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Least Energy Solutions and Group Invariant Solutions of the Hénon Equation

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Abstract

In this paper we study the generalized Hénon equation in the unit ball, where the coefficient function may or may not change its sign. We prove that the least energy solution is not radial and moreover we show the existence of a group invariant positive solution without radial symmetry.

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1 Introduction

In this paper, we prove the existence of nonradial positive solutions and group invariant positive solutions for the generalized Hénon equation

$$-\Delta u = h(|x|)u^p, \quad u > 0 \quad \text{in } B, \tag{1.1}$$

$$u = 0, \quad \text{on } \partial B, \tag{1.2}$$

where B is a unit ball of $\mathbb{R}^N$ with $N \geq 2$, $1 < p < \infty$ if $N = 2$ and $1 < p < (N+2)/(N-2)$ if $N \geq 3$, $h \in L^\infty(B)$, $h(|x|) \not\equiv 0$, $h(|x|)$ is radially symmetric and may or may not change its sign. If $h(|x|) \leq 0$ for all x, then no positive solution exists because of the maximum principle. Hence we always assume that $h_+(r) \not\equiv 0$, where $h_+(r) := \max(h(r), 0)$. Most typical examples of $h(|x|)$ are $h(|x|) = |x|^\lambda$ (the Hénon equation), $h(|x|) = e^{\lambda|x|}$, $(|x|/(1+|x|))^\lambda$ with $\lambda > 0$ large enough, $h(|x|) = (|x| - a)(1 - a)$ (sign-changing weight) and $h(|x|) = -\alpha \leq 0$ for $|x| < a$ and $h(|x|) = 1$ for $a < |x| < 1$, etc. Since h

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is assumed to be radially symmetric, there exists a positive radial solution, which can be proved by the mountain pass lemma in the space of radial functions. However we look for a nonradial positive solution. It will be obtained as a least energy solution. To define it, we need the *Rayleigh quotient*

$$R(u) := \left(\int_B |\nabla u|^2 dx\right) \Big/ \left(\int_B h(|x|)|u|^{p+1} dx\right)^{2/(p+1)},$$

with the definition domain

$$D(R) := \{u \in H_0^1(B) : \int_B h(|x|)|u|^{p+1} dx > 0\}.$$

Here $H_0^1(B)$ is a usual Sobolev space. Since $h \in L^\infty(B)$ with $h_+(r) \not\equiv 0$, the definition domain $D(R)$ is not empty. For the proof, we refer the readers to [15, Lemma 3.1]. We define the *least energy* L by

$$L := \inf\{R(u) : u \in D(R)\}.$$

Because of the Sobolev imbedding theorem, L is well defined and positive. Next, we define the *Nehari manifold*

$$\mathcal{N} := \{u \in H_0^1(B) \setminus \{0\} : \int_B (|\nabla u|^2 - h(|x|)|u|^{p+1}) dx = 0\}.$$

It is obvious that $\mathcal{N} \subset D(R)$. Observe that for any $u \in D(R)$, there is a $\lambda > 0$ such that $\lambda u \in \mathcal{N}$. Furthermore, $R(\lambda u) = R(u)$ for any $\lambda > 0$. Therefore we have

$$L = \inf\{R(u) : u \in D(R)\} = \inf\{R(u) : u \in \mathcal{N}\}.$$

Then the infimum is achieved at a certain point $u \in \mathcal{N}$. Moreover, u satisfies (1.1), (1.2) and it is positive or negative in Ω. For the proof of the result above, we refer the readers to [14] or [15]. Since $h \in L^\infty(B)$, u belongs to $W^{2,q}(\Omega)$ for all $q < \infty$ because of the elliptic regularity theorem. We call u a *least energy solution* if $u \in \mathcal{N}$ and $R(u) = L$. As stated above, a least energy solution is positive or negative. Throughout the paper, a least energy solution means a positive one because we can replace u by $-u$, if necessary.

When h is nonnegative, there are some results on the existence of nonradial positive solutions. When $h(|x|) = |x|^\lambda$ is a power coefficient, (1.1), (1.2) becomes the original Hénon equation

$$-\Delta u = |x|^\lambda u^p, \quad u > 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B, \tag{1.3}$$

which was introduced by Hénon [10] to study spherically symmetric clusters of stars. Smets, Willem and Su [20] have proved that if λ is large enough, then a least energy solution of (1.3) is nonradial. There are many contributions which have studied the Hénon equation ([2, 3, 4, 5, 6, 7, 8, 9, 11, 18, 19]).

On the other hand, Moore and Nehari [16, pp.32–33] have studied the two point boundary value problem

$$u''(t) + h(t)u^p = 0, \quad u > 0 \quad \text{in } (-1, 1), \tag{1.4}$$

$$u(-1) = u(1) = 0. \tag{1.5}$$

Here $h(t) = 0$ for $|t| < a$ and $h(t) = 1$ for $a < |t| < 1$. They have constructed at least three positive solutions of (1.4), (1.5) with a suitable $a(< 1)$ sufficiently close to 1: the first one is even, the second one $u(t)$ is non-even and the third one is the reflection $u(-t)$. Tanaka [21, 22] has extended the results to the p-Laplace equation and to radial solutions of a semilinear elliptic equation. Moore and Nehari's non-even solution can be obtained as a least energy solution, which has been proved in our paper [13].

