \right\} & \text{in} \quad \Omega\,, \\ u = g & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.8}$$

proving that (1.8) has a unique solution if it has both a subsolution and a supersolution. The solutions of (1.8) converge as $h \to 0$ to the unique p-harmonic function u on Ω that equals g on $\partial\Omega$, where $p \geq 2$, q is determined by p and N, and $N \geq 2$. Solutions of (1.8) are thus p-harmonious for $p \geq 2$.

It is a pleasure to dedicate this article to Klaus Schmitt and to connect our results to some of his work. The method of sub- and supersolutions has been a recurrent theme in his long and prolific career. Indeed, many of his earliest publications involve lower and upper solutions ([41]–[49] and [11]). Work on this subject has spanned his professional life and includes some of his most recent papers; see, for example, [50], [2], [34], [13], [23], [24], [51], [29] and [30]. Furthermore, a large portion of his work has involved the study of equations involving the p-Laplacian or more general quasilinear equations from a variety of perspectives. We mention (in addition to some of the articles already cited) the following selection of publications, listed chronologically: [4], [8], [31], [5], [9], [14], [7], [10], [3], [22], [35], [52], [20] and [21].

The rest of the paper is organized as follows. In Section 2, we present the definition of the median of a function, review its basic properties and establish further properties needed for the study of the Dirichlet problem (1.7). The definitions of sub and supersolutions for (1.7), maximum and comparison principles, and the proof of Theorem 3.2 guaranteeing the existence of a $C(\overline{\Omega})$ solution when a sub and supersolution exist are in Section 3. This section also contains the proof that when Ω is strictly convex sub and supersolutions can always be found. The following section concerns the approximation of p-harmonic functions by p-harmonious ones. Problem (1.8) is discussed in Section 5.

2 Medians and the median operator

The Lebesgue measure of a measurable set $E \subset \mathbb{R}^k$ will be denoted $|E|$, where the dimension k is clear by context. We begin with the following basic definition [55, p. 269]:

Definition 2.1 *If u is a real-valued integrable function on the measurable set $E \subset \mathbb{R}^k$ and $0 < |E| < \infty$, then m is a median of u over E if and only if*

$$|\{x \in E \,:\, u(x) < m\}| \leq \frac{1}{2}|E| \qquad \textit{and} \qquad |\{x \in E \,:\, u(x) > m\}| \leq \frac{1}{2}|E|\,.$$

$\mathrm{med}_E\{u\}$ *denotes the set of all medians of u over E. We will often abbreviate sets such as $\{x \in E : u(x) < m\}$ and $\{x \in E : u(x) > m\}$ by $\{u < m\}$ and $\{u > m\}$, respectively, with the understanding that the function u is restricted to the set over which the median is taken.*

As pointed out in [55, p. 269], $\mathrm{med}_E\{u\}$ is a non-empty compact interval,

$$\operatorname*{med}_E\{\alpha u + \beta\} = \alpha \operatorname*{med}_E\{u\} + \beta \tag{2.9}$$

for any constants α and β, and

$$|\{u \geq m\}| \geq \frac{1}{2}|E| \qquad \text{and} \qquad |\{u \leq m\}| \geq \frac{1}{2}|E|$$

whenever $m \in \mathrm{med}_E\{u\}$.

Alternatively, one could define the median of a function $u : E \to \mathbb{R}$ by $\operatorname{med}_E u := u^*(|E|/2)$, where u^* is the non-increasing one-dimensional rearrangement of u. This approach involves choosing a specific value for the median (just as one does with medians of finite sets) instead of allowing the median to be multi-valued as in Definition 2.1. For much more on rearrangements, we refer the reader to, for example, [17].

Before stating the fundamental Lemma 2.1 below, we recall some facts about semicontinuous functions for the reader's convenience [6]. By definition, $u : E \to \mathbb{R}$ is lower semicontinuous if and only if $\{x \in E : u > c\}$ is relatively open for every $c \in \mathbb{R}$; $LSC(E)$ denotes the set of lower semicontinuous functions on E. Similarly, $u : E \to \mathbb{R}$ is upper semicontinuous if and only if $\{x \in E : u < c\}$ is relatively open for every $c \in \mathbb{R}$; $USC(E)$ denotes the set of upper semicontinuous functions on E. A function u clearly belongs to $C(E)$ if and only if $u \in LSC(E) \cap USC(E)$. We rely on the basic facts that the supremum of a subset of $LSC(E)$ is an element of $LSC(E)$, while the infimum of a subset of $USC(E)$ is an element of $USC(E)$.

