

Existence Results for the Prescribed Webster Scalar Curvature on Higher Dimensional CR Manifolds

Ridha Yacoub *

Department of Mathematics and Computer Science,
University of Monastir, I.P.E.I.M. 5019 Monastir, Tunisia.
e-mail: ridha.yacoub@ipeim.rnu.tn

Received 03 June 2012

Communicated by Abbas Bahri

Abstract.

This paper is about prescribing the Webster scalar curvature on a compact CR manifold of dimension $2n+1 \geq 5$, which is locally CR equivalent to the standard CR unit sphere \mathbb{S}^{2n+1} . The associated variational problem being noncompact we approach this issue with techniques from the critical points at infinity theory, combined with topological tools. Our existence results are based on a generalized Bahri-Coron type criteria. We also give an upper bound on the Morse index of the obtained solutions.

2000 Mathematics Subject Classification. 18G35, 35A15, 35A25, 35J65, 53C15, 53C21, 58E05.

Key words. Critical point at infinity, Floer-Milnor homology, Intersection number, Morse index, Morse theory, Pseudogradient, Webster scalar curvature.

Contents

1	Introduction. Main results.	626
2	Defect of compactness.	630
3	Expansion of the functional and its gradient.	631
3.1	Expansion of J .	631
3.2	Expansion of ∂J .	632

*Unity of Research : UR 11ES50, Algebra, Number Theory and Nonlinear Analysis. University of Monastir.

4 Estimates on the v-part. 634

5 More about critical points at infinity. 636

5.1 Morse lemma at infinity. 637

5.2 Characterization of the critical points at infinity. 638

5.3 Morse index of a critical point at infinity. 638

5.4 Stable and unstable manifold of a critical point at infinity 639

5.5 Proof of Lemma 639

6 Proofs of Theorems. 646

6.1 Proof of Theorem 646

6.2 Proof of Theorem 647

6.3 Proof of Theorem 649

Appendix A. 650

Appendix B. 656

Appendix C. 658

1 Introduction. Main results.

Let M be an orientable compact real $(2n+1)$ -dimensional CR manifold and θ a given contact form on M . We denote by $L = \frac{2(n+1)}{n}\Delta_\theta + R_\theta$ the CR invariant laplacian of M , where Δ_θ is the sublaplacian operator, R_θ the Webster scalar curvature associated to θ .

Let K be a strictly positive C^2 function on M . We try to find conditions under which K is equal to the Webster scalar curvature $R_{\tilde{\theta}}$ of some contact form $\tilde{\theta}$ on M , which is CR equivalent to θ . This is the Webster scalar curvature problem on CR manifolds. This problem amounts to finding a function u such that:

$$\begin{cases} Lu = Ku^{1+\frac{2}{n}} \\ u > 0 \text{ on } M. \end{cases} \tag{1.1}$$

Notice that if K is taken to be constant, then (1.1) is the CR Yamabe problem, which was dealt with in [16], [17], [18] and [13].

The main difficulty of problem (1.1) comes from the presence of the critical Sobolev exponent $\frac{2(n+1)}{n}$. Indeed, our problem enjoys a variational version. Let S_1^2 be the completion of $C^\infty(M)$ for the norm $\|u\| = (\int_M Lu u \theta \wedge d\theta^n)^{\frac{1}{2}}$. Let $\Sigma = \{u \in S_1^2 / \|u\| = 1\}$ and $\Sigma^+ = \{u \in \Sigma / u > 0\}$. The Euler-Lagrange functional associated to problem (1.1), denoted by J , is then defined on S_1^2 by

$$J(u) = \frac{\|u\|^2}{\left(\int_M K|u|^{2+\frac{2}{n}}\theta \wedge d\theta^n\right)^{\frac{n}{n+1}}}.$$

One knows that if v is a critical point of J in Σ^+ , then the function $u = J(v)^{\frac{n}{2}}v$ is a solution for (1.1), and thus the contact form $\tilde{\theta} = u^{\frac{2}{n}}\theta$ has its Webster scalar curvature $R_{\tilde{\theta}} = K$.

Problem (1.1) therefore amounts to finding the critical points of the functional J . However, the functional J does not satisfy the Palais-Smale condition, that is to say there exist critical points at infinity (or asymptotes), which are the limits of noncompact orbits for the gradient flow of $-J$. In

addition, as in the conformal case, in the CR case there are topological obstructions of Kazden-Warner condition type to solve (1.1) (see e.g. [14]).

With collaboration of H. Chtioui and M. Ould Ahmedou, we have obtained in [10] some existence results to problem (1.1) for $n \geq 2$ through topological methods initiated by Aubin-Bahri in [2], and in [11] we have obtained some existence and multiplicity results to problem (1.1) for $n = 1$, using some ‘‘Morse inequalities at infinity’’. Among other articles devoted to this subject, one can cite [20] where A. Malchiodi and F. Uguzzoni established some existence results for problem (1.1) when $M = \mathbb{S}^{2n+1}$ using a perturbative method due to Ambrosetti and Badiale (c.f. [1]), and also [12] where V. Felli and F. Uguzzoni obtained solutions to this problem on the Heisenberg group and CR spheres, in the case of presence of symmetry, as well as in the perturbative case.

In the present paper, generalized Morse theory is used in combination with the theory of critical points at infinity. More precisely, one knows that a critical point at infinity, as a usual critical point, induces a change in the topology of the descendent level sets of J . Our main idea is to compute the contribution of some critical points at infinity into some changes of topology, and, since arguing by contradiction we are assuming that (1.1) has no solution, our main problem consists to finding conditions on K such that one observes some change of topology, that cannot be explained by the only contribution of the critical points at infinity. Such a situation, if it occurs, leads to the existence of (genuine) critical points for the functional J . Notice that once a Morse type reduction is performed around critical points at infinity, there is no difficulty to compute their Morse index and to define their stable and unstable manifolds (see Section 5).

To state our results, we need to recall the construction of the family of ‘‘almost solutions’’ $\varphi_{(a,\lambda)}$ of the Yamabe problem on M . Since M is compact and locally CR equivalent to the standard CR sphere \mathbb{S}^{2n+1} , any point a of M has a neighborhood containing a ball $B(a, r)$, where pseudohermitian normal coordinates can be defined (see [16], [17]), $r > 0$ is independent of a , and there exists a positive function u_a on $B(a, r)$ such that $\theta_0 = u_a^{\frac{2}{n}} \theta$, where $\theta_0 = dt + \sum_{j=1}^n iz_j d\bar{z}_j - i\bar{z}_j dz_j$ is the standard contact form on the Heisenberg group \mathbb{H}^n . Let $\chi : \mathbb{R}_+ \rightarrow [0, 1]$ be a cut off function such

$$\text{that: } \begin{cases} \chi(t) = 1 & \text{if } 0 \leq t \leq \frac{r}{2} \\ \chi(t) = 0 & \text{if } t \geq r. \end{cases}$$

Define on $B(a, r)$ the function $\chi_a(x) = \chi(\|\exp_a^{-1}(x)\|_{\mathbb{H}^n})$ where \exp_a^{-1} is the parabolic exponential map based in a (see [17]), and where, putting $\exp_a^{-1}(x) = (z, t)$, $\|(z, t)\|_{\mathbb{H}^n} = (|z|^4 + t^2)^{\frac{1}{4}}$ is the norm of the Heisenberg group.

Let λ be a positive large parameter. We introduce on $B(a, r)$ the function:

$$\delta(a, \lambda)(x) = c_n \lambda^n |1 + \lambda^2(|z|^2 - it)|^{-n} \tag{1.2}$$

where the constant c_n is such that the equation $L_0 \delta(a, \lambda) = \delta(a, \lambda)^{1+\frac{2}{n}}$ is satisfied on $B(a, r)$. Here $L_0 = \frac{2(n+1)}{n} \Delta_{\theta_0}$ is the CR invariant laplacian of \mathbb{H}^n . Observe that

$$\delta(a, \lambda) = c_n \omega_{(0,\lambda)} \circ \exp_a^{-1} \tag{1.3}$$

where $\omega_{(0,\lambda)}$ is a solution of Yamabe problem on \mathbb{H}^n (see (A.2) in Appendix A).

We define a family of ‘‘almost solutions’’ $\varphi_{(a,\lambda)}$ to be the unique solutions of:

$$L\varphi_{(a,\lambda)} = \delta'(a, \lambda)^{1+\frac{2}{n}} \quad \text{on } M, \tag{1.4}$$

with

$$\begin{cases} \delta'(a, \lambda)(x) = \chi_a(x) u_a(x) \delta(a, \lambda)(x) & \text{on } B(a, r) \\ \delta'(a, \lambda)(x) = 0 & \text{on } {}^c B(a, r). \end{cases} \tag{1.5}$$

Let $K : M \rightarrow \mathbb{R}$ be a C^2 function having a finite set of critical points denoted by I :

$$I = \{y \in M; \partial K(y) = 0\}.$$

Let

$$I^+ = \{y \in I \text{ such that, } -\Delta_\theta K(y) > 0\}$$

and, denoting by $\#I^+$ the cardinality of I^+ , let F^+ be the set

$$F^+ = \{(y_{i_1}, \dots, y_{i_p}) \in (I^+)^p / y_{i_j} \neq y_{i_k} \text{ if } j \neq k, 1 \leq p \leq \#I^+\}. \tag{1.6}$$

We say that K satisfies **(N.D)** a non-degeneracy condition if it satisfies:

$$\text{(N.D)} \forall y \in I, \Delta_\theta K(y) \neq 0.$$

We prove in section 5 (Propositions 5.2 and 5.3) the following result:

Proposition 1.1 *Assume that (1.1) has no solution, $n \geq 2$, and let $K : M \rightarrow \mathbb{R}$ be C^2 be positive and satisfy **(N.D)**. Then, the critical points at infinity (see definition 2.1 below) of J in Σ^+ correspond to*

$$\sum_{j=1}^p \frac{1}{K(y_{i_j})^{n/2}} \varphi_{(y_{i_j}, +\infty)} := (y_{i_1}, \dots, y_{i_p})_\infty.$$

where $(y_{i_1}, \dots, y_{i_p}) \in F^+$. The Morse index of such a critical point at infinity is:

$$i((y_{i_1}, \dots, y_{i_p})_\infty) = p - 1 + \sum_{j=1}^p (2n + 1 - \text{ind}(K, y_{i_j}))$$

where $\text{ind}(K, y_{i_j})$ is the Morse index of K at y_{i_j} .

Let F_∞^+ be the set of critical points at infinity of J . According to Proposition 1.1,

$$F_\infty^+ = \{\phi_\infty^p = (y_{i_1}, \dots, y_{i_p})_\infty; (y_{i_1}, \dots, y_{i_p}) \in F^+\}. \tag{1.7}$$

If $\phi_\infty^p \in F_\infty^+$, then $W_u(\phi_\infty^p)$ denotes its unstable manifold and $W_s(\phi_\infty^p)$ its stable manifold, with respect of the C^1 vector field $-\partial J$. We have,

$$\dim W_u(\phi_\infty^p) = \text{codim} W_s(\phi_\infty^p) = i(\phi_\infty^p).$$

Now, for any $k \in \mathbb{N}$ and any subset X_k of $\{\phi_\infty^p \in F_\infty^+; i(\phi_\infty^p) \leq k\}$, we define the following set

$$X_k^\infty = \bigcup_{\phi_\infty^p \in X_k} W_u(\phi_\infty^p). \tag{1.8}$$

X_k^∞ is then a stratified set of dimension at most k . Without loss of generality we can assume it equal to k . Observe that since Σ^+ is a contractible set, then X_k^∞ is contractible in Σ^+ . Let $\psi(X_k^\infty)$ be a contraction of X_k^∞ in Σ^+ . Then we have the following existence result:

Theorem 1.1 *Assume $n \geq 2$. Let $K : M \rightarrow \mathbb{R}$ be a C^2 positive function satisfying **(N.D)**. If there exists $k_0 \in \mathbb{N}$ and $X_{k_0} \subset \{\phi_\infty^p \in F_\infty^+; i(\phi_\infty^p) \leq k_0\}$ such that*

$$\text{(H}_1) \quad \sum_{\phi_\infty^p \in X_{k_0}} (-1)^{i(\phi_\infty^p)} \neq 1$$

$$\text{(H}_2) \quad \psi(X_{k_0}^\infty) \cap W_s(\phi_\infty^p) = \emptyset, \text{ for all } \phi_\infty^p \in F_\infty^+ \setminus X_{k_0} \text{ such that } i(\phi_\infty^p) \leq k_0 + 1,$$

then there exists a solution w of (1.1) such that $i(w) \leq k_0 + 1$, where $i(w)$ is the Morse index of the solution w .

As an example of functions satisfying the assumptions of Theorem 1.1 we give the following perturbative result, in the case where M is the standard CR sphere \mathbb{S}^{2n+1} . So we let $K : \mathbb{S}^{2n+1} \rightarrow \mathbb{R}$ a C^2 function, and we introduce the following perturbative assumption:

$$(C.C) \quad K(\zeta) = 1 + \varepsilon K_0(\zeta) \text{ , } \forall \zeta \in \mathbb{S}^{2n+1} \text{ , where } K_0 \in C^2(\mathbb{S}^{2n+1}) \text{ and } |\varepsilon| \text{ small.}$$

We introduce the following set:

$$\mathfrak{N} = \{ k \in \mathbb{N} / \forall y \in I^+ \text{ , } 2n + 1 - \text{ind}(K, y) \neq k + 1 \} . \tag{1.9}$$

For example if $k \geq 2n + 1$ then $k \in \mathfrak{N}$.

The next existence result generalizes the perturbative result of Malchiodi-Uguzzoni [20]:

Theorem 1.2 *Let $n \geq 1$, $K : \mathbb{S}^{2n+1} \rightarrow \mathbb{R}$ a C^2 positive function satisfying (N.D) and (C.C). If*

$$\max_{k \in \mathfrak{N}} \left| 1 - \sum_{\substack{y \in I^+ \\ 2n+1 - \text{ind}(K, y) \leq k}} (-1)^{2n+1 - \text{ind}(K, y)} \right| \neq 0 , \tag{1.10}$$

then, for $|\varepsilon|$ small enough, there exists a solution w of (1.1) such that $i(w) \leq k_0 + 1$, where k_0 realizes the maximum in (1.10).

As a corollary we find the result of [20]:

Corollary 1.1 *Let $n \geq 1$. $K : \mathbb{S}^{2n+1} \rightarrow \mathbb{R}$ a C^2 positive function satisfying (N.D) and (C.C). If*

$$\sum_{y \in I^+} (-1)^{2n+1 - \text{ind}(K, y)} \neq 1 ,$$

then, for $|\varepsilon|$ small enough, there exists a solution for problem (1.1).

Now we return to a general CR manifold M . To state the existence result that follows, let us assume, without loss of generality, that $I = \{y_0, y_1, y_2, \dots\}$, the critical points being ordered such that

$$K(y_0) \geq K(y_1) \geq K(y_2) \geq \dots$$

$$(A_1) \text{ Assume that } y_0, y_1 \in I^+ \text{ and } y_2 \in I \setminus I^+ .$$

For $y \in I$ we denote by $W_s(y)$ the stable manifold of y with respect of the C^1 vector field $-\partial K$, and by $\overline{W_s(y)}$ its closure. Let

$$N_1 = \overline{W_s(y_1)} \quad \text{and} \quad N_2 = \overline{W_s(y_2)} .$$

Let ∂ be the boundary operator of the Floer-Milnor Homology associated to the functional K , (see [5], [22]).

$$(A_2) \text{ Assume that } \partial N_2 = N_1 .$$

Let y_ℓ be the first critical point of K below y_2 , which belongs to I^+ , and:

$$(A_3) \text{ Assume that } K(y_2) > K(y_\ell) .$$

then, we have:

Theorem 1.3 *Assume $n \geq 2$. Let $K : M \rightarrow \mathbb{R}$ be a C^2 positive function satisfying (N.D). Under assumptions (A₁), (A₂) and (A₃), there exists at least a solution w of (1.1) such that $i(w) \leq k_1 + 1$, where $k_1 = 2n + 1 - \text{ind}(K, y_1)$.*

Our paper will be organized as follows. In Sections 2 we study the lack of compactness in the variational framework. Sections 3 and 4 are devoted to estimate the behavior of J in the neighborhoods of its potential critical points at infinity. In section 5 we state the Morse lemma at infinity, which is a crucial result, and we give the available information on the critical points at infinity of the functional J . Section 6 is devoted to proving our main results.

2 Defect of compactness.

Since the exponent $\frac{2(n+1)}{n}$ is critical for the Sobolev embedding $S_1^2 \hookrightarrow L^{\frac{2(n+1)}{n}}(M)$, which is continuous but not compact, the functional J does not satisfy the Palais-Smale condition. In order to describe the sequences violating the Palais-Smale condition (see [4], [9] [19], [21]), we introduce in a natural way, for $p \geq 1$, the set

$$\mathcal{E}^p = (0, +\infty)^p \times M^p \times (0, +\infty)^p, \tag{2.1}$$

which is the space of the variables $(\alpha, a, \lambda) = (\alpha_1, \dots, \alpha_p, a_1, \dots, a_p, \lambda_1, \dots, \lambda_p)$, and in particular we consider, for a small enough $\epsilon > 0$, the following subset of \mathcal{E}^p :

$$B_\epsilon = \left\{ (\alpha, a, \lambda) \in \mathcal{E}^p, \text{ s.t. } \left| \frac{\alpha_i^{\frac{2}{n}} K(a_i)}{\alpha_j^{\frac{2}{n}} K(a_j)} - 1 \right| < \epsilon, \epsilon_{ij} < \epsilon, \lambda_i > \frac{1}{\epsilon} \right\}, \tag{2.2}$$

where $\epsilon_{ij} = \epsilon_{ij}(a_i, a_j, \lambda_i, \lambda_j) = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \tilde{d}(a_i, a_j)^2 \right)^{-n}$ where $\tilde{d}(x, y) = \| \exp_x^{-1}(y) \|_{\mathbb{H}^n}$ if x and y are in a small ball of M of radius r , and $\tilde{d}(x, y)$ is equal to $\frac{r}{2}$ otherwise. Now, we define the set $V(p, \epsilon)$ of potential critical points at infinity to be:

$$V(p, \epsilon) = \left\{ u \in \Sigma^+ / \exists (\alpha, a, \lambda) \in B_\epsilon \text{ s.t. } \left\| u - \sum_{i=1}^{i=p} \alpha_i \varphi_i \right\| < \epsilon \right\} \tag{2.3}$$

where $\varphi_i = \varphi_{(a_i, \lambda_i)}$ is the ‘‘almost solution’’ defined by (1.4) in the introduction. Then, the following result gives a complete description of the noncompact sequences along which the functional J is bounded and its gradient goes to zero.