In the present paper, we study the sign-changing weight $h(|x|)$ and prove the same result as above. Moreover, we prove the existence of nonradial $O(n) \times O(N-n)$ invariant positive solutions for $1 \le n \le N-1$. Here

$$O(n) \times O(N-n) := \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} : g \in O(n),\ h \in O(N-n) \right\},$$

and $O(n)$ denotes the orthogonal group of degree n. We call u an $O(n) \times O(N-n)$ *invariant solution* if it satisfies (1.1), (1.2) and $u(gx) = u(x)$ for all $g \in O(n) \times O(N-n)$. Such a solution exists. Indeed, a radial positive solution is clearly $O(n) \times O(N-n)$ invariant. We look for a nonradial $O(n) \times O(N-n)$ invariant solution. Such a solution has already been obtained by Badiale and Serra [2] for the Hénon equation (1.3) when $N/2 \le n \le N-2$, $1 < p < (N+1)/(N-3)$ and $N \ge 4$. We extend this result to more general weights $h(x)$ and cover the case where $n = N-1$ and $N \ge 2$, but p is in the subcritical range $(1, (N+2)/(N-2))$. Thus our result is valid when $1 \le n \le N-1$ and $1 < p < \infty$ with $N = 2$, $1 < p < (N+2)/(N-2)$ with $N \ge 3$. We shall show that if $h(x) \le 0$ in $|x| \le a$ with a sufficiently close to 1, then there exists a nonradial $O(n) \times O(N-n)$ invariant positive solution. It seems to the author that little is known about the sign-changing weight, except for our recent result [15]. However [15] deals with the one dimensional equation only. In the present paper, we shall study sign-changing weights as well as positive weights in the higher dimensional case.

We denote the set of radial functions in $H_0^1(B)$ by $H_{0,r}^1(B)$, i.e.,

$$H_{0,r}^1(B) := \{u \in H_0^1(B) : \ u = u(|x|)\}.$$

We define

$$D_r(R) := D(R) \cap H_{0,r}^1(B), \quad \mathcal{N}_r := \mathcal{N} \cap H_{0,r}^1(B),$$
$$L_r := \inf\{R(u) : \ u \in D_r(R)\} = \inf\{R(u) : \ u \in \mathcal{N}_r\}.$$

We call L_r a *radial least energy* and u a *radial least energy solution* if $u \in \mathcal{N}_r$ and $R(u) = L_r$. There exists a radial least energy solution, which can be proved in the standard argument (we refer the readers to [14]). To avoid confusion, a usual least energy solution is called a *global least energy solution*.

Our strategy is as follows. For a radial least energy solution u, we construct a function v deformed slightly from u such that v is $O(n) \times O(N-n)$ invariant and the energy $R(v)$ is less than that of u. A minimizer of R in the set of $O(n) \times O(N-n)$ invariant functions in $H_0^1(B)$ becomes an $O(n) \times O(N-n)$ invariant solution because of the principle of symmetric criticality (see Palais [17]). Moreover, it is not radial because its energy level is less than the radial least energy.

Our paper is organized in five sections. In Section 2, we state the main results and give some examples. In Section 3, we construct an $O(n) \times O(N-n)$ invariant function whose energy is less than the radial least energy. In Section 4, we investigate the properties of radial least energy solutions and give some estimates on them. In Section 5, we prove the main theorems.

2 Main results

In this section, we state the main results and give examples of h. First, we define constants ν, d and $\mu(h, a)$ as bellow. Since $p < (N+2)/(N-2)$, we define $\nu \in (0, 1)$ which satisfies

$$p < \frac{N+2-2\nu}{N-2} < \frac{N+2}{N-2}. \tag{2.1}$$

If $N = 2$, we put $\nu := 1$. Define

$$d := \frac{p+1}{N+2-(N-2)p-2\nu} > 0. \tag{2.2}$$

Moreover, we define

$$\mu(h,a) := (1-a)^{(p+1)/2}\left(\int_0^a h(r)r^{-1+\nu}dr\right)\left(\int_a^1 h(r)(1-r)^{p+1}r^{N-1}dr\right)^{-1}. \tag{2.3}$$

Write $h_+(r) := \max(h(r), 0)$. We introduce two types of assumptions on h depending on a parameter $a \in (0,1)$: one is a positive case and another is a sign-changing case.

$(A)_a$ $h(r) \geq 0$ in $(0,1)$, $h_+(r) \not\equiv 0$ in $(a,1)$ and $\mu(h,a) \leq (2d)^{-(p+1)/2}$.

$(B)_a$ $h(r) \leq 0$ in $(0,a)$ and $h_+(r) \not\equiv 0$ in $(a,1)$.

Our main result is as follows.

Theorem 2.1 *Let $N \geq 2$ and $1 \leq n \leq N-1$. Then there exists an $\varepsilon \in (0,1)$ depending only on N, n and p such that if $h \in L^\infty(B)$ satisfies $(A)_a$ or $(B)_a$ with a certain $a \in (1-\varepsilon, 1)$, then a least energy solution is not radial and there exists a nonradial $O(n) \times O(N-n)$ invariant positive solution.*

Remark 2.1 For solutions u and v of (1.1), (1.2), we say that u is equivalent to v if $u(gx) = v(x)$ with some $g \in O(N)$. Hence an $O(n) \times O(N-n)$ invariant solution is equivalent to an $O(N-n) \times O(n)$ invariant solution. In our paper [12, Theorem 9.5], it has been proved that any nonradial $O(n) \times O(N-n)$ invariant solution is not equivalent to a nonradial $O(m) \times O(N-m)$ invariant solution if $1 \leq n < m \leq N/2$. Therefore (1.1), (1.2) has nonequivalent positive solutions: a positive radial solution and nonradial $O(n) \times O(N-n)$ invariant positive solutions with $1 \leq n \leq N/2$.

Corollary 2.1 *Let $g(r)$ be a continuous function on $[0,1]$ such that $0 \leq g(r) < g(1)$ for $r < 1$. Put $h(|x|) := g(|x|)^\lambda$. If $\lambda > 0$ is large enough, then a least energy solution is not radial and there exists a nonradial $O(n) \times O(N-n)$ invariant positive solution for each $1 \leq n \leq N-1$.*

In $(A)_a$, the condition $\mu(h,a) \leq (2d)^{-(p+1)/2}$ is fulfilled if $h(|x|)$ in $(0,a)$ is sufficiently smaller than that in $(a,1)$. If $h(|x|)$ is negative in $(0,a)$, then $(B)_a$ is satisfied. Under such a condition, if a is sufficiently close to 1, then a least energy solution is not radial and moreover, a nonradial $O(n) \times O(N-n)$ invariant positive solution exists. We give some examples of h.