Lemma 2.1 *Suppose that $u \in LSC(E)$, $U \in USC(E)$, $u \geq U$, and that $E \subset \mathbb{R}^k$ is compact and connected. If $m \in \operatorname{med}_E u$ and $M \in \operatorname{med}_E U$, then $m \geq M$.*

Proof. Suppose to the contrary that $m < M$. Since $u \geq U$,

$$\{u \leq m\} \cap \{U \geq M\} = \emptyset\,, \quad \{u \leq m\} \subset \{U < M\}\,, \quad \text{and} \quad \{U \geq M\} \subset \{u > m\}\,. \tag{2.10}$$

It follows from these containments and the definition of medians that

$$\frac{1}{2}|E| \geq |\{U < M\}| \geq |\{u \leq m\}| \geq \frac{1}{2}|E|$$

and, similarly, that

$$\frac{1}{2}|E| \geq |\{u > m\}| \geq |\{U \geq M\}| \geq \frac{1}{2}|E|\,.$$

These inequalities must therefore be identities, and we have

$$|\{u > m\}| = |\{u \leq m\}| = |\{U \geq M\}| = |\{U < M\}| = \frac{1}{2}|E|\,. \tag{2.11}$$

Since $u \in LSC(E)$ and $U \in USC(E)$, the sets $\{u > m\}$ and $\{U < M\}$ are both open. Consequently, the sets $\{u \leq m\}$ and $\{U \geq M\}$ are compact subsets of E which, by (2.10) and (2.11), respectively, are disjoint and have measure $\frac{1}{2}|E|$. Because E is connected this is impossible, so m cannot be less than M.

Many of the basic properties of medians follow directly from Lemma 2.1. First, we show that medians of continuous functions on connected domains are unique. Using a variational characterization of the median, Noah proved this result in [37]. Using different methods, this result was established for planar domains in [54].

Proposition 2.1 *Suppose that $E \subset \mathbb{R}^k$ is compact and connected. If $v \in C(E)$, then the median of v over E is unique.*

Proof. Suppose that m and M both belong to $\operatorname{med}_E v$. Letting $u = v$ and $U = v$ in Lemma 2.1 yields both $m \geq M$ and $M \geq m$, so that $m = M$ as claimed.

Combining Lemma 2.1 and Proposition 2.1 shows that the median functional is monotone on continuous functions on connected domains.

Corollary 2.1 *Suppose that $E \subset \mathbb{R}^k$ is compact and connected. If $u \in C(E)$ and $v \in C(E)$ satisfy $u \geq v$, then $\operatorname{med}_E u \geq \operatorname{med}_E v$.*

We now use the median to define the median operator on $C(\overline{\Omega})$; recall from the introduction that $\Omega \subset \mathbb{R}^N$ is bounded, open and connected. Note that the following definition involves medians over spheres, so the measure being used is $(N-1)$-dimensional surface measure.

Definition 2.2 *Given $h \geq 0$, define the median operator $\operatorname{med}_h : C(\overline{\Omega}) \to C(\overline{\Omega})$ by*

$$\left(\operatorname*{med}_h u\right)(x) := \begin{cases} \operatorname{med}_{\partial B_x^h} u & \text{if} \quad x \in \Omega\,, \\ u(x) & \text{if} \quad x \in \partial\Omega\,. \end{cases}$$

For $h = 0$ we take $\partial B_x^h = \{x\}$ and define $\operatorname{med}_{\partial B_x^h} u = u(x)$.

To show that this operator is well-defined, we must verify that it maps continuous functions to continuous functions. This claim is trivial when $h = 0$. We remark that the operator med_h can be applied to (some) functions that fail to be continuous in $\overline{\Omega}$; if u is such a function, $\operatorname{med}_h u$ need not be continuous and can, in fact, be multi-valued.

Proposition 2.2 *If $u \in C(\overline{\Omega})$, then $\operatorname{med}_h u \in C(\overline{\Omega})$.*

Proof. Let $\varepsilon > 0$ be given. Since $u \in C(\overline{\Omega})$ and $\overline{\Omega}$ is compact, u is uniformly continuous, so there exists a $\delta > 0$ such that $|u(y) - u(z)| < \varepsilon$ whenever $y, z \in \overline{\Omega}$ and $|y - z| < \delta$.