Proposition 2.1 *Assume that (1.1) has no solution. Let (v_k) be a sequence in Σ^+ such that $J'(v_k) \rightarrow 0$ and $J(v_k)$ is bounded. Then, there exists an integer $p \in \mathbb{N}^*$, a sequence (ϵ_k) , $\epsilon_k > 0$, $\epsilon_k \rightarrow 0$, such that, up to an extracted subsequence, $v_k \in V(p, \epsilon_k)$.*

Proof. One can operate as in section 4 of [13], making some minor modifications. □

Now, for $u \in V(p, \epsilon)$, we consider the minimization problem:

$$\min_{(\alpha, a, \lambda) \in B_\epsilon} \left\| u - \sum_{i=1}^p \alpha_i \varphi_i \right\|. \tag{2.4}$$

The following proposition follows from the corresponding results of [3] and [4]:

Proposition 2.2 *Let ϵ small enough. For any u in $V(p, \epsilon)$, the minimization problem (2.4) has a unique solution (α, a, λ) in B_ϵ up to permutation.*

Thus, for $\epsilon > 0$ sufficiently small, any function $u \in V(p, \epsilon)$ can be uniquely written as: $u = \sum_{i=1}^p \alpha_i \varphi_i + v$, where $(\alpha, a, \lambda) \in B_\epsilon$ and $v \in H_\epsilon(a, \lambda)$, where

$$H_\epsilon(a, \lambda) = \left\{ v \in \{ \varphi_i, \frac{\partial}{\partial a_i} \varphi_i, \frac{\partial}{\partial \lambda_i} \varphi_i ; 1 \leq i \leq p \}^\perp, \|v\| < \epsilon \right\}, \tag{2.5}$$

where orthogonality is relative to the inner product $\langle u, v \rangle = \int_M Lu.v \theta \wedge d\theta^n$. Furthermore,

$$\forall i \neq j, \quad \frac{\alpha_i^{\frac{2}{n}} K(a_i)}{\alpha_j^{\frac{2}{n}} K(a_j)} \longrightarrow 1 \quad \text{when } \epsilon \rightarrow 0. \tag{2.6}$$

Definition 2.1 A critical point at infinity of J in Σ^+ is a limit of a flow line $u(s)$ of the equation

$$\begin{cases} \frac{\partial u(s)}{\partial s} = -J(u(s)) \\ u(0) = u_0 \end{cases}$$

such that $u(s)$ remains in a set $V(p, \epsilon(s))$ for all $s \geq s_0$, where $\epsilon(s) \rightarrow 0$ when $s \rightarrow +\infty$, and where u_0 is an initial condition. Using Proposition 2.2, $u(s)$ can be written as $u(s) = \sum_{i=1}^p \alpha_i(s) \varphi_{(a_i(s), \lambda_i(s))} + v(s)$. If we let $\alpha_i = \lim_{s \rightarrow \infty} \alpha_i(s)$ and $a_i = \lim_{s \rightarrow \infty} a_i(s)$, then such a critical point at infinity is denoted by $\sum_{i=1}^p \alpha_i \varphi_{(a_i, +\infty)}$ or $(a_1, \dots, a_p)_\infty$.

Characterization, strictly speaking, of the critical points at infinity of J in Σ^+ is provided by Proposition 5.2 in Section 5.

3 Expansion of the functional and its gradient.

3.1 Expansion of J .

We want to understand the behavior of J near infinity, that is when a flow line of ∂J , the gradient of J , enters into a set $V(p, \epsilon)$. We provide a local expansion of J in this set:

Proposition 3.1 For $\epsilon > 0$ small enough, let $u = \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + v$ in $V(p, \epsilon)$, with $(\alpha, a, \lambda) \in B_\epsilon$ and $v \in H_\epsilon(a, \lambda)$. Then we have the following expansion of J :

$$J(u) = \gamma_0(\alpha, a) \left\{ 1 + \gamma_1(a, \lambda) + \gamma_2(a, \lambda) - \langle f, v \rangle + Q(v) + O(\|v\|^q) \right\}$$

where

$$\gamma_0(\alpha, a) = \frac{S^{\frac{n}{n+1}} (\sum_i \alpha_i^2)}{(\sum_i \alpha_i^q K(a_i))^{\frac{n}{n+1}}}, \quad \gamma_1(a, \lambda) = - \sum_i c_i \frac{\Delta_\theta K(a_i)}{\lambda_i^2} + o\left(\sum_i \frac{1}{\lambda_i^2}\right),$$

$$\gamma_2(a, \lambda) = O\left(\sum_{i \neq j} \varepsilon_{ij} (1 + o(1)) + \varepsilon_{ij}^{1+\frac{1}{n}} \log \varepsilon_{ij}^{-\frac{1}{n}}\right)$$

$$\langle f, v \rangle = \frac{2}{S^n \sum_i \alpha_i^q K(a_i)} \int_M K \left(\sum_i \alpha_i \varphi_i \right)^{q-1} v \tag{3.1}$$

$$Q(v) = \frac{1}{S^n \sum_i \alpha_i^2} \left\{ \|v\|^2 - \frac{2(n+2) \sum_i \alpha_i^2}{n \sum_i \alpha_i^q K(a_i)} \int_M K \left(\sum_i \alpha_i \varphi_i \right)^{\frac{2}{n}} v^2 \right\}, \tag{3.2}$$

where c_i are positive constants, S is the Sobolev constant given in (A.5), and $q = 2 + \frac{2}{n}$.

Proof. Let $J(u) = J(\sum_{i=1}^p \alpha_i \varphi_i + v) = \frac{\|u\|^2}{(\int_M Ku^{2+\frac{2}{n}})^{\frac{n}{n+1}}} = \frac{N}{D}$.

Since $v \in H_\epsilon(a, \lambda)$, using (A.9) and (A.10), we have

$$N = \sum_i \alpha_i^2 \int_M L\varphi_i \varphi_i + \|v\|^2 + \sum_{i \neq j} \alpha_i \alpha_j \int_M L\varphi_i \varphi_j$$

$$\begin{aligned}
 &= S^n \sum_i \alpha_i^2 + O\left(\sum_i \frac{1}{\lambda_i^{2n}}\right) + \sum_{i \neq j} \alpha_i \alpha_j O(\varepsilon_{ij}) + \|v\|^2. \\
 &= S^n \sum_i \alpha_i^2 \left\{ 1 + \sum_{i \neq j} \alpha_i \alpha_j O(\varepsilon_{ij}) + O\left(\sum_i \frac{1}{\lambda_i^{2n}}\right) + \frac{1}{S^n \sum_i \alpha_i^2} \|v\|^2 \right\}
 \end{aligned}$$

Next, using Lemma A.2 we have:

$$\begin{aligned}
 D^{\frac{n+1}{n}} = \int_M K(\sum_{i=1}^p \alpha_i \varphi_i + v)^q &= \int_M K(\sum_i \alpha_i \varphi_i)^q + q \int_M K(\sum_i \alpha_i \varphi_i)^{q-1} v \\
 &\quad + q(q-1) \int_M K(\sum_i \alpha_i \varphi_i)^{q-2} v^2 + O(\|v\|^q).
 \end{aligned}$$

Using Lemmas A.2–A.7 we obtain

$$\begin{aligned}
 D^{\frac{n+1}{n}} &= \sum_i \alpha_i^q K(a_i) S^n + \sum_i \alpha_i^q \tilde{c} \frac{\Delta_\theta K(a_i)}{\lambda_i^2} + o\left(\sum_i \frac{1}{\lambda_i^2}\right) + O\left(\sum_i \frac{1}{\lambda_i^{2n}}\right) \\
 &\quad + q \sum_{i \neq j} \alpha_i^{q-1} \alpha_j c_{ij} \varepsilon_{ij} (1 + o(1)) + O\left(\sum_{i \neq j} \varepsilon_{ij}^{1+\frac{1}{n}} \log \varepsilon_{ij}^{-\frac{1}{n}}\right) \\
 &\quad + q \int_M K\left(\sum_i \alpha_i \varphi_i\right)^{q-1} v + q(q-1) \int_M K\left(\sum_i \alpha_i \varphi_i\right)^{\frac{2}{n}} v^2 + O(\|v\|^q).
 \end{aligned}$$

Hence

$$\begin{aligned}
 D &= S^{n \frac{n}{n+1}} \left(\sum_i \alpha_i^q K(a_i) \right)^{\frac{n}{n+1}} \left\{ 1 + \sum_i \frac{n \alpha_i^q \tilde{c}}{(n+1) S^n \sum_i \alpha_i^q K(a_i)} \frac{\Delta_\theta K(a_i)}{\lambda_i^2} + o\left(\sum_i \frac{1}{\lambda_i^2}\right) \right. \\
 &\quad + O\left(\sum_i \frac{1}{\lambda_i^{2n}}\right) + O\left(\sum_{i \neq j} \varepsilon_{ij} (1 + o(1))\right) + O\left(\sum_{i \neq j} \varepsilon_{ij}^{1+\frac{1}{n}} \log \varepsilon_{ij}^{-\frac{1}{n}}\right) + O(\|v\|^q) \\
 &\quad + \frac{nq}{(n+1) \sum_i \alpha_i^q K(a_i) S^n} \int_M K\left(\sum_i \alpha_i \varphi_i\right)^{q-1} v \\
 &\quad \left. + \frac{nq(q-1)}{(n+1) \sum_i \alpha_i^q K(a_i) S^n} \int_M K\left(\sum_i \alpha_i \varphi_i\right)^{\frac{2}{n}} v^2 \right\}.
 \end{aligned}$$

Finally, computing the quotient $J(u) = \frac{D}{Q}$ we obtain the expected expansion of J , where Q , given in (3.2), is a quadratic form on $H_\epsilon(a, \lambda)$, and f , given in (3.1), represents the continuous linear form $v \mapsto \frac{2}{S^n \sum_i \alpha_i^q K(a_i)} \int_M K(\sum_i \alpha_i \varphi_i)^{q-1} v$, with respect to the inner product $\langle \cdot, \cdot \rangle$. Proposition 3.1 follows, with $c_i = \frac{n \alpha_i^q \tilde{c}}{S^{n(n+1)} \sum_i \alpha_i^q K(a_i)}$.

3.2 Expansion of ∂J .

Now, we provide the following expansions of the gradient of J , in the set $V(p, \epsilon)$.

Lemma 3.1 *Let $n \geq 2$. For $\epsilon > 0$ small enough and $u = \sum_j \alpha_j \varphi_{(a_j, \lambda_j)} \in V(p, \epsilon)$, the following expansions hold:*

$$\langle \partial J(u), \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \rangle = c \frac{\Delta_\theta K(a_i)}{\lambda_i^2} + \sum_{j \neq i} c_{ij} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\left(\sum_j \frac{1}{\lambda_j^2} + \sum_{j \neq i} \varepsilon_{ij}\right) \tag{3.3}$$

$$\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \rangle = -\tilde{c} \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + \sum_{j \neq i} \frac{\tilde{c}_{ij}}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} + O\left(\frac{1}{\lambda_i^2}\right) + o\left(\sum_j \frac{1}{\lambda_j^2} + \sum_{j \neq i} \varepsilon_{ij}\right) \tag{3.4}$$

where $c > 0$, $\tilde{c} > 0$, and c_{ij}, \tilde{c}_{ij} are bounded constants. (For definition of ∇_θ please see lemma 4.1 below).

Proof. We have,

$$\langle \partial J(u), \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \rangle = J(u) \langle \sum_j \alpha_j \varphi_j, \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \rangle - J(u)^{\frac{2n+1}{n}} \int_M K(\sum_j \alpha_j \varphi_j)^{1+\frac{2}{n}} \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i}.$$

Using the following estimates

$$\begin{aligned} \langle \varphi_i, \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \rangle &= o\left(\frac{1}{\lambda_i^2}\right) \\ \langle \varphi_j, \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \rangle &= c_{ij}^1 \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_j^2} + \varepsilon_{ij}\right) \\ \int_M K \varphi_i^{1+\frac{2}{n}} \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} &= -\frac{1}{q} \tilde{c} \frac{\Delta_\theta K(a_i)}{\lambda_i^2} + o\left(\frac{1}{\lambda_i^2}\right) \\ \int_M K \varphi_j^{1+\frac{2}{n}} \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} &= c_{ij}^2 \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_j^2} + \varepsilon_{ij}\right) \end{aligned}$$

then, using the results of Appendix A, it yields,

$$\langle \partial J(u), \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \rangle = J(u)^{\frac{2n+1}{n}} \left[\tilde{c} \alpha_i^{1+\frac{2}{n}} \frac{\Delta_\theta K(a_i)}{\lambda_i^2} + \sum_{j \neq i} c_{ij}^3 \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\left(\sum_j \frac{1}{\lambda_j^2} + \sum_{j \neq i} \varepsilon_{ij}\right) \right].$$

Thus, the first expansion follows. For the second, we have

$$\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \rangle = J(u) \langle \sum_j \alpha_j \varphi_j, \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \rangle - J(u)^{\frac{2n+1}{n}} \int_M K(\sum_j \alpha_j \varphi_j)^{1+\frac{2}{n}} \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i}.$$

Using the following estimates,

$$\begin{aligned} \langle \varphi_i, \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \rangle &= o\left(\frac{1}{\lambda_i^2}\right) \\ \langle \varphi_j, \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \rangle &= \frac{c_{ij}^4}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} + o\left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_j^2} + \varepsilon_{ij}\right) \\ \int_M K \varphi_i^{1+\frac{2}{n}} \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} &= c_1 \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + o\left(\frac{1}{\lambda_i^2}\right) \\ \int_M K \varphi_j^{1+\frac{2}{n}} \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} &= \frac{c_{ij}^5}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} + o\left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_j^2} + \varepsilon_{ij}\right) \end{aligned}$$

we derive, using the results of Appendix A, that

$$\begin{aligned} \langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \rangle &= -J(u)^{\frac{2n+1}{n}} \left[c_1 \alpha_i^{1+\frac{2}{n}} \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + o\left(\frac{1}{\lambda_i^2}\right) \right. \\ &\quad \left. + \sum_{j \neq i} \frac{c_{ij}^6}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} + o\left(\sum_j \frac{1}{\lambda_j^2} + \sum_{j \neq i} \varepsilon_{ij}\right) \right]. \end{aligned}$$

The second expansion follows and the proof of lemma 3.1 is thereby completed. □

4 Estimates on the v-part.

In the expansion of $J(u)$ above, the v -part is : $-\langle f, v \rangle + Q(v) + O(\|v\|^q)$.

We first estimate the linear part $\langle f, v \rangle$:

Lemma 4.1 *Let $\epsilon > 0$ sufficiently small. There exists a nonnegative constant c such that for all v in $H_\epsilon(a, \lambda)$ we have :*

$$\langle f, v \rangle \leq C\|v\| \left(\sum_i \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n+2}} + \sum_{i \neq j} \epsilon_{ij}^{\frac{n+2}{2n}} (\log \epsilon_{ij}^{-\frac{1}{n}})^{\frac{n+2}{2n+2}} \right)$$

where ∇_θ is the so called subelliptic gradient on M , which can be characterized by the identity: $\int_M (\nabla_\theta u, \nabla_\theta v)_\theta = \int_M \Delta_\theta u \cdot v$, where $(\cdot, \cdot)_\theta$ is the Levi form of θ .

Proof. By (3.1) we have $\langle f, v \rangle = \frac{2}{S^{n \sum_i \alpha_i^q K(a_i)}} \int_M K(\sum_i \alpha_i \varphi_i)^{q-1} v$. Using Lemma A.9,

$$\int_M K(\sum_i \alpha_i \varphi_i)^{q-1} v = \sum_i \alpha_i^{1+\frac{2}{n}} \int_M K \varphi_i^{1+\frac{2}{n}} v + O\left(\sum_{i \neq j} \alpha_i^{\frac{2}{n}} \alpha_j \int_M K \varphi_i^{\frac{2}{n}} \inf(\varphi_i, \varphi_j) v\right).$$

Then, using Lemma A.8,

$$\sum_i \alpha_i^{1+\frac{2}{n}} \int_M K \varphi_i^{1+\frac{2}{n}} v \leq C\|v\| \left(\sum_i \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + O\left(\sum_i \frac{1}{\lambda_i^2}\right) + O\left(\sum_i \frac{1}{\lambda_i^{n+2}}\right) \right).$$

On the other hand, K being bounded we have

$$\begin{aligned} \int_M K \varphi_i^{\frac{2}{n}} \inf(\varphi_i, \varphi_j) v &\leq \left(\int_M v^{2+\frac{2}{n}} \right)^{\frac{n}{2n+2}} \left(\int_M (K \varphi_i^{\frac{2}{n}} \inf(\varphi_i, \varphi_j))^{\frac{2n+2}{2n}} \right)^{\frac{n+2}{2n+2}} \\ &\leq c'\|v\| \left(\int_M \varphi_i^{\frac{4n+4}{n^2+2n}} \inf(\varphi_i, \varphi_j)^{\frac{2n+2}{n+2}} \right)^{\frac{n+2}{2n+2}} \\ &\leq c\|v\| \left(\int_M \varphi_i^{\frac{4n+4}{n^2+2n}} \varphi_j^{1+\frac{1}{n}} \varphi_i^{\frac{2n+2}{n+2} - \frac{n+1}{n}} \right)^{\frac{n+2}{2n+2}} \\ &\leq c\|v\| \left(\int_M \varphi_i^{1+\frac{1}{n}} \varphi_j^{1+\frac{1}{n}} \right)^{\frac{n+2}{2n+2}} \leq \|v\| O\left(\epsilon_{ij}^{1+\frac{1}{n}} \log \epsilon_{ij}^{-\frac{1}{n}}\right)^{\frac{n+2}{2n+2}} \end{aligned}$$

since $\frac{2n+2}{n+2} \geq 1 + \frac{1}{n}$, and using Lemma A.6, (c, c' are suitable constants). Thus

$$O\left(\sum_{i \neq j} \alpha_i^{\frac{2}{n}} \alpha_j \int_M K \varphi_i^{\frac{2}{n}} \inf(\varphi_i, \varphi_j) v\right) = \|v\| O\left(\sum_{i \neq j} \epsilon_{ij}^{\frac{n+2}{2n}} (\log \epsilon_{ij}^{-\frac{1}{n}})^{\frac{n+2}{2n+2}}\right)$$

which completes the proof of Lemma 4.1. □

In order to refine our estimates on the quadratic form Q , given by (3.2), we claim

Lemma 4.2

$$\begin{aligned} \int_M K \left(\sum_i \alpha_i \varphi_i \right)^{\frac{2}{n}} v^2 &= \sum_i \alpha_i^{\frac{2}{n}} K(a_i) \int_M \varphi_i^{\frac{2}{n}} v^2 + \\ &\|v\|^2 O\left(\sum_i \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \epsilon_{ij}^{\frac{1}{n}} (\log \epsilon_{ij}^{-\frac{1}{n}})^{\frac{1}{n+1}}\right). \end{aligned}$$

Proof. First, we proceed as in the proof of Lemma A.8, and we obtain

$$\int_M K \varphi_i^{\frac{2}{n}} v^2 = K(a_i) \int_M \varphi_i^{\frac{2}{n}} v^2 + \|v\|^2 O\left(\frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right). \tag{4.1}$$

Next, by Hölder, Lemmas A.9 and A.6, and since $n + 1 \geq 1 + \frac{1}{n}$, we have:

$$\begin{aligned} & \left| \int_M K \left(\sum_i \alpha_i \varphi_i\right)^{\frac{2}{n}} v^2 - \sum_i \alpha_i^{\frac{2}{n}} \int_M K \varphi_i^{\frac{2}{n}} v^2 \right| \\ & \leq \|v\|^2 \left(\int_M \left| \left(\sum_i \alpha_i \varphi_i\right)^{\frac{2}{n}} - \sum_i \alpha_i^{\frac{2}{n}} \varphi_i^{\frac{2}{n}} \right|^{n+1} \right)^{\frac{1}{n+1}} \\ & \leq \|v\|^2 \left(\int_M \sum_{i \neq j} \left(\sup(\alpha_i \varphi_i, \alpha_j \varphi_j)^{\frac{2-n}{n}} \inf(\alpha_i \varphi_i, \alpha_j \varphi_j) \right)^{n+1} \right)^{\frac{1}{n+1}} \\ & \leq \|v\|^2 \left(\int_M \sum_{i \neq j} (\alpha_i \varphi_i)^{1+\frac{1}{n}} (\alpha_j \varphi_j)^{1+\frac{1}{n}} \right)^{\frac{1}{n+1}} \\ & \leq \|v\|^2 \sum_{i \neq j} (\alpha_i \alpha_j)^{\frac{1}{n}} \epsilon_{ij}^{\frac{1}{n}} (\log \epsilon_{ij}^{-\frac{1}{n}})^{\frac{1}{n+1}}. \end{aligned}$$

Lemma 4.2 follows from this inequality and (4.1).

