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Heteroclinic Solutions to Asymptotically Autonomous Equations via Continuation Methods

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Abstract

By means of a continuation argument, we prove the existence of at least one increasing heteroclinic solution to a scalar equation of the kind $\ddot{x} = a(t)V'(x)$, where V is a non-negative double well potential, and $a(t)$ is a positive, measurable coefficient, which is definitively monotone with respect to $|t|$, converges to l as $|t|$ diverges and fulfils one of the two following assumptions: $a(t) \geq l$ everywhere, or $a(t) - l$ converges to 0, as $|t| \to +\infty$, more slowly than a suitable exponential term.

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1 Introduction

As is known, if a non-negative potential V has two consecutive, non-degenerate wells ξ^- and ξ^+ at level zero, the autonomous equation $\ddot{x} = V'(x)$ admits infinitely many heteroclinic solutions which connect the two wells, and differ each other by a shift in time. It looks interesting to know what happens if we multiply the right-hand side of the equation by a positive coefficient $a(t)$ which converges to a positive limit l as $|t|$ diverges. In this case, we say that the corresponding equation

$$\ddot{x} = a(t)V'(x) \tag{1.1}$$

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is "asymptotically autonomous". For instance, when $V(x) = 1 + \cos x$ and $\xi^{\pm} = \pm\pi$, (1.1) could represent the motion of a pendulum whose pivot is subject to a vertical perturbation $a(t) - l$, which becomes negligible for large values of $|t|$: a similar model, but in the more familiar case of a periodic perturbation, can be found in §4.2a of [19]. From now on, for the sake of simplicity, we suppose $\xi^{\pm} = \pm 1$: then we ask whether (1.1) presents at least one heteroclinic trajectory, namely a solution of (2.5). In spite of the many papers which concern periodic or almost periodic coefficients $a(t)$ (see, for instance, [16],[17]), this particular situation was not widely studied in the literature, and most results are rather recent. In order to give a short summary of them, we begin to recall the case $a(t) \leq l$, where, without further assumptions, a solution always exists, and is found by minimizing the corresponding action functional F : see [4] (Chapter 2, Thm. 2.2), and also a related result in [6] (§5, Example 1). On the contrary, in the case $a(t) \geq l$, some minimax arguments on F show the existence of a "mountain pass" solution under the additional assumption that $a(t)$ is bounded from above by a suitable number, which depends on l and the shape of V [8], [12]. We also quote a recent paper where this result is improved, by allowing $a(t)$ to go below l [18]. A case which can be handled more easily is when $a(t)$ and $V(x)$ are both even (on this subject, see also [10], [11], where a third intermediate well is considered).

Since problem (2.5) is one-dimensional, another possible approach is provided by shooting methods, on the ground of the usual representation of (1.1) on the phase plane $x\dot{x}$: in this way, in [9], we are able to deal with the more general case in which $a(t)$ converges from above to two possibly different limits l_* and l^*. In this case, however, we need that $a(t)$ is definitively monotone, and also an exponential bound on the speed at which $a(t)$ converges to $\max(l_*, l^*)$. Furthermore, V is supposed to fulfil, in a slightly stronger form, condition (V_2) of §2, which is indeed typical of the most common double-well potentials, but is not required in [8], [12]. In [9] we also deal with the definitively autonomous case, when $a(t) \equiv l$ outside a compact interval J: in this framework, solutions are proved to exist without further assumptions. The latter result exploits a peculiar technique, which is based on the property that a hamiltonian flow on the plane preserves area.

In this paper we are going to present a further way for tackling the problem: by means of a continuation method, we try to relate it to a well-known situation, where a is replaced by a new coefficient α_0, in such a way that the corresponding problem surely admits a solution, say x_0. More precisely, as explained by Theorem 2.9, it is possible to build a "homotopy" $\lambda \mapsto \alpha_\lambda$ on the coefficient of the equation which joins α_0 with the given coefficient $\alpha_1 = a$, and x_0 to a solution of (2.5), in the usual sense: if $\mathcal{P}(\alpha_\lambda)$ denotes the problem we get from (2.5) by putting $a = \alpha_\lambda$, then the space $C^1 \times [0, 1]$ contains a connected set $\mathcal{M}_0$ of pairs (x, λ) such that x solves $\mathcal{P}(\alpha_\lambda)$, and the projection of $\mathcal{M}_0$ on $[0, 1]$ is onto. For the sake of simplicity, α_λ is defined as in (2.10), but other kinds of homotopies could be considered (see Remark 4.5). The existence results which follow from the procedure above are summed up in Theorem 2.10, where two alternative sets of assumptions are presented. In particular, when (A_0)-(A_3) hold, the corresponding claim improves the result of [9] in the case $l_* = l^* = l$, since it removes the exponential estimate we quoted before. On the other hand, when (A_1)-(A_5) are satisfied (so that estimates of the previous kind hold both for $t \to -\infty$ and $t \to +\infty$) and the coefficient $a(t)$ converges to l from above, it is allowed to go below l on a bounded set. With respect to the results of [8], [12] and [18], the main advantage of Theorem 2.10 is that it does not require any bound from above on the coefficient $a(t)$. On the other hand, the quoted papers need no conditions about monotonicity or convergence rate of $a(t)$.

The main results of the paper are explained in §2. In particular, Theorem 2.9 contains the continuation argument which, together with Prop. 2.6, allows to get quickly our existence result for

problem (2.5), that is Theorem 2.10. In §3 we put forward some information about the stable and unstable manifold of the perturbed equation near the two wells ±1, which is summed up in Prop. 3.7. The proof of this result combines the classical method of the Implicit Function Theorem (which allows to get the regularity properties of the functions $x^{\pm}$, and is explained in §6) with a variational approach, which seems more suitable for getting an upper and lower bound on the initial times of solutions asymptotic to, respectively, -1 and $+1$ (see Prop. 3.7c). In §4 we are going to reduce the perturbed problem to a suitable nonlinear system of two scalar equations of the form $\phi_\lambda(\sigma, \tau) = 0$, where σ and τ represent, respectively, the initial times of solutions asymptotic to -1 and $+1$. Indeed, as is known, continuation methods are based on degree theory in functional spaces, but it looks rather difficult to apply such a tool when the given equation is to be considered on the whole real line, rather than on a compact interval. Then we adopt the following, well-known argument: problem $\mathcal{P}(\alpha_\lambda)$ is solved if and only if the unstable manifold at $(-1, 0)$ of the corresponding equation and its stable manifold at $(1, 0)$ meet in the space $txx\dot{x}$. Of course, this problem can be tackled in several ways (see, for instance, §4 of [15] for a similar approach, even if under different assumptions): in our case, it looks suitable to describe it by means of two degrees of freedom, namely two values σ and τ of the time axis. This new setting allows to represent the solution set in the three-dimensional space $\sigma\tau\lambda$: then the behaviour of its connected component which starts from $\lambda = 0$, say Γ_0, can be studied by means of the more simple Brouwer degree. In order to reach our goal, that is the level $\lambda = 1$, we need that Γ_0 is bounded: it is exactly at this point that we exploit the crucial assumption of definitive monotonicity on $a(t)$ given in (A_3), together with (A_0) or (A_5), so as to get the required estimates on Γ_0.

The starting point of our continuation method, namely Prop. 2.6, is proved in §5, by means of Prop. 5.2, which provides a particular bifurcation from the autonomous case. For the sake of completeness, we also give the obvious proof of Prop. 2.11, which combines Prop. 5.2 with Theorem 2.9 in order to get a global extension of the previous bifurcation. At the end of the section, in Remarks 5.3-4, we recall that Prop. 5.2 could be partially inserted in a more general framework. Finally, the already mentioned §6 is an appendix where we prove some results which were exploited in §3.

In order to conclude, we point out that, in spite of the great variety of possible approaches and alternative conditions which were introduced for solving (2.5), this problem seems to remain open under the only basic assumptions (A_1), (A_2) and (A_4) on the coefficient $a(t)$: actually, we do not know of any counter-examples on this subject.

2 Definitions and main results

We begin to introduce the main conditions we need on the potential $V(x)$ and the coefficient $a(t)$ of (1.1), which we denote respectively by (V_1)-(V_2), (A_0)-(A_6). Throughout the paper (V_1)-(V_2) will be fixed, while the assumptions on $a(t)$ may change here and there. Some of them, indeed, are alternative to other ones, as can be easily understood. For the sake of simplicity, we suppose that the number $l > 0$ is given.

(V_1) $V \in C^2(\boldsymbol{R}; [0, +\infty[)$, $V(\pm 1) = 0$, $V''(\pm 1) > 0$.

(V_2) On $]-1, 1[$ V' vanishes at just one point x^*.

(A_0) $a(t) \geq l$ for $t \in \boldsymbol{R}$

(A_1) $a(t) \to l$ as $t \to \pm\infty$.

(A_2) $a \in L^\infty(\boldsymbol{R})$.

(A_3) $a(t)$ is definitively monotone with respect to $|t|$, that is to say: for some $T \geq 0$, one of the two following conditions holds, where the arrows $\uparrow$ and $\downarrow$ mean respectively *increasing* and *decreasing* (not necessarily in a strict sense).

$$a(t) \uparrow \text{ for } t \leq -T \text{ and } a(t) \downarrow \text{ for } t \geq T \tag{2.1}$$

$$a(t) \downarrow \text{ for } t \leq -T \text{ and } a(t) \uparrow \text{ for } t \geq T. \tag{2.2}$$

(A_4) $\inf a > 0$

(A_5) For some $\beta < \sqrt{lV''(-1)}$, $\gamma > -\sqrt{lV''(1)}$ it is

$$|a(t) - l|e^{\beta t} \to +\infty \qquad \text{as} \qquad t \to +\infty, \tag{2.3}$$

$$|a(t) - l|e^{\gamma t} \to +\infty \qquad \text{as} \qquad t \to -\infty. \tag{2.4}$$

(A_6) For some $T \geq 0$ it is $a(t) \equiv l$ for $|t| \geq T$.

Remark 2.1 We point out that, by virtue of (V_1) and (V_2), the strict inequality $V > 0$ actually holds on $]-1, 1[$. Furthermore, when (A_0) and (A_1) hold, (A_3) is obviously equivalent to (2.1).