Let $x \in \Omega$. For any $y \in \Omega$, note that

$$\operatorname*{med}_{\partial B_y^h} \{u\} = \operatorname*{med}_{s \in \partial B_y^h} \{u(s)\} = \operatorname*{med}_{s \in \partial B_x^h} \left\{u\left(\rho_y(s)\right)\right\},$$

where $\rho_y : \partial B_x^h \to \partial B_y^h$ is defined by

$$\rho_y(s) := y + \frac{r_h(y)}{r_h(x)}(s - x)\ .$$

Using the compactness of $\overline{\Omega}$ and the continuity of $r_h(\cdot)$, there exists a $\delta_1 > 0$ such that

$$\left| s - \rho_y(s) \right| < \delta$$

for any $s \in \partial B_x^h$ as long as $|x - y| < \delta_1$, which we assume henceforth. For any $s \in \partial B_x^h$, it follows that

$$\left| u(s) - u(\rho_y(s)) \right| < \varepsilon\,,$$

so that

$$u(s) - \varepsilon < u(\rho_y(s)) < u(s) + \varepsilon\ .$$

By (2.9) and the monotonicity of the median, we obtain

$$\operatorname*{med}_{\partial B_x^h} \{u(s)\} - \varepsilon < \operatorname*{med}_{\partial B_x^h} \{u(\rho_y(s))\} < \operatorname*{med}_{\partial B_x^h} \{u(s)\} + \varepsilon$$

and conclude that

$$|\operatorname*{med}_{\partial B^h_x}\{u\} - \operatorname*{med}_{\partial B^h_y}\{u\}| < \varepsilon\,,$$

verifying the continuity of $\mathrm{med}_h\, u$ at x.

Now let $x \in \partial\Omega$. For all $y \in \overline{\Omega}$ sufficiently close to x, $u(x) - \varepsilon < u(z) < u(x) + \varepsilon$ for all $z \in \partial B^h_y$. Monotonicity of the median and (2.9) then imply

$$|\operatorname*{med}_{\partial B^h_y}\{u\} - u(x)| = |\operatorname*{med}_h\{u\}(y) - \operatorname*{med}_h\{u\}(x)| < \varepsilon,$$

so that $\mathrm{med}_h\, u$ is continuous at x.

Proposition 2.3 *1. Suppose that $\{u_n\} \subset C(\overline{\Omega})$ is a nondecreasing sequence that converges pointwise to $u \in LSC(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$. At each $x \in \Omega$,*

$$\operatorname*{med}_{\partial B^h_x} u_n \to \min\left(\operatorname*{med}_{\partial B^h_x} u\right).$$

2. Suppose that $\{u_n\} \subset C(\overline{\Omega})$ is a nonincreasing sequence that converges pointwise to $u \in USC(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$. At each $x \in \Omega$,

$$\operatorname*{med}_{\partial B^h_x} u_n \to \max\left(\operatorname*{med}_{\partial B^h_x} u\right).$$

Proof. The proofs of both claims are similar, so we only prove the first statement. Fix a point $x \in \Omega$. Since $u_n \le u_{n+1}$ for all n and $u_n \to u$ pointwise, we know *a priori* that u must be lower semicontinuous. Since u is also bounded by assumption, we know that its medians over ∂B^h_x are finite; let m_* denote the smallest element of $\operatorname*{med}_{\partial B^h_x} u$. Lemma 2.1 and Corollary 2.1 show, respectively, that

$$\operatorname*{med}_{\partial B^h_x} u_n \le m_* \quad \text{and} \quad \operatorname*{med}_{\partial B^h_x} u_n \le \operatorname*{med}_{\partial B^h_x} u_{n+1}$$

for all n, so the nondecreasing sequence $\{\operatorname*{med}_{\partial B^h_x} u_n\}$ is bounded above and must converge to $m \in \mathbb{R}$, where $m \le m_*$.

If m were strictly less than m_*, then there would exist an $\varepsilon > 0$ such that $m < m_* - \varepsilon$ and we would have $\operatorname*{med}_{\partial B^h_x} u_n < m_* - \varepsilon$ for each n. Consequently,

$$|\{u_n \le m_* - \varepsilon\}| \ge |\{u_n < m_* - \varepsilon\}| \ge \frac{1}{2}|\partial B^h_x|$$

for each n, and thus

$$\lim_{n\to\infty} |\{u_n \le m_* - \varepsilon\}| \ge \frac{1}{2}|\partial B^h_x|\,.$$

On the other hand,

$$\lim_{n\to\infty} |\{u_n \le m_* - \varepsilon\}| = \left|\bigcap_{n=1}^{\infty}\{u_n \le m_* - \varepsilon\}\right| = |\{u \le m_* - \varepsilon\}|\,,$$

so we see that

$$|\{u \le m_* - \varepsilon\}| \ge \frac{1}{2}|\partial B_x^h| \,. \tag{2.12}$$

Since $\{u \le m_* - \varepsilon\} \subset \{u < m_*\}$,

$$|\{u \le m_* - \varepsilon\}| \le |\{u < m_*\}| \,.$$

On the other hand, m_* is a median for u, so $|\{u < m_*\}| \le \frac{1}{2}|\partial B_x^h|$. Combining this fact with (2.12), we conclude that

$$\frac{1}{2}|\partial B_x^h| = |\{u < m_*\}| = |\{u \le m_* - \varepsilon\}|.$$

As a consequence, the sets $\{u < m_* - \varepsilon\}$ and $\{u > m_* - \varepsilon\}$ both have measure less than or equal to $(1/2)|\partial B_x^h|$, so that $m_* - \varepsilon$ is a median for u over ∂B_x^h, contradicting the definition of m_*. Therefore, $m = m_*$ as claimed.