Let $\tilde{\delta}(1) = O\left(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \epsilon_{ij}^{\frac{1}{n}} (\log \epsilon_{ij}^{-\frac{1}{n}})^{\frac{1}{n+1}}\right)$. Using Lemma 4.2 we have

$$Q(v) = \frac{1}{S^n \sum_i \alpha_i^2} \left\{ \|v\|^2 - 2\binom{n+2}{n} \sum_i \alpha_i^{\frac{2}{n}} K(a_i) \frac{\sum_i \alpha_i^2}{\sum_i \alpha_i^q K(a_i)} \int_M \varphi_i^{\frac{2}{n}} v^2 + \|v\|^2 \tilde{\delta}(1) \right\}.$$

Since $u \in V(p, \epsilon)$, $u = \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + v$ with $v \in H_{\epsilon}(a, \lambda)$. By (2.6), $\frac{\alpha_i^{\frac{2}{n}} K(a_i)}{\alpha_j^{\frac{2}{n}} K(a_j)} \rightarrow 1$ when $\epsilon \rightarrow 0$, for

$i \neq j$. So we can assume the coefficients $\alpha_i^{\frac{2}{n}} K(a_i) \frac{\sum_i \alpha_i^2}{\sum_i \alpha_i^q K(a_i)} = 1$ and therefore Q is very close to the following quadratic form, denoted also by Q :

$$Q(v) = \frac{1}{S^n \sum_i \alpha_i^2} \left\{ \|v\|^2 - 2\binom{n+2}{n} \sum_i \int_M \varphi_i^{\frac{2}{n}} v^2 + \|v\|^2 \tilde{\delta}(1) \right\}. \tag{4.2}$$

Q is bounded on $H_{\epsilon}(a, \lambda)$, furthermore, arguing as in the proof of Proposition 3.1 in [6], we derive the following result

Lemma 4.3 *Assume the ϵ_{ij} are small enough. Then there exists $\alpha_0 = \alpha_0(p) > 0$ such that for all v in $H_{\epsilon}(a, \lambda)$: $Q(v) \geq \alpha_0 \|v\|^2$.*

On the other hand one has:

Lemma 4.4 *There exists a C^1 -map $\bar{v} : B_{\epsilon} \rightarrow H_{\epsilon}(a, \lambda)$, $(\alpha, a, \lambda) \mapsto \bar{v}(\alpha, a, \lambda)$, such that $\bar{v} = \bar{v}(\alpha, a, \lambda)$ is the unique minimum of the functional J on $H_{\epsilon}(a, \lambda)$:*

$$J\left(\sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + \bar{v}\right) = \min_{v \in H_{\epsilon}(a, \lambda)} J\left(\sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + v\right) \tag{4.3}$$

and we have the estimate

$$\|\bar{v}\| = O(\|f\|). \tag{4.4}$$

Proof. Observe that from Section 2, the parameterization of $V(p, \epsilon)$ is given by the map

$$B_\epsilon \times H_\epsilon(a, \lambda) \longrightarrow V(p, \epsilon)$$

$$(\alpha, a, \lambda, v) \longmapsto u = \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + v$$

where (α, a, λ) is the solution in B_ϵ of the minimization problem (2.4), and where $v = u - \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)}$ is in $H_\epsilon(a, \lambda)$. Since $(\alpha, a, \lambda) \in B_\epsilon$, ϵ_{ij} are small enough, then by Lemma 4.3, Q is definite, positive, and lower bounded on $H_\epsilon(a, \lambda)$. Thus, there exists a continuous self adjoint, positive and invertible operator A , such that $Q(v) = \frac{1}{2} \langle Av, v \rangle$ on $H_\epsilon(a, \lambda)$ and $\beta_0 Id \leq A \leq \beta_1 Id$ where $\beta_1 > \beta_0 > 0$. Then, the expansion of J provided by Proposition 3.1 becomes

$$J(u) = \gamma_0 \left\{ 1 + \gamma_1 + \gamma_2 - \langle f, v \rangle + \frac{1}{2} \langle Av, v \rangle + O(\|v\|^q) \right\}$$

where γ_0, γ_1 and γ_2 are independent of v , and f is small, since we derive from Lemma 4.1

$$\|f\| = O\left(\sum_{i=1}^p \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n+2}} + \sum_{i \neq j} \epsilon_{ij}^{\frac{n+2}{2n}} \left(\log \epsilon_{ij}^{-\frac{1}{n}}\right)^{\frac{n+2}{2n+2}}\right). \tag{4.5}$$

Observe that $O(\|v\|^q)$ is, as J is, twice differentiable in v , and it's differential at the origin is $O(\|v\|^{1+\frac{2}{n}})$. The expansion of J' along an increment h near the origin in $H_\epsilon(a, \lambda)$ is $J'(u)(h) = \gamma_0 \langle -f + Av + O(\|v\|^{1+\frac{2}{n}}), h \rangle$. The second differential of $O(\|v\|^q)$ is $O(\|v\|^{\frac{2}{n}})$. Thus the functional $-\langle f, v \rangle + \frac{1}{2} \langle Av, v \rangle + O(\|v\|^q)$ is coercive in a neighborhood of the origin where $\|v\|^{\frac{2}{n}} < c_0$, a small enough fixed constant, and therefore, using the implicit function theorem, this functional has a unique minimum \bar{v} in a neighborhood of zero in $H_\epsilon(a, \lambda)$, which satisfies:

$$-f + A\bar{v} + O(\|\bar{v}\|^{1+\frac{2}{n}}) = 0. \tag{4.6}$$

Now, since the operator $A + O(\|\bar{v}\|^{1+\frac{2}{n}})$ is positive and invertible in a neighborhood of the origin (depending only on the smallness of v), with inverse denoted A^{-1} and satisfying $\frac{2}{\beta_0} Id \geq A^{-1} \geq \frac{1}{2\beta_1} Id$, there exist $c, c' > 0$, such that $\|\bar{v}\| \leq c' \|A^{-1} f\| \leq c \|f\|$. Thus, estimate (4.4) follows, and Lemma 4.4 is thereby proved. \square

For such a \bar{v} satisfying (4.3) we set in the sequel

$$\bar{u} = \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + \bar{v}. \tag{4.7}$$

Since \bar{v} verify (4.6), we have : $-\langle f, v \rangle + Q(v) + O(\|v\|^q) = Q(v - \bar{v}) - Q(\bar{v}) + O(\|\bar{v}\|^q)$, and since Q is definite positive, we have $-Q(\bar{v}) + O(\|\bar{v}\|^q) = O(\|\bar{v}\|^2) = O(\|f\|^2)$. Using then estimate (4.5) on f , one has

$$J(u) = \gamma_0 \{1 + \gamma_1 + \gamma_2 + \gamma_3\} + \gamma_0 Q(v - \bar{v}), \tag{4.8}$$

$$J(\bar{u}) = \gamma_0 \{1 + \gamma_1 + \gamma_2 + \gamma_3\} \tag{4.9}$$

where $\gamma_3 = \gamma_3(f) = O(\|f\|^2)$, and (setting $v = 0$),

$$J\left(\sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)}\right) = \gamma_0 \{1 + \gamma_1 + \gamma_2\}. \tag{4.10}$$

5 More about critical points at infinity.

This section is devoted to characterize the critical points at infinity associated to problem (1.1) under (N.D) condition. The characterization is obtained through the construction of a suitable pseudogradient at infinity in the set $V(p, \epsilon)$, depending on a delicate expansion of the gradient of J near infinity.

5.1 Morse lemma at infinity.

To establish our Morse Lemma, we need to use the following two technical lemmas. The first lemma provides some pseudogradient for J which have a very nice behavior near the critical points at infinity.

We recall that $F^+ = \{(y_{i_1}, \dots, y_{i_p}) \in (I^+)^p / y_{i_j} \neq y_{i_k} \text{ if } j \neq k, 1 \leq p \leq \#I^+\}$.

Lemma 5.1 . *There exists on $V(p, \epsilon)$ a vector field w , for which there exists constants $c, c', C > 0$ independent of $\sum_i \alpha_i \varphi_i$ in $V(p, \epsilon)$ such that :*

- (i) $-J'(\bar{u})(w + \frac{\partial \bar{v}}{\partial W}) \geq c(\sum_i \frac{|\nabla_{\bar{g}} K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \epsilon_{ij})$.
- (ii) $-J'(\sum_i \alpha_i \varphi_i) w \geq c(\sum_i \frac{|\nabla_{\bar{g}} K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \epsilon_{ij})$.
- (iii) $\|w\| \leq C$.
- (iv) $|\mathrm{d}\lambda_i(w)| \leq c' \lambda_i, \forall i \in \{1, \dots, p\}$.
- (v) *The only region where the λ_i are not bounded along the decreasing flow lines of w , is where (a_1, \dots, a_p) is close to some $(y_{i_1}, \dots, y_{i_p}) \in F^+$, and the λ_i are comparable.*

The proof is given below in subsection 5.5.

With the second lemma we focus on the behavior of $J(\bar{u})$ in the neighborhood of some critical point at infinity, and we show that there exists $(\tilde{a}, \tilde{\lambda})$ such that $J(\bar{u}) = J(\sum_i \alpha_i \tilde{\varphi}_i)$, where $\tilde{\varphi}_i = \varphi_{(\tilde{a}_i, \tilde{\lambda}_i)}$. More precisely, we have

Lemma 5.2 . *Let $\sum_i \alpha_i \varphi_i \in V(p, \epsilon)$. There exists on $V(p, \epsilon)$ a change of variables $(a, \lambda) \rightarrow (\tilde{a}, \tilde{\lambda})$, such that $J(\sum_i \alpha_i \varphi_i + \bar{v}(\alpha, a, \lambda)) = J(\sum_i \alpha_i \tilde{\varphi}_i)$ and such that:*

- (*) $\tilde{d}(a_i, \tilde{a}_i) \rightarrow 0$ as $\sum_{i \neq j} \epsilon_{ij} + \sum_i \frac{1}{\lambda_i^2} \rightarrow 0$,
- (**) $\sum_{i \neq j} \tilde{\epsilon}_{ij} + \sum_i \frac{1}{\tilde{\lambda}_i^2} \rightarrow 0$ if and only if $\sum_{i \neq j} \epsilon_{ij} + \sum_i \frac{1}{\lambda_i^2} \rightarrow 0$.

Proof. See Appendix C.

Here $\tilde{\epsilon}_{ij} = \epsilon_{ij}(\tilde{a}_i, \tilde{a}_j, \tilde{\lambda}_i, \tilde{\lambda}_j) = (\frac{\tilde{\lambda}_i}{\tilde{\lambda}_j} + \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i} + \tilde{\lambda}_i \tilde{\lambda}_j \tilde{d}(\tilde{a}_i, \tilde{a}_j)^2)^{-n}$.

The following Morse lemma at infinity establishes in $V(p, \epsilon)$ a change of the variables (α, a, λ, v) into $(\tilde{\alpha}, \tilde{a}, \tilde{\lambda}, V)$, ($\tilde{\alpha} = \alpha$), where V is a variable completely independent of \tilde{a} and $\tilde{\lambda}$, which lives around zero in some fixed Hilbert subspace, and such that $J(\sum_i \alpha_i \varphi_{(a_i, \lambda_i)} + v)$ behaves like $J(\sum_i \alpha_i \varphi_{(\tilde{a}_i, \tilde{\lambda}_i)}) + \|V\|^2$. Namely we prove

Proposition 5.1 (Morse Lemma at infinity). *Let $u = (\sum_i \alpha_i \varphi_i + v)$ in $V(p, \epsilon)$. There exists a C^1 change of variables : $(\alpha, a, \lambda, v) \rightarrow (\alpha, \tilde{a}, \tilde{\lambda}, V)$, such that*

$$J(\sum_i \alpha_i \varphi_i + v) = J(\sum_i \alpha_i \tilde{\varphi}_i) + \|V\|^2.$$

where V is independent of $(\tilde{a}, \tilde{\lambda})$ and belongs to a neighborhood of zero in some fixed Hilbert subspace.

Proof. By lemma 5.2, there exists $(\tilde{a}, \tilde{\lambda})$ such that $J(\bar{u}) = J(\sum_i \alpha_i \tilde{\varphi}_i)$, and using (4.8)-(4.9), we derive that $J(u) = J(\bar{u}) + \gamma_0 Q(v - \bar{v})$. But, for ϵ small enough, Q is definite positive on $H_\epsilon(a, \lambda)$. Therefore, there exists a C^1 change of variables $v - \bar{v} \rightarrow V$, from a neighborhood of \bar{v} in $H_\epsilon(a, \lambda)$, into a neighborhood of zero in the subspace orthogonal to $\{\tilde{\varphi}_i, \frac{\partial}{\partial \tilde{\lambda}_i} \tilde{\varphi}_i, \frac{\partial}{\partial \tilde{a}_i} \tilde{\varphi}_i ; 1 \leq i \leq p\}$, such that $J(u) = J(\sum_i \alpha_i \tilde{\varphi}_i) + \|V\|^2$. q.e.d. □

5.2 Characterization of the critical points at infinity.

According to the notations of Definition 2.1 and (1.7), we prove:

Proposition 5.2 . Assume that (1.1) has no solution. Then, the set of critical points at infinity of J in Σ^+ is

$$F_\infty^+ = \{ \phi_\infty^p = (y_{i_1}, \dots, y_{i_p})_\infty ; (y_{i_1}, \dots, y_{i_p}) \in F^+ \}.$$

Proof. In fact we will prove that critical points at infinity occur only in the case (v) of Lemma 5.1, since by Lemma 5.1 the Palais-Smale condition is satisfied along the flow lines of w in the other cases (see the proof of Lemma 5.1 below). So we assume for a function $u = \sum_j \alpha_j \varphi_j + v$ in $V(p, \epsilon)$ that the concentration points a_j converge to distinct critical point y_{i_j} in I^+ and that all speeds of concentrations λ_j are comparable. Using the Morse lemma (proposition 5.1 above) and (4.10), it yields:

$$J(u) = \gamma_0(\alpha, \tilde{a}) \{ 1 + \gamma_1(\tilde{a}, \tilde{\lambda}) + \gamma_2(\tilde{a}, \tilde{\lambda}) \} + \|V\|^2. \tag{5.1}$$

The variable V is completely independent from the others, and close to zero in a fixed Hilbert subspace. Then, minimizing with respect to V , the problem is reduced in a finite dimensional problem. In fact we can do as V was nil. Indeed, one can define on the V -variable the pseudogradient $\frac{\partial V}{\partial s} = -V$, and then $V(s) = e^{-s}V(0)$ will go to zero as s goes to $+\infty$. Observe that since a_j , and hence (by Lemma 5.2) \tilde{a}_j , converge to distinct critical points y_{i_j} we have $|\nabla_\theta K(\tilde{a}_j)| = o(1)$, and since all the concentrations are comparable, each $\tilde{\epsilon}_{i_j} = o(\frac{1}{\lambda_j^2}) = o(\frac{1}{\tilde{\lambda}_j^2})$. Therefore $\gamma_1(\tilde{a}, \tilde{\lambda})$ and $\gamma_2(\tilde{a}, \tilde{\lambda})$ can easily be seen as of order $\sum_{j=1}^p o(\frac{1}{\tilde{\lambda}_j^2})$. Now, since $\Delta_\theta K(\tilde{a}_j) = \Delta_\theta K(y_{i_j})(1 + o(1))$, and after a change of variable $\tilde{\lambda}_j \mapsto \tilde{\lambda}_j = \frac{\tilde{\lambda}_j}{\sqrt{c_j(1+o(1))}}$, it yields (we drop the tilde for simplicity):

$$J(u) = \gamma_0 \left\{ 1 - \sum_{j=1}^p \frac{\Delta_\theta K(y_{i_j})}{\lambda_j^2} \right\}. \tag{5.2}$$

Recall that $\gamma_0 = \gamma_0(\alpha, a) = \frac{S^{\frac{n}{n+1}} (\sum_i \alpha_i^2)}{(\sum_i \alpha_i^q K(a_i))^{\frac{n}{n+1}}}$. Equation (5.2) then, yields a split of the variables a and λ .

Thus it is easy to see that when $a = (a_1, \dots, a_p)$ is equal to $y = (y_{i_1}, \dots, y_{i_p})$, only the variables λ_j can move. Since $-\Delta_\theta K(y_{i_j}) > 0$, in order to decrease $J(u)$, the only way is to increase the variables λ_j . When $\lambda_j = +\infty$ for $j = 1, \dots, p$, we obtain the critical point at infinity $(y_{i_1}, \dots, y_{i_p})_\infty$. \square

5.3 Morse index of a critical point at infinity.

Proposition 5.3 Let $\phi_\infty^p = (y_{i_1}, \dots, y_{i_p})_\infty \in F_\infty^+$. Let $i(\phi_\infty^p)$ be the Morse index of J at its critical point at infinity ϕ_∞^p . Then,

$$i(\phi_\infty^p) = p - 1 + \sum_{j=1}^p (2n + 1 - \text{ind}(K, y_{i_j}))$$

where $\text{ind}(K, y_{i_j})$ is the Morse index of K at y_{i_j} .

Proof. Using (5.2), it is easy to see that the Morse index of J at the critical point at infinity $(y_{i_1}, \dots, y_{i_p})_\infty$ is equal to the Morse index of the functional $\gamma_0 = \gamma_0(\alpha, a)$ at its critical point. Observe that, on the one side γ_0 is homogeneous in the variable α , and has an absolute degenerate maximum $\tilde{\alpha} = (\frac{1}{K(a_1)^{\frac{n}{2}}}, \dots, \frac{1}{K(a_p)^{\frac{n}{2}}})$ with one dimensional nullity space, and (since one can assume that $\frac{\alpha_i^{\frac{n}{2}} K(a_i)}{\tilde{\alpha}_j^{\frac{n}{2}} K(a_j)} = 1$)

with a critical value equal to $S^{\frac{n}{n+1}} (\sum_{i=1}^p \frac{1}{K(a_i)^n})^{\frac{1}{n+1}}$. On the other hand, γ_0 has a single critical point $y = (y_{i_1}, \dots, y_{i_p})$ in the α -variable. Thus, the critical point of γ_0 has an index equal to $p - 1$ in the α -variable. Let $\alpha^- \in \mathbb{R}^{p-1}$ be the coordinate of α , and, denote by a_i^+ , (respectively a_i^-), the coordinate of a_i along $W_s(y_i)$ the stable manifold of y_i , (respectively the coordinate of a_i along $W_u(y_i)$ the unstable manifold of y_i), with respect to a Morse-Smale pseudogradient for K . Then, using a Morse Lemma for the function γ_0 around its critical point $(\bar{\alpha}, y)$, we obtain, after a changing of variables, the following normal form:

$$J(u) = S^{\frac{n}{n+1}} \left(\sum_{i=1}^p \frac{1}{K(y_i)^n} \right)^{\frac{1}{n+1}} \left\{ 1 - |\alpha^-|^2 + \sum_{i=1}^p (|a_i^+|^2 - |a_i^-|^2) \right\}. \tag{5.3}$$

Noticing that unstable manifolds for K coincide with stable manifolds for $\frac{1}{K}$, the Morse index searched is equal to $p - 1 + \sum_{j=1}^p (2n + 1 - \text{ind}(K, y_{i_j}))$. Hence the result.