We define $\mathcal{A}$ as the family of those functions a which fulfil the basic assumptions (A_1), (A_2) and (A_4). For any $a \in \mathcal{A}$, we are interested in the following problem:

$$\mathcal{P}(a) \begin{cases} \ddot{x}(t) = a(t)V'(x(t)) & (i) \\ x(-\infty) = -1,\ x(+\infty) = 1 & (ii) \\ -1 < x(t) < 1 & (iii) \end{cases} \tag{2.5}$$

Remark 2.2 Since a is only supposed to be measurable, (2.5i) is required to hold almost everywhere, according to the usual definition of weak solution. More precisely: x is C^1, $\dot{x}$ is absolutely continuous and (2.5i) holds at any Lebesgue point of a.

Remark 2.3 Clearly the behaviour of V outside the interval $[-1, 1]$ does not affect problem (2.5). In particular, without loss of generality, we may suppose that $xV'(x) > 0$ for $|x| > 1$, and also that V' is bounded, so that (iii) follows from (i-ii), and all solutions to (2.5i) are global. We also point out that a solution of (2.5) needs not be monotone: nevertheless, our proof will actually provide a monotone solution.

Definition 2.4 We denote by $\mathcal{S}(a)$ the family of all solutions to $\mathcal{P}(a)$. By virtue of the previous remark, these solutions are supposed to be global: in particular, we shall regard $\mathcal{S}(a)$ as a subset of $C^1(\boldsymbol{R})$. We also define $\mathcal{M}(a)$ as the subset of $\mathcal{S}(a)$ made up by monotone functions. We say that $x \in \mathcal{S}(a)$ is *regular* or, respectively, *degenerate* according to whether the following linear problem admits only the trivial solution $z \equiv 0$ or not:

$$\begin{cases} \ddot{z}(t) = a(t)V''(x(t))z(t) & (i) \\ z(\pm\infty) = 0 & (ii) \end{cases} \tag{2.6}$$

Definition 2.5 The script $\mathcal{R}(a)$ will denote the set of all regular solutions to (2.5). We say that problem $\mathcal{P}(a)$ is regular if it admits a unique monotone solution, which is also regular, and denote by $\mathcal{A}^*$ the set of those coefficients $a \in \mathcal{A}$ such that $\mathcal{P}(a)$ is regular.

Proposition 2.6 *Both of the following families are non-empty:*

$$\begin{aligned} \mathcal{A}^- &:= \{a \in \mathcal{A}^* : (A_6) \text{ holds, and } a \le l\}, \\ \mathcal{A}^+ &:= \{a \in \mathcal{A}^* : (A_6) \text{ holds, and } a \ge l\}. \end{aligned}$$

Proof. see §5.

Remark 2.7 If we take $a(t) \equiv l$, we know that $\mathcal{S}(l)$, and therefore $\mathcal{M}(l)$, is made up by the functions $t \mapsto x^s(t) := x^0(t+s)$, $s \in \boldsymbol{R}$, where x^0 is the solution of the first order equation $\dot{x} = \sqrt{2lV(x)}$ which fulfils the initial condition $x(0) = 0$. In particular, $\mathcal{P}(l)$ is not regular, so that $l \notin \mathcal{A}^*$.

Definition 2.8 A map $\lambda \mapsto \alpha_\lambda$ from $[0,1]$ into $\mathcal{A}$ is said to be a *homotopy* between α_0 and α_1 if it is continuous from $[0,1]$ to $L^\infty(\boldsymbol{R})$, that is:

$$\|\alpha_\lambda - \alpha_\mu\|_\infty \to 0 \quad \text{as} \quad \lambda \to \mu. \tag{2.7}$$

If $\lambda \mapsto \alpha_\lambda$ is a given homotopy, we associate to it the sets

$$\mathcal{S} = \{(x,\lambda) \in C^1_b(\boldsymbol{R}) \times [0,1] : x \in \mathcal{S}(\alpha_\lambda)\}, \tag{2.8}$$

$$\mathcal{M} = \{(x,\lambda) \in \mathcal{S} : x \in \mathcal{M}(\alpha_\lambda)\}, \tag{2.9}$$

where $C^1_b(\boldsymbol{R})$ denotes the space of those functions $x \in C^1(\boldsymbol{R})$ which are bounded together with their derivative. $C^1_b(\boldsymbol{R})$ is endowed with its usual norm, and both $\mathcal{S}$ and $\mathcal{M}$ will inherit the product topology of $C^1_b(\boldsymbol{R}) \times [0,1]$. A standard example of homotopy is given by:

$$\alpha_\lambda = (1-\lambda)\alpha_0 + \lambda\alpha_1, \quad \lambda \in [0,1]. \tag{2.10}$$

Theorem 2.9 *Let (V_1) and (V_2) hold, and suppose that $a \not\equiv l$ fulfils (A_0)-(A_3) or (A_1)-(A_5). Let $\alpha_0 \in \mathcal{A}^+$ or $\alpha_0 \in \mathcal{A}^-$ according to whether, respectively, a satisfies (2.1) or (2.2), and consider the homotopy (2.10), where $\alpha_1 = a$. Let $\mathcal{M}(\alpha_0) = \{x_0\}$: then there exists a compact, connected subset $\mathcal{M}_0$ of (2.9) which joins $(x_0, 0)$ with $\mathcal{M}(a) \times \{1\}$.*

The theorem above will be proved in §4: together with Prop. 2.6, it allows to get easily our existence result for problem (2.5), which follows below.

Theorem 2.10 *Let conditions (V_1)-(V_2) hold, and also (A_0)-(A_3) or (A_1)-(A_5). Then problem (2.5) admits a monotone solution.*

Proof. By virtue of Prop. 2.6, we can find α_0 as in Theorem 2.9. Furthermore, we can obviously exclude the trivial case $a(t) \equiv l$, where we find the infinitely many solutions of Remark 2.7: then we can apply Theorem 2.9, which ensures, in particular, that $\mathcal{M}(\alpha_1) \times \{1\} \neq \emptyset$. ∎

Theorem 2.9 can be combined with the results of §5, so as to get a branch of solutions which bifurcates from the autonomous case and reaches a solution of the given problem (see §5 for more details).

Proposition 2.11 *Let conditions (V_1)-(V_2) hold, and also (A_0)-(A_3) or (A_1)-(A_5). Let x^s be as in Remark 2.7: then there exist a homotopy $\lambda \mapsto \alpha_\lambda$ between $\alpha_0 \equiv l$ and $\alpha_1 = a$, and a compact, connected subset $\mathcal{M}_0$ of (2.9) which joins $(x^s, 0)$ with $\mathcal{M}(a) \times \{1\}$.*

3 The stable and unstable manifolds

As we can easily see, (1.1) gives rise, in the phase plane $xy = x\dot{x}$, to the hamiltonian field $f(t, x, y) = (y, a(t)V'(x))$, whose equilibria $(\pm 1, 0)$ turn out to be saddle points. In this section we are going to study, in particular, the unstable manifold of f at $(-1, 0)$ and its stable manifold at $(1, 0)$. Actually, we are going to deal with the former, since the latter can be derived from it through suitable changes of variable, as we are going to show in the final part of the proof of Prop. 3.7. As we mentioned in the introduction, we adopt a variational point of view. We recall that $V''(-1) > 0$, so that we can find $r > 0$ and a function $Q : \boldsymbol{R} \to \boldsymbol{R}$ such that

$$Q \in C^2(\boldsymbol{R}), \quad Q(0) = 0, \quad \inf Q'' > 0, \quad \sup Q'' < +\infty, \tag{3.1}$$

$$Q(u) = V(-1 + u) \quad \text{for} \quad |u| \le r. \tag{3.2}$$

Proposition 3.1 *Let $a \in L^\infty(\boldsymbol{R})$, $\inf a > 0$. Let $Q \in C^2(\boldsymbol{R})$ fulfil (3.1). For any $\tau, h \in \boldsymbol{R}$, let us denote by $W^-(\tau, h)$ the space of those functions $u \in H^1(]-\infty, \tau])$ such that $u(\tau) = h$. Then the problem*

$$\min\left\{ \int_{-\infty}^{\tau} \left(\frac{1}{2}\dot{u}(t)^2 + a(t)Q(u(t)) \right) dt;\ u \in W^-(\tau, h) \right\} \tag{3.3}$$

admits a unique minimizer u, which is also the unique solution on $]-\infty, \tau]$ of the following problem:

$$\begin{cases} \ddot{u} = a\, Q'(u) & (i) \\ u(-\infty) = 0,\ u(\tau) = h & (ii) \end{cases} \tag{3.4}$$

Proof. Existence and uniqueness of the minimum in (3.3) follow from standard arguments in Calculus of Variations, since the integral functional Φ which appears in (3.3) is strictly convex, weakly lower semicontinuous and cohercive on H^1. On the other hand, (3.4i) is the Euler equation of (3.3), while the first condition in (3.4ii) is satisfied by any function $u \in H^1(]-\infty, \tau])$ ([5], Cor. VIII.8), so that any solution of (3.3) fulfils (3.4). Finally, thanks to the convexity of Φ, the inverse implication holds true as well. ∎

From now on, let $\lambda \mapsto \alpha_\lambda \in \mathcal{A}$ be a given homotopy of the form (2.10), where α_0 and $a = \alpha_1$ are as in Theorem 2.9. For any $c \in \boldsymbol{R}$ we denote respectively by $\Gamma^-(c)$ and $\Gamma^+(c)$ the two half-lines $]-\infty, c]$ and $[c, +\infty[$. Furthermore, $W_c^\pm$ will stand for the space of those functions $x \in C^1(\Gamma^\pm(c))$ such that x and $\dot{x}$ vanish at $\pm\infty$, on which we put the C^1-norm. For any function $u : \boldsymbol{R} \to \boldsymbol{R}$ we denote by $R_c^\pm u$ its restriction to $\Gamma^\pm(c)$. The following result concerns the dependence of the solution of (3.3) on τ, h and a, and will be proved in §6.