3 The Dirichlet Problem

Recall from the introduction that $h > 0$ has been given and that $\Omega \subset \mathbb{R}^N$ is bounded, open and connected. In addition, suppose throughout this section that $q \in (0, 1)$ has been given. Using the results of the previous section, we can now address the Dirichlet problem described in the introduction.

3.1 Statement of the problem and definitions

Define the operator $M_q^h : C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$\left(M_q^h u\right)(x) := \begin{cases} q \operatorname*{mean}_{\partial B_x^h}\{u\} + (1-q)\operatorname*{med}_{\partial B_x^h}\{u\}, & \text{if} \quad x \in \Omega, \\ u(x), & \text{if} \quad x \in \partial\Omega. \end{cases} \tag{3.13}$$

$M_q^h u(x)$ is therefore a convex combination of the mean and median of u over the boundary of the ball B_x^h. The uniform continuity of u, combined with the continuity of r_h, implies that the map $x \to \operatorname{mean}_{\partial B_x^h}\{u\}$ is continuous on $\overline{\Omega}$, so that by Proposition 2.2, $M_q^h u \in C(\overline{\Omega})$ for any $u \in C(\overline{\Omega})$. Furthermore, we see from the linearity of the mean and the results of the previous section that M_q^h is monotone, homogeneous, and invariant under translation by constants.

We now restate the Dirichlet problem that is our main concern: given $g \in C(\partial\Omega)$, we seek a function $u \in C(\overline{\Omega})$ such that

$$\begin{cases} u = M_q^h u & \text{in} \quad \Omega\,, \\ u = g & \text{on} \quad \partial\Omega\,. \end{cases} \tag{3.14}$$

Observe that this problem is very well understood when $q = 1$; as long as the boundary $\partial\Omega$ is sufficiently regular, the Dirichlet problem for Laplace's equation has a unique continuous solution that is characterized by the mean-value property. Problem (3.14) is also well understood when $q = 0$ and $\Omega \subset \mathbb{R}^2$; in this case, we generally expect either nonuniqueness or nonexistence, depending on the interaction of the boundary data g and the geometry of the boundary $\partial\Omega$ [40]. Previously, no results were available for this problem when $0 < q < 1$.

Since we rely on subsolutions and supersolutions to solve (3.14), we conclude this section with the following natural definitions.

Definition 3.1 *The function $v \in C(\overline{\Omega})$ is a subsolution of (3.14) if and only if*

$$\begin{cases} v \le M_q^h v & \text{in} \quad \Omega\,, \\ v = g & \text{on} \quad \partial\Omega\,. \end{cases}$$

Similarly, $w \in C(\overline{\Omega})$ is a supersolution to (3.14) if and only if

$$\begin{cases} w \ge M_q^h w & \text{in} \quad \Omega\,, \\ w = g & \text{on} \quad \partial\Omega\,. \end{cases}$$

Note that we require subsolutions and supersolutions to be continuous and to attain the prescribed boundary data. Continuity guarantees that their medians are single-valued and thus that the inequalities defining them make sense.

3.2 Comparison and maximum principles

We begin with a comparison principle which immediately yields uniqueness for solutions of (3.14).

Theorem 3.1 *If v is a subsolution of (3.14) and w is a supersolution of (3.14), then $v \le w$.*

Proof. By definition, $v = w$ on $\partial\Omega$. If the claim were not true, then $v - w$ would attain a positive maximum S in the interior of Ω; define $\Omega_S := \{x \in \Omega \,:\, v(x) - w(x) = S\}$. Then Ω_S is closed and $\Omega_S \cap \partial\Omega = \emptyset$. By the definition of S, $v \le S + w$ throughout $\overline{\Omega}$, so monotonicity and invariance under translation by constants yield

$$M_q^h v \le M_q^h(w + S) = M_q^h w + S \quad \text{in} \quad \Omega\,.$$

There must exist a point $x \in \Omega_S$ such that $\partial B_x^h \not\subset \Omega_S$. By continuity, $v - w < S$ in a nonempty relatively open subset of ∂B_x^h, and it follows that

$$\operatorname*{mean}_{\partial B_x^h}(v - w) < S, \quad \text{i.e.,} \quad \operatorname*{mean}_{\partial B_x^h} v < S + \operatorname*{mean}_{\partial B_x^h} w. \tag{3.15}$$