5.4 Stable and unstable manifold of a critical point at infinity

Denote by ϕ_∞^p the critical point at infinity $(y_{i_1}, \dots, y_{i_p})_\infty$. Since a Morse lemma at infinity is available around ϕ_∞^p , one can define its stable manifold as usual, that is $W_s(\phi_\infty^p)$ is the set of all points attracted by the asymptote ϕ_∞^p through the decreasing flow lines of w .

The unstable manifold is more delicate to define. Rigorously speaking, it is a shadow object, since it exists only as the limit of $W_u(u_\lambda)$ the unstable manifold of the critical point u_λ of the reduced problem in a section relative to λ . Since there is a Morse lemma in this section and since the flow of w splits the variable λ from the other variables near ϕ_∞^p , one can therefore think of $W_u(\phi_\infty^p)$ to be represented by some $W_u(u_\lambda)$ for λ very large. To be more precise, using the notations of 5.3 above, the unstable manifold of ϕ_∞^p can be defined as:

$$W_u(\phi_\infty^p) = W_u(\bar{\alpha}, y) \times [A, +\infty) \tag{5.4}$$

where $W_u(\bar{\alpha}, y)$ is the unstable manifold of the critical point $(\bar{\alpha}, y)$ of the functional γ_0 and A a large enough positive constant. $[A, +\infty)$ being the domain of the variable λ .

5.5 Proof of Lemma 5.1

We will use here the notations and results of Appendix B. Observe that, since the proof of (ii) can be easily derived from the proof of (i), we will then prove (i).

Since \bar{v} minimizes $J(\sum_{i=1}^p \alpha_i \varphi_i + v)$ in $H_\epsilon(a, \lambda)$, $J'(\bar{u}) = J'(\sum_i \alpha_i \varphi_i + \bar{v}) = \pi_E(J'(\bar{u}))$, and therefore

$$J'(\bar{u})(w + \frac{\partial \bar{v}}{\partial w}) = \pi_E(J'(\bar{u}))(w + \frac{\partial \bar{v}}{\partial w}) = J'(\bar{u})(w + \pi_E(\frac{\partial \bar{v}}{\partial w})). \tag{5.5}$$

Since $H_\epsilon(a, \lambda)$ is orthogonal to E , we have $\langle \bar{v}, \frac{\partial \varphi_i}{\partial w} \rangle = 0$, and then $\langle \frac{\partial \bar{v}}{\partial w}, \varphi_i \rangle = 0$ for $1 \leq i \leq p$. Thus $\pi_E(\frac{\partial \bar{v}}{\partial w}) \in F$, that is

$$\pi_E(\frac{\partial \bar{v}}{\partial w}) = \pi_F(\frac{\partial \bar{v}}{\partial w}). \tag{5.6}$$

Using (5.5), (5.6) and Proposition B.1, we get

$$-J'(\bar{u})(w + \frac{\partial \bar{v}}{\partial w}) \geq -J'(\bar{u})(w) - \bar{c} \left(\sup_{\phi \in F} \frac{J'(\bar{u})(\phi)}{\|\phi\|} \right) \|\bar{v}\| \|w\|, \tag{5.7}$$

then, using Lemma B.1, we have

$$\begin{aligned}
 -J'(\bar{u})(w + \frac{\partial \bar{v}}{\partial w}) &\geq -J'(\sum_i \alpha_i \varphi_i)w - J''(\sum_i \alpha_i \varphi_i)\bar{v}w \\
 &+ O(\|\bar{v}\|^2 \|w\|) + O(\sup_{\phi \in F} \frac{J'(\bar{u})(\phi)}{\|\phi\|}) \|\bar{v}\| \|w\|.
 \end{aligned}
 \tag{5.8}$$

By Lemmas B.2 and B.3, (5.8) becomes

$$\begin{aligned}
 -J'(\bar{u})(w + \frac{\partial \bar{v}}{\partial w}) &\geq -J'(\sum_i \alpha_i \varphi_i)w - \\
 &- C\|w\|(\sum_i \frac{|\nabla_{\theta} K(a_i)|^2}{\lambda_i^2} + \sum_i \frac{1}{\lambda_i^4} + \sum_{i \neq j} \epsilon_{ij}^{\frac{n+2}{n}} (\log \epsilon_{ij}^{-\frac{1}{n}})^{\frac{n+2}{n+1}}),
 \end{aligned}
 \tag{5.9}$$

where C is a universal constant.

We need to define w so that the (P.S) condition should be satisfied on its decreasing flow lines. w has no action on the α_i 's variables, and, moving the a_i 's contains no risk for (P.S) condition, since they lie in a compact set, so we only need

$$\forall s \geq 0 \quad \frac{\partial}{\partial s} (\sup_{1 \leq i \leq p} \lambda_i) \leq 0,
 \tag{5.10}$$

where s is the time along a flow line of w .

Here, we resort to the estimates given by Lemma 3.1. Since $|\lambda_i \frac{\partial \epsilon_{ij}}{\partial a_i}| \leq c \epsilon_{ij}$, we derive from (3.4)

$$|J'(\sum_j \alpha_j \varphi_j) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i}| \geq c \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} - \frac{1}{c} (\sum_{j \neq i} \epsilon_{ij} + \frac{1}{\lambda_i^2})
 \tag{5.11}$$

where c is a suitable positive constant.

Assuming that

$$\sum_{j \neq i} \epsilon_{ij} \leq \frac{C}{\lambda_i^2},
 \tag{5.12}$$

where C is a suitable constant, (5.11) becomes

$$|J'(\sum_j \alpha_j \varphi_j) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i}| \geq c \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} - \frac{1}{c'} \frac{1}{\lambda_i^2}
 \tag{5.13}$$

c' a suitable constant. We claim that lemma 5.1 follows from (3.3) and (5.13), if (5.13) holds for any i . We will prove this later. Assuming that (5.12) does not hold for some indexes, we choose the index i so that λ_i is the largest concentration with

$$\sum_{j \neq i} \epsilon_{ij} > \frac{C}{\lambda_i^2},
 \tag{5.14}$$

then, for $\lambda_j \geq \lambda_i$, we have

$$\sum_{k \neq j} \epsilon_{kj} \leq \frac{C}{\lambda_j^2}.
 \tag{5.15}$$

Observe that, if λ_j and λ_i are comparable, or if $\lambda_i \geq \lambda_j$, then

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -n \varepsilon_{ij} (1 + o(1)). \tag{5.16}$$

If they are not, and $\lambda_i = o(\lambda_j)$, then

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = O(\varepsilon_{ij}) \leq \frac{C}{\lambda_j^2} = o\left(\frac{1}{\lambda_i^2}\right). \tag{5.17}$$

Thus (5.14), (5.16) and (5.17) imply

$$-\sum_{j \neq i} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \geq \frac{n}{2} \sum_{j \neq i} \varepsilon_{ij} \geq \frac{C}{2\lambda_i^2}. \tag{5.18}$$

hence, choosing C large enough, we derive from (3.3)

$$J'(\sum_j \alpha_j \varphi_j) \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \geq \frac{C}{4\lambda_i^2}. \tag{5.19}$$

(5.11) and (5.19) imply

$$\left| J'(\sum_j \alpha_j \varphi_j) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \right| + \frac{\tilde{c}}{c} J'(\sum_j \alpha_j \varphi_j) \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \geq c \frac{|\nabla_\theta K(a_i)|}{\lambda_i} - \frac{1}{c\lambda_i^2} \tag{5.20}$$

for suitable positive constants c, \tilde{c} . Observe that (5.13) is obtained from (5.20) by taking $\tilde{c} = 0$. Thus we established (5.20) for an index i with λ_i the largest satisfying (5.14).

Assume now, that we have another index i such that (5.14) holds, but λ_i is not the largest. Then we introduce

$$I_i = \left\{ k / \lambda_k \geq \lambda_i, \sum_{j \neq k} \varepsilon_{kj} > \frac{c}{\lambda_k^2} \right\}. \tag{5.21}$$

Observe that for $\lambda_k \geq \lambda_i$ we have

$$2\lambda_k \frac{\partial \varepsilon_{kj}}{\partial \lambda_k} + \lambda_j \frac{\partial \varepsilon_{kj}}{\partial \lambda_j} \leq -\frac{n}{2} \varepsilon_{kj} (1 + o(1)). \tag{5.22}$$

We derive, using (5.16) (5.17), the existence of suitable bounded constants c_k , (one can take $c_k = \frac{\tilde{c}}{c} 2^k$), such that

$$\left| J'(\sum_j \alpha_j \varphi_j) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \right| + \sum_{k \in I_i} c_k J'(\sum_j \alpha_j \varphi_j) \lambda_k \frac{\partial \varphi_k}{\partial \lambda_k} \geq c \frac{|\nabla_\theta K(a_i)|}{\lambda_i} - \frac{1}{c\lambda_i^2}. \tag{5.23}$$

We need to prove lemma 5.1 under (3.3), (5.21), (5.22) and (5.23). For this purpose, and for sake of simplicity, we assume that

$$\lambda_1 \leq \dots \leq \lambda_p. \tag{5.24}$$

If

$$\frac{|\nabla_\theta K(a_1)|}{\lambda_1} \geq \frac{1}{c^2 \lambda_1^2}, \tag{5.25}$$

then, (either by (5.13) or by (5.23), we derive

$$\left| J' \left(\sum_j \alpha_j \varphi_j \right) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \right| + \sum_{k \in I_1} c_k J' \left(\sum_j \alpha_j \varphi_j \right) \lambda_k \frac{\partial \varphi_k}{\partial \lambda_k} \geq \frac{1}{4c\lambda_1^2} + \frac{|\nabla_\theta K(a_1)|}{4\lambda_1}. \tag{5.26}$$

This, combined with (5.23) again and (3.3) yields, with a suitable constant \bar{c} , for any i ,

$$\begin{aligned} & \left| J' \left(\sum_j \alpha_j \varphi_j \right) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \right| + \bar{c} \left| J' \left(\sum_j \alpha_j \varphi_j \right) \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} \right| + J' \left(\sum_j \alpha_j \varphi_j \right) \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \\ & + \sum_{k \in I_1} c_k J' \left(\sum_j \alpha_j \varphi_j \right) \lambda_k \frac{\partial \varphi_k}{\partial \lambda_k} + \sum_{k \in I_1} c_k J' \left(\sum_j \alpha_j \varphi_j \right) \lambda_k \frac{\partial \varphi_k}{\partial \lambda_k} \geq \\ & \text{(since } \lambda_1 \text{ is the smallest)} \geq - \sum_{j \neq i} c_{ij} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + \frac{\bar{c}}{\lambda_i^2}. \end{aligned} \tag{5.27}$$

We would like to construct a combination with given bounded nonnegative constants β_i, γ_i , such that, using (5.22) and (5.27), we have :

$$\sum_i \beta_i \left| J' \left(\sum_j \alpha_j \varphi_j \right) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \right| + J' \left(\sum_j \alpha_j \varphi_j \right) \sum_i \gamma_i \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \geq \bar{c} \left(\sum_i \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij} \right).$$

We can also require that

$$\beta_i = 0 \quad \text{if} \quad \left| J' \left(\sum_j \alpha_j \varphi_j \right) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \right| < \frac{\bar{c}}{10p} \sum_i \frac{1}{\lambda_i^2}. \tag{5.28}$$

Setting then

$$w = - \sum_i \beta_i \text{sign} \left(J' \left(\sum_j \alpha_j \varphi_j \right) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \right) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} - \sum_i \gamma_i \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i}, \tag{5.29}$$

w is then a C^1 vector field ($\beta_i = 0$, by (5.28), when $J'(\sum_j \alpha_j \varphi_j) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i}$ is small). Since the β_i, γ_i are bounded, and since $\| \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \| \leq C$ and $\| \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \| \leq C$, we derive that

$$\|w\| \leq C. \tag{5.30}$$

On the other hand, $d\lambda_i(w) = \dot{\lambda}_i = -\alpha_i \gamma_i \lambda_i$; thus

$$\left| d\lambda_i(w) \right| \leq c' \lambda_i, \quad \forall i = 1, \dots, p. \tag{5.31}$$

where c' is a constant, and hence (iii) and (iv) are satisfied in this case.

A similar proof can be repeated if

$$\sum_{j \neq i} \varepsilon_{ij} \geq \frac{C}{\lambda_1^2}. \tag{5.32}$$

even if we assume

$$\frac{|\nabla_\theta K(a_1)|}{\lambda_1} \leq \frac{2}{c^2 \lambda_1^2}. \tag{5.33}$$

Now, we will extend the above proof as follows.

Assume that instead of $\lambda_1 \leq \dots \leq \lambda_p$, we single out a subpack

$$\lambda_{i_1} \leq \lambda_{i_1+1} \dots \leq \lambda_{i_1+r}. \tag{5.34}$$

Therefore, we will construct a vector field $w_{(i_1,r)}$ in $\text{Span}\left\{\frac{\partial\varphi_i}{\partial a_i}, \frac{\partial\varphi_i}{\partial\lambda_i}\right\}_{i_1\leq i\leq i_1+r}$ such that

$$\|w_{(i_1,r)}\| \leq C \tag{5.35}$$

and

$$0 \leq -d\left(\sup_{i_1\leq i\leq i_1+r} \lambda_i\right)(w_{(i_1,r)}) \leq C \sup_{i_1\leq i\leq i_1+r} \lambda_i. \tag{5.36}$$

Under the hypothesis

$$\frac{|\nabla_\theta K(a_{i_1})|}{\lambda_{i_1}} \geq \frac{2}{c^2\lambda_{i_1}^2} \quad \text{or} \quad \sum_{s=0}^r \sum_{\substack{j\leq i_1+r \\ j\neq i_1+s}} \varepsilon_{i_1+s,j} \geq \frac{1}{\lambda_{i_1}^2} \tag{5.37}$$

we obtain

$$\begin{aligned} -J'\left(\sum_j \alpha_j \varphi_j\right)(w_{(i_1,r)}) &\geq C\left(\sum_{s=0}^r \sum_{\substack{j\leq i_1+r \\ j\neq i_1+s}} \varepsilon_{i_1+s,j} + \sum_{s=1}^r \frac{1}{\lambda_{i_1+s}^2} + \right. \\ &\quad \left. + \sum_{s=1}^r \frac{|\nabla_\theta K(a_{i_1+s})|}{\lambda_{i_1+s}}\right) - \frac{1}{\bar{c}} \sum_{j\geq i_1+r+1} \sum_{s=1}^r \varepsilon_{i_1+s,j}. \end{aligned} \tag{5.38}$$

We first assume that such indexes i_1 satisfying (5.37) exist (we will assume the opposite later on), and we denote again by i_1 the index satisfying (5.37) such that λ_{i_1} is the smallest concentration. Since $\lambda_{i_1} \leq \dots \leq \lambda_p$, we derive

$$-J'\left(\sum_j \alpha_j \varphi_j\right)(w_{(i_1,p-i_1)}) \geq \bar{c}\left(\sum_{i\geq i_1} \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + \frac{1}{\lambda_{i_1}^2} + \sum_{k=i_1}^p \varepsilon_{ik}\right). \tag{5.39}$$

If $i_1 = 1$, we have the result. Otherwise, for any $\ell < i_1$, we have

$$\frac{|\nabla_\theta K(a_\ell)|}{\lambda_\ell} \leq \frac{2}{c^2\lambda_\ell^2} \quad \text{and} \quad \sum_{k=\ell}^p \sum_{i\neq k} \varepsilon_{ik} \leq \frac{c}{\lambda_\ell^2}. \tag{5.40}$$

We claim that lemma 5.1 follows from (5.39) and (5.40), unless:

$$\sum_{i\geq i_1} \frac{|\nabla_\theta K(a_i)|}{\lambda_i} + \frac{1}{\lambda_{i_1}^2} + \sum_{k=i_1}^p \sum_{i\neq k} \varepsilon_{ik} = o\left(\frac{1}{\lambda_\ell^2}\right) = o\left(\frac{1}{\lambda_1^2}\right) \tag{5.41}$$

for some $\ell \leq i_1 - 1$. Assuming that (5.41) holds, then by (5.40), for $i \leq i_1 - 1$, we have $\frac{|\nabla_\theta K(a_i)|}{\lambda_i} \leq \frac{2}{c^2\lambda_i^2}$, then

$$\lambda_i |\nabla_\theta K(a_i)| \leq \frac{2}{c^2} \tag{5.42}$$

and thus, since λ_i is very large, $|\nabla_\theta K(a_i)| = o(1)$. So, for $i \leq i_1 - 1$, a_i is close to a critical point of K which we denote by y_i . Using then a pseudohermitian normal coordinates chart centered at a_i , or at y_i , (y_i is nondegenerate, i.e. the Hessian matrix of K at y_i , in this chart, is invertible), we derive that $\tilde{d}(a_i, y_i) \leq \kappa |\nabla_\theta K(a_i)|$ for some suitable positive constant κ , and then $\lambda_i \tilde{d}(a_i, y_i) \leq C$ for each $i \leq i_1 - 1$. Consequently, if for $i, j \leq i_1 - 1$, a_i and a_j are close to the same critical point y_i , $\frac{\lambda_j}{\lambda_i} \rightarrow +\infty$ or $\frac{\lambda_j}{\lambda_i} \rightarrow +\infty$, and then $\varepsilon_{ij} = o\left(\frac{1}{\inf(\lambda_i, \lambda_j)^2}\right) = o\left(\frac{1}{\lambda_i^2}\right)$. Now, if a_i and a_j are close to distinct critical

points y_i and y_j , $\varepsilon_{ij} = O\left(\frac{1}{(\lambda_i \lambda_j)^n}\right) = o\left(\frac{1}{\inf(\lambda_i, \lambda_j)^2}\right) = o\left(\frac{1}{\lambda_1^2}\right)$, because $n \geq 2$. Thus, for all $i, j \leq i_1 - 1$, the ε_{ij} 's are $o\left(\frac{1}{\lambda_1^2}\right)$, as well as they are for $i, j \geq i_1$ by (5.41). Thus all the ε_{ij} 's, in this case, are $o\left(\frac{1}{\lambda_1^2}\right)$, and therefore

$$\sum_{i \neq j} \varepsilon_{ij} = o\left(\frac{1}{\lambda_1^2}\right). \tag{5.43}$$

This implies

$$\sum_{i \geq i_1} \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} = o\left(\frac{1}{\lambda_1^2}\right). \tag{5.44}$$

Using (5.42) we have

$$\sum_{i < i_1} \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} \leq \frac{c}{\lambda_i^2} \leq \frac{c}{\lambda_1^2}, \tag{5.45}$$

and thus

$$\sum_{i=1}^p \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} \leq \frac{c}{\lambda_1^2}. \tag{5.46}$$

Since a_1 is close to a critical point y_1 , ($1 < i_1$), and since y_1 is nondegenerate, $\Delta_{\theta} K(y_1) \neq 0$, and $\Delta_{\theta} K(a_1) = \Delta_{\theta} K(y_1)(1 + o(1)) \neq 0$, we derive then by (3.3),

$$-J'\left(\sum_j \alpha_j \varphi_j\right) \lambda_1 \frac{\partial \varphi_1}{\partial \lambda_1} = -c \frac{\Delta_{\theta} K(y_1)}{\lambda_1^2} + o\left(\frac{1}{\lambda_1^2}\right) + \sum_{i \neq 1} \tilde{c}_{1i} \lambda_1 \frac{\partial \varepsilon_{1i}}{\partial \lambda_1}. \tag{5.47}$$

Since $\lambda_1 \frac{\partial \varepsilon_{1i}}{\partial \lambda_1} = O(\varepsilon_{1i}) = o\left(\frac{1}{\lambda_1^2}\right)$, from (5.40) and (5.41) we derive that,

$$-J'\left(\sum_j \alpha_j \varphi_j\right) \lambda_1 \frac{\partial \varphi_1}{\partial \lambda_1} = -c \frac{\Delta_{\theta} K(y_1)}{\lambda_1^2} + o\left(\frac{1}{\lambda_1^2}\right). \tag{5.48}$$

Observe that by (5.43) and (5.46),

$$\frac{1}{\lambda_1^2} \geq c^2 \left(\sum_{i=1}^p \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right). \tag{5.49}$$

where $c > 0$ is as we please.