Proposition 3.2 *Let Q fulfil (3.1). Put $a = \alpha_\lambda$ in (3.3), denote by $t \mapsto u(t; \tau, h, \lambda)$ the corresponding minimizer, and extend it to the whole real line as a solution of $\ddot{u} = \alpha_\lambda Q'(u)$. Apply the same procedure after replacing $Q(u)$ by its quadratic part $Q_0(u) = \frac{1}{2}Q''(0)u^2$, and denote by $t \mapsto u_0(t; \tau, h, \lambda)$ the corresponding function. Let us also put $\eta(\tau, h, \lambda) = \dot{u}_t(\tau; \tau, h, \lambda)$, $p_\lambda(t) = u_0(t; 0, 1, \lambda)$, $w_\lambda(t) = \dot{p}_\lambda(t)/p_\lambda(t)$. Finally, for any $h \ge 0$, let us denote respectively by $\kappa(h)$ and $K(h)$ the minimum and the maximum value of $Q''(u)$ on $[0, h]$, and put $m(h) = (m\kappa(h))^{1/2}$, $M(h) = (MK(h))^{1/2}$, where, for any $\lambda \in [0, 1]$, $0 < m \le \alpha_\lambda \le M$. Then:*

(a) for any $c \in \boldsymbol{R}$, the map $(s, h, \lambda) \mapsto R_c^- u(\cdot; s, h, \lambda)$ is continuous from $\boldsymbol{R}^2 \times [0, 1]$ to W_c^-, and continuously differentiable with respect to (s, h).

(b) $u''_{th}(\tau;\tau,0,\lambda) = \eta'_h(\tau,0,\lambda) = w_\lambda(\tau)$.

(c) $h\exp(M(h)(t-\tau)) \le u(t;\tau,h,\lambda) \le h\exp(m(h)(t-\tau))$ *for any* $t \le \tau$.

(d) $\dot w_\lambda(t) = \alpha_\lambda(t)Q''(0) - w_\lambda(t)^2$ *a.e. on* $\boldsymbol{R}$.

(e) $m(0) = \sqrt{mQ''(0)} \le w_\lambda(t) \le M(0) = \sqrt{MQ''(0)}$ *for any* $t \in \boldsymbol{R}$. *Furthermore, the first inequality (or, respectively, the last one) is definitively strict to the right according to whether, on a set of positive measure, it is, respectively:* $\alpha_\lambda > m$ *or* $\alpha_\lambda < M$.

Remark 3.3 Of course, it is $u(t;\tau,h,\lambda) = u(t;\sigma,u(\sigma;\tau,h,\lambda),\lambda)$ for any $\sigma,\tau,t,h \in \boldsymbol{R}$, $\lambda \in [0,1]$. Indeed, both sides of the previous equality, as functions of t, solve problem (3.4), where we put $a = \alpha_\lambda$. In particular, they have the same derivative at $t = \sigma$, so that: $u'_t(\sigma;\tau,h,\lambda) = \eta(\sigma;u(\sigma;\tau,h,\lambda),\lambda)$.

If we put $q(t) = (Q''(0)\alpha_\lambda(t))^{1/2}$, from Prop. 3.2d we argue that the function $w = w_\lambda$ solves the Riccati equation $\dot w = q^2 - w^2$, which we denote by $(\mathcal{R})$. In the next result we are going to study, under suitable assumptions, the asymptotic behaviour of its solutions. To this end, we adopt the following notations: whenever a function z is definitively monotone to the right and converges, as $t \to +\infty$, to a number $\theta \in \boldsymbol{R}$, we write $z(t) \downarrow \theta$ or $z(t) \uparrow \theta$ according to whether the convergence to θ is from above or from below.

Lemma 3.4 *Let* $\sigma \in \boldsymbol{R}$, $J = [\sigma,+\infty[$, $q : J \to]0,+\infty[$ *be monotone,* $\theta := q(+\infty)$, $0 < \theta < +\infty$. *Let* w *be a positive solution of* $(\mathcal{R})$: *then* $w(t) \downarrow \theta$ *or* $w(t) \uparrow \theta$.

Proof. 1) If w is definitively monotone, it converges to θ. Indeed, in this case, w admits a limit $\bar w \ge 0$. Of course, $\bar w < +\infty$: otherwise, $(\mathcal{R})$ would imply the contradiction $\dot w(+\infty) = -\infty$. On the other hand, if $\bar w \ne l$, from $(\mathcal{R})$ we should argue that $\dot w$ converge to a non-zero limit, which is not possible, since $\bar w$ is finite.

2) If $w(\tau) \ge q(\tau) \ge \theta$ for some $\tau \ge \sigma$, then $w(t) \ge q(t)$ for $t \ge \tau$. We point out that, in this case, q is decreasing. As can be easily guessed, we can replace q by q^+ in the inequality to be proved, where $q^+(t)$ denotes the right limit of q at t. So, by contradiction, let $\tau_1 > \tau$ be such that $w(\tau_1) < q^+(\tau_1)$, A the closed subset of $[\tau,\tau_1]$ where $w \ge q^+$, $\sigma_1 = \max A$, $J_1 = [\sigma_1,\tau_1]$, so that $w \le q^+$ on J_1: then, according to $(\mathcal{R})$, $\dot w \ge 0$ on J_1, in contrast with the inequalities $w(\tau_1) < q^+(\tau_1) \le q^+(\sigma_1)$.

3) If $w(\tau) \le q(\tau) \le \theta$ for some $\tau \ge \sigma$, then $w(t) \uparrow \theta$. We can argue in a quite similar way to step 2).

4) Conclusion. Thanks to steps 2) and 3), we only need to prove our claim when $w \le q$ on J, or $w \ge q$ on J. In both cases, $(\mathcal{R})$ ensures that w is monotone, so as to apply 1). ∎

Lemma 3.5 *Let* Q *satisfy (3.1) and (3.2). Let* α_λ *be given by (2.10), where* α_0 *and* $a = \alpha_1$ *are as in Theorem 2.9, and* w_λ *be as in Prop. 3.2. Let* $T \in \boldsymbol{R}$ *be as in (2.1) or (2.2), and suppose also* $\alpha_0(t) \equiv l$ *for* $t \ge T$. *Then, for any* $\lambda \in [0,1]$*:*

(a) $w_\lambda \to \theta$ *"from the same side" as* q_λ, *as* $t \to +\infty$.

(b) The equality $w_\lambda = \theta$ *cannot hold definitively to the right.*

(c) there exists $\tau = \tau(\lambda) \ge T$ *such that* $\dot w_\lambda(\tau) < 0$ *or* $\dot w_\lambda(\tau) > 0$ *according to whether, respectively, (2.1) or (2.2) holds.*

Proof. We split the proof into the following steps.

1) $(A_0) \vee [(\lambda = 0) \wedge (2.1)] \Rightarrow (a) \wedge (b)$. Of course, on the assumptions of this step, it is $q_\lambda \downarrow \theta$. Furthermore, $\alpha_\lambda \geq l$ on $\boldsymbol{R}$: then we can put $m = l$ in Prop. 3.2, so that the number $m(0)$ in claim (e) is nothing but θ. Hence $w_\lambda \geq \theta$, so that $w_\lambda \to \theta^+$. As regards (b), we point out that $\alpha_0 \geq l$, $\alpha_1 \geq l$ and the equalities $\alpha_0 = l$ and $\alpha_1 = l$ cannot hold everywhere (the former because $\alpha_0 \in \mathcal{A}^*$, the latter by an explicit assumption). Then (2.10) entails that $\alpha_\lambda > l$ on a set of positive measure, and we can put $m = l$ in the last claim of Prop. 3.2e, so that $w_\lambda(t) > m(0) = \theta$ for t large enough.

2) $(\lambda = 0) \wedge (2.2) \Rightarrow (a) \wedge (b)$. Now $q_\lambda \uparrow \theta$, and $\alpha_\lambda = \alpha_0 \leq l$ on $\boldsymbol{R}$, so that we can exploit Prop. 3.2e again, where we put $M = l$.

3) $(A_5) \wedge (2.1) \wedge (\lambda > 0) \Rightarrow (a) \wedge (b)$. Thanks to (2.10) and (A_6):

$$|q_\lambda(t) - \theta| = (q_\lambda(t) + \theta)^{-1} |q_\lambda(t)^2 - \theta^2| \geq$$

$$\theta^{-1} Q''(0) |\alpha_\lambda(t) - l| = \lambda \sqrt{Q''(0)/l}\, |a(t) - l|, \quad t \geq T.$$

Then, as $t \to +\infty$, $|q_\lambda(t) - \theta| e^{\beta t} \to +\infty$ for $\beta < \theta$: on this subject, we recall that $Q''(0) = V''(-1)$ (so that β actually fulfils (2.3)), and $\lambda > 0$. Now, let us argue by contradiction, so as to deny that $w \downarrow \theta$: then, by virtue of Lemma 3.4, it is $w_\lambda \uparrow \theta$, so that $w_\lambda \leq \theta$ on $[\tau, +\infty[$, for some $\tau \geq \sigma$. Now we can apply Lemma 6.3a, where we put $f(t, w) = q(t)^2 - w^2$, $\psi(w) = \theta^2 - w^2$. Then $\delta(t) = q(t)^2 - \theta^2 \geq 0$, in contrast with the thesis of the lemma. Of course, since α_λ can replace a in (2.3), it cannot be definitively costant to the right, and the same we can say of w_λ, thanks to Prop. 3.2d: hence (b) is proved as well.

4) $(A_5) \wedge (2.2) \wedge (\lambda > 0) \Rightarrow (a) \wedge (b)$. We can argue in the same way as in 3): now, of course, we exploit Lemma 6.3b.

5) Conclusion. Under the assumptions of Theorem 2.9, $\alpha_1 = a$ always fulfils $(2.1) \vee (2.2)$ and $(A_0) \vee (A_5)$, so that the four cases we considered in the previous steps are exhaustive, and both claims (a) and (b) are proved. Finally, (c) is a consequence of (a) and (b), by virtue also of Lemma 3.4. ∎

Lemma 3.6 *Let us put $E_t(x; \lambda) := \frac{1}{2}\dot{x}(t)^2 - a_\lambda(t) V(x(t))$ for any $\lambda \in [0, 1]$, $x \in \mathcal{S}(\alpha_\lambda)$, $t \in \boldsymbol{R}$, and take T from (2.2) or (2.1). Then, for $|t| \geq T$, it is $E_t(x; \lambda) \leq 0$ or $E_t(x; \lambda) \geq 0$ according to whether, respectively, (2.1) or (2.2) holds.*

Proof. As a simple argument shows, whenever $x \in \mathcal{S}(a_\lambda)$, it is $\dot{x}(\pm\infty) = 0$, so that $E_t(x, \lambda) \to 0$ as $t \to \pm\infty$. On the other hand, if (A_3) holds, it is easy to infer, from the equation $\ddot{x} = \alpha_\lambda V'(x)$, that the map $t \mapsto E_t(x; \lambda)$ is monotone, for $|t| \geq T$, in an opposite way with respect to $\alpha_\lambda(t)$, and therefore to $a(t)$. Hence our claim is proved. ∎

Proposition 3.7 *Let $\lambda \mapsto \alpha_\lambda$ be as in Theorem 2.9. Then there exist $r > 0$ and functions $x^\pm : \boldsymbol{R}^2 \times [0, r] \times [0, 1] \to \boldsymbol{R}$ which satisfy the following conditions, according to the notations which come before Prop. 3.2.*

(a) For any $c \in \boldsymbol{R}$ the map $(s, h, \lambda) \mapsto R_c^\pm x^\pm(\cdot; s, h, \lambda)$ is continuous from $\boldsymbol{R} \times [0, r] \times [0, 1]$ to $\pm 1 + W_c^\pm$, and also continuously differentiable with respect to (s, h).