Using the definitions of subsolution and supersolution, the definition of S, translation invariance, and monotonicity, we obtain

$$S + w(x) = v(x) \le M_q^h v(x) \le S + M_q^h w(x) \le S + w(x)\,,$$

and we see that these inequalities must be equalities. In particular, $M_q^h v(x) = S + M_q^h w(x)$, so that

$$\begin{aligned} M_q^h v(x) &= (1-q) \operatorname*{med}_{\partial B_x^h} v + q \operatorname*{mean}_{\partial B_x^h} v \\ &= (1-q)\left(S + \operatorname*{med}_{\partial B_x^h} w\right) + q\left(S + \operatorname*{mean}_{\partial B_x^h} w\right). \end{aligned}$$

Combining this with (3.15), we must have

$$\operatorname*{med}_{\partial B_x^h} v > \operatorname*{med}_{\partial B_x^h}(w + S),$$

contradicting the monotonicity of the median and the fact that $v \le w + S$ in Ω. We conclude that $v \le w$ in Ω.

The following maximum principle is both obvious and useful. Among other things, it provides uniform bounds on iterates that are used in the proof of Theorem 3.2.

Proposition 3.1 *A subsolution of (3.14) cannot have a strict maximum inside Ω; a supersolution of (3.14) cannot have a strict minimum inside Ω. Furthermore if v is a subsolution of (3.14),* $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$*, and if w is a supersolution,* $\min_{\overline{\Omega}} w = \min_{\partial\Omega} w$.

3.3 Existence of solutions on general domains

Next, we establish the existence of a solution of problem (3.14) when it has both a subsolution and a supersolution. The uniqueness of this solution follows from Theorem 3.1.

Theorem 3.2 *If there exist a subsolution v and a supersolution w of (3.14), then problem (3.14) has a unique solution $u \in C(\overline{\Omega})$.*

Proof. Define the sequences $\{u_n\} \subset C(\overline{\Omega})$ and $\{U_n\} \subset C(\overline{\Omega})$ iteratively as follows:

$$u_0 := v\,, \quad u_{n+1} := M_q^h u_n \quad \text{for} \quad n \geq 0\,, \quad U_0 := w\,, \quad \text{and} \quad U_{n+1} := M_q^h U_n \quad \text{for} \quad n \geq 0\,.$$

It follows directly from monotonicity and the definitions of subsolution and supersolution that

$$u_n \leq u_{n+1} \quad \text{and} \quad U_n \geq U_{n+1} \quad \text{for all} \quad n,$$

u_n is a subsolution of (3.14) for each n, and U_n is a supersolution of (3.14) for each n. Theorem 3.1 implies that

$$u_m \leq U_n \quad \text{for all} \quad m, n\,, \tag{3.16}$$

and Proposition 3.1 shows that

$$u_n \leq \max_{\partial\Omega} g \quad \text{and} \quad U_n \geq \min_{\partial\Omega} g \quad \text{for all} \quad n.$$

Consequently, the bounded, monotone sequences $\{u_n\}$ and $\{U_n\}$ converge pointwise to the functions u and U defined, respectively, by

$$u(x) := \lim_{n\to\infty} u_n(x) \quad \text{and} \quad U(x) := \lim_{n\to\infty} U_n(x), \quad x \in \overline{\Omega},$$

and it follows from (3.16) that

$$u \leq U \quad \text{throughout} \quad \overline{\Omega}.$$

Consequently, $U - u \geq 0$ in $\overline{\Omega}$. Moreover, since $u_0 \leq u_n \leq \max_{\partial\Omega} g$ for all n, and u_0 and the constant function $\max_{\partial\Omega} g$ are integrable on ∂B_x^h for any $x \in \Omega$, $u_n \to u$ in $L^1(\partial B_x^h)$ by the dominated convergence theorem. Similarly $U_n \to U$ in $L^1(\partial B_x^h)$ for any $x \in \Omega$.

Since u is the supremum of continuous functions, $u \in LSC(\overline{\Omega})$; similarly, since U is the infimum of continuous functions, $U \in USC(\overline{\Omega})$. Also, since $u_n = U_n = g$ on $\partial\Omega$ for every n, $u = U = g$ on $\partial\Omega$ as well. In fact, u and U are both continuous on $\partial\Omega$, as the following argument shows. Since each u_n is a subsolution of (3.14), Theorem 3.1 gives that $u_n(y) \leq w(y)$ for all n and for all $y \in \overline{\Omega}$, and, as a result, $u(y) \leq w(y)$ for all $y \in \overline{\Omega}$. Now let $x \in \partial\Omega$ and $\epsilon > 0$. There exists $\delta > 0$ such that $|u_0(y) - g(x)|$ and $|w(y) - g(x)|$ are both less than ϵ when $y \in \Omega$ and $|x - y| < \delta$. For such y we get

$$g(x) - \epsilon < u_0(y) \leq u(y) \leq w(y) < g(x) + \epsilon$$

concluding that $|u(y) - g(x)| < \epsilon$. The proof that U is continuous on $\partial\Omega$ is similar.