First, let us assume that

$$-\Delta_{\theta} K(y_1) \geq c' > 0. \tag{5.50}$$

So, we have $-c \frac{\Delta_{\theta} K(y_1)}{\lambda_1^2} \geq \frac{1}{c \lambda_1^2}$, and then

$$-J'\left(\sum_j \alpha_j \varphi_j\right) \lambda_1 \frac{\partial \varphi_1}{\partial \lambda_1} \geq c \left(\sum_{i=1}^p \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right). \tag{5.51}$$

In this case, we are free to take

$$w = \lambda_1 \frac{\partial \varphi_1}{\partial \lambda_1}. \tag{5.52}$$

w will increase λ_1 ($\dot{\lambda}_1 = \lambda_1$), but w will be bounded and will not increase $\sup_i \lambda_i$, that is, $\overline{(\sup_i \lambda_i)} \leq 0$.

Next, assume

$$-\Delta_\theta K(y_1) < 0, \tag{5.53}$$

we then take

$$w = -\lambda_1 \frac{\partial \varphi_1}{\partial \lambda_1}. \tag{5.54}$$

Therefore, the vector field w given by (5.52) or (5.54), is bounded, satisfies (5.51) and is such that $d(\sup_i \lambda_i)(w) \leq 0$, that is to say, it satisfies all the required properties. Lemma 5.1 then follows as soon as i_1 exists.

Assume now that such i_1 satisfying (5.37) does not exist, that is the case where $\ell = p$. Then, (5.40) holds for any index i , and we get $\forall i \in \{1, \dots, p\}$,

$$\frac{|\nabla_\theta K(a_i)|}{\lambda_i} \leq \frac{2}{c^2 \lambda_i^2} \quad \text{and} \quad \sum_{j \neq i} \varepsilon_{ij} \leq \frac{c}{\lambda_i^2}. \tag{5.55}$$

Thus a_i is close to a critical point of K which we denote by y_i , and then $\lambda_i \tilde{d}(a_i, y_i) = O(1)$ for all i . If for $i \neq j$, a_i and a_j are close to the same critical point y_i , then $\frac{\lambda_i}{\lambda_j}$ or $\frac{\lambda_j}{\lambda_i} \rightarrow +\infty$, and $\varepsilon_{ij} = o\left(\frac{1}{\inf(\lambda_i, \lambda_j)^2}\right)$. Now, if for all $i \neq j$, a_i and a_j are close to distinct critical points y_i and y_j , then, since $n \geq 2$, $\varepsilon_{ij} = o\left(\frac{1}{\inf(\lambda_i, \lambda_j)^2}\right)$. One can assume that

$$\inf \tilde{d}(a_i, a_j) < \frac{1}{2} \inf \tilde{d}(y_k, y_l). \tag{5.56}$$

(If not, the proof is straightforward). Under (5.56), a_i and a_j are close to the same critical point y_i , then $\inf(\lambda_i, \lambda_j) = o(\sup(\lambda_i, \lambda_j))$, so $\lambda_1 = o(\sup_i \lambda_i)$, and therefore

$$\frac{|\nabla_\theta K(a_i)|}{\lambda_i} \leq \frac{2}{c^2 \lambda_1^2} \quad \text{and} \quad \sum_{j \neq i} \varepsilon_{ij} = o\left(\frac{1}{\lambda_1^2}\right). \tag{5.57}$$

The same argument used in the previous case can be repeated here, since we obtain once more (5.48), with $\lambda_1 = o(\sup_i \lambda_i)$.

Observe that the above argument is available as soon as two concentrations are not comparable. So we will assume now that $\inf \tilde{d}(a_i, a_j) \geq d_0 > 0$, and all the concentrations are comparable, that is $\frac{1}{c} \leq \frac{\lambda_i}{\lambda_j} \leq c$. If for some index i , a_i is not close to a critical point y_i , then $\varepsilon_{ij} = o\left(\frac{1}{\lambda_i^2}\right) = o\left(\frac{1}{\lambda_j^2}\right)$, and $\frac{|\nabla_\theta K(a_i)|}{\lambda_i} \geq \frac{c}{\lambda_i^2}$, so we have $|J'(\sum_j \alpha_j \varphi_j) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i}| \geq c \frac{|\nabla_\theta K(a_i)|}{4\lambda_i} - \frac{1}{c\lambda_i^2} \geq \frac{c}{2\lambda_i}$, and thus

$$\left| J' \left(\sum_j \alpha_j \varphi_j \right) \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \right| \geq c \left(\sum_{k=1}^p \frac{|\nabla_\theta K(a_k)|}{\lambda_k} + \sum_{k=1}^p \frac{1}{\lambda_k^2} + \sum_{k \neq l} \varepsilon_{kl} \right)$$

because all the concentrations are comparable. Hence w is also constructed in this case. Now, we are left with the case where each point a_i is close to a critical point y_i , (always with comparable concentrations). We know that

$$-J' \left(\sum_j \alpha_j \varphi_j \right) \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} = -c \frac{\Delta_\theta K(y_i)}{\lambda_i^2} + o\left(\frac{1}{\lambda_i^2}\right).$$

If $-\Delta_\theta K(y_i) < 0$, $\lambda_i = -\lambda_i$, so, $d(\sup_i \lambda_i)(w) \leq 0$, and in this case again w is constructed.

Our final pseudogradient w will be then a convex combination of the previous constructed pseudogradients, since all the involved estimates are compatible with convex combinations.

Finally, it remains the last case, which is the case where the points a_i are close to distinct critical points y_{j_i} , having $-\Delta_\theta K(y_{j_i}) > 0$, and such that all the concentrations are comparable. One can argue as in the proof of proposition 5.2, to conclude that in this region the decreasing flow lines of w are attracted by the critical point at infinity $(y_{j_1}, \dots, y_{j_p})_\infty$. Thus condition (v) is satisfied, which ends the proof of lemma 5.1.

6 Proofs of Theorems.

6.1 Proof of Theorem 1.1

In order to prove our Theorem, we introduce the following definition

Definition 6.1 *Let N be a submanifold of dimension k of Σ^+ , and let ϕ_∞^p be a critical point at infinity of Morse index $i(\phi_\infty^p) \leq k$. We say that ϕ_∞^p is dominated by N , and we write $\phi_\infty^p < N$, if*

$$N \cap W_s(\phi_\infty^p) \neq \emptyset.$$

Now, for any $k \in \mathbb{N}$ and any subset X_k of $\{\phi_\infty^p \in F_\infty^+ ; i(\phi_\infty^p) \leq k\}$, we recall the following set given by (1.8)

$$X_k^\infty = \bigcup_{\phi_\infty^p \in X_k} W_u(\phi_\infty^p).$$

X_k^∞ defines a stratified set of top dimension k . Without loss of generality we can assume it equal to k . Observe that since Σ^+ is a contractible set, then X_k^∞ is contractible in Σ^+ . More precisely, there exists a contraction $h : [0, 1] \times X_k^\infty \rightarrow \Sigma^+$, i.e. h continuous and such that $\forall u \in X_k^\infty, h(0, u) = u$ and $h(1, u) = \tilde{u}$ a fixed point in X_k^∞ . Let

$$\psi(X_k^\infty) = h([0, 1] \times X_k^\infty).$$

$\psi(X_k^\infty)$ is a contractible stratified set of dimension $k + 1$. Now, we use the flow lines of $-\partial J$, to deform $\psi(X_k^\infty)$. By transversality arguments, we can assume that the deformation avoids all critical points at infinity of Morse index greater than or equal to $k + 2$.

Recall that F_∞^+ is the set of the critical points at infinity of J , and Let \simeq denotes retracts by deformation. Using the result of Bahri-Rabinowitz (Proposition 7.24 and Theorem 8.2 of [7]), which states that each manifold retracts by deformation onto the union of the unstable manifolds of the critical points and the critical points at infinity of J dominated by the manifold, it follows that

$$\begin{aligned} \psi(X_k^\infty) &\simeq \bigcup_{\{\phi_\infty^p < \psi(X_k^\infty), i(\phi_\infty^p) \leq k+1\}} W_u(\phi_\infty^p) \\ &\simeq X_k^\infty \cup \bigcup_{\{\phi_\infty^p \in F_\infty^+ \setminus X_k, \phi_\infty^p < \psi(X_k^\infty), i(\phi_\infty^p) \leq k+1\}} W_u(\phi_\infty^p). \end{aligned}$$

Now, taking $k = k_0$, where k_0 is the integer defined in Theorem 1.1, and since k_0 verifies the assumption **(H₂)** we derive that

$$\psi(X_{k_0}^\infty) \simeq X_{k_0}^\infty. \tag{6.1}$$

Recall that if N is a finite CW complex in dimension k , then the Euler-Poincaré characteristic is given by

$$\chi(N) = \sum_{j=0}^k (-1)^j n_j \tag{6.2}$$

where n_j is the number of cells of dimension j of N (see e.g. [15]). In our context, $X_{k_0}^\infty$ is a finite CW complex in dimension k_0 , and the j -dimensional cells of $X_{k_0}^\infty$ are the unstable manifolds of critical points at infinity of Morse index $i(\phi_\infty^p) = j$.

According to (6.2), we derive that

$$\chi(X_{k_0}^\infty) = \sum_{\{\phi_\infty^p \in X_{k_0}\}} (-1)^{i(\phi_\infty^p)}.$$

On the other hand, since $\psi(X_{k_0}^\infty)$ is a contractible set, we derive that $\chi(\psi(X_{k_0}^\infty)) = 1$. Taking the Euler-Poincaré characteristic of both sides of (6.1), we obtain

$$1 = \sum_{\{\phi_\infty^p \in X_{k_0}\}} (-1)^{i(\phi_\infty^p)}$$

which contradicts the assumption **(H₁)**. Hence, there is a solution w , and it is easy to derive from the above arguments that its Morse index $i(w) \leq k_0 + 1$.

6.2 Proof of Theorem 1.2

Since $K = 1 + \varepsilon K_0$, the functional is

$$J(u) = \frac{\|u\|^2}{\left(\int_{\mathbb{S}^{2n+1}} (1 + \varepsilon K_0) |u|^{2+\frac{2}{n}} \theta \wedge d\theta^n\right)^{\frac{n}{n+1}}}$$

and for $\varepsilon = 0$ we obtain the Yamabe functional

$$J_0(u) = \frac{\|u\|^2}{\left(\int_{\mathbb{S}^{2n+1}} |u|^{2+\frac{2}{n}} \theta \wedge d\theta^n\right)^{\frac{n}{n+1}}},$$

which possesses a $(2n + 2)$ -dimensional manifold of critical points

$$\mathcal{Z} = \{\delta_{(a,\lambda)}, (a, \lambda) \in \mathbb{S}^{2n+1} \times (0, \infty)\}.$$

Here $\delta_{(a,\lambda)}$ are the solutions of Yamabe problem on \mathbb{S}^{2n+1} (see [12], [20]). They correspond to the “almost solutions $\varphi_{(a,\lambda)}$ ”, except that they satisfy the Yamabe equation $L\delta_{(a,\lambda)} = \delta_{(a,\lambda)}^{1+\frac{2}{n}}$ on \mathbb{S}^{2n+1} . The sphere is globally CR equivalent to itself, then standard arguments and results of the previous sections are valid here.

Let σ be the minimum of J_0 on Σ^+ , that is

$$\sigma = J_0(\delta_{(a,\lambda)}) = S^{\frac{n}{n+1}}$$

where S is the Sobolev constant, see (A.5). Let $J^\beta = \{u \in \Sigma^+, J(u) \leq \beta\}$, $\beta \in \mathbb{R}$. We need the following Lemma:

Lemma 6.1 *Let $\eta > 0$. For $|\varepsilon|$ small enough, we have $J^{\sigma+\eta} \subset J_0^{\sigma+2\eta} \subset J^{\sigma+3\eta}$.*

Proof. Observe that, for $u \in \Sigma^+$, we have

$$J(u) = J_0(u) \frac{1}{\left(1 + \varepsilon \left(\int_{\mathbb{S}^{2n+1}} u^{2+\frac{2}{n}}\right)^{-1} \int_{\mathbb{S}^{2n+1}} K_0 u^{2+\frac{2}{n}}\right)^{\frac{n}{n+1}}}.$$

Since K_0 is bounded on \mathbb{S}^{2n+1} , we derive that

$$J(u) = J_0(u)(1 + O(\varepsilon))$$

where $O(\varepsilon)$ is independent of u and $O(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$. Whence our Lemma.

On the other hand, using (5.3), the critical level corresponding to a critical point at infinity made of p masses is

$$J((y_1, \dots, y_p)_\infty) = S^{\frac{n}{n+1}} \left(\sum_{i=1}^p \frac{1}{K(y_i)^n}\right)^{\frac{1}{(n+1)}}$$

which tends to $S^{\frac{n}{n+1}} p^{1/(n+1)}$ as $\varepsilon \rightarrow 0$, since $K(y_i) = 1 + \varepsilon K_0(y_i)$.

Taking $\eta = \frac{\sigma}{4}$, we can assume $|\varepsilon|$ sufficiently small that:

(i) critical points at infinity made of two bubbles or more are above the level $\sigma + 3\eta$,

and

(ii) critical points at infinity made of a single bubble are below the level $\sigma + \eta$.

Therefore, J has no critical points at infinity in the set

$$J_{\sigma+\eta}^{\sigma+3\eta} = \{u \in \Sigma^+, \sigma + \eta \leq J(u) \leq \sigma + 3\eta\}.$$

Since, arguing by contradiction, we assume that (1.1) has no solution. It follows that $J^{\sigma+3\eta} \simeq J^{\sigma+\eta}$, where \simeq denotes retracts by deformation. Using Lemma 6.1, we have that $J_0^{\sigma+2\eta} \simeq J^{\sigma+\eta}$. Now, we claim that $J^{\sigma+\eta}$ is a contractible set. Indeed, from what precedes, it is sufficient to prove that $J_0^{\sigma+2\eta}$ is a contractible set. Let $u_0 \in J_0^{\sigma+2\eta}$, and $s \mapsto u(s, u_0)$ the Yamabe flow line. It satisfies the following equation

$$\begin{cases} \frac{\partial u}{\partial s} = -\partial J_0(u) \\ u(0) = u_0. \end{cases}$$

We know that the Palais-Smale condition is satisfied for the above equation for all $s > 0$. When $s \rightarrow +\infty$, $u(s, u_0)$ converges to a single mass in \mathcal{Z} . Thus, $J_0^{\sigma+2\eta} \simeq \mathcal{Z}$. It follows that $J_0^{\sigma+2\eta}$ is a contractible set, since \mathcal{Z} is a contractible set. Our claim follows.

Now, let k_0 be the integer for which the maximum is achieved in (1.10). Then

$$\sum_{\{y \in I^+, 2n+1-ind(K,y) \leq k_0\}} (-1)^{2n+1-ind(K,y)} \neq 1. \tag{6.3}$$

We choose the set

$$X_{k_0} = \{(y)_\infty, y \in I^+, i(y)_\infty \leq k_0\}$$

which is the subset of the critical points at infinity made out of a single mass, and having a Morse index $\leq k_0$. Now, since the Morse index of $(y)_\infty$ is $i(y)_\infty = 2n + 1 - ind(K, y)$, equation (6.3) becomes

$$\sum_{\{(y)_\infty \in X_{k_0}\}} (-1)^{i(y)_\infty} \neq 1, \tag{6.4}$$

which signifies that k_0 satisfies assumption (\mathbf{H}_1) of Theorem 1.1. We introduce

$$X_{k_0}^\infty = \bigcup_{\{(y)_\infty \in X_{k_0}\}} W_u(y)_\infty.$$

It is a stratified set of top dimension k_0 , and, since it is made of unstable manifolds of critical points at infinity of a single mass, we derive from (\mathbf{ii}) that $X_{k_0}^\infty \subset J^{\sigma+\eta}$. Observe that $X_{k_0}^\infty$ is contractible in $J^{\sigma+\eta}$, since $J^{\sigma+\eta}$ is a contractible set. More precisely, there exists a contraction $h : [0, 1] \times X_{k_0}^\infty \rightarrow J^{\sigma+\eta}$, i.e. h continuous and $\forall u \in X_{k_0}^\infty, h(0, u) = u$ and $h(1, u) = \tilde{u}$ a fixed point in $X_{k_0}^\infty$. Let

$$\psi(X_{k_0}^\infty) = h([0, 1] \times X_{k_0}^\infty).$$

$\psi(X_{k_0}^\infty)$ is a contractible stratified set of dimension $k_0 + 1$.

Now let $\phi_\infty^p \in F_\infty^+ \setminus X_{k_0}$ such that $i(\phi_\infty^p) \leq k_0 + 1$. If ϕ_∞^p is made of two masses or more, i.e. $p \geq 2$, then, using the fact that $\psi(X_{k_0}^\infty) \subset J^{\sigma+\eta}$, and using (\mathbf{i}) , we derive that $\psi(X_{k_0}^\infty) \cap W_s(\phi_\infty^p) = \emptyset$. By using (\mathbf{ii}) , it remains to consider the critical points at infinity made out of a single mass $(y)_\infty \notin X_{k_0}$, and thus, such that $i(y_\infty) = k_0 + 1$. But then, since $k_0 \in \mathfrak{N}$, there are no critical points at infinity made out of a single mass, of Morse index equal to $k_0 + 1$. Therefore assumption (\mathbf{H}_2) of Theorem 1.1 is satisfied by $\psi(X_{k_0}^\infty)$, and hence Theorem 1.1 applies.

6.3 Proof of Theorem 1.3

We have assumed that K has a set of critical points $I = \{y_0, y_1, y_2, \dots\}$, ordered such that $K(y_0) \geq K(y_1) \geq K(y_2) \geq \dots$, and that $y_0, y_1 \in I^+, y_2 \in I \setminus I^+$, and $y_\ell \in I^+ \setminus \{y_0, y_1\}$.