Example 2.1 The following h satisfies the assumption of Theorem 2.1 or Corollary 2.1.

(i) Let $h(r) = \alpha$ in $(0,a)$ and $h(r) = 1$ in $(a,1)$ with a close to 1. If $\alpha > 0$ is small enough, h satisfies $(A)_a$. If $\alpha \leq 0$, h satisfies $(B)_a$.

(ii) Let $h(|x|) = |x|^\lambda, e^{\lambda|x|}, (|x|/(1+|x|))^\lambda$, $(\sin(\pi|x|/2))^\lambda$, etc. Then h satisfies the assumption of Corollary 2.1.

(iii) Let $h(|x|) = (|x|-a)/(1-a)$ with a sufficiently close to 1. Then h satisfies $(B)_a$.

3 Construction of a lower energy function

In this section, we shall construct an $O(n) \times O(N-n)$ invariant function whose energy is less than the radial least energy. We use the similar idea to our paper [1]. First, we deal with the case where $N \geq 3$. We introduce the N dimensional polar coordinate:

$$\begin{aligned} x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1\cos\theta_2, \\ &\vdots \\ x_N &= r\sin\theta_1\cdots\sin\theta_{N-2}\sin\theta_{N-1}, \end{aligned}$$

for $r \in (0,1)$, $\theta_i \in [0,\pi]$ for $1 \leq i \leq N-2$ and $\theta_{N-1} \in [0,2\pi]$. The Jacobian of the transformation is computed as

$$\frac{\partial(x_1, x_2, \ldots, x_N)}{\partial(r, \theta_1, \ldots, \theta_{N-1})} = r^{N-1}\mathrm{Jac}(\theta) \quad \text{with } \mathrm{Jac}(\theta) = \prod_{i=1}^{N-2} \sin^{N-1-i}\theta_i.$$

Definition 3.1 *Let $N \geq 3$ and $1 \leq n \leq N-1$. Let $u(r)$ be a positive radial solution of* (1.1)*,* (1.2)*. Using the polar coordinate, we define for $\varepsilon > 0$,*

$$v(x) := \phi(\theta)u(r), \quad \phi(\theta) := 1 + \varepsilon S(\theta), \tag{3.1}$$

$$S(\theta) := \prod_{i=1}^{n} \sin\theta_i - S_n \quad \text{for } \theta = (\theta_1, \ldots, \theta_{N-1}),$$

where S_n is defined by

$$S_n := \frac{\Gamma(N/2)\Gamma((N-n+1)/2)}{\Gamma((N+1)/2)\Gamma((N-n)/2)} \quad \text{for } n \leq N-2, \tag{3.2}$$

and $S_{N-1} := 0$. Here $\Gamma(\cdot)$ denotes the Gamma function.

We write

$$\theta = (\theta_1, \ldots, \theta_{N-1}), \quad d\theta = d\theta_1 \cdots d\theta_{N-1}.$$

We define Θ by the set of $\theta = (\theta_1, \ldots, \theta_{N-1})$ such that $\theta_i \in [0,\pi]$ for $1 \leq i \leq N-2$ and $\theta_{N-1} \in [0, 2\pi]$. The constant S_n is so chosen that the integral of $S(\theta)\mathrm{Jac}(\theta)$ vanishes. Indeed, we have the next lemma.

Lemma 3.1

$$\int_\Theta S(\theta)\mathrm{Jac}(\theta)d\theta = 0. \tag{3.3}$$

Proof. To prove the lemma, we use the formula

$$\int_0^\pi \sin^n t\, dt = \frac{\sqrt{\pi}\,\Gamma((n+1)/2)}{\Gamma((n+2)/2)}. \tag{3.4}$$

First, we deal with the case $n \leq N-2$. Hereafter, if $n > m$, we mean $\sum_{i=n}^m a_i = 0$ and $\prod_{i=n}^m a_i = 1$ for any sequence $\{a_i\}$. Note that the intervals of integrations in (3.3) are $[0,\pi]$ for θ_i with $i \leq N-2$ but

$[0, 2\pi]$ for θ_{N-1}. From a direct computation, it follows that

$$\begin{aligned}\int_\Theta \left(\prod_{i=1}^n \sin\theta_i\right) \mathrm{Jac}(\theta) d\theta &= 2\pi \prod_{i=1}^n \int_0^\pi \sin^{N-i}\theta_i d\theta_i \prod_{i=n+1}^{N-2} \int_0^\pi \sin^{N-1-i}\theta_i d\theta_i \\ &= 2\pi \prod_{i=1}^n \frac{\sqrt{\pi}\Gamma((N-i+1)/2)}{\Gamma((N-i+2)/2)} \prod_{i=n+1}^{N-2} \frac{\sqrt{\pi}\Gamma((N-i)/2)}{\Gamma((N-i+1)/2)} \\ &= \frac{2\pi^{N/2}\Gamma((N-n+1)/2)}{\Gamma((N+1)/2)\Gamma((N-n)/2)}.\end{aligned}$$

From similar computation, it follows that

$$\int_\Theta \mathrm{Jac}(\theta) d\theta = \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$

Therefore (3.3) follows. Let $n = N - 1$. Then we see

$$\int_\Theta \left(\prod_{i=1}^{N-1} \sin\theta_i\right) \mathrm{Jac}(\theta) d\theta = \prod_{i=1}^{N-2} \int_0^\pi \sin^{N-i}\theta_i d\theta_i \int_0^{2\pi} \sin\theta_{N-1} d\theta_{N-1} = 0.$$