(b) $t \mapsto x^{\pm}(t; s, h, \lambda)$ *is the unique function* $x(t)$ *which is increasing on* $\Gamma^{\pm}(s)$ *and such that, according to the sign,*

$$\begin{cases} \ddot{x}(t) = a_\lambda(t)V'(x(t)), \quad t \in \boldsymbol{R}, & (i) \\ x(\pm\infty) = \pm 1,\ x(s) = \pm(1-h) & (ii) \end{cases} \tag{3.5}$$

(c) The sets $A^{\pm} := \{s \in \boldsymbol{R} :\ x^{\pm}(\cdot; s, r, \lambda) \in \mathcal{S}(\alpha_\lambda)$ *for some* $\lambda \in [0,1]\}$ *are such that* $\sup A^- < +\infty$ *and* $\inf A^+ > -\infty$.

Proof. Let r and Q be as in (3.1-2), and let u, Q_0, u_0, w_λ, η be given by Prop. 3.2: it is easy to see that the map $z(t) := u(t; s, h, \lambda)$ and its derivative $\dot{z}(t)$ are positive for $t \le s$, so that $0 \le z(t) \le h$ on that half-line. Then, for $h \in [0, r]$, the values that Q takes outside $[0, r]$ do not affect u: hence the map $t \mapsto u(t; s, h, \lambda)$ solves the equation $\ddot{z} = \alpha_\lambda(t)V'(-1+z)$ for $t \le s$. By virtue of Remark 2.1, this map can be extended to the whole real line as a solution of the latter equation, so as to become a new map $\tilde{u}$. Now we are going to prove that the function $x^-(t; s, h, \lambda) := -1 + \tilde{u}(t; s, h, \lambda)$ fulfils all the required properties which concern it, that we denote by (a^-)-(c^-), (3.5^-). In a similar way, (a^+)-(c^+) and (3.5^+) will stand for the corresponding properties of x^+.

(a^-) The regularity properties of u that we get from Prop. 3.2a are obviously inherited by $\tilde{u}$, since standard arguments about dependence on initial data [1] ensure that, for any compact interval J, the map $(s, h, \lambda) \mapsto \tilde{u}(\cdot; s, h, \lambda)_{|J}$ is continuous from $\boldsymbol{R} \times [0, r] \times [0, 1]$ to $C^1(J)$, and continuously differentiable with respect to (s, h). In particular, $\tilde{u}$ and $\tilde{u}'_t$ admit continuous partial derivatives with respect to s and h.

(b^-) By construction, the function $t \mapsto x^-(t; s, h, \lambda)$ solves (3.5^-). As regards uniqueness, let $x(t)$ be a solution of (3.5^-) which is monotone for $t \le s$: then, according to the previous arguments, $x(]-\infty, s]) \subseteq [0, h]$ so that, by the uniqueness result of Prop. 3.1, $x(t) = x^-(t; s, h, \lambda)$ for $t \le s$. Of course, thanks to (3.5i), the two functions agree for $t \ge s$ as well.

(c^-) Suppose, by contradiction, $\sup A^- = +\infty$: then we can find points $s_i \to +\infty$ and $\lambda_i \in [0, 1]$ such that $s_i \to +\infty$ as $i \to +\infty$, and $x_i(+\infty) = 1$, where $x_i(t) = x^-(t; s_i, r, \lambda_i)$. Suppose also, as we may do up to a subsequence, that $\lambda_i \to \lambda \in [0, 1]$. For such a λ, take $\tau = \tau(\lambda)$ as in Lemma 3.5c: then, for large i's, when putting $h = r$, $t = \tau$ and $\tau = s_i$ in the second inequality of Prop. 3.2c we get

$$0 \le h_i := x_i(\tau) + 1 = u(\tau; r, s_i, \lambda_i) \le r \exp(m(r)(\tau - s_i)).$$

In particular, $h_i \to 0$ as $i \to +\infty$. Now, let $\eta_i := \dot{x}_i(\tau)$, and put $\sigma = \tau$, $\tau = s_i$, $h = r$, $\lambda = \lambda_i$ in the final claim of Remark 3.3, so as to get $\eta_i = \eta(\tau, h_i, \lambda_i)$: then Prop. 3.2b implies that $\eta_i/h_i \to w(\tau)$ as $i \to +\infty$. Indeed, since u''_{th} is continuous, the ratio between $\eta(\tau, h, \lambda)$ and h converges uniformly with respect to $\lambda \in [0, 1]$, as $h \to 0$. Now, let us recall the function E_t from Lemma 3.6: then, as $i \to +\infty$,

$$\frac{E_\tau(x_i; \lambda_i)}{h_i^2} \to \frac{1}{2}(w_\lambda(\tau)^2 - \alpha_\lambda(\tau)Q''(0)) = -\frac{1}{2}\dot{w}_\lambda(\tau). \tag{3.6}$$

Indeed, we have $E_\tau(x_i; \lambda_i) = \frac{1}{2}\dot{x}_i(\tau)^2 - \alpha_{\lambda_i}(\tau)V(x_i(\tau)) = \frac{1}{2}\eta_i^2 - \alpha_{\lambda_i}(\tau)Q(h_i)$, so that (3.6) follows easily: in particular, the last equality can be deduced from Prop. 3.2d. Now, if (2.1) holds, from Lemma 3.5c we get $\dot{w}_\lambda(\tau) < 0$. Then, from (3.6), we infer that $E_\tau(x_i; \lambda_i) > 0$ for large i's, in contrast with Lemma 3.6. A completely similar argument can be applied if (2.2) holds: in this case, we get $E_\tau(x_i; \lambda_i) < 0$ for large i's, again in contrast with Lemma 3.6.

(a^+)-(c^+) If we put $\tilde{V}(x) = V(-x)$ and $\tilde{a}(t) = a(-t)$, it is straightforward to check that $\tilde{V}$ and $\tilde{a}$ fulfil the same assumptions as V and a in Theorem 2.9. Then we can apply the previous arguments

to $\tilde{V}$ and $\tilde{a}$, so as to get a function $\tilde{x}^-$ together with the corresponding set $\tilde{A}^-$. Now, let us put $x^+(t;s,h,\lambda) := -\tilde{x}^-(-t;-s,h,\lambda)$, so that $A^+ = -\tilde{A}^-$: then (a^+)-(c^+) follow easily. In particular, when (A_5) holds instead of (A_0), we remark that only (2.3) was used for building x^-. Now, of course, we exploit (2.4), which plays the role of (2.3) for $\tilde{x}^-$. ■

4 Reduction to a two-dimensional problem

In this section we are going to reduce problem $\mathcal{P}(\alpha_\lambda)$ to the equation

$$\phi_\lambda(\sigma,\tau) = 0, \quad \sigma,\ \tau \in \boldsymbol{R}, \tag{4.1}$$

where ϕ_λ is a suitable function from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$. To this end, let us take r and $x^\pm$ as in Prop. 3.7, put $y^\pm := (x^\pm)'_t$, $v^\pm := (x^\pm, y^\pm)$ and also:

$$\phi_\lambda(\sigma,\tau) = v^+(0;\tau,r,\lambda) - v^-(0;\sigma,r,\lambda), \quad \lambda \in [0,1],\ \sigma,\tau \in \boldsymbol{R}, \tag{4.2}$$

$$\Omega(\lambda) = \{(\sigma,\tau) \in \boldsymbol{R}^2 :\ \phi_\lambda(\sigma,\tau) = 0\}, \quad \lambda \in [0,1],$$

$$\Omega = \{(\sigma,\tau,\lambda) \in \boldsymbol{R}^2 \times [0,1] :\ (\sigma,\tau) \in \Omega(\lambda)\},$$

$$x(\cdot;\omega) = x^-(\cdot;\sigma,r,\lambda) = x^+(\cdot;\tau,r,\lambda), \quad \omega = (\sigma,\tau,\lambda) \in \Omega, \tag{4.3}$$

$$\Theta(\omega) = (x(\cdot;\omega),\lambda), \quad \omega = (\sigma,\tau,\lambda) \in \Omega. \tag{4.4}$$

Proposition 4.1 *Definition (4.3) is well posed on Ω, and the map Θ in (4.4) is a continuous bijection from Ω to the set $\mathcal{S} \subseteq C^1_b(\boldsymbol{R}) \times [0,1]$ of (2.8). In particular, for any $\lambda \in [0,1]$, problem $\mathcal{P}(\alpha_\lambda)$ admits a solution if and only if so does (4.1).*

Proof. 1) We are going to show that, for any $\lambda \in [0,1]$, the map $s \mapsto x^-(\cdot;s,r,\lambda)$ is a bijection between $\boldsymbol{R}$ and the family $\mathcal{S}^-_\lambda$ of those solutions x of the equation $\ddot{x} = \alpha_\lambda V'(x)$ such that $x(t) \to -1^+$ as $t \to -\infty$. First of all, it is obvious that $x^-(\cdot;s,r,\lambda) \in \mathcal{S}^-_\lambda$. Now, let $x \in \mathcal{S}^-_\lambda$, $s = \min x^{-1}(\{-1+r\})$, $\bar{x} = x^-(\cdot;s,r,\lambda)$: then both functions $\bar{x}$ and x solve (3.5). Furthermore, thanks to the choice of s, $x(]-\infty,s]) \subseteq]-1,-1+r]$, so that $x(t)$ is increasing for $t \le s$: according to Prop. 3.7b, these properties characterize $\bar{x}$, so that $x \equiv \bar{x}$, and $s \mapsto x^-(\cdot;s,r,\lambda)$ is onto. On the other hand, the same map is obviously one-to-one, because $x^-(\cdot;s,r,\lambda)$ is strictly increasing on $]-\infty,s]$.