We recall that $k_1 = 2n + 1 - \text{ind}(K, y_1)$.

Observe that for $X_{k_1} = \{(y_0)_\infty, (y_1)_\infty\}$, the hypothesis (\mathbf{H}_1) of Theorem 1.1 is satisfied. Let

$$X_{k_1}^\infty = W_u(y_0)_\infty \cup W_u(y_1)_\infty.$$

We are going to construct a contraction $\psi(X_{k_1}^\infty)$ of $X_{k_1}^\infty$, which satisfies the hypothesis (\mathbf{H}_2) of Theorem 1.1. For this, using the assumption (\mathbf{A}_3) of Theorem 1.3, we can choose a positive real c , very close to $K(y_\ell)$, such that $K(y_2) > c > K(y_\ell)$, and we define

$$\check{c} = S \frac{n}{n+1} \frac{1}{c^{\frac{n}{n+1}}}.$$

It follows from Proposition 1.1, that under assumption (\mathbf{A}_3) , $(y_0)_\infty$ and $(y_1)_\infty$ are the only critical points at infinity of J , which are below the energy level \check{c} .

Notice that, in view of (5.3)-(5.4) (or [2], p. 535), the unstable manifold of a critical point at infinity made of a single mass $(y)_\infty$, can be described as the product of $W_s(y)$ the stable manifold of y with respect to the vector field $-\partial K$, with $[A, +\infty)$ the domain of the variable λ , for some large $A > 0$, i.e. $W_u(y)_\infty = W_s(y) \times [A, +\infty)$.

Let $J^{\check{c}} = \{u \in \Sigma^+; J(u) \leq \check{c}\}$. We claim that

Lemma 6.2 $X_{k_1}^\infty$ is contractible in $J^{\check{c}}$.

Indeed, using the assumption (\mathbf{A}_2) of Theorem 1.3, it is straightforward to see that $N_1 = \overline{W_s}(y_1) = W_s(y_0) \cup W_s(y_1) = \partial N_2$ is contractible in $N_2 = \overline{W_s}(y_2)$. It follows that there exists a contraction

$$\begin{aligned} g : [0, 1] \times N_1 &\longrightarrow N_2 \\ (t, x) &\longmapsto g(t, x) \end{aligned}$$

where g is continuous and such that for all $x \in N_1$, $g(0, x) = x$ and $g(1, x) = x_0$ some fixed point of N_1 . Such a contraction gives rise to the following contraction

$$G : [0, 1] \times X_{k_1}^\infty \longrightarrow \Sigma^+ \\ (t, x, \lambda) \longmapsto \frac{\varphi_{(g(t,x),\lambda)}}{\|\varphi_{(g(t,x),\lambda)}\|}$$

since $X_{k_1}^\infty$ can be parameterized by $N_1 \times [A, +\infty)$. So G is continuous, $G(0, x, \lambda) = \frac{\varphi_{(x,\lambda)}}{\|\varphi_{(x,\lambda)}\|} \in X_{k_1}^\infty$, and $G(1, x, \lambda) = \frac{\varphi_{(x_0,\lambda)}}{\|\varphi_{(x_0,\lambda)}\|}$ a fixed point in $X_{k_1}^\infty$.

Using the expansion given by (4.10) with $p = 1$, and since J is homogeneous, we have

$$J(\varphi_{(g(t,x),\lambda)}) = S^{\frac{n}{n+1}} \frac{1}{K(g(t,x))^{\frac{n}{n+1}}} \left\{ 1 + O\left(\frac{1}{\lambda^2}\right) \right\}$$

for all $(t, x, \lambda) \in [0, 1] \times N_1 \times [A, +\infty)$. Since $g(t, x) \in N_2$, $K(g(t, x)) \geq K(y_2)$. Then, for $\lambda \geq A$ large enough,

$$J(\varphi_{(g(t,x),\lambda)}) \leq S^{\frac{n}{n+1}} \frac{1}{K(y_2)^{\frac{n}{n+1}}} \left\{ 1 + O\left(\frac{1}{A^2}\right) \right\} < \check{c}.$$

It follows that the contraction G is performed under the level \check{c} , that is to say, if we let $\psi(X_{k_1}^\infty) = G([0, 1] \times X_{k_1}^\infty)$, we then have $\psi(X_{k_1}^\infty) \subset J^{\check{c}}$. Whence our Lemma.

Observe now that, under the assumption **(A₃)** of Theorem 1.3, the lowest level of energy of the critical points at infinity not belonging to X_{k_1} is $J(y_\infty)$, and it is above the energy level \check{c} . Thus,

$$\psi(X_{k_1}^\infty) \cap \bigcup_{\{\phi_\infty^p \in F_\infty^+ \setminus X_{k_1}\}} W_s(\phi_\infty^p) = \emptyset.$$

In particular $\psi(X_{k_1}^\infty)$ satisfies the assumption **(H₂)** of Theorem 1.1. Applying Theorem 1.1, Theorem 1.3 follows.

Appendix A.

We recall that

$$\omega(z, t) = |1 + |z|^2 - it|^{-n} \tag{A.1}$$

denotes the solution of Yamabe problem on \mathbb{H}^n . For $(a, \lambda) \in \mathbb{H}^n \times \mathbb{R}_+$ we obtain the other solutions $\omega_{(a,\lambda)}$ defined $\forall (z, t) \in \mathbb{H}^n$ by :

$$\omega_{(a,\lambda)}(z, t) = \lambda^n \omega(\lambda a^{-1}(z, t)) \tag{A.2}$$

$\omega_{(a,\lambda)}$ are the translated and dilated of $\omega = \omega_{(0,1)}$. Then $\omega_{(a,\lambda)}$ verify on \mathbb{H}^n :

$$L_{\theta_0} \omega_{(a,\lambda)} = \omega_{(a,\lambda)}^{1+\frac{2}{n}} \tag{A.3}$$

where $\theta_0 = dt + \sum_{j=1}^n iz_j d\bar{z}_j - i\bar{z}_j dz_j$ is the standard contact form on \mathbb{H}^n . Recall that the nonisotropic dilation by $\lambda > 0$ in \mathbb{H}^n is the map

$$(z, t) \longmapsto \lambda(z, t) = (\lambda z, \lambda^2 t), \tag{A.4}$$

and the Sobolev constant S is

$$S = \left(\int_{\mathbb{H}^n} \omega(z, t)^{2+\frac{2}{n}} \theta_0 \wedge d\theta_0^n \right)^{\frac{1}{n}}. \tag{A.5}$$

Lemma A.1 For all $c > 0$, $v = \frac{1}{c^{\frac{2}{n}}} \omega_{(a,\lambda)}$ is the solution on \mathbb{H}^n of the Yamabe equation $L_{\theta_0} v = c v^{1+\frac{2}{n}}$.

In what follows λ_i, λ_j are large enough positive parameters, and a_i, a_j are any distinct points of M . Then, from the results of section 3 and Appendix 1 of [13], we easily derive the following estimates, provided that $n \geq 2$.

$$\int_M \delta'(a_i, \lambda_i)^{2+\frac{2}{n}} \theta \wedge d\theta^n = S^n + O\left(\frac{1}{\lambda_i^{2n+2}}\right) \tag{A.6}$$

$$\int_M \varphi_{(a_i, \lambda_i)}^{2+\frac{2}{n}} \theta \wedge d\theta^n = S^n + O\left(\frac{1}{\lambda_i^{2n}}\right) \tag{A.7}$$

$$\int_M \delta'(a_i, \lambda_i)^{1+\frac{2}{n}} \theta \wedge d\theta^n = O\left(\frac{1}{\lambda_i^n}\right) \tag{A.8}$$

$$\int_M L\varphi_{(a_i, \lambda_i)} \varphi_{(a_i, \lambda_i)} \theta \wedge d\theta^n = \|\varphi_{(a_i, \lambda_i)}\|^2 = S^n + O\left(\frac{1}{\lambda_i^{2n}}\right) \tag{A.9}$$

$$\int_M L\varphi_{(a_i, \lambda_i)} \varphi_{(a_j, \lambda_j)} \theta \wedge d\theta^n = c_{ij} \varepsilon_{ij} \tag{A.10}$$

$$\int_M \varphi_{(a_i, \lambda_i)}^{1+\frac{2}{n}} \varphi_{(a_j, \lambda_j)} \theta \wedge d\theta^n = c_{ij} \varepsilon_{ij} (1 + o(1)) \tag{A.11}$$

where ε_{ij} is given below (2.2), and c_{ij} are positive bounded constants. The next two Lemmas, useful to the proof of Theorem 3.1, are derived from [6]. We will write φ_i for $\varphi_{(a_i, \lambda_i)}$ in the sequel, and $q = 2 + \frac{2}{n}$.

Lemma A.2 We have

$$\begin{aligned} \int K\left(\sum_i \alpha_i \varphi_i + v\right)^q &= \int K\left(\sum_i \alpha_i \varphi_i\right)^q + q \int K\left(\sum_i \alpha_i \varphi_i\right)^{q-1} v \\ &\quad + q(q-1) \int K\left(\sum_i \alpha_i \varphi_i\right)^{q-2} v^2 + O(\|v\|^q). \end{aligned}$$

Lemma A.3 We have

$$\begin{aligned} \int K\left(\sum_i \alpha_i \varphi_i\right)^q &= \sum_i \alpha_i^q \int K \varphi_i^q + q \sum_{i \neq j} \alpha_i^{q-1} \alpha_j \int K \varphi_i^{q-1} \varphi_j \\ &\quad + O\left(\sum_{i \neq j} \int K \sup(\alpha_i \varphi_i, \alpha_j \varphi_j)^{q-2} \inf(\alpha_i \varphi_i, \alpha_j \varphi_j)^2\right). \end{aligned}$$

The following Lemma corresponds to Lemma A.1 of [10]:

Lemma A.4 Assume $n \geq 2$. Let $a \in M$ and $\lambda > 0$ very large. There exists a constant $\bar{c} > 0$ such that

$$\int_M K \varphi_{(a,\lambda)}^{2+\frac{2}{n}} = K(a) S^n + \bar{c} \frac{\Delta_\theta K(a)}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right) + O\left(\frac{1}{\lambda^4}\right) + O\left(\frac{1}{\lambda^{2n}}\right) + O\left(\frac{1}{\lambda^{2n+2}}\right).$$

Proof. We give the proof for sake of completeness. We have

$$\int_M K \varphi_{(a,\lambda)}^{2+\frac{2}{n}} = K(a) \int_M \varphi_{(a,\lambda)}^{2+\frac{2}{n}} + \int_M (K(x) - K(a)) \varphi_{(a,\lambda)}^{2+\frac{2}{n}}. \tag{A.12}$$

Denoting by B the ball $B(a, \frac{r}{2})$, we derive that

$$\int_M (K(x) - K(a)) \varphi_{(a,\lambda)}^{2+\frac{2}{n}} = \int_B (K - K(a)) \varphi_{(a,\lambda)}^{2+\frac{2}{n}} + \int_{c_B} (K - K(a)) \varphi_{(a,\lambda)}^{2+\frac{2}{n}}.$$

Using (iii) of Lemma 4 and Lemma A.1 in [13], we get

$$\int_{c_B} (K(x) - K(a)) \varphi_{(a,\lambda)}^{2+\frac{2}{n}} = O\left(\frac{1}{\lambda^{2n+2}}\right). \tag{A.13}$$

On the other hand, using Lemma 3 in [13], with (1.2)–(1.5), we derive that

$$\begin{aligned} \int_B (K - K(a)) \varphi_{(a,\lambda)}^{2+\frac{2}{n}}(x) \theta \wedge d\theta^n &= \int_B (K - K(a)) \delta'(a, \lambda)^{2+\frac{2}{n}}(x) \theta \wedge d\theta^n + O\left(\frac{1}{\lambda^{2n}}\right) \\ &= \int_{B(0,r')} (K - K(a)) c_n \omega_{(0,\lambda)}^{2+\frac{2}{n}}(z, t) \theta_0 \wedge d\theta_0^n + O\left(\frac{1}{\lambda^{2n}}\right) \end{aligned}$$

where $x = \exp_a(z, t)$, $B(0, r') = \exp_a^{-1} B$, $\omega_{(0,\lambda)} = \lambda^n |1 + \lambda^2(|z|^2 - it)|^{-n}$, and where $(z, t) = (z(x), t(x))$ are pseudohermitian normal coordinates centered at a , i.e. such that $z(a) = 0$, and $t(a) = 0$. Let $\{Z_j, \bar{Z}_j, T\}$ be the standard CR structure of the Heisenberg group \mathbb{H}^n , where $Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}$, $\bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - iz_j \frac{\partial}{\partial t}$, ($1 \leq j \leq n$), and $T = \frac{\partial}{\partial t}$. By virtue of Lemma 3.10 in [17], the Taylor series of the function K around a to the second order is: $K(x) = K(a) + K_{(1)}(x) + K_{(2)}(x) + o(\rho^2)$ where $\rho = \sqrt{|z|^4 + t^2}$, and $K_{(1)}(x)$, resp. $K_{(2)}(x)$, is the homogeneous part (in terms of the nonisotropic dilations) of order 1, resp. 2, of this expansion; more precisely: $K_{(1)}(x) = \sum_{j=1}^n Z_j K(a) \cdot z_j + \bar{Z}_j K(a) \cdot \bar{z}_j$ and $K_{(2)}(x) = T K(a) \cdot t + \frac{1}{2} \sum_{j,k=1}^n (Z_j \bar{Z}_k K(a) z_j \bar{z}_k + \bar{Z}_j Z_k K(a) \bar{z}_j z_k + Z_j Z_k K(a) z_j z_k + \bar{Z}_j \bar{Z}_k K(a) \bar{z}_j \bar{z}_k)$. Hence :

$$\int_{B(0,r')} (K(x) - K(a)) \omega_{(0,\lambda)}^{2+\frac{2}{n}} = \int_{B(0,r')} (K_{(1)}(x) + K_{(2)}(x) + o(\rho^2)) \omega_{(0,\lambda)}^{2+\frac{2}{n}}. \tag{A.14}$$

Notice that most of the integrals vanish by oddness, so (A.14) becomes :

$$\begin{aligned} \int_{B(0,r')} (K(x) - K(a)) \omega_{(0,\lambda)}^{2+\frac{2}{n}} &= \frac{1}{2} \int_{B(0,r')} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j) K(a) |z_j|^2 \omega_{(0,\lambda)}^{2+\frac{2}{n}} \\ &\quad + \int_{B(0,r')} o(\rho^2) \omega_{(0,\lambda)}^{2+\frac{2}{n}}. \end{aligned} \tag{A.15}$$

Now, combining the results of Theorem 3.1 and Lemma 3.5 in [17], we derive that for any integer $m \geq 2$, there exists a choice of contact form θ' , such that, in a pseudohermitian normal coordinates chart centered at a , we have:

$$\Delta_{\theta'} = \Delta_{\theta} + O(\rho^{m-2}), \tag{A.16}$$

and we have also in this same chart

$$\Delta_{\theta'} = \frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + O(\rho^2). \tag{A.17}$$

Taking then $m \geq 4$ will suffice to derive from (A.16) and (A.17) the relation

$$\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j) = \Delta_\theta + O(\rho^2). \tag{A.18}$$

Inserting (A.18) in (A.15), and using the change of variable $(z, t) \mapsto \lambda(z, t) = (\lambda z, \lambda^2 t)$, equation (A.15) becomes:

$$\int_{B(0,r')} (K(x) - K(a)) \omega_{(0,\lambda)}^{2+\frac{2}{n}} = \bar{c} \frac{\Delta_\theta K(a)}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right) + O\left(\frac{1}{\lambda^4}\right) + O\left(\frac{1}{\lambda^{2n+2}}\right) \tag{A.19}$$

where

$$\bar{c} = \int_{\mathbb{H}^n} |z_j|^2 \omega^{2+\frac{2}{n}} \tag{A.20}$$

and using the estimates: $\int_{B(0,r')} |z_j|^2 \omega_{(0,\lambda)}^{2+\frac{2}{n}} = O\left(\frac{1}{\lambda^{2n+2}}\right)$, $\int_{B(0,r')} o(\rho^2) \omega_{(0,\lambda)}^{2+\frac{2}{n}} = o\left(\frac{1}{\lambda^2}\right)$,

and $\int_{B(0,r')} O(\rho^2) K(a) |z_j|^2 \omega_{(0,\lambda)}^{2+\frac{2}{n}} = O\left(\frac{1}{\lambda^4}\right)$. On the other hand, using (A.7),

$$K(a) \int_M \varphi_{(a,\lambda)}^{2+\frac{2}{n}} = K(a) S^n + O\left(\frac{1}{\lambda^{2n}}\right). \tag{A.21}$$

Collecting then our estimates in (A.13), (A.19), (A.21), and inserting in (A.12), Lemma A.4 follows.

Lemma A.5 For λ_i, λ_j very large we have: $\int_M K \varphi_i^{1+\frac{2}{n}} \varphi_j = K(a_i) c_{ij} \epsilon_{ij} (1 + o(1))$.

Proof. We have

$$\int_M K \varphi_i^{1+\frac{2}{n}} \varphi_j = K(a_i) \int_M \varphi_i^{1+\frac{2}{n}} \varphi_j + \int_M (K(x) - K(a_i)) \varphi_i^{1+\frac{2}{n}} \varphi_j.$$

By (A.11) we have $K(a_i) \int_M \varphi_i^{1+\frac{2}{n}} \varphi_j = K(a_i) c_{ij} \epsilon_{ij} (1 + o(1))$. We turn to the second integral:

$$\int_M (K(x) - K(a_i)) \varphi_i^{1+\frac{2}{n}} \varphi_j = \int_{B_i} (K(x) - K(a_i)) \varphi_i^{1+\frac{2}{n}} \varphi_j + \int_{cB_i} (K(x) - K(a_i)) \varphi_i^{1+\frac{2}{n}} \varphi_j,$$

where B_i is the ball $B(a_i, \frac{r}{2})$. We have for the first term

$$\begin{aligned} \int_{B_i} (K(x) - K(a_i)) \varphi_i^{1+\frac{2}{n}} \varphi_j &= O\left(\frac{1}{\lambda_j^n}\right) \int_{B_i} (K - K(a_i)) \varphi_i^{1+\frac{2}{n}} \\ &= O\left(\frac{1}{\lambda_j^n}\right) \int_{B(a_i,r)} (K - K(a_i)) \delta_i'^{1+\frac{2}{n}} + O\left(\frac{1}{\lambda_i^{n+2} \lambda_j^n}\right) \\ &= O\left(\frac{1}{\lambda_j^n}\right) \int_{B(0,r')} O(\| (z, t) \|_{\mathbb{H}^n}) \omega_{(0,\lambda_i)}^{1+\frac{2}{n}} + O\left(\frac{1}{\lambda_i^{n+2} \lambda_j^n}\right) \\ &= O\left(\frac{1}{\lambda_j^n}\right) \frac{1}{\lambda_i^{n+1}} \int_{B(0,\lambda_i r')} O(\rho) \omega^{1+\frac{2}{n}} + O\left(\frac{1}{\lambda_i^{n+2} \lambda_j^n}\right) \\ &= O\left(\frac{\epsilon_{ij}}{\lambda_i^2}\right) = o(\epsilon_{ij}) \end{aligned}$$

where we used polar coordinates to show that $\int_{B(0,\lambda_i r')} O(\rho)\omega^{1+\frac{2}{n}} = O(\frac{1}{\lambda_i})$, and the fact that $\frac{1}{\lambda_i^n \lambda_j^n} = O(\epsilon_{ij})$. Now, we turn to the second term

$$\begin{aligned} \int_{c_{B_i}} (K(x) - K(a_i))\varphi_i^{1+\frac{2}{n}}\varphi_j &\leq 2\|K\|_\infty \int_{c_{B(a_i, \frac{r}{2})}} \varphi_i^{1+\frac{2}{n}}\varphi_j \leq \frac{c}{\lambda_i^{n+2}} \int_{c_{B(a_i, \frac{r}{2})}} \varphi_j \\ &\leq \frac{c'}{\lambda_i^{n+2}} \left(\int_{B(a_j, \frac{r}{2})} \varphi_j + O\left(\frac{1}{\lambda_j^n}\right) \right) \leq \frac{c'}{\lambda_i^{n+2}} \left(\int_{B(a_j, \frac{r}{2})} \delta'_j + O\left(\frac{1}{\lambda_j^n}\right) \right) \\ &\leq \frac{c'}{\lambda_i^{n+2}} \left(\int_{B(0,\lambda_j)} \omega_{(0,\lambda_j)} + O\left(\frac{1}{\lambda_j^n}\right) \right) \leq \frac{c'}{\lambda_i^{n+2}} \left(\frac{1}{\lambda_j^{n+2}} \int_{B(0,\lambda_j r')} \omega + O\left(\frac{1}{\lambda_j^n}\right) \right) \\ &\leq O\left(\frac{\epsilon_{ij}}{\lambda_i^2}\right) = o(\epsilon_{ij}). \end{aligned}$$

Lemma A.5 follows.