Since $S_{N-1} = 0$, (3.3) holds. The proof is complete. □

Lemma 3.2 *Let $v(x)$ be as in* (3.1). *Then it is $O(n) \times O(N-n)$ invariant and belongs to $D(R)$ when $\varepsilon > 0$ is small enough.*

Proof. In the polar coordinate, the point $(r, 0, \theta_2, \ldots, \theta_{N-1})$ with $\theta_1 = 0$ for any $\theta_2, \ldots, \theta_{N-1}$ corresponds to the common point $(x_1 \ldots, x_N) = (r, 0, \ldots, 0)$. Therefore $v(r, 0, \theta_2, \ldots, \theta_{N-1})$ should be independent of $\theta_2, \ldots, \theta_{N-1}$. This fact holds for $\theta_1 = \pi$ also. In the same reason, $v(r, \theta_1, \ldots, \theta_{N-1})$ with $\theta_i = 0, \pi$ should be independent of $\theta_{i+1}, \ldots, \theta_{N-1}$. Moreover, v should be 2π periodic in θ_{N-1}. Our definition of $v(x)$ obeys these rules and so it is well defined.

Since

$$\sqrt{x_{n+1}^2 + \cdots + x_N^2} = r \sin\theta_1 \cdots \sin\theta_n,$$

we have

$$S(\theta) = \prod_{i=1}^n \sin\theta_i - S_n = \frac{1}{r}|(x_{n+1}, \ldots, x_N)| - S_n.$$

Therefore $v(x)$ depends only on $|(x_1, \ldots, x_n)|$ and $|(x_{n+1}, \ldots, x_N)|$, i.e., it is $O(n) \times O(N-n)$ invariant. Note that any nontrivial solution of (1.1), (1.2) belongs to $\mathcal{N}$. Indeed, multiplying (1.1) by u and integrating it over B, we get

$$0 < \int_B |\nabla u|^2 dx = \int_B h|u|^{p+1} dx. \tag{3.5}$$

Thus u belongs to $\mathcal{N}$, and to $D(R)$. Then v also belongs to $D(R)$ for $\varepsilon > 0$ small enough, because as $\varepsilon \to 0$,

$$\int_B h(|x|)|v|^{p+1} dx \to \int_B h(|x|)|u|^{p+1} dx > 0.$$

□

We consider the inequality

$$\int_0^1 u(r)^2 r^{N-3} dr < (p-1)(N-n-NS_n^2)n^{-1} \int_0^1 u'(r)^2 r^{N-1} dr, \tag{3.6}$$

where $u'(r) = du/dr$. In the proof of the next proposition, it will be shown that the coefficient $(p-1)(N-n-NS_n^2)n^{-1}$ is positive. We shall show that if a radial solution u satisfies (3.6), then the energy of v is lower than that of u. Therefore u is not a local minimizer of R because $v \to u$ in $H_0^1(B)$ as $\varepsilon \to 0$.

Proposition 3.1 *Let $N \geq 3$, $1 \leq n \leq N-1$ and u be a radial solution satisfying* (3.6). *Define v by* (3.1). *Then $R(v) < R(u)$ for $\varepsilon > 0$ small enough. Therefore u is not a local minimizer of R, and hence it is not a global least energy solution.*

Proof. By (3.6), we choose a constant $P > 0$ slightly less than $(p-1)(N-n-NS_n^2)n^{-1}$ such that

$$\int_0^1 u(r)^2 r^{N-3} dr < P \int_0^1 u'(r)^2 r^{N-1} dr. \tag{3.7}$$

Since $nP < (p-1)(N-n-NS_n^2)$, we choose a constant α slightly less than 1 such that

$$0 < \alpha < 1, \quad nP < (p\alpha - 1)(N-n-NS_n^2). \tag{3.8}$$

Under the N-dimensional polar coordinate, $|\nabla v|^2$ is represented as

$$\begin{aligned} |\nabla v|^2 &= \left|\frac{\partial v}{\partial r}\right|^2 + \sum_{j=1}^{N-1} \frac{1}{r^2 \prod_{i=1}^{j-1} \sin^2\theta_i} \left|\frac{\partial v}{\partial \theta_j}\right|^2 \\ &= \phi(\theta)^2 u'(r)^2 + \varepsilon^2 \sum_{j=1}^{n} \cos^2\theta_j (\prod_{i=j+1}^{n} \sin^2\theta_i) \frac{u(r)^2}{r^2}. \end{aligned}$$

Therefore we have

$$\int_B |\nabla v|^2 dx = \int_0^1 u'(r)^2 r^{N-1} dr \int_\Theta \phi(\theta)^2 \mathrm{Jac}(\theta) d\theta + \varepsilon^2 \mu_n \int_0^1 u(r)^2 r^{N-3} dr, \tag{3.9}$$

where μ_n is defined by

$$\mu_n := \sum_{j=1}^{n} \int_\Theta \cos^2\theta_j \left(\prod_{i=j+1}^{n} \sin^2\theta_i\right) \mathrm{Jac}(\theta) d\theta.$$

Let us compute μ_n. We first deal with the case where $n \leq N-2$. Rewrite the integrand as

$$\begin{aligned} &\cos^2\theta_j \left(\prod_{i=j+1}^{n} \sin^2\theta_i\right) \mathrm{Jac}(\theta) \\ &= \left(\prod_{i=1}^{j-1} \sin^{N-1-i}\theta_i\right) \left(\sin^{N-1-j}\theta_j \cos^2\theta_j\right) \\ &\quad \times \left(\prod_{i=j+1}^{n} \sin^{N+1-i}\theta_i\right) \left(\prod_{i=n+1}^{N-2} \sin^{N-1-i}\theta_i\right). \end{aligned}$$