To see that $u = U$ in $\overline{\Omega}$ and thus that u is a continuous solution of (3.14), observe first that, thanks to the continuity of the mean operator with respect to L^1 convergence and Proposition 2.3,

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \lim_{n\to\infty} M_q^h u_n(x) = q \operatorname*{mean}_{\partial B_x^h} \{u\} + (1-q) \min \left(\operatorname*{med}_{\partial B_x^h} u \right) \tag{3.17}$$

at every $x \in \Omega$. Similarly, we find that

$$U(x) = q \operatorname*{mean}_{\partial B_x^h} \{U\} + (1-q) \max \left(\operatorname*{med}_{\partial B_x^h} U \right) \tag{3.18}$$

at each $x \in \Omega$.

Since the nonnegative function $U - u$ is upper semicontinuous, it attains its (nonnegative) maximum somewhere in $\overline{\Omega}$. If this maximum is zero, then $U = u$ throughout $\overline{\Omega}$ and the proof is complete. Suppose that $U - u$ attains its maximum at $z \in \Omega$. (2.9), (3.17) and (3.18) then yield

$$0 = q \operatorname*{mean}_{\partial B_z^h} \{u - u(z)\} + (1-q) \min \left(\operatorname*{med}_{\partial B_z^h} \{u - u(z)\} \right)$$

and

$$0 = q \operatorname*{mean}_{\partial B_z^h} \{U - U(z)\} + (1-q) \max \left(\operatorname*{med}_{\partial B_z^h} \{U - U(z)\} \right).$$

By definition of z,

$$U(z) - u(z) \geq U(x) - u(x) \quad \Longrightarrow \quad u(x) - u(z) \geq U(x) - U(z)$$

for any other point $x \in \Omega$. If $u - u(z)$ were strictly greater than $U - U(z)$ on a subset of ∂B_z^h of positive measure, then it would follow that

$$\operatorname*{mean}_{\partial B_z^h}(u - u(z)) > \operatorname*{mean}_{\partial B_z^h}(U - U(z)),$$

thereby forcing

$$\min \left(\operatorname*{med}_{\partial B_x^h} \{u - u(z)\} \right) < \max \left(\operatorname*{med}_{\partial B_x^h} \{U - U(z)\} \right)$$

and contradicting Lemma 2.1. Consequently, $u - u(z) = U - U(z)$ almost everywhere on ∂B_z^h and the set $\{y \in \partial B_z^h \,:\, u(y) - u(z) > U(y) - U(z)\}$ has surface measure zero. Since this set equals $\{y \in \partial B_z^h \,:\, U(z) - u(z) > U(y) - u(y)\}$ and $U - u$ is upper semicontinuous, this set is also relatively open. It must therefore be empty, so $u - u(z) = U - U(z)$ at every point on ∂B_z^h. Repeating this argument, we find that $u - u(z) = U - U(z)$ at some point on $\partial\Omega$, from which we conclude that $u = U$ throughout $\overline{\Omega}$.

3.4 Existence of solutions on strictly convex domains

Theorem 3.2 in the previous section establishes that whenever problem (3.14) has a subsolution and a supersolution, it has a unique solution. A natural problem, then, is to determine when sub- and supersolutions can be found. In this section, we show that if Ω is strictly convex, (3.14) has a subsolution and a supersolution for any $g \in C(\partial\Omega)$.

Theorem 3.3 *Suppose that $\Omega \subset \mathbb{R}^N$ is strictly convex and that $g \in C(\partial\Omega)$. Then problem (3.14) has a unique solution.*

Proof. We use Theorem 3.2 and solve Dirichlet problems for the Monge-Ampère equation to produce sub- and supersolutions.

Let v be the unique convex solution for the following problem:

$$\begin{cases} \det D^2 u = 1 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases},$$

where the notation $D^2\varphi$ denotes the Hessian matrix of the function φ. This problem has a unique solution in $C(\overline{\Omega})$ for any $g \in C(\partial\Omega)$ (see, for example, [12, Theorem1.6.2]).