Lemma A.6 For λ_i, λ_j very large we have $\int_M \varphi_i^{1+\frac{1}{n}}\varphi_j^{1+\frac{1}{n}} = O(\epsilon_{ij}^{1+\frac{1}{n}} \log \epsilon_{ij}^{-\frac{1}{n}})$.

Proof. Using polar coordinates we have

$$\begin{aligned} \int_{B_i} \delta_i'^{1+\frac{1}{n}} &= \int_{B(0,r')} \omega_{(0,\lambda_i)}^{1+\frac{1}{n}} = \frac{1}{\lambda_i^{n+1}} \int_{B(0,r'\lambda_i)} \omega^{1+\frac{1}{n}} \\ &\leq \frac{c'}{\lambda_i^{n+1}} \int_0^{r'\lambda_i} \frac{\rho^{2n+1}d\rho}{(1+\rho^4)^{\frac{n+1}{2}}} \leq \frac{c}{\lambda_i^{n+1}} \log \lambda_i \end{aligned}$$

where $\rho = (|z|^4 + t^2)^{\frac{1}{4}}$. Therefore we have

$$\begin{aligned} \int_M \varphi_i^{1+\frac{1}{n}}\varphi_j^{1+\frac{1}{n}} &= \frac{1}{\lambda_j^{n+1}} \int_{B_i} \delta_i'^{1+\frac{1}{n}} + \frac{1}{\lambda_i^{n+1}} \int_{B_j} \delta_j'^{1+\frac{1}{n}} + \frac{\tilde{c}}{(\lambda_i\lambda_j)^{n+1}} \\ &= O(\epsilon_{ij}^{1+\frac{1}{n}} \log \epsilon_{ij}^{-\frac{1}{n}}). \end{aligned}$$

Lemma A.7 If λ_i, λ_j are very large then,

$$\int_M K \sup(\alpha_i\varphi_i, \alpha_j\varphi_j)^{\frac{2}{n}} \inf(\alpha_i\varphi_i, \alpha_j\varphi_j)^2 = O(\epsilon_{ij}^{1+\frac{1}{n}} \log \epsilon_{ij}^{-\frac{1}{n}}).$$

Proof. It is straightforward. Since $2 \geq 1 + \frac{1}{n}$ we have

$$\begin{aligned} \int_M K \sup(\alpha_i\varphi_i, \alpha_j\varphi_j)^{\frac{2}{n}} \inf(\alpha_i\varphi_i, \alpha_j\varphi_j)^2 &\leq C' \int_M \alpha_i^{1+\frac{1}{n}}\varphi_i^{1+\frac{1}{n}}\alpha_j^{1+\frac{1}{n}}\varphi_j^{1+\frac{1}{n}} \\ &\leq C(\alpha_i\alpha_j)^{1+\frac{1}{n}}\epsilon_{ij}^{1+\frac{1}{n}} \log \epsilon_{ij}^{-\frac{1}{n}}. \end{aligned}$$

Lemma A.8 There exists a nonnegative constant C such that for all v in $H_\epsilon(a, \lambda)$:

$$\int_M K\varphi_i^{1+\frac{2}{n}}v \leq C\|v\| \left(\frac{|\nabla_\theta K(a_i)|}{\lambda_i} + O\left(\frac{1}{\lambda_i^2}\right) + O\left(\frac{1}{\lambda_i^{n+2}}\right) \right).$$

Proof. We have

$$\int_M K\varphi_i^{1+\frac{2}{n}}v = K(a_i) \int_M \varphi_i^{1+\frac{2}{n}}v + \int_M (K - K(a_i))\varphi_i^{1+\frac{2}{n}}v.$$

First we compute the integral:

$$\begin{aligned} \int_M \varphi_i^{1+\frac{2}{n}} v &= \int_{B_i} \delta_i^{\prime 1+\frac{2}{n}} v + O\left(\frac{1}{\lambda_i^{n+2}}\right) \|v\| = \int_M L\varphi_i v + O\left(\frac{1}{\lambda_i^{n+2}}\right) \|v\| \\ &= \langle \varphi_i, v \rangle + O\left(\frac{1}{\lambda_i^{n+2}}\right) \|v\| = O\left(\frac{1}{\lambda_i^{n+2}}\right) \|v\| \end{aligned}$$

where we used the fact that $\langle \varphi_i, v \rangle = 0$, (since $v \in H_\epsilon(a, \lambda)$, see (2.5)), and the continuity of the inclusion $S_1^2(M) \hookrightarrow L^{2+\frac{2}{n}}$. Next, we turn to

$$\int_M (K - K(a_i)) \varphi_i^{1+\frac{2}{n}} v = \int_{B_i} (K - K(a_i)) \varphi_i^{1+\frac{2}{n}} v + \int_{cB_i} (K - K(a_i)) \varphi_i^{1+\frac{2}{n}} v.$$

Outside of the ball B_i , φ_i is $O(\frac{1}{\lambda_i^n})$, then $\int_{cB_i} (K - K(a_i)) \varphi_i^{1+\frac{2}{n}} v = O(\frac{1}{\lambda_i^{n+2}}) \|v\|$.

Now we are left with

$$\begin{aligned} \int_{B_i} (K - K(a_i)) \varphi_i^{1+\frac{2}{n}} v &= \int_{B_i} (K - K(a_i)) \delta_i^{\prime 1+\frac{2}{n}} v + O\left(\frac{1}{\lambda_i^{n+2}}\right) \|v\| \\ &= \int_M (K - K(a_i)) \delta_i^{\prime 1+\frac{2}{n}} v + O\left(\frac{1}{\lambda_i^{n+2}}\right) \|v\| \\ &\leq \left(\int_M v^{2+\frac{2}{n}}\right)^{\frac{n}{2n+2}} \left(\int_M (K - K(a_i))^{\frac{2n+2}{n+2}} \delta_i^{\prime 2+\frac{2}{n}}\right)^{\frac{n+2}{2n+2}} + O\left(\frac{1}{\lambda_i^{n+2}}\right) \|v\| \\ &\leq \|v\| \left(\int_{B_i} (K - K(a_i))^{\frac{2n+2}{n+2}} \delta_i^{\prime 2+\frac{2}{n}}\right)^{\frac{n+2}{2n+2}} + O\left(\frac{1}{\lambda_i^{n+2}}\right) \|v\|. \end{aligned}$$

Using the notations of Lemma A.4, with $(z, t) = \exp_{a_i}^{-1}(x)$, the Taylor series of K around a_i at first order is

$$\begin{aligned} K(x) - K(a_i) &= K_{(1)}(x) + O(|z|^2) = \sum_{j=1}^n Z_j K(a_i) z_j + \bar{Z}_j K(a_i) \bar{z}_j + O(|z|^2) \\ &= \nabla_{\theta_0} K(a_i) \cdot z + O(|z|^2) \end{aligned}$$

where $\nabla_{\theta_0} = (Z_1, \dots, Z_n, \bar{Z}_1, \dots, \bar{Z}_n)$ is the CR gradient of \mathbb{H}^n , $z = (z_j, \bar{z}_j)$, and \cdot is the inner product of \mathbb{R}^{2n} . Denoting by ∇_{θ} the CR gradient associated to Δ_{θ} , and using the relation derived from section 3 of [17],

$$\nabla_{\theta} K(a_i) \cdot z = \nabla_{\theta_0} K(a_i) \cdot z + O(|z|^2)$$

we derive that

$$\begin{aligned} \int_{B_i} (K - K(a_i))^{\frac{2n+2}{n+2}} \delta_i^{\prime 2+\frac{2}{n}} &\leq \int_{B(0, r')} (|\nabla_{\theta} K(a_i)| |z| + O(|z|^2))^{\frac{2n+2}{n+2}} \omega_{(0, \lambda_i)}^{2+\frac{2}{n}} \\ &\leq \int_{B(0, \lambda_i r')} \left(\frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} |z| + O\left(\frac{|z|^2}{\lambda_i^2}\right)\right)^{\frac{2n+2}{n+2}} \omega^{2+\frac{2}{n}}. \end{aligned}$$

Taking then everything into account, we find

$$\left(\int_{B_i} (K - K(a_i))^{\frac{2n+2}{n+2}} \delta_i^{\prime 2+\frac{2}{n}}\right)^{\frac{n+2}{2n+2}} \leq C \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + O\left(\frac{1}{\lambda_i^2}\right),$$

where $C = \left(\int_{\mathbb{R}^n} |z|^{\frac{2n+2}{n+2}} \omega^{2+\frac{2}{n}}\right)^{\frac{n+2}{2n+2}}$, the remainder terms being absorbed by the $O(\frac{1}{\lambda_i^2})$ since $n \geq 2$. Therefore

$$\int_{B_i} (K - K(a_i)) \varphi_i^{1+\frac{2}{n}} v \leq C \|v\| \left(\frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + O\left(\frac{1}{\lambda_i^2}\right) + O\left(\frac{1}{\lambda_i^{n+2}}\right) \right).$$

Collecting then our estimates, Lemma A.8 follows.

Lemma A.9 (Lemma 1.2 [6]). *There exists $M > 0$ such that for any $(a_1, \dots, a_p) \in \mathbb{R}^p$ and $\alpha > 0$:*
 $|(\sum_i a_i)^\alpha - \sum_i a_i^\alpha| \leq M \sum_{i \neq j} |a_i|^{\alpha-1} \inf(|a_i|, |a_j|).$

Appendix B.

In this Appendix, we fix some notations, and we establish some basic results, used in the proof of Lemma 5.1. Let

$$E = \text{Span}\left\{ \varphi_i, \frac{\partial \varphi_i}{\partial a_i}, \frac{\partial \varphi_i}{\partial \lambda_i}, 1 \leq i \leq p \right\}, \quad F = \text{Span}\left\{ \frac{\partial \varphi_i}{\partial a_i}, \frac{\partial \varphi_i}{\partial \lambda_i}, 1 \leq i \leq p \right\}$$

$\pi_E : S_1^2 \rightarrow E$ (resp. $\pi_F : S_1^2 \rightarrow F$) the orthogonal projection on E (resp. on F).

Let w be an increment in the space of the variables (a, λ) . Then the variation of \bar{v} along w is denoted by $\frac{\partial \bar{v}}{\partial(\alpha, a, \lambda)}(w) = \frac{\partial \bar{v}}{\partial W}$. Observe that, since w is a variation in (a, λ) , or equivalently in F , and denoting by $da_i(w) = \dot{a}_i$ (resp. $d\lambda_i(w) = \dot{\lambda}_i$) the action of w on a_i (resp. on λ_i), we have:

$$w = \sum_i \frac{\partial \varphi_i}{\partial a_i} \dot{a}_i + \sum_i \frac{\partial \varphi_i}{\partial \lambda_i} \dot{\lambda}_i.$$

Proposition B.1 *Let w be an increment in the space of the variables (a, λ) . Then, provided $(\alpha, a, \lambda) \in B_\epsilon$, ϵ adequate, there exists a constant $\bar{c} > 0$ independent of w and of (α, a, λ) , such that*

$$\forall \phi \in F \quad \langle \pi_F\left(\frac{\partial \bar{v}}{\partial W}\right), \phi \rangle \leq \bar{c} \|\bar{v}\| \|w\| \|\phi\|.$$

Proof. Since $\bar{v} \in H_\epsilon(a, \lambda)$, we have $\forall \phi \in E$, thus a fortiori $\forall \phi \in F$

$$\langle \bar{v}, \phi \rangle = 0. \tag{B.1}$$

Differentiating (B.1) with respect to w one has $\langle \frac{\partial \bar{v}}{\partial W}, \phi \rangle = -\langle \bar{v}, \frac{\partial \phi}{\partial W} \rangle$ where

$$\frac{\partial \phi}{\partial W} = \sum_i \frac{\partial \phi}{\partial a_i} \dot{a}_i + \sum_i \frac{\partial \phi}{\partial \lambda_i} \dot{\lambda}_i = \sum_i \left(\frac{1}{\lambda_i} \frac{\partial \phi}{\partial a_i} \lambda_i \dot{a}_i + \lambda_i \frac{\partial \phi}{\partial \lambda_i} \frac{\dot{\lambda}_i}{\lambda_i} \right).$$

Since ϕ is a linear combination of $\frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i}$ and $\lambda_i \frac{\partial \varphi_i}{\partial \lambda_i}$ which are almost orthogonal, it is easy to see that there is a positive constant c such that $\|\frac{1}{\lambda_i} \frac{\partial \phi}{\partial a_i}\|, \|\lambda_i \frac{\partial \phi}{\partial \lambda_i}\| \leq c \|\phi\|$, hence $\|\frac{\partial \phi}{\partial W}\| \leq c \|\phi\| \sum_i (\lambda_i |\dot{a}_i| + \frac{|\dot{\lambda}_i|}{\lambda_i})$. On the other hand, there is a uniform positive constant c_0 such that $\|\frac{1}{\lambda_i} \frac{\partial \phi}{\partial a_i}\|, \|\lambda_i \frac{\partial \phi}{\partial \lambda_i}\| \geq c_0$, and we have also $\langle \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i}, \lambda_j \frac{\partial \varphi_j}{\partial \lambda_j} \rangle = O(\epsilon_{ij})$ which is small. Then, noting that $w = \sum_i \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \lambda_i \dot{a}_i + \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \frac{\dot{\lambda}_i}{\lambda_i}$ we derive that $\|w\| \geq \frac{1}{c} \sum_i (\lambda_i |\dot{a}_i| + \frac{|\dot{\lambda}_i|}{\lambda_i})$ and thus

$$\left| \langle \bar{v}, \frac{\partial \phi}{\partial W} \rangle \right| \leq c \|\bar{v}\| \|\phi\| \sum_i \left(\lambda_i |\dot{a}_i| + \frac{|\dot{\lambda}_i|}{\lambda_i} \right) \leq \bar{c} \|\bar{v}\| \|\phi\| \|w\|.$$

Therefore one has $\left| \langle \frac{\partial \bar{v}}{\partial W}, \phi \rangle \right| \leq \bar{c} \|\bar{v}\| \|\phi\| \|w\|$. Proposition B.1 follows.

To prove lemma 5.1 we also need the following sequence of Lemmas.

Lemma B.1 $J'(\bar{u})w = J'(\sum_i \alpha_i \varphi_i)w + J''(\sum_i \alpha_i \varphi_i)\bar{v}w + O(\|\bar{v}\|^2\|w\|)$.

Proof. Recall that $\bar{u} = \sum_i \alpha_i \varphi_i + \bar{v}$. We have

$$\begin{aligned} & \left| |\bar{u}|^{\frac{2}{n}} \bar{u} - \left(\sum_i \alpha_i \varphi_i \right)^{1+\frac{2}{n}} - \left(1+\frac{2}{n}\right) \left(\sum_i \alpha_i \varphi_i \right)^{\frac{2}{n}} \bar{v} \right| \leq \\ & C \left(|\bar{v}|^{1+\frac{2}{n}} \mathbf{1}_{|\sum_i \alpha_i \varphi_i| \leq |\bar{v}|} + \mathbf{1}_{|\sum_i \alpha_i \varphi_i| \geq |\bar{v}|} \left(\sum_i \alpha_i \varphi_i \right)^{\frac{2}{n}-1} |\bar{v}|^2 \right) \end{aligned}$$

where $\mathbf{1}_X$ is the function equal to 1 on the set X . Thus,

$$\begin{aligned} \int K |\bar{u}|^{\frac{2}{n}} \bar{u} w &= \int K \left(\sum_i \alpha_i \varphi_i \right)^{1+\frac{2}{n}} w + \left(1+\frac{2}{n}\right) \int K \left(\sum_i \alpha_i \varphi_i \right)^{\frac{2}{n}} \bar{v} w + \\ & O \left(\int_{|\sum_i \alpha_i \varphi_i| \leq |\bar{v}|} |\bar{v}|^{1+\frac{2}{n}} |w| + \int \left(\sum_i \alpha_i \varphi_i \right)^{\frac{2}{n}-1} |\bar{v}|^2 |w| \right). \end{aligned}$$

Observe now that, for any $w \in F$ we have pointwise $|w(x)| \leq C\|w\|(\sum_i \alpha_i \varphi_i)(x)$, since any $\left| \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i}(x) \right|$ and $\left| \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i}(x) \right|$ are bounded by $C\varphi_i(x)$ and since the components of w are upperbounded by $\|w\|$. Therefore

$$\int_{|\sum_i \alpha_i \varphi_i| \leq |\bar{v}|} |\bar{v}|^{1+\frac{2}{n}} |w| + \int \left(\sum_i \alpha_i \varphi_i \right)^{\frac{2}{n}-1} |\bar{v}|^2 |w| \leq C(\|\bar{v}\|^{2+\frac{2}{n}} + \|\bar{v}\|^2)\|w\|,$$

and hence,

$$\int K |\bar{u}|^{\frac{2}{n}} \bar{u} w = \int K \left(\sum_i \alpha_i \varphi_i \right)^{1+\frac{2}{n}} w + \left(1+\frac{2}{n}\right) \int K \left(\sum_i \alpha_i \varphi_i \right)^{\frac{2}{n}} \bar{v} w + O(\|\bar{v}\|^2\|w\|),$$

and then Lemma B.1 readily follows.