2) In the same way we can prove that, for any $\lambda \in [0,1]$, the map $s \mapsto x^+(\cdot;s,r,\lambda)$ is a bijection between $\boldsymbol{R}$ and the family $\mathcal{S}^+_\lambda$ of those solutions x of the equation $\ddot{x} = \alpha_\lambda V'(x)$ such that $x(t) \to 1^-$ as $t \to +\infty$.

3) Conclusion. Both functions $x^-(\cdot;\sigma,r,\lambda)$ and $x^+(\cdot;\tau,r,\lambda)$ solve (2.5i) and, thanks to (4.1), fulfil the same initial conditions at $t = 0$. Hence the second equality in (4.3) actually holds. Furthermore, from Prop. 3.7a, we obviously infer the continuity of Θ, while 1) and 2) entail that, for any $\lambda \in [0,1]$, the map $(\sigma,\tau) \mapsto x(\cdot;\sigma,\tau,\lambda)$ is a bijection between $\Omega(\lambda)$ and the set $\mathcal{S}^-_\lambda \cap \mathcal{S}^+_\lambda$, which can be thought to agree with $\mathcal{S}(\alpha_\lambda)$, according to Remark 2.3. Then $\Theta : \Omega \to \mathcal{S}$ is also one-to-one. ■

Lemma 4.2 *A sufficient condition in order that problem (2.6) admits only the trivial solution is that we can find two independent solution z^- and z^+ to (2.6i) such that $z^-(-\infty) = 0$, $z^+(+\infty) = 0$.*

Proof. Let z^- and z^+ as above.

1) None of the two conditions $z^{\mp}(\pm\infty) = 0$ can hold. Indeed, let us suppose, for instance, $z^-(+\infty) = 0$: by solving, with respect to z, the equation $(z^+\dot{z} - z\dot{z}^+)^{\cdot} = 0$, we easily get a solution z of (2.6i) such that z^+ and z are independent and $z(+\infty) = +\infty$. In particular, $z^- = c_1 z + c_2 z^+$ for suitable $c_1, c_2 \in \boldsymbol{R}$, and the condition $z^-(+\infty) = 0$ would entail $c_1 = 0$, so that $z^- = c_2 z^+$. In a similar way, of course, we could argue if $z^+(-\infty) = 0$, so as to infer that z^- and z^+ are linearly dependent, a contradiction.

2) Any solution z of (2.6i) takes the form $c_1 z^- + c_2 z^+$, for some $c_1, c_2 \in \boldsymbol{R}$. Now, since $z(\pm\infty) = 0$, from 1) we get $c_1 = c_2 = 0$. ∎

Proposition 4.3 *According to Def. 2.4, let R be the set of those $\omega \in \Omega$ such that $x(\cdot;\omega)$ is regular. Then:*

$$R = \{(\sigma, \tau, \lambda) \in \Omega : \det \nabla\phi_\lambda(\sigma, \tau) \neq 0\}. \tag{4.5}$$

Proof. Let $\omega = (\sigma, \tau, \lambda) \in \Omega$, $x = x(\cdot;\omega)$ and, according to the notations of Prop. 3.7, $z^-(t) = (x^-)'_s(t;\sigma,r,\lambda)$, $z^+(t) = (x^+)'_s(t;\tau,r,\lambda)$, $w = z^-\dot{z}^+ - z^+\dot{z}^-$.

1) $\det \nabla\phi_\lambda(\sigma,\tau) = w(0)$: it is enough to perform an exchange between $\partial/\partial t$ and $\partial/\partial s$ at the left-hand side.

2) $\ddot{z}^- = \alpha_\lambda V''(x) z^-$. In the proof of Prop. 3.7, the function $x^-(\cdot;s,h,\lambda)$ can be seen as that solution of (3.5i) which fulfils the initial conditions $x(0) = -1 + \tilde{u}(0;s,h,\lambda)$, $\dot{x}(0) = \tilde{u}'_t(0;s,h,\lambda)$: then we only need to apply well-known results about the dependence on initial data (see, for instance, [1]).

3) $z^-(-\infty) = 0$. On this subject, it is enough to point out that the derivative $(\tilde{u}^-)'_s$ is performed in a space of functions which vanish at $-\infty$.

4) In the same way as in the two previous steps, we can show that $\ddot{z}^+ = \alpha_\lambda V''(x) z^+$ and $z^+(+\infty) = 0$.

5) If x is degenerate, then $\det \nabla\phi_\lambda(\sigma,\tau) = 0$. Indeed, thanks to 2)-3)-4) and Lemma 4.2, z^- and z^+ are linearly dependent, so that $w(0) = 0$, and our claim follows from step 1).

6) If $\det \nabla\phi_\lambda(\sigma,\tau) = 0$, then x is degenerate. Again by virtue of step 1), our assumption entails that $c_1 z^- + c_2 z^+ \equiv 0$ for a suitable pair $(c_1, c_2) \neq (0,0)$. Actually, $c_1 \neq 0$ and $c_2 \neq 0$, since it is easy to understand that both z^- and z^+ are non trivial. Then the conditions $z^{\pm}(\pm\infty) = 0$ imply $z^{\pm}(\mp\infty) = 0$, so that z^- and z^+ both solve problem (2.6), where we put $a = \alpha_\lambda$.

7) Conclusion. Now (4.5) obviously follows from 5) and 6). ∎

Proposition 4.4 *In the same assumptions of Theorem 2.9, let $\omega_0 = (\sigma_0, \tau_0, 0)$ be such that $\mathcal{M}(\alpha_0) = \{x(\cdot;\omega_0)\}$. Then the connected component Γ_0 of Ω which contains ω_0 is a compact subset of $M := \{\omega \in \Omega : x(\cdot;\omega)$ is monotone$\}$.*

Proof. 1) $\Gamma_0 \subseteq M$. Let us recall the point x^* of condition (V_2), denote by U^* the set of those functions $x \in C^1(\boldsymbol{R})$ such that the pair $(x - x^*, \dot{x})$ never vanishes, and put

$$\begin{aligned} \theta(x;\sigma,\tau) &= \int_\sigma^\tau \frac{\alpha_\lambda V'(x)(x-x^*) - \dot{x}^2}{(x-x^*)^2 + \dot{x}^2}\, dt, \quad x \in U^*,\ \sigma,\tau \in \boldsymbol{R}, \\ \chi(\omega) &= \theta(x(\cdot;\omega);\sigma,\tau), \quad \omega = (\sigma,\tau,\lambda) \in \Omega. \end{aligned} \tag{4.6}$$

The function θ is continuous from $U^* \times \boldsymbol{R}^2$ to $\boldsymbol{R}$, where U^* is endowed with the C^1_{loc}-topology. Furthermore, χ is well defined on Ω, since the only solution $x \notin U^*$ of (3.5i) is $x \equiv x^*$, and continuous as well. If x solves (3.5i), (4.6) can be seen, in the phase plane $x\dot{x}$, as the angular variation, with respect to the point $(x^*, 0)$, that $(x, \dot{x})$ undergoes in the interval $[\sigma, \tau]$: then it is easy to understand that $\chi(\omega)$ is less or bigger than π according to whether, respectively, the solution $x(\cdot; \omega)$ is monotone or not: in particular, $\chi(\omega_0) < \pi$. Now, let us suppose, by contradiction, that there exists $\omega^* \in \Gamma_0$ such that the corresponding solution $x(\cdot; \omega^*)$ is not monotone, that is to say $\chi(\omega^*) > \pi$: since χ is continuous and Γ_0 is connected, we should find $\omega = (\sigma, \tau, \lambda) \in \Gamma_0$ such that $\chi(\omega) = \pi$: this is a contradiction, since $x(\cdot; \omega)$ has a positive derivative at σ and τ.

2) Γ_0 is bounded. Let us put $J = [-1 + r, 1 - r]$, where $r > 0$ is taken from Prop. 3.7, and $\Delta := \int_J (2lV(x))^{-1/2} dx$. Let Γ^* be the projection of Γ_0 on the plane $\sigma\tau$: since $x(\cdot, \omega)$ is monotone and (V_2) holds, the same arguments as in [9] show that $\tau - \sigma \leq \Delta$ for any $(\sigma, \tau) \in \Gamma^*$, that is to say: $\Gamma^* \subseteq S$, where S is the strip of the plane $\sigma\tau$ which lies between the two lines $\tau = \sigma$ and $\tau = \sigma + \Delta$. On the other hand, according to Prop. 3.7c, the intersection between S and the set $A := A^- \times A^+$ is bounded, and turns out to contain Γ^*: since $\Gamma_0 \subseteq \Gamma^* \times [0, 1]$, our claim follows.

3) Γ_0 is closed. Of course, it is enough to show that Ω is closed, as we can argue from the continuity of the map $(\sigma, \tau, \lambda) \mapsto \phi_\lambda(\sigma, \tau)$. ∎

Proof of Theorem 2.9. Let us consider the homotopy (2.10) and the corresponding function ϕ_λ of (4.2), define Γ_0 as above and put $\mathcal{M}_0 = \Theta(\Gamma_0)$, where Θ is the map of (4.4): thanks to Propositions 4.1 and 4.4 $\mathcal{M}_0$ is a compact, connected subset of $C^1_b(\boldsymbol{R}) \times [0, 1]$, and $\mathcal{M}_0 \subseteq \mathcal{M}$. Since obviously $(x_0, 0) \in \mathcal{M}_0$, we only need to prove that $\mathcal{M}_0$ meets $\mathcal{M}(\alpha_1) \times \{1\}$, that is to say: $1 \in \Lambda_0 := P(\Gamma_0)$, where P is the projection $(\sigma, \tau, \lambda) \mapsto \lambda$. So, by contradiction, suppose $1 \notin \Lambda_0$, and recall Prop. 4.3: since $(\sigma_0, \tau_0, 0) \in R$, the Implicit Function Theorem ensures the existence of an arc γ from an interval $J = [0, \delta]$ to an open neighbourhood V_0 of (σ_0, τ_0) such that $\gamma(0) = (\sigma_0, \tau_0)$ and, for any $\lambda \in J$, $\gamma(\lambda) =: (\sigma(\lambda), \tau(\lambda))$ is the only solution of (4.1) in V_0, and $\det \nabla\phi_\lambda(\gamma(\lambda)) \neq 0$. We also recall, according to Def. 2.5, that problem $\mathcal{P}(\alpha_0)$ is regular, so that $(\sigma_0, \tau_0, 0)$ is the only point where M meets the plane $\lambda = 0$. Then, if we put $W = [0, \delta[\times V_0$ and consider the closed sets $\Gamma = \Gamma_0 \setminus W$, $\Gamma_1 = \Omega \setminus \Gamma_0$, we get $P(\Gamma) \subseteq]0, 1[$ and $\Gamma \cap \Gamma_1 = \emptyset$: since Γ is compact, we can find an open neighbourhood A of Γ such that $\bar{A} \cap \Gamma_1 = \emptyset$ and $P(\bar{A}) \subseteq]0, 1[$. Now, let λ_*, $\lambda^* \in]0, 1[$ be such that the intervals $[0, \lambda_*]$, $[\lambda^*, 1]$ do not meet $P(\bar{A})$, put $U = A \cup W^\circ$ and define a homotopy $h : U \times [\lambda_*, \lambda^*] \to \boldsymbol{R}^3$ as follows:

$$h(\omega, \mu) = (\phi_\lambda(\sigma, \tau), \lambda - \mu), \quad \omega = (\tau, \sigma, \lambda) \in U, \, \mu \in [\lambda_*, \lambda^*].$$