Because v is convex, it is subharmonic; thus $v(x)$ is less than or equal to its mean on the boundary of any ball centered at x contained in Ω. The convexity of v also implies that $v(x)$ is less than or equal to its median on ∂B_x^h for any $x \in \Omega$, as we now demonstrate. Convex functions have convex sublevel sets. If x is on the boundary of the sublevel set $\{y \in \Omega : v(y) \le v(x)\}$, then on at least half of ∂B_x^h, $v > v(x)$, so $\operatorname{med}_{\partial B_x^h} v \ge v(x)$. If x is not on the boundary of this sublevel set, then $v(x)$ is equal to the minimum of v in Ω and the claim follows in this case as well. Combining these two observations, we obtain that $v(x) \le M_q^h v(x)$ and since $v = g$ on $\partial\Omega$, v is a subsolution.

A supersolution is obtained similarly. Let W be the unique convex solution of the following problem.

$$\begin{cases} \det D^2 u = 1 & \text{in } \Omega \\ u = -g & \text{on } \partial\Omega \end{cases}$$

Then $w = -W \in C(\bar{\Omega})$ is concave and hence superharmonic. So, by the same argument showing that convex functions (with the right boundary values) are subsolutions w is a supersolution.

4 p-harmonious functions for $1 < p \le 2$

In the previous section, values of $h > 0$ and $q \in (0, 1)$ were given and remained fixed. For these values, Theorems 3.1 and 3.2 show that there exists a unique solution $u_h \in C(\overline{\Omega})$ of the Dirichlet problem (3.14) if both a subsolution and supersolution are known to exist; we suppose throughout this section that this condition holds for all h sufficiently small. (This is the case, for example, if Ω is strictly convex, as demonstrated in Section 3.4.) In addition, Proposition 3.1 yields the bounds

$$\min_{\partial\Omega} g \le u_h(x) \le \max_{\partial\Omega} g \quad \text{for any} \quad x \in \overline{\Omega},$$

so $\|u_h\|_\infty$ is controlled by the L^∞ norm of g and does not depend on h.

The goal of this section is to explore the convergence of the solutions u_h as $h \to 0$; to do so, we recall some well-known facts that we will need. As discussed in the introduction, the continuous function u is p-harmonic, for $p > 1$, if and only if it is a viscosity solution of

$$-\Delta_p u = -\operatorname{div}\left(|Du|^{p-2} Du\right) = 0. \tag{4.19}$$

It is more common to define p-harmonic functions variationally as weak (Sobolev) solutions of (4.19); fortunately, Juutinen, Lindqvist and Manfredi proved in [16] that these definitions are equivalent. Moreover, they showed that one only has to consider test functions with nonvanishing gradient when working with (4.19). Using the operator A_p defined by

$$A_p\varphi := (1-p)\,\Delta\varphi + (p-2)\,|D\varphi|\,\operatorname{div}\left(\frac{D\varphi}{|D\varphi|}\right)$$

for smooth functions φ with $D\varphi \neq 0$ (sometimes referred to as the game-theoretic p-Laplacian), u is therefore p-harmonic if and only if it is a viscosity solution of

$$-A_p u = 0\,.$$

When $\partial\Omega$ is sufficiently regular, there is a unique p-harmonic function $u \in C(\overline{\Omega})$ with $u = g$ on $\partial\Omega$; we assume that the reader is familiar with these facts and refer to [16] and its references for more details.

To analyze the convergence of the solutions u_h as $h \to 0$, we implement the framework developed by Barles and Souganidis in [1] and exploited frequently, as in the papers [18], [19], [38] and [39]. Recalling the form of problem (3.14),

$$\begin{cases} u_h &= M_q^h u \quad \text{in} \quad \Omega\,, \\ u_h &= g \quad \text{on} \quad \partial\Omega\,, \end{cases}$$

we must now check that the operators M_q^h have the following properties:

1. $M_q^0 v = v$ for any $v \in C(\overline{\Omega})$,

2. $M_q^h(v + c) = M_q^h v + c$ for any $v \in C(\overline{\Omega})$ and any constant c,

3. $M_q^h v \leq M_q^h w$ whenever $v, w \in C(\overline{\Omega})$ satisfy $v \leq w$, and

4. for any smooth function φ and any $x \in \mathbb{R}^2$ where $D\varphi(x) \neq 0$,

$$\lim_{h\to 0}\left(\frac{\varphi(x) - \left(M_q^h\varphi\right)(x)}{h}\right) = A_p\varphi(x)\,. \tag{4.20}$$

The first three of these conditions (stability and monotonicity) clearly hold in any dimension. (The uniform bound on $\|u_h\|_\infty$ mentioned above also yields stability.) As for the fourth, the consistency condition, observe carefully the requirement that x belong to $\mathbb{R}^2$; with this restriction on the dimension, consistency follows directly from the work in [15]. Whether this condition holds in higher dimensions is an open problem.