We use (3.4) and the formula

$$\int_0^\pi \sin^n t \cos^2 t\, dt = \frac{\sqrt{\pi}\,\Gamma((n+1)/2)}{(n+2)\Gamma((n+2)/2)}.$$

Noting that $0 \le \theta_i \le \pi$ for $i \le N-2$ and $0 \le \theta_{N-1} \le 2\pi$, we have

$$\mu_n = \sum_{j=1}^n \frac{2\pi^{N/2}\Gamma((N-j)/2)\Gamma((N-n+2)/2)}{(N-j+1)\Gamma(N/2)\Gamma((N-j+2)/2)\Gamma((N-n)/2)}.$$

We use the relation $\Gamma(n+1) = n\Gamma(n)$ to get

$$\begin{aligned} \mu_n &= \sum_{j=1}^n \frac{2\pi^{N/2}(N-n)}{(N-j+1)(N-j)\Gamma(N/2)} \\ &= \frac{2\pi^{N/2}(N-n)}{\Gamma(N/2)} \sum_{j=1}^n \left(\frac{1}{N-j} - \frac{1}{N-j+1} \right) \\ &= \frac{2n\pi^{N/2}}{N\Gamma(N/2)}. \end{aligned}$$

The equation above holds for $n = N-1$ also from the same computation.

Since u is radial, we reduce (3.5) to

$$\int_0^1 u'(r)^2 r^{N-1} dr = \int_0^1 h(r)|u|^{p+1} r^{N-1} dr = \frac{1}{\omega_N} R(u)^{(p+1)/(p-1)}. \tag{3.10}$$

Here ω_N denotes the surface area of the unit sphere of $\mathbb{R}^N$, i.e.,

$$\omega_N := \int_\Theta \mathrm{Jac}(\theta) d\theta = \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{3.11}$$

Then μ_n is rewritten as

$$\mu_n = \frac{2n\pi^{N/2}}{N\Gamma(N/2)} = \frac{n}{N}\omega_N. \tag{3.12}$$

Let us compute the integral of $\phi(\theta)^2\mathrm{Jac}(\theta)$ in (3.9). We expand $\phi(\theta)^2$ to get

$$\int_\Theta \phi(\theta)^2 \mathrm{Jac}(\theta) d\theta = \int_\Theta \mathrm{Jac}(\theta) d\theta + 2\varepsilon \int_\Theta S(\theta)\mathrm{Jac}(\theta) d\theta + \varepsilon^2 \int_\Theta S(\theta)^2 \mathrm{Jac}(\theta) d\theta. \tag{3.13}$$

We shall compute the last integral. When $n \le N-2$, we use (3.4) to get

$$\begin{aligned} \int_\Theta \left(\prod_{i=1}^n \sin^2 \theta_i \right) \mathrm{Jac}(\theta) d\theta &= \int_\Theta \left(\prod_{i=1}^n \sin^2 \theta_i \right) \left(\prod_{i=1}^{N-2} \sin^{N-1-i} \theta_i \right) d\theta \\ &= \frac{2\pi^{N/2}(N-n)}{N\Gamma(N/2)} = \frac{N-n}{N}\omega_N. \end{aligned}$$

The above computation is still valid for $n = N-1$. By (3.3), we see

$$\int_\Theta \left(\prod_{i=1}^n \sin\theta_i \right) \mathrm{Jac}(\theta) d\theta = S_n \int_\Theta \mathrm{Jac}(\theta) d\theta = S_n \omega_N.$$

Therefore

$$\begin{aligned}\int_\Theta S(\theta)^2 \mathrm{Jac}(\theta)d\theta &= \int_\Theta \left(\prod_{i=1}^n \sin^2\theta_i\right)\mathrm{Jac}(\theta)d\theta - 2S_n\int_\Theta\left(\prod_{i=1}^n \sin\theta_i\right)\mathrm{Jac}(\theta)d\theta \\ &\quad + S_n^2\int_\Theta \mathrm{Jac}(\theta)d\theta \\ &= \left((N-n)/N - S_n^2\right)\omega_N = T_n\omega_N. \end{aligned} \tag{3.14}$$

Here we have put $T_n := (N-n)/N - S_n^2$. Since the left had side of (3.14) is positive, T_n is positive, and so is the coefficient of (3.6). Substituting (3.11), (3.3) and (3.14) into (3.13), we obtain

$$\int_\Theta \phi(\theta)^2\mathrm{Jac}(\theta)d\theta = (1+\varepsilon^2 T_n)\omega_N. \tag{3.15}$$

Using (3.7), (3.10) and (3.12), we evaluate the last term in (3.9) as

$$\varepsilon^2\mu_n\int_0^1 u(r)^2 r^{N-3}dr \le \varepsilon^2 nN^{-1}PR(u)^{(p+1)/(p-1)}.$$

Substituting the inequality above, (3.10) and (3.15) into (3.9), we obtain

$$\int_B |\nabla v|^2 dx \le \left[1+\varepsilon^2 T_n + \varepsilon^2 nN^{-1}P\right]R(u)^{(p+1)/(p-1)}. \tag{3.16}$$

On the other hand, from the definition of v and (3.10), it follows that

$$\begin{aligned}\int_B h(|x|)|v|^{p+1}dx &= \int_0^1 h(r)|u|^{p+1}r^{N-1}dr\int_\Theta \phi(\theta)^{p+1}\mathrm{Jac}(\theta)d\theta \\ &= \frac{1}{\omega_N}R(u)^{(p+1)/(p-1)}\int_\Theta \phi(\theta)^{p+1}\mathrm{Jac}(\theta)d\theta. \end{aligned} \tag{3.17}$$