Actually, $h(\cdot, \mu)$ cannot vanish on ∂U: indeed, on ∂U the mapping $(\sigma, \tau, \lambda) \mapsto \phi_\lambda(\sigma, \tau)$ can vanish only on ω_0, where the last component of h takes the value $-\mu < 0$. Hence h is an admissible homotopy. On the other hand, $\omega_* := (\gamma(\lambda_*), \lambda_*)$ is the only point at which $h(\cdot, \lambda_*)$ vanishes and, of course, the determinant of $\nabla h(\cdot, \lambda_*)$ at ω_* agrees with $\det \nabla\phi_{\lambda_*}(\sigma(\lambda_*), \tau(\lambda_*)) \neq 0$. On the contrary, $h(\cdot, \lambda^*)$ cannot vanish in U, since otherwise we should argue that $\lambda^* \in P(U) = P(A) \cup P(W^\circ) = P(A) \cup]0, \delta[$. Since $\lambda \notin P(A)$, we should get $0 < \lambda^* < \delta$, in contrast with $[\lambda^*, 1] \cap P(A) = \emptyset$. Now we get the following contradiction, which concludes the proof:

$$\deg(h(\cdot, \lambda_*), U) \neq 0 = \deg(h(\cdot, \lambda^*), U).$$ ∎

Remark 4.5 For the sake of simplicity, Theorem 2.9 only deals with the particular homotopy (2.10). Actually, if we analyze all the results which lead to the previous proof, we argue easily that (2.10)

can be replaced by any other homotopy such that, for any $\lambda \in]0,1[$: $\alpha_\lambda \not\equiv l$ and, whenever $\alpha_1 = a$ fulfils condition (A_i) for some $i = 0, 1, ..., 5$, so does α_λ.

Remark 4.6 Actually, there is a further case to which Theorem 2.9 could be extended. Indeed, let us suppose $a \le l$, $\alpha_0 \in \mathcal{A}^-$, and that (A_1), (A_2) and (2.2) hold: then we can easily check that Lemma 3.5c still works, so as to infer, also in this case, that the set Γ_0 of Prop. 4.4 is bounded. We point out, however, that the existence result which would follow for (2.5) is not very interesting in this case, since the variational methods of [4] (Chapter 2, Thm. 2.2) allow to get more easily a stronger result.

5 Bifurcation from the autonomous case

In this section we are going to prove Propositions 2.6 and 2.11. To this end, we are going to perform a simple perturbation of the autonomous equation $\ddot{x} = lV'(x)$, by modifying the constant value l on a compact interval. According to condition (V_2), let ρ be the unique function from the interval $[-1, 1]$ into itself such that $V(\rho(\xi)) = V(\xi)$ and $\rho(\xi) \neq \xi$ for $\xi \neq x^*$. For any $\xi \in]-1, x^*[$, $\delta > -l$, we define $t \mapsto \tilde{x}(t, \xi, \delta))$ as the function $x(t)$ such that: $\ddot{x} = lV'(x)$ on $]-\infty, 0[$, $\ddot{x} = (l+\delta)V'(x)$ on $]0, +\infty[$ and $(x(0), \dot{x}(0)) = w(\xi)$, where $w(\xi) := (\xi, \sqrt{2lV(\xi)})$. We also put $u := (\tilde{x}, \tilde{x}'_t)$.

Lemma 5.1 *Let* $\sigma(\xi, \delta) := \int_\xi^{\rho(\xi)} (2(l+\delta)V(x) - 2\delta V(\xi))^{-1/2} dx$, *where* $|\xi| < 1$, $\delta > -l$. *Then the following properties hold:*

(a) $u(\sigma(\xi, \delta), \xi, \delta) = w(\rho(\xi))$.

(b) For $\xi \neq x^*$: $\rho'(\xi) = V'(\xi)/V'(\rho(\xi))$.

(c) For $\xi \neq x^*$: $\sigma'_\xi(\xi, 0) = -(2lV(\xi))^{-1/2}(1 - \rho'(\xi))$.

(d) For any $T > 0$ *there exists* $\delta_T \in]0, l[$ *and* $\xi_T \in C^1([-\delta_T, \delta_T])$ *such that, for* $|\delta| \le \delta_T$, $\xi_T(\delta)$ *is the unique solution of the equation* $\sigma(\xi, \delta) = T$ *in the whole interval* $]-1, x^*[$. *Furthermore,* $\sigma'_\xi(\xi_T(\delta), \delta) < 0$.

Proof. (a) It is enough to analyze the given equation in the phase plane.

(b)-(c) From the equality $V(\rho(\xi)) = V(\xi)$, the expression of σ and the chain rule we easily argue that both formulas hold true.

(d) Let $T > 0$ be given, and fix $\delta_0 \in]0, l[$: we remark that $\sigma(\xi, \delta) \to 0$ as $\xi \to x^*$ and $\sigma(\xi, \delta) \to +\infty$ as $\xi \to -1$. Furthermore, both behaviours are uniform with respect to $\delta \in [-\delta_0, \delta_0]$: then we can find a compact interval $J \subseteq]-1, x^*[$ such that the image of $]-1, x^*[\setminus J$ through the map $\sigma(\cdot, \delta)$ does not contain T, for $|\delta| \le \delta_0$. On the other hand, (b) and (c) show that $\sigma'_\xi(\xi, 0) < 0$, since $V'(\xi)$ and $V'(\rho(\xi))$ have opposite signs: then we can find $\delta_T \in]0, \delta_0]$ such that $\sigma'_\xi(\xi, \delta) < 0$ for $\xi \in J$, $|\delta| \le \delta_T$. Now we only need to put $\xi_T(\delta) = \sigma(\cdot, \delta)^{-1}(T)$ for $|\delta| \le \delta_T$: of course, the map ξ_T is C^1 by virtue of the Implicit Function Theorem. ∎

Now, let us introduce the field $f_\delta(x, y) = (y, (l+\delta)V'(x))$, and derive with respect to ξ both sides of the equality in Lemma 5.1a, as we are allowed to do by well-known results about differentiability with respect to initial data. Since $u'_t = f_\delta(u)$ on $]0, +\infty[$ (in particular for $t = \sigma(\xi, \delta)$), we get:

$$u'_\xi(\sigma(\xi, \delta), \xi, \delta) = \rho'(\xi) w'(\rho(\xi)) - \sigma'_\xi(\xi, \delta) f_\delta(w(\rho(\xi))). \tag{5.1}$$

Proposition 5.2 *Let* $T > 0$, γ *be the characteristic function of* $[0, T]$ *and put, for any* $\delta > -l$, $a(t, \delta) = l + \delta\gamma(t)$. *Then there exists* $\delta^* \in]0, l[$ *such that:*

(a) $\mathcal{P}(a(\cdot,\delta))$ admits a solution x_δ for any $\delta \in [-\delta^*, \delta^*]$.
(b) The map $\delta \mapsto x_\delta$ is continuous from $[-\delta^*, \delta^*]$ to $C^1_b(\boldsymbol{R})$.
(c) x_δ is regular for any $\delta \in [-\delta^*, \delta^*]$.
(d) x_δ is the unique monotone solution to $\mathcal{P}(a(\cdot,\delta))$.

Proof. (a) Let δ_T and ξ_T be as in Lemma 5.1d, and put $\delta^* = \delta_T$. Let us define $x_\delta(t) = \tilde{x}(t, \xi_T(\delta), \delta)$ for $t \le T$, and extend x_δ to $]T, +\infty[$ as a solution of the equation $\ddot{x} = lV'(x)$. Let us denote by $(\mathcal{E}_\delta)$ the equation we get from (2.5i) when putting $a = a(\cdot,\delta)$. Then surely x_δ solves $(\mathcal{E}_\delta)$. Finally, both points $(x_\delta(0), \dot{x}_\delta(0))$ and $(x_\delta(T), \dot{x}_\delta(T))$ lie on the curve $\Gamma := w(]-1, 1[)$, where $\xi \mapsto w(\xi)$ was defined before: then $x_\delta(\pm\infty) = \pm 1$.

(b) Thanks to the continuity of ξ_T and standard arguments on the dependence on initial data, we can easily check that the map $\delta \mapsto (x_\delta)_{|J}$ is continuous, with respect to the required topology, whenever we put $J =]-\infty, 0]$, $J = [0, T]$, $J = [T, +\infty[$.

(c) We are going to show the existence of functions z^- and z^+ as in Lemma 4.2, where we replace the function a of (2.6i) by $a(\cdot,\delta)$, and x by x_δ. To this end, we denote by $(\mathcal{L}_\delta)$ the corresponding equation, and point out that it admits the solution $z^-(t) := \tilde{x}'_\xi(t, \xi_T(\delta), \delta)$, by virtue of well-known results about dependence on initial data. Furthermore, when deriving with respect to t the equality $\ddot{x}_\delta = lV'(x_\delta)$, we see that $\dot{x}_\delta$ solves $(\mathcal{L}_\delta)$ outside $]0, T[$: then it is easy to check that $z^- = (2lV(\xi))^{-1/2}\dot{x}_\delta$ on $]-\infty, 0]$, and we are also allowed to define $z^+(t)$ as that solution of $(\mathcal{L}_\delta)$ which agrees with $\dot{x}_\delta(t)$ for $t \ge T$. In particular, we get $z^\pm(\pm\infty) = 0$, since $\dot{x}_\delta(\pm\infty) = 0$. Now we only need to show that z^- and z^+ are independent, that is to say: $\nu^-(\sigma) \wedge \nu^+(\sigma) \neq 0$ at some point σ, where $\nu^\pm = (z^\pm, \dot{z}^\pm)$. To this end, we take $\sigma = \sigma_\delta := \sigma(\xi_T(\delta), \delta)$, and remark that, when putting $\xi = \xi_T(\delta)$ in (5.1), the left-hand side just becomes $\nu^-(\sigma_\delta)$, while $\nu^+(\sigma_\delta)$ is nothing but $w'(\rho(\xi_T(\delta)))$: hence the previous determinant is the product of the three numbers δ, $V'(\rho(\xi_T(\delta)))$ and $\sigma'_\xi(\xi_T(\delta), \delta)$: since $\delta \neq 0$, we only need to recall that $V' \neq 0$ on $]x^*, 1[$, and the last claim in Lemma 5.1d.