Lemma B.2

$$J''\left(\sum_i \alpha_i \varphi_i\right)\bar{v}w = \|w\| O\left(\sum_i \frac{|\nabla_{\theta} K(a_i)|^2}{\lambda_i^2} + \sum_i \frac{1}{\lambda_i^4} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2}{2n}} \left(\log \varepsilon_{ij}^{-\frac{1}{n}}\right)^{\frac{n+2}{2n+2}}\right).$$

Proof. Observe that, since $\bar{v} \in H_c(a, \lambda)$ and $w \in F$ (that is, w has no component on φ_i), we have $\langle w, \varphi_i \rangle = \langle w, \bar{v} \rangle = 0$, and thus

$$\begin{aligned} J''\left(\sum_i \alpha_i \varphi_i\right)\bar{v}w &= -J(\bar{u})^{\frac{2n+1}{n}} \left(1+\frac{2}{n}\right) \int K \left(\sum_i \alpha_i \varphi_i \right)^{\frac{2}{n}} \bar{v} w + \\ & O\left(\int K \left(\sum_i \alpha_i \varphi_i \right)^{1+\frac{2}{n}} w\right) O\left(\int K \left(\sum_i \alpha_i \varphi_i \right)^{1+\frac{2}{n}} \bar{v}\right). \end{aligned}$$

For the second term of the right hand side, we operate as in the proof of lemma 4.1, and we find

$$\int K \left(\sum_i \alpha_i \varphi_i \right)^{1+\frac{2}{n}} w = \|w\| O\left(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_i \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2}{2n}} \left(\log \varepsilon_{ij}^{-\frac{1}{n}}\right)^{\frac{n+2}{2n+2}}\right)$$

(B.2)

$$\int K \left(\sum_i \alpha_i \varphi_i \right)^{1+\frac{2}{n}} \bar{v} = \|\bar{v}\| O\left(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_i \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2}{2n}} \left(\log \varepsilon_{ij}^{-\frac{1}{n}}\right)^{\frac{n+2}{2n+2}}\right).$$

Now, to estimate the first term, we will estimate $\int K(\sum_i \alpha_i \varphi_i)^{\frac{2}{n}} \bar{v} \phi$ for $\phi = \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i}$ or $\phi = \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i}$, the result will be true for w since w is in F . Using the fact that such ϕ 's are orthogonal to \bar{v} , and that $|\phi(x)| \leq C \varphi_i(x)$ where C is a universal constant, we have

$$\begin{aligned} \int K(x) \left(\sum_i \alpha_i \varphi_i\right)^{\frac{2}{n}} \bar{v} \phi &= \int (K - K(a_i)) (\alpha_i \varphi_i)^{\frac{2}{n}} \bar{v} \phi + O\left(\int |\bar{v}| \varphi_i^{\frac{2}{n}} \sum_{j \neq i} \inf(\alpha_i \varphi_i, \alpha_j \varphi_j)\right) \\ &\quad + O\left(\int_{\alpha_i \varphi_i \leq \sum_{j \neq i} \alpha_j \varphi_j} |\bar{v}| \varphi_i \left(\sum_{j \neq i} \alpha_j \varphi_j\right)^{\frac{2}{n}}\right) \\ &= \int (K - K(a_i)) (\alpha_i \varphi_i)^{\frac{2}{n}} \bar{v} \phi + O\left(\int |\bar{v}| \varphi_i^{\frac{2}{n}} \sum_{j \neq i} \inf(\alpha_i \varphi_i, \alpha_j \varphi_j)\right) \\ &\quad + O\left(\|\bar{v}\| \left\{ \int_{\alpha_i \varphi_i \leq \sum_{j \neq i} \alpha_j \varphi_j} \varphi_i^{\frac{2n+2}{n+2}} \left(\sum_{j \neq i} \alpha_j \varphi_j\right)^{\frac{4(n+1)}{n(n+2)}} \right\}^{\frac{n+2}{2n+2}}\right), \end{aligned}$$

and, using estimates as in lemma 4.1, we derive that

$$\int K \left(\sum_i \alpha_i \varphi_i\right)^{\frac{2}{n}} \bar{v} \phi = \|\bar{v}\| O\left(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_i \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2}{2n}} (\log \varepsilon_{ij}^{-\frac{1}{n}})^{\frac{n+2}{2n+2}}\right). \tag{B.3}$$

Observe that by (4.4) and (4.5) we have the following estimate on \bar{v}

$$\|\bar{v}\| = O\left(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_i \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2}{2n}} (\log \varepsilon_{ij}^{-\frac{1}{n}})^{\frac{n+2}{2n+2}}\right). \tag{B.4}$$

Lemma B.2 follows by (B.2)–(B.4).

Lemma B.3 $\sup_{\phi \in F} \frac{|J'(\bar{u})(\phi)|}{\|\phi\|} = O\left(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_i \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2}{2n}} (\log \varepsilon_{ij}^{-\frac{1}{n}})^{\frac{n+2}{2n+2}}\right).$

Proof. We recall that $\langle \bar{v}, \lambda_i \frac{\partial \varphi_i}{\partial \lambda_i} \rangle = \langle \bar{v}, \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial a_i} \rangle = 0$; thus, for $\phi \in F$, we have:

$$J'(\bar{u}).\phi = J(\bar{u})\langle \bar{u}, \phi \rangle - \int K J(\bar{u})^{2+\frac{1}{n}} \bar{u}^{1+\frac{2}{n}} \text{sign } \bar{u} \phi = -J(\bar{u})^{2+\frac{1}{n}} \int K \bar{u}^{1+\frac{2}{n}} \text{sign } \bar{u} \phi.$$

Arguing as in the proof of lemma 4.1, one has

$$J'(\bar{u}).\phi \leq \|\phi\| O\left(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_i \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2}{2n}} (\log \varepsilon_{ij}^{-\frac{1}{n}})^{\frac{n+2}{2n+2}}\right),$$

from which we derive lemma B.3.

Appendix C.

Proof of Lemma 5.2. The vector field w , provided by Lemma 5.1, is C^1 and therefore integrable. It satisfies the (P.S) condition on the decreasing flow lines of J , away from the critical points at infinity. Let η_s be the 1-parameter group generated by w , we have

$$\begin{cases} \frac{\partial}{\partial s} \eta_s(\sum_i \alpha_i \varphi_i) = w(\eta_s(\sum_i \alpha_i \varphi_i)), \\ \eta_0(\sum_i \alpha_i \varphi_i) = \sum_i \alpha_i \varphi_i. \end{cases}$$

By lemma 5.1, $J(\eta_s)$ is a decreasing function of s . Observe that since \bar{v} is a minimizer, we have $J(\bar{u}) \leq J(\sum_i \alpha_i \varphi_i)$. Now, using the estimates (4.9), (4.10), we have $J(\bar{u}) = J(\sum_i \alpha_i \varphi_i) + o(1)$. Since the flow line started from $\sum_i \alpha_i \varphi_i$, which is not a critical point at infinity (a critical point at infinity occurs only when $\lambda_i = +\infty$), the flow line should have been down the level $J(\bar{u})$ before it exits from $V(p, \epsilon)$, which is the region where the vector field w is defined. Thus, there is at most one solution to the equation

$$J(\eta_s(\sum_i \alpha_i \varphi_i)) = J(\bar{u}). \tag{C.1}$$

Indeed, assume that the initial data $\sum_i \alpha_i \varphi_i$ is in $V(p, \epsilon_0)$ with $\epsilon_0 < \frac{\epsilon}{2}$. Then the flow line will move from $\partial V(p, \frac{\epsilon}{2})$ to $\partial V(p, \epsilon)$, and during this traveling we have

$$\begin{cases} -J'(\eta_s(\sum_i \alpha_i \varphi_i)) & \geq c(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \epsilon_{ij}) \\ \text{dist}(\partial V(p, \frac{\epsilon}{2}), \partial V(p, \epsilon)) & = a(\epsilon) > 0, \text{ and } \|w\| \leq C. \end{cases}$$

If we denote by Δs the duration of the trip from $\partial V(p, \frac{\epsilon}{2})$ to $\partial V(p, \epsilon)$, then we have $a(\epsilon) \leq C \Delta s$. If we let $\gamma(\epsilon) = \frac{c(\epsilon)a(\epsilon)}{C}$, then $J(\eta_s(\sum_i \alpha_i \varphi_i))$ should have decreased at least of $\gamma(\epsilon)$ during this crossing. Using estimates (4.9) and (4.10), we derive that $J(\bar{u}) - J(\sum_i \alpha_i \varphi_i) \rightarrow 0$ as $\epsilon \rightarrow 0$. Then, if we choose ϵ small enough, we have $J(\bar{u}) > J(\sum_i \alpha_i \varphi_i) - \gamma(\epsilon)$, that is to say, $J(\eta_s(\sum_i \alpha_i \varphi_i))$ crosses the level $J(\bar{u})$ before $\eta_s(\sum_i \alpha_i \varphi_i)$ leaves $V(p, \epsilon)$, and therefore, equation (C.1) possesses a unique solution.

Conversely, starting from $\sum_i \alpha_i \tilde{\varphi}_i \in V(p, \epsilon')$, ϵ' as small as we please, we would like to find $\sum_i \alpha_i \varphi_i = \eta_{-s}(\sum_i \alpha_i \tilde{\varphi}_i)$ so that

$$J(\bar{u}) = J(\sum_i \alpha_i \tilde{\varphi}_i). \tag{C.2}$$

Observe that by lemma 5.1, $J(\bar{u}(-s)) = J(\eta_{-s}(\sum_i \alpha_i \tilde{\varphi}_i) + \bar{v}(\eta_{-s}(\sum_i \alpha_i \tilde{\varphi}_i)))$ is an increasing function of s ; so at time $s = 0$ we have $J(\sum_i \alpha_i \tilde{\varphi}_i + \bar{v}(\alpha, \tilde{a}, \tilde{\lambda})) < J(\sum_i \alpha_i \tilde{\varphi}_i)$. Therefore, there is at most one solution for (C.2). Note that, along an increasing flow line for J , one has $\frac{\partial}{\partial s}(\sum_i \alpha_i \varphi_i) = -w(\sum_i \alpha_i \varphi_i)$, so one might exit from $V(p, \epsilon')$; but the former argument applies here again, and we would have gone higher than $J(\sum_i \alpha_i \tilde{\varphi}_i)$ before exiting from $V(p, \epsilon')$.

The other possibility is that the increasing flow line goes to infinity before reaching the level $J(\sum_i \alpha_i \tilde{\varphi}_i)$. If this happens the φ_i corresponding to the highest concentration λ_i will go weakly to zero, thus λ_i goes to $+\infty$. However, since, by lemma 5.1, $|d\lambda_i(w)| \leq c' \lambda_i$, this will happen at an infinite time. Observe that since

$$-J'(\bar{u})(w + \frac{\partial \bar{v}}{\partial w}) \geq c(\sum_{i \neq j} \epsilon_{ij} + \sum_i \frac{1}{\lambda_i^2}),$$

we must have, on such a flow line, at least for a subsequence (s_k) , $s_k \rightarrow +\infty$, $\sum_{i \neq j} \epsilon_{ij}(-s_k) + \sum_i \frac{1}{\lambda_i^2}(-s_k) \rightarrow 0$. This implies that $\bar{v}(\eta_{-s_k}(\sum_i \alpha_i \tilde{\varphi}_i)) \rightarrow 0$, and then $J(\bar{u}(-s_k)) - J(\sum_i \alpha_i \varphi_i(-s_k)) \rightarrow 0$. Thus,

$$\begin{aligned} \limsup_{s \rightarrow +\infty} J(\bar{u}(-s)) &= \limsup_{s \rightarrow +\infty} J(\eta_{-s}(\sum_i \alpha_i \tilde{\varphi}_i)) > J(\sum_i \alpha_i \tilde{\varphi}_i) \\ &\geq J(\sum_i \alpha_i \tilde{\varphi}_i + \bar{v}(\alpha, \tilde{a}, \tilde{\lambda})). \end{aligned}$$

Hence, by continuity, (C.2) must have a solution. The required diffeomorphism is then given by η_s which solves (C.1)-(C.2).

To complete the proof of Lemma 5.2 we need also to show that (*) and (**) hold. Set $\sum_i \alpha_i \varphi_i(s) = \sum_i \alpha_i \varphi_{(a_i(s), \lambda_i(s))} = \eta_s(\sum_i \alpha_i \varphi_i)$. Since w has no action on the variable α , we have

$$w = \sum_i \alpha_i \frac{1}{\lambda_i(s)} \frac{\partial \varphi_i(s)}{\partial a_i} (\lambda_i(s) \dot{a}_i(s)) + \sum_i \alpha_i \lambda_i(s) \frac{\partial \varphi_i(s)}{\partial \lambda_i} \left(\frac{\dot{\lambda}_i(s)}{\lambda_i(s)} \right),$$

where $\dot{a}_i(s)$ and $\dot{\lambda}_i(s)$ denote the action of w on the variables a_i and λ_i . Observe that $\varepsilon_{ij} = o(1)$, and $\frac{1}{\lambda_i} \frac{\partial \varphi_i(s)}{\partial a_i}$ and $\lambda_i \frac{\partial \varphi_i(s)}{\partial \lambda_i}$ are nearly orthogonal, both of order $O(\varphi_i)$, and such that $0 < c_0 \leq \left\| \frac{1}{\lambda_i} \frac{\partial \varphi_i(s)}{\partial a_i} \right\| \leq C$ and $0 < c_0 \leq \left\| \lambda_i \frac{\partial \varphi_i(s)}{\partial \lambda_i} \right\| \leq C$. Since w is bounded, it yields that $|\lambda_i \dot{a}_i| \leq C$ and $|\frac{\dot{\lambda}_i}{\lambda_i}| \leq C$. On the other hand, $\frac{\partial}{\partial s} \varepsilon_{ij} \leq C \varepsilon_{ij}$, and $\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}, \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i}$ are $O(\varepsilon_{ij})$, then we have,

$$\begin{cases} e^{-Cs} \varepsilon_{ij}(0) \leq \varepsilon_{ij}(s) \leq e^{Cs} \varepsilon_{ij}(0) \\ e^{-Cs} \leq \frac{\lambda_i(s)}{\lambda_i(0)} \leq e^{Cs} \\ \tilde{d}(a_i, a_i(s)) \leq \frac{e^{Cs}}{\lambda_i(0)}. \end{cases} \tag{C.3}$$

Since s satisfies equation (C.2), the time s needed to have $J(\bar{u})=J(\eta_s(\sum_i \alpha_i \varphi_i))$ is bounded because we have $-J'(\bar{u})(w + \frac{\partial \bar{v}}{\partial W}) \geq c(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} + \sum_i \frac{1}{\lambda_i^2})$, and, by (4.9)-(4.10), $J(\bar{u}) - J(\sum_i \alpha_i \varphi_i) = o(\sum_i \frac{|\nabla_{\theta} K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} + \sum_i \frac{1}{\lambda_i^2})$. Using (C.3), we see that $\tilde{d}(a_i, \tilde{a}_i) = \tilde{d}(a_i(0), a_i(s))$ tends to zero when $\sum_i \frac{1}{\lambda_i^2}(0)$ tends to zero, which establishes (*). Since s is bounded, $(\sum_{i \neq j} \varepsilon_{ij} + \sum_i \frac{1}{\lambda_i^2})(0)$ tends to zero is equivalent to $(\sum_{i \neq j} \varepsilon_{ij} + \sum_i \frac{1}{\lambda_i^2})(s) = \sum_{i \neq j} \tilde{\varepsilon}_{ij} + \sum_i \frac{1}{\tilde{\lambda}_i^2}$ tends to zero. Thus, we derive (**), and lemma 5.2 is thereby proved. □

References

- [1] A. Ambrosetti and M. Badiale, *Homoclinics : Poincaré-Melnikov type results via a variational approach*, Ann. Inst. H. Poincaré Anal. Nonlinéaire **15** (1998), 233-252.
- [2] T. Aubin and A. Bahri, *Méthodes de topologie algébrique pour le problème de la courbure scalaire prescrite*, J. Math. Pures Appl. **76** (1997), n°6, 525-549.
- [3] A. Bahri, H. Brezis, *Nonlinear Elliptic Equations*, Topics in Geometry in memory of J. D’Atri, Simon Gindikin editor, Birkhäuser, Boston, Basel, Berlin, (1996), 1-100, Zbl 863.35037.
- [4] A. Bahri, J.M. Coron, *On a non linear elliptic equation involving the critical Sobolev exponent. The effect of the topology of the domain*, Comm. on pure and applied Math. Vol. XLI, (1988), 253-294.
- [5] A. Bahri, *An invariant for Yamabe-type flows with applications to scalar-curvature problems in high dimension*, Duke Math. J. **81** (1996), 323-466.
- [6] A. Bahri, *Critical Points at Infinity in Some Variational Problems*, Pitman Research Notes in Mathematics Series N° **182**, Longman, (1989), MR 91h:58022, Zbl 676.58021.
- [7] A. Bahri, P.H. Rabinowitz, *Periodic solutions of Hamiltonian systems of 3-body type*, Annales de l’institut H. Poincaré. Analyse non linéaire. **8**, N°6, (1991), 561-649.
- [8] L. Brandolini, M. Rigoli and A. G. Setti, *Positive solutions of Yamabe-Type equations on the Heisenberg Group*, Duke Math. Journal, **91**, N°2. (1998), 241-296.
- [9] H. Brezis and J. M. Coron, *Convergence of solutions of H-systems or how to blow bubbles*, Arch. Rational Mech. Anal. **81** (1985), 21-56.

- [10] H. Chtioui, M. O. Ahmedou , R. Yacoub, *Topological methods for the prescribed Webster Scalar Curvature problem on CR manifolds*, Differential Geometry and its Applications **28** (2010), 264-281.
- [11] H. Chtioui, M. O. Ahmedou , R. Yacoub, *Existence and Multiplicity results for the prescribed Webster Scalar Curvature Problem on three CR manifolds*, Journal of Geometric Analysis, Vol. 23, Number 2 (2013), 878-894. DOI: 10.1007/s12220-011-9267-z.
- [12] V. Felli, F. Uguzzoni, *Some existence results for the Webster scalar curvature problem in presence of symmetry*, Annali di Matematica **183** (2004), 469-493.
- [13] N. Gamara, R. Yacoub, *CR Yamabe Conjecture-The Conformally flat case*, Pacific Journal of Mathematics **201**, No. 1, (2001), 121-175.
- [14] N. Garofalo and E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group*, Indiana Univ. Math. J. **41** (1992), 71-98.
- [15] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2002.
- [16] D. Jerison–J.M. Lee, *The Yamabe problem on CR manifolds*, J.Differential Geometry **25** (1987), 167-197.
- [17] D. Jerison–J.M. Lee, *Intrinsic CR normal coordinates and the CR Yamabe problem*, J. Differential Geometry **29** (1989), 303-343.
- [18] D. Jerison–J.M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem* J. Amer. Math. Soc. **1** (1988), N° 11-13 (H.Jacobowitz).
- [19] P.L. Lions, *The Concentration Compactness Principle in the calculus of variations. The limit case*, Rev. Math. Iberoamericana **1** (1985), I: 165-201, II: 45-121.
- [20] A. Malchiodi, F. Uguzzoni, *A perturbation result for the Webster scalar curvature problem on the CR sphere*, J. Math. Pures Appl.**81** (2002), 983-997.
- [21] M. Struwe, *A Global Compactness Result for Elliptic Boundary Value Problems Involving Limiting Nonlinearities*, Mathematische Zeitschrift **187**, Springer Verlag (1984), 511–517.
- [22] R. Yacoub, *On the Scalar Curvature equations in high dimension*, Advanced Nonlinear Studies, Vol.2, N. 4, (2002), 373–393.