Let us estimate the last integral. For $t > -1$, there exists a $\delta \in (0,1)$ by the Taylor theorem such that

$$(1+t)^{p+1} = 1 + (p+1)t + \frac{p(p+1)}{2}(1+\delta t)^{p-1}t^2. \tag{3.18}$$

Recall that $\alpha > 0$ has been defined by (3.8). Then we choose an $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ so small that $(1-\varepsilon_0)^{p-1} \ge \alpha$. Therefore it holds that

$$(1+\delta t)^{p-1} \ge (1-\varepsilon_0)^{p-1} \ge \alpha \quad \text{if } |t| < \varepsilon_0.$$

Putting $t = \varepsilon S(\theta)$ in (3.18), for $\varepsilon > 0$ small enough, we have

$$(1+\varepsilon S(\theta))^{p+1} \ge 1 + (p+1)\varepsilon S(\theta) + \frac{p(p+1)}{2}\alpha\varepsilon^2 S(\theta)^2.$$

Using (3.11), (3.3) and (3.14), we see

$$\int_\Theta \phi(\theta)^{p+1}\mathrm{Jac}(\theta)d\theta \ge \omega_N + \varepsilon^2\omega_N Q,$$

where Q is given by

$$Q := \alpha p(p+1)T_n/2.$$

Hence (3.17) is rewritten as

$$\int_B h(|x|)|v|^{p+1}dx \geq R(u)^{(p+1)/(p-1)}(1+\varepsilon^2 Q),$$

which leads to

$$\left(\int_B h(|x|)|v|^{p+1}dx\right)^{-2/(p+1)} \leq R(u)^{-2/(p-1)}(1+\varepsilon^2 Q)^{-2/(p+1)}.$$

For $t > 0$, we use the mean value theorem to get a $\delta \in (0,1)$ such that

$$\begin{aligned}(1+t)^{-2/(p+1)} &= 1 - \frac{2}{p+1}(1+\delta t)^{-(p+3)/(p+1)}t \\ &\leq 1 - \frac{2}{p+1}(1+t)^{-(p+3)/(p+1)}t. \qquad (3.19)\end{aligned}$$

Substituting $t = \varepsilon^2 Q$, we have

$$(1+\varepsilon^2 Q)^{-2/(p+1)} \leq 1 - 2(p+1)^{-1}\beta\varepsilon^2 Q = 1 - \varepsilon^2\alpha\beta p T_n,$$

where

$$\beta := (1+\varepsilon^2 Q)^{-(p+3)/(p+1)}.$$

Thus we obtain

$$\left(\int_B h(|x|)|v|^{p+1}dx\right)^{-2/(p+1)} \leq R(u)^{-2/(p-1)}(1-\varepsilon^2\alpha\beta p T_n).$$

Using this inequality with (3.16), we find

$$\begin{aligned}R(v) &= \left(\int_B |\nabla v|^2 dx\right)\left(\int_B h|v|^{p+1}dx\right)^{-2/(p+1)} \\ &\leq R(u)\left\{1+\varepsilon^2 T_n + \varepsilon^2 nN^{-1}P\right\}\left\{1-\varepsilon^2\alpha\beta p T_n\right\} \\ &\leq R(u)\left\{1-\varepsilon^2(\alpha\beta p T_n - T_n - nN^{-1}P)\right\}.\end{aligned}$$

By (3.8), $(\alpha p - 1)T_n - nN^{-1}P > 0$. If $\varepsilon > 0$ is small enough, then β is sufficiently close to 1, and hence $(\alpha\beta p - 1)T_n - nN^{-1}P > 0$. Consequently, $R(v) < R(u)$ for $\varepsilon > 0$ small enough. The proof is complete. □

Remark 3.1 Under the assumption that $h(|x|)$ is positive, Smets, Willem and Su [20, Theorem 2.1] have proved that if u is a local minimizer of $R(\cdot)$ in $H_0^1(B) \setminus \{0\}$ and it is radially symmetric, then it satisfies

$$\int_B |\nabla u|^2 dx \leq \frac{N-1}{p-1}\int_B \frac{u^2}{|x|^2}dx.$$

Since u is radial, the inequality above is rewritten as

$$\int_0^1 u'(r)^2 r^{N-1}dr \leq \frac{N-1}{p-1}\int_0^1 u(r)^2 r^{N-3}dr.$$

In other words, their result means that if a radial solution u does not satisfy the inequality above, or equivalently, it satisfies

$$\int_0^1 u(r)^2 r^{N-3}dr < \frac{p-1}{N-1}\int_0^1 u'(r)^2 r^{N-1}dr, \qquad (3.20)$$

then it cannot be a local minimizer in $H^1_0(B)$, of course it is not a global least energy solution. Since $S_{N-1} = 0$ by Definition 3.1, our assumption (3.6) with $n = N-1$ coincides with (3.20). Hence Proposition 3.1 gives another proof of the result above. However, our proposition provides a stronger conclusion. Indeed, it ensures the existence of a nonradial $O(N-1) \times O(1)$ invariant solution under the assumption (3.20) because $n = N-1$. It will be proved in Section 5.

We deal with the case $N = 2$. Since the left hand side of (3.6) with $N = 2$ diverges, we need another inequality. We use the two dimensional polar coordinate:

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta.$$

For a radial solution u and $\varepsilon > 0$, we define

$$v(r,\theta) := \phi(r,\theta)u(r), \quad \phi(r,\theta) := 1 + \varepsilon r\cos 2\theta. \tag{3.21}$$

Then $v(r,\theta) = v(r,-\theta) = v(r,\pi-\theta)$, hence v is axially symmetric, i.e., it is $O(1) \times O(1)$ invariant.