(d) Let $x \in \mathcal{M}(a(\cdot,\delta))$, $x(0) = \xi$: a simple analysis on the phase plane shows that both conditions $\dot{x}(0) = \sqrt{2lV(\xi)}$ and $\sigma(\xi, \delta) = T$ must be satisfied. Then Lemma 5.1d implies that $\xi = \xi_T(\delta)$, so that $x = x_\delta$. ∎

Proof of Prop. 2.6. From the previous result we argue, in particular, that $x_\delta \in \mathcal{A}^-$ for $-\delta^* \le \delta < 0$ and $x_\delta \in \mathcal{A}^+$ for $0 < \delta \le \delta^*$. ∎

Proof of Prop. 2.11. Let us put, for any $\lambda \in [0, 1]$, $\beta_\lambda = a(\cdot, \lambda\delta^*)$, and call now γ_λ the homotopy of Theorem 2.9: if we join, according to the usual definition, the two paths $\lambda \mapsto \beta_\lambda$ and $\lambda \mapsto \gamma_\lambda$, we get a new homotopy $\lambda \mapsto \alpha_\lambda$ which obviously fulfils the statement of Prop. 2.11. ∎

Remark 5.3 Prop. 5.2a is known to hold not only when γ is the characteristic function of an interval, but, more generally, under the following assumptions: $\gamma \in L^\infty(\boldsymbol{R})$, $\gamma(\pm\infty) = 0$, $\gamma \not\equiv 0$. Indeed, let us recall the functions x^s of Remark 2.7, and define on the real line the Melnikov function $M(s)$ as the derivative of the map $s \mapsto \int_R \gamma V(x^s)dt$: then M must vanish at some point s_0 in such a way that its local Brouwer degree is non-zero, so that standard results in bifurcation theory imply Prop. 5.2a (see also [2] for a variational approach to the problem). Furthermore, if the stronger condition $M'(s_0) \neq 0$ is satisfied, claims (b) and (c) hold as well: on this matter see, for instance, Theorem I.5.1 of [14] (where the roles of λ and s are exchanged).

Remark 5.4 Actually, it is Prop. 5.2d which may fail to hold in the general setting of the previous remark: this was the reason of our choice of γ. In particular, our proof of Prop. 5.2c is confined

to that specific framework, but it is obvious to expect a relation between the two transversality conditions $M'(s_0) \neq 0$ and $\sigma'_\xi(\xi, 0) \neq 0$, since the latter was the key point for getting that claim. Indeed, let $\xi_0 \in]-1, x^*[$, $s_0 \in \boldsymbol{R}$ be such that $\sigma(\xi_0, 0) = T$, $x^0(s_0) = \xi_0$. Then:

$$M'(s_0) = \sqrt{2lV(\xi_0)}(V'(\rho(\xi_0)) - V'(\xi_0)) = -2lV(\xi_0)V'(\rho(\xi_0))\sigma'_\xi(\xi_0, 0).$$

6 Appendix

In this section we are going to prove two results which were exploited in §3: Prop. 3.2 and Lemma 6.3. Since most techniques we employ are well-known, we shortly outline the proves. Let us begin with Prop. 3.2: fix a number c as in its statement and suppose, without loss of generality, $c \geq 0$. We denote by E^i the Sobolev space $H^i(]-\infty, c])$, for $i = 1, 2$, and also put $E^0 = L^2(]-\infty, c])$, while the script E^i_0 ($i = 1, 2$) stands for the subspace of E^i where $z(c) = 0$. We denote respectively by $\|\cdot\|_2$, $\|\cdot\|_{1,2}$ and $\|\cdot\|_{2,2}$ the standard norms on E^0, E^1 and E^2, while the script $z \cdot u$ stands for the usual inner product between $z, u \in E^0$. We also consider the subset A of the normed space $L^\infty(\boldsymbol{R})$ whose functions are positive and bounded away from zero. For any $b \in A$, $\gamma \in L^2(\boldsymbol{R})$, $\tau \leq c$, $h \in \boldsymbol{R}$, we are going to study the linear problem

$$\begin{cases} \ddot{z} = bz + \gamma & (i) \\ z(-\infty) = 0,\ z(\tau) = h. & (ii) \end{cases} \tag{6.1}$$

Lemma 6.1 *Let all conditions above hold. Then:*

(a) Problem (6.1) admits a unique solution $z = L(\gamma; b, \tau, h)$.

(b) The operator L is continuous from $E^0 \times A \times \Gamma^-(c) \times \boldsymbol{R}$ to E^2, and continuously differentiable with respect to (γ, τ, h).

Proof. (a) Let us put $[u, v]_b = \dot{u} \cdot \dot{v} + (bu) \cdot v$ for any $u, v \in E^1_0$. Since the linear functional $u \mapsto \gamma \cdot u$ is bounded on $(E^1_0, [\cdot, \cdot]_b)$, the Riesz Representation Theorem yields a unique $z := P(\gamma; b) \in E^1_0$ such that, for any $u \in E^1_0$, it is $\gamma \cdot u = [z, u]_b$: then z solves (6.1) for $\tau = c$, $h = 0$, and the general solution of (6.1) takes the form $P(\gamma; b) + k\, U(b)$, where k is a suitable constant and $z = U(b)$ solves (6.1) for $\tau = c$, $h = 1$, $\gamma = 0$. $U(b)$ can be got as the unique minimizer of (3.3), where we put $a = b$, $Q(z) = \frac{1}{2}z^2$, $\tau = c$, $h = 1$, while k comes out from the condition $z(\tau) = h$. Then:

$$L(\gamma; b, \tau, h) = P(\gamma; b) + J(\gamma; b, \tau, h)U(b), \tag{6.2}$$

where $J(\gamma; b, \tau, h) = (h - [P(\gamma; b)](\tau))/[U(b)](\tau)$.

(b) Let $0 < m < 1 < M$, $\tilde{A}$ the set of those functions $b \in A$ such that $m \leq b(t) \leq M$ a. e.: since m and M are arbitrary, we may replace A by $\tilde{A}$. By standard arguments, $P(\cdot; b)$ is linear, and bounded in $\mathcal{L}(E^0; E^1)$ by $1/m$. On the other hand, (6.1i) easily yields an evaluation on $\|\ddot{z}\|_2$ as well, so that $P(\cdot; b)$ is bounded also in $\mathcal{L}(E^0; E^2)$, by $C := (1 + m + M)/m$. Furthermore

$$\begin{aligned} &P(\gamma; b) - P(\gamma; a) = P((b - a)P(\gamma; a); b), \\ &U(b) - U(a) = P((b - a)U(a); b), \quad \text{so that} \\ &\|P(\cdot; b) - P(\cdot; a)\| \leq C^2\|b - a\|_\infty, \qquad (6.3) \\ &\|U(b) - U(a)\|_{2,2} \leq C\|U(a)\|_{2,2}\,\|b - a\|_\infty, \quad a, b \in \tilde{A}, \qquad (6.4) \end{aligned}$$

where the first norm in (6.3) comes from $\mathcal{L}(E^0; E^2)$. Now, (6.3) implies that the map $(\gamma, b) \mapsto P(\gamma; b)$ is continuous, and continuously differentiable with respect to γ. On the other hand, thanks to (6.4), U is continuous, and we can easily infer also on the functional J of (6.2) all the required properties, so as to prove our claim. ∎

Lemma 6.2 *Let $\lambda \mapsto \alpha_\lambda$, Q and u be as in Prop. 3.2. For any $\lambda \in [0,1]$ let $N_\lambda : E^1 \to E^0$ be defined by $N_\lambda(z) = \alpha_\lambda(Q'(z) - Q''(0)z)$. For any $z \in E^1$, $\tau, h \in \boldsymbol{R}$ let us also put $G(z; \tau, h, \lambda) = L(N_\lambda(z); b_\lambda, \tau, h) - z$, where $b_\lambda = \alpha_\lambda Q''(0)$. Then:*

(a) $[N'_\lambda(z)]w = \alpha_\lambda(Q''(z) - Q''(0))w$ for any $\lambda \in [0,1]$, $z, w \in E^1$.

(b) The map G is continuous from $E^1 \times \Gamma^-(c) \times \boldsymbol{R} \times [0,1]$ to E^2, and continuously differentiable with respect to $(z; \tau, h)$

(c) G'_z is everywhere invertible as a linear operator from E^1 to E^2.

(d) $G(u(\cdot; \tau, h, \lambda); \tau, h, \lambda) = 0$.

Proof. (a) Let $z, w \in E^1$, $R = \|z\|_\infty$, $K = \sup_\lambda \|\alpha_\lambda\|_\infty$, and denote by Γw the right-hand side of the relation to be proved. For any $r > 0$, let $\omega(r)$ be the supremum of $|Q''(\xi) - Q''(p)|$ for $|p| \le R$, $|\xi - p| \le r$. Then $\omega(r) \to 0$ as $r \to 0$, and the following estimate shows our claim:

$$\|N_\lambda(z + w) - N_\lambda(z) - \Gamma w\|_2 \le K\omega(\|w\|_\infty)\|w\|_2 \le K\omega(\|w\|_{1,2})\|w\|_{1,2}.$$

(b) From (a) and the continuity of Q'' we argue easily that the map $z \mapsto N'_\lambda(z)$ is continuous from L^∞ to $\mathcal{L}(E^1; E^0)$. Then, actually, $N_\lambda \in C^1(E^1; E^0)$, since $\|\cdot\|_\infty \le \|\cdot\|_{1,2}$. On the other hand, N_λ is obviously lipschitzian from E^0 to E^0, while, for any $\rho > 0$, the maps $\lambda \mapsto N_\lambda(z)$ and $\lambda \mapsto N'_\lambda(z)$, by virtue of (2.7), are continuous, uniformly with respect to the condition $\|z\|_{1,2} \le \rho$. Then, again by virtue of (2.7) and Lemma 6.1, both maps $(\lambda, z) \mapsto N_\lambda(z)$ and $(\gamma; \tau, h, \lambda) \mapsto L(\gamma; b_\lambda, \tau, h)$ are continuous with respect to the prescribed norms, and also continuously differentiable with respect to z (the former) and (z, τ, h) (the latter). Now our claim follows easily.