Combining these facts and the results of [1] (cf. also [39]) yields a proof of the following convergence result:

Theorem 4.1 *In addition to the earlier assumptions on Ω, suppose that $\Omega \subset \mathbb{R}^2$. Suppose that $1 < p \leq 2$, let u denote the unique p-harmonic function on Ω such that $u = g$ on $\partial\Omega$, and define $q := 2 - \frac{2}{p}$. As $h \to 0$, the solutions u_h of (3.14) converge locally uniformly to u.*

In light of this result and the work in [33], we call the solution u_h of (3.14) p-harmonious (for $1 < p \leq 2$), and we can restate Theorem 3.2 as follows:

Theorem 4.2 *Suppose that $\Omega \subset \mathbb{R}^2$, $g \in C(\partial\Omega)$ and $1 < p \leq 2$. If for each sufficiently small $h > 0$, there exist a subsolution v_h and a supersolution w_h of (3.14), with $q = 2 - \frac{2}{p}$, then there is a unique p-harmonious function $u_h \in C(\overline{\Omega})$ such that $u_h = g$ on $\partial\Omega$.*

5 p-harmonious functions for $p \geq 2$

Let $p \geq 2$ be fixed throughout this section. We now consider the following variant of the problem considered in [33]: given $h > 0$ and $g \in C(\partial\Omega)$, and letting $q = (2 + N)/(p + N)$, we seek a solution $u_h \in C(\overline{\Omega})$ of the Dirichlet problem

$$\begin{cases} u_h = q \operatorname*{mean}_{B_x^h} \{u_h\} + \dfrac{1-q}{2} \left\{ \max\limits_{\overline{B_x^h}} u_h + \min\limits_{\overline{B_x^h}} u_h \right\} & \text{in} \quad \Omega , \\ u_h = g & \text{on} \quad \partial\Omega . \end{cases} \tag{5.21}$$

A solution of this problem is p-harmonious, a term that will be justified by the convergence result below.

If we define the operator $\mathbb{M}_q^h : C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$\left(\mathbb{M}_q^h u\right)(x) := \begin{cases} q \operatorname*{mean}_{B_x^h} \{u\} + \dfrac{1-q}{2} \left\{ \max\limits_{\overline{B_x^h}} u + \min\limits_{\overline{B_x^h}} u \right\}, & \text{if} \quad x \in \Omega, \\ u(x), & \text{if} \quad x \in \partial\Omega, \end{cases}$$

then it is easy to see that, like the operator M_q^h used earlier, $\mathbb{M}_q^h$ is monotone, homogeneous, and invariant under translation by constants. Consequently, the techniques used to prove Theorems 3.1 and 3.2 will work for the operator $\mathbb{M}_q^h$ if we can prove an appropriate version of Lemma 2.1. The following is the result we need:

Lemma 5.1 *Suppose that $u \in LSC(E)$, $U \in USC(E)$, $u \geq U$, and that $E \subset \mathbb{R}^N$ is compact. Then* $\min_E \{u\} \geq \inf_E \{U\}$ *and* $\sup_E \{u\} \geq \max_E \{U\}$.

Proof. Since $u \in LSC(E)$, u achieves its minimum at some point $x^* \in E$. It follows that $U(x^*) \leq u(x^*) = \min_E u$ and thus that $\inf_E U \leq \min_E u$. The proof of the other inequality is similar, using the fact that U achieves its maximum on E.

Using this lemma and monotone iteration as in Section 3 and defining subsolutions and supersolutions of (5.21) in the obvious way, we obtain both

Theorem 5.1 *If v_h is a subsolution of (5.21) and w_h is a supersolution of (5.21), then $v_h \leq w_h$.*

and

Theorem 5.2 *If there exist a subsolution v_h and a supersolution w_h of (5.21), then problem (5.21) has a unique solution $u_h \in C(\overline{\Omega})$.*

As in the previous section we now assume that for all positive h sufficiently small, (5.21) has a unique continuous solution u_h, and we apply the Barles-Souganidis machinery to analyze the limit of these solutions as $h \to 0$. Manfredi et al. verified the consistency condition (4.20) in [33] and the other conditions obviously hold, so we have

Theorem 5.3 *Let u denote the unique p-harmonic function on Ω such that $u = g$ on $\partial\Omega$, and define $q := \frac{2+N}{p+N}$. As $h \to 0$, the solutions u_h of (5.21) converge locally uniformly to u.*

In other words, for any $p \geq 2$, for each h there is a unique p-harmonious function $u_h \in C(\overline{\Omega})$ such that $u_h = g$ on $\partial\Omega$.

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