Proposition 3.2 *Let $N = 2$ and u be a radial solution satisfying*

$$\int_0^1 u(r)^2 r dr < \frac{(p-1)}{2p+3}\int_0^1 u'(r)^2 r^3 dr. \tag{3.22}$$

Define v by (3.21). *Then $R(v) < R(u)$ for $\varepsilon > 0$ small enough. Therefore u is not a local minimizer of R, and so it is not a global least energy solution.*

Proof. From an easy calculation, it follows that

$$\begin{aligned}\int_B |\nabla v|^2 dx &= \int_0^1 \left(\int_0^{2\pi} (|v_r|^2 + |v_\theta|^2/r^2)d\theta\right) r dr \\ &= 2\pi\int_0^1 u_r^2 r dr + \varepsilon^2\pi\int_0^1 u_r^2 r^3 dr \\ &\quad +5\varepsilon^2\pi\int_0^1 u^2 r dr + 2\varepsilon^2\pi\int_0^1 u_r u r^2 dr.\end{aligned}$$

Integrating by parts in the last term and using (3.10), we find

$$\int_B |\nabla v|^2 dx = R(u)^{(p+1)/(p-1)} + \varepsilon^2\pi\int_0^1 u_r^2 r^3 dr + 3\varepsilon^2\pi\int_0^1 u^2 r dr.$$

Observing the last two terms, we put

$$P := R(u)^{-(p+1)/(p-1)}\left(\pi\int_0^1 u_r^2 r^3 dr + 3\pi\int_0^1 u^2 r dr\right).$$

Then we have

$$\int_B |\nabla v|^2 dx = R(u)^{(p+1)/(p-1)}(1 + \varepsilon^2 P). \tag{3.23}$$

Now, we shall compute the integral of $h|v|^{p+1}$. To this end, we estimate $\phi(r,\theta)$. Define $\psi(r,\theta)$ by

$$\varepsilon^2\psi(r,\theta) := (1+\varepsilon r\cos 2\theta)^{p+1} - 1 - (p+1)\varepsilon r\cos 2\theta - \frac{p(p+1)}{2}\varepsilon^2 r^2\cos^2 2\theta. \tag{3.24}$$

From the Taylor theorem (put $t = \varepsilon r \cos 2\theta$ in (3.18)), it follows that

$$\psi(r,\theta) \to 0 \quad \text{uniformly on } (r,\theta) \in [0,1]\times[0,2\pi] \text{ as } \varepsilon \to 0.$$

Rewrite (3.24) as

$$\phi(r,\theta)^{p+1} = 1 + (p+1)\varepsilon r \cos 2\theta + \frac{p(p+1)}{2}\varepsilon^2 r^2 \cos^2 2\theta + \varepsilon^2 \psi(r,\theta).$$

Using the equation above, we find

$$\begin{aligned} \int_B h|v|^{p+1}dx &= 2\pi \int_0^1 h|u|^{p+1} r dr + (p(p+1)/2)\varepsilon^2 \pi \int_0^1 h|u|^{p+1} r^3 dr \\ &\quad + \varepsilon^2 \int_0^1 \int_0^{2\pi} h|u|^{p+1} \psi r d\theta dr. \end{aligned}$$

We put

$$Q := R(u)^{-(p+1)/(p-1)} \left((p(p+1)/2)\pi \int_0^1 h|u|^{p+1} r^3 dr + \int_0^1 \int_0^{2\pi} h|u|^{p+1} \psi r d\theta dr \right).$$

Then by (3.10), we see

$$\int_B h|v|^{p+1}dx = R(u)^{(p+1)/(p-1)}(1 + \varepsilon^2 Q),$$

which with (3.19) implies

$$\left(\int_B h|v|^{p+1}dx \right)^{-2/(p+1)} \le R(u)^{-2/(p-1)}(1 - (2/(p+1))\varepsilon^2 \beta Q). \tag{3.25}$$

Here we have put

$$\beta := (1 + \varepsilon^2 Q)^{-(p+3)/(p+1)}.$$

Combining (3.23) and (3.25), we obtain

$$\begin{aligned} R(v) &= \left(\int_B |\nabla v|^2 dx \right) \left(\int_B h|v|^{p+1} dx \right)^{-2/(p+1)} \\ &\le R(u)(1 + \varepsilon^2 P)(1 - (2/(p+1))\varepsilon^2 \beta Q) \\ &\le R(u)\left\{ 1 + \varepsilon^2 \left(P - (2/(p+1))\beta Q \right) \right\} \\ &< R(u), \end{aligned}$$

provided that $P < (2/(p+1))\beta Q$. We shall verify this inequality. Since $\beta \to 1$ as $\varepsilon \to 0$, we have only to prove $P < (2/(p+1))Q$ for $\varepsilon > 0$ small. Since ψ converges to 0 uniformly on r and θ as $\varepsilon \to 0$, we find

$$Q \to (p(p+1)/2)\pi R(u)^{-(p+1)/(p-1)} \int_0^1 h|u|^{p+1} r^3 dr \quad \text{as } \varepsilon \to 0.$$

Therefore it is enough to show that

$$\int_0^1 u_r^2 r^3 dr + 3 \int_0^1 u^2 r dr < p \int_0^1 h|u|^{p+1} r^3 dr. \tag{3.26}$$

Now, since u is radial, (1.1) with $N = 2$ is reduced to

$$\frac{d}{dr}(u_r r) + h(r)u^p r = 0.$$

Multiplying the both sides by ur^2, integrating it over $[0, 1]$ and using integration by parts, we get

$$\int_0^1 u_r^2 r^3 dr - 2\int_0^1 u^2 r dr = \int_0^1 h|u|^{p+1} r^3 dr.$$

This relation means that (3.26) is equivalent to

$$(2p+3)\int_0^1 u^2 r dr < (p-1)\int_0^1 u_r^2 r^3 dr.$$

This inequality coincides with our assumption (3.22) and the proof is complete. □

4 Radial least energy solution

Observe Propositions 3.1 and 3.2. To prove that a global least energy solution is not radial, it is enough to show that a radial least energy solution satisfies (3.6) for $N \geq 3$ or (3.22) for $N = 2$. To this end, we investigate the properties of radial least energy solutions. Recall the notations $H^1_{0,r}(B)$, $D_r(R)$, $\mathcal{N}_r$ and L_r defined in Introduction. For $u \in H^1_{0,r}(B)$, we define

$$R_r(u) := \left(\int_0^1 u'(r)^2 r^{N-1} dr\right) \Big/ \left(\int_0^1 h(r)|u|^{p+1} r^{N-1} dr\right)^{2/(p+1)}. \tag{4.1}$$

Then $R(u)$ is rewritten as

$$R(u) = \omega_N^{(p-1)/(p+1)} R_r(u) \quad \text{for } u \in H^1_{0,r}(B). \tag{4.2}$$

Here ω_N is the surface area of the unit sphere of $\mathbb{R}^N$, which has been computed in (3.11). We first deal with the case where h is nonnegative.