(c) Let $z \in E^1$, $\phi \in E^2$ and look for $w \in E^1$ such that $G'_z(z; \tau, h, \lambda)w = \phi$: if we put $v = \phi + w$ we get $v = L(-N'_\lambda(z)\phi; \alpha_\lambda Q''(z), \tau, h)$, where the use of the operator L is justified by the inequality $\inf Q'' > 0$, so as to yield a unique solution v. Since it must be $w = v - \phi$, G'_z is bijective.

(d) $G(z; \tau, h, \lambda) = 0 \Leftrightarrow \ddot{z} = \alpha_\lambda Q'(z)$, $z(-\infty) = 0$, $z(\tau) = h \Leftrightarrow z = u(\cdot; \tau, h, \lambda)$. ∎

Proof of Proposition 3.2. (a) Thanks to Lemma 6.2 and the Implicit Function Theorem [3], the map $(s, h, \lambda) \mapsto u(\cdot; s, h, \lambda)$ enjoys, with respect to the norm $\|\cdot\|_{2,2}$, the same properties we stated in Prop. 3.2 with respect to the weaker C^1-norm.

(b) Let us fix $\tau \in \boldsymbol{R}$, $\lambda \in [0,1]$ and put, for any $h \in \boldsymbol{R}$, $\varepsilon(h) = u(\cdot; \tau, h, \lambda)$. We point out that $\varepsilon(0) = 0 = N_\lambda(0)$, and that $[U(b_\lambda)](t) = kp_\lambda(t)$, where $k = [U(b_\lambda)](0)$: then from the chain rule we get

$$0 = G'_z(0; \tau, 0, \lambda)\varepsilon'(0) + G'_h(0; \tau, 0, \lambda) = -\varepsilon'(0) + J'_h(0; \tau, 0, \lambda)U(b_\lambda) \Rightarrow$$
$$\Rightarrow [\varepsilon'(0)](t) = \frac{p_\lambda(t)}{p_\lambda(\tau)} \Rightarrow \frac{1}{h}\left[\frac{\partial u}{\partial t}(t; \tau, h, \lambda) - \frac{\partial u}{\partial t}(t; \tau, 0, \lambda)\right] \to \frac{\dot{p}_\lambda(t)}{p_\lambda(\tau)}$$

as $h \to 0$, uniformly with respect to $t \le c$. Indeed, the last implication holds because the derivative $\varepsilon'(0)$ is performed with respect to $\|\cdot\|_{2,2}$, which is stronger than the C^1-norm. From the previous relation we argue that $u''_{th}(t; \tau, 0, \lambda) = \dot{p}_\lambda(t)/p_\lambda(\tau)$: in particular, for $t = \tau$, we get our claim.

(c) Let τ, h, λ be fixed, and denote respectively by $v^-(t)$ and $v^+(t)$ the left-hand and right-hand side of the given inequalities. Then v^- and v^+ turn out to be, respectively, a lower and an upper solution for (3.4i), where we put $a = \alpha_\lambda$: according to a well-known theory (see, for instance, [7]), for any $n \in \mathbf{Z}^+$, $n \geq -\tau$, (3.4) must admit on $[-n, \tau]$ a solution v_n such that $v^- \leq v_n \leq v^+$. On the other hand, since $v^-(\tau) = v^+(\tau) = h$, it must also be $v_n(\tau) = h$, and $\dot{v}^-(\tau) \leq \dot{v}_n(\tau) \leq \dot{v}^+(\tau)$, so that the sequence $(\dot{v}_n(\tau))_n$ admits a limit point η: now, let us put again $a = \alpha_\lambda$ in (3.4i), and denote by v its solution which fulfils the conditions $v(\tau) = h$, $\dot{v}(\tau) = \eta$: it is easy to understand that $v(-\infty) = 0$, so that $v(t) = u(t; \tau, h, \lambda)$. Now it is enough to point out that $v^- \leq v \leq v^+$ on $]-\infty, \tau]$.

(d) Let us put in (3.3) $a = \alpha_\lambda$, $Q_0(u) := \frac{1}{2}Q''(0)u^2$ in place of Q, and denote by (Γ_λ) the corresponding Euler equation, that is $\ddot{u} = Q''(0)\alpha_\lambda u$: since $u = p_\lambda$ solves (Γ_λ), and $w_\lambda = \dot{p}_\lambda / p_\lambda$, our claim follows easily.

(e) First of all, we point out that the function w_λ can be expressed as the ratio between $(u_0)'_t(\cdot; \tau, h, \lambda)$ and $u_0(\cdot, \tau, h, \lambda)$ not only for $\tau = 0$, $h = 1$ (as by definition), but for *any* $\tau \in \boldsymbol{R}$, $h > 0$: indeed, since the equation (Γ_λ) of the previous step is linear, it is easy to understand that $u_0(\cdot; \tau, h, \lambda)$ is a scalar multiple of $u_0(\cdot; 0, 1, \lambda)$. Now, for the sake of convenience, let us denote m by m^- and M by m^+. Let us put again in (3.3) Q_0 in place of Q and, respectively, $a = \alpha_\lambda$, $a \equiv m^\pm$. Let us suppose that τ is fixed, for now, and call $u \mapsto \Phi_\lambda(u)$, $u \mapsto \Phi^\pm(u)$ the corresponding functionals and $\mu_\lambda(h)$, $\mu^\pm(h)$ the respective minima on $W^-(\tau, h)$. Let us also put, according to the notations of Prop. 3.2, $u_\lambda(t) := u_0(t; \tau, 1, \lambda)$, and denote by $u^\pm$ the function we get in the same way when replacing α_λ by $m^\pm$. Then:

$$\begin{aligned} \mu^-(1) &= \Phi^-(u^-) \leq \Phi^-(u_\lambda) \leq \Phi_\lambda(u_\lambda) = \\ &= \mu_\lambda(1) \leq \Phi_\lambda(u^+) \leq \Phi^+(u^+) = \mu^+(1). \end{aligned} \tag{6.5}$$

Now, an integration by parts in the first term of $\Phi_\lambda(u_\lambda)$, together with the corresponding Euler equation, yields $\mu_\lambda(1) = \frac{1}{2}u_\lambda(\tau)\dot{u}_\lambda(\tau)$, while the arguments at the beginning of this step imply that $w_\lambda(\tau) = \dot{u}_\lambda(\tau)/u_\lambda(\tau)$. On the other hand, by definition, $u_\lambda(\tau) = 1$, so that $\mu_\lambda(1) = \frac{1}{2}w_\lambda(\tau)$. Finally, explicit computations show that $\mu^\pm(1)$ is given by $\frac{1}{2}\sqrt{m^\pm Q''(0)}$, so that (6.5) contains the two inequalities to be proved, for $t = \tau$: since τ was arbitrary, the first part of our claim holds. As regards the last one, it is enough to point out that, in (6.5), the first or the last inequality turns out to be strict according to whether, respectively, $\alpha_\lambda > m$ or $\alpha_\lambda < M$ on a set of positive measure. ■

Lemma 6.3 *Let $\tau \in \boldsymbol{R}$, $J = [\tau, +\infty[$, $f : J \times \boldsymbol{R} \to \boldsymbol{R}$ be locally bounded, $f(t, w)$ measurable with respect to t and continuous with respect to w, $\psi \in C^1(\boldsymbol{R})$, $\theta \in \boldsymbol{R}$ be such that $\psi(\theta) = 0$ and $\psi'(\theta) < 0$. Let $w \in AC(J)$ a solution of $\dot{w} = f(t, w)$ such that $w(+\infty) = \theta$ and, for some $\beta < -\psi'(\theta)$, $|\delta(t)|e^{\beta t} \to +\infty$ as $t \to +\infty$, where $\delta(t) := f(t, w(t)) - \psi(w(t))$. Then:*

(a) $w \leq \theta$ on $J \Rightarrow \delta(\sigma) < 0$ for some $\sigma \geq \tau$.

(b) $w \geq \theta$ on $J \Rightarrow \delta(\sigma) > 0$ for some $\sigma \geq \tau$.

Proof. (a) Suppose, by contradiction, $w \leq \theta$ and $\delta \geq 0$ on J. Take $\gamma \in]\beta, -\psi'(\theta)[$, $\zeta < \theta$ and $\sigma_0 \geq \tau$ such that $-\psi'(w) > \gamma$ for $\zeta \leq w \leq \theta$ and $w(t) \geq \zeta$ for $t \geq \sigma_0$. Then:

$$\dot{w}(t) = f(t, w(t) \geq \psi(w(t)) \geq \gamma(\theta - w(t)), \quad t \geq \sigma_0. \tag{6.6}$$

Indeed, the first inequality holds because $\delta(t) \geq 0$, while the second one follows from the mean value theorem, since $\psi(\theta) = 0$ and $\zeta \leq w(t) \leq \theta$. In particular, from (6.6) and the inequality $\beta < \gamma$, we

argue that the function $\phi(t) := (\theta - w(t))e^{\beta t}$ has a non-positive derivative for $t \geq \sigma_0$: since $\phi(t) \geq 0$, we get $\phi(+\infty) = k$ for some finite value $k \geq 0$, so that $\psi(w(t))e^{\beta t}$ should converge to $-k\psi'(\theta)$. On the other hand, $\phi(t)$ can be seen as the ratio between $\theta - w(t)$ and $e^{-\beta t}$, so that, by virtue of de l'Hopital's rule, $\dot{w}(t)e^{\beta t} \to k\beta$. But $\delta(t) = \dot{w}(t) - \psi(w(t))$, so that $\delta(t)e^{\beta t}$ should converge, as $t \to +\infty$, to the finite limit $k(\beta + \psi'(\theta))$, a contradiction.

(b) It is enough to apply the previous proof to $2\theta - w$, $-f(t, 2\theta - w)$, $-\psi(2\theta - w)$ in place of w, $f(t, w)$, $\psi(w)$. ∎

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