Lemma 4.1 *Assume that $h(r) \geq 0$ in $(0, 1)$ and $h_+(r) \not\equiv 0$. Let u be a positive radial solution. Then $u'(r) \leq 0$ for $0 \leq r \leq 1$ and $u(r)$ attains its maximum at $r = 0$. Moreover, $u(r)$ satisfies*

$$R_r(u)^{(p+1)/(p-1)} = \int_0^1 u'(r)^2 r^{N-1} dr = \int_0^1 h u^{p+1} r^{N-1} dr. \tag{4.3}$$

Proof. Since u is radial, we rewrite (1.1), (1.2) as

$$\begin{aligned} &(u'(r)r^{N-1})' = -h(r)u(r)^p r^{N-1} \leq 0, \quad u > 0 \quad \text{in } (0,1), \\ &u'(0) = u(1) = 0. \end{aligned} \tag{4.4}$$

Therefore $u'(r)r^{N-1}$ is nonincreasing, and hence $u'(r) \leq 0$ for $r > 0$. Combining (3.10) with (4.2), we obtain (4.3). □

The next two lemmas has been shown in our paper [14] , however we give a proof for the reader's convenience.

Lemma 4.2 *Suppose that Assumption* $(A)_a$ *holds and let* u *be a positive radial least energy solution. Then*

$$R_r(u) \le (1-a)\left(\int_a^1 h(r)(1-r)^{p+1} r^{N-1} dr\right)^{-2/(p+1)}.$$

Proof. Put

$$v(r) := \begin{cases} 1 & \text{if } 0 \le r \le a, \\ (1-r)/(1-a) & \text{if } a \le r \le 1. \end{cases}$$

Then $v \in D_r(R)$. From the definition of v, we have

$$\int_0^1 v'(r)^2 r^{N-1} dr \le \frac{1}{1-a},$$

$$\int_0^1 h|v|^{p+1} r^{N-1} dr \ge (1-a)^{-(p+1)} \int_a^1 h(r)(1-r)^{p+1} r^{N-1} dr.$$

Therefore we obtain

$$R_r(v) \le (1-a)\left(\int_a^1 h(r)(1-r)^{p+1} r^{N-1} dr\right)^{-2/(p+1)}.$$

Since u is a radial least energy solution, $R_r(u) \le R_r(v)$ and the proof is complete. □

Lemma 4.3 *Impose the same assumption as in Lemma 4.2. Then* $u(0) \le 2u(a)$.

Proof. Integrating (4.4) over $[0, r]$, we have

$$-u'(r) r^{N-1} = \int_0^r h(t) u(t)^p t^{N-1} dt.$$

By Lemma 4.1, $u(0)$ is the maximum of $u(r)$ on $[0, 1]$. Dividing both sides by r^{N-1} and integrating it over $[0, a]$, we have

$$\begin{aligned} u(0) - u(a) &= \int_0^a r^{-(N-1)} \left(\int_0^r h u^p t^{N-1} dt\right) dr \\ &\le u(0) \int_0^a r^{-(N-1)} \left(\int_0^r h u^{p-1} t^{N-1} dt\right) dr. \end{aligned}$$

Using the Hölder inequality and applying Lemma 4.1, we estimate the integral above as

$$\begin{aligned} \int_0^r h u^{p-1} t^{N-1} dt &\le \left(\int_0^r h u^{p+1} t^{N-1} dt\right)^{(p-1)/(p+1)} \left(\int_0^r h t^{N-1} dt\right)^{2/(p+1)} \\ &\le R_r(u) \left(\int_0^r h t^{N-1} dt\right)^{2/(p+1)}. \end{aligned}$$

Using the constant ν defined by (2.1), we have

$$\int_0^r h t^{N-1} dt \le r^{N-\nu} \int_0^r h(t) t^{-1+\nu} dt.$$

Combine all the estimates above with Lemma 4.2 and (2.2). In $(A)_a$, we assumed the condition that $\mu(h,a)^{2/(p+1)} \le 1/(2d)$. Then we find

$$
\begin{aligned}
&u(0) - u(a) \\
&\le u(0)R_r(u)\int_0^a r^{-(N-1)+2(N-\nu)/(p+1)}dr\left(\int_0^a h(t)t^{-1+\nu}dt\right)^{2/(p+1)} \\
&\le u(0)d(1-a)\left(\int_a^1 h(r)(1-r)^{p+1}r^{N-1}dr\right)^{-2/(p+1)}\left(\int_0^a h(t)t^{-1+\nu}dt\right)^{2/(p+1)} \\
&= \mu(h,a)^{2/(p+1)}u(0)d \le u(0)/2.
\end{aligned}
$$

This completes the proof. □

The next lemma ensures that a radial least energy solution satisfies (3.6) or (3.22) if a is sufficiently close to 1.

Lemma 4.4 *Under the same assumption as in Lemma 4.2, we have*

$$\int_0^1 u(r)^2 r^{N-3}dr < \frac{4a^{-(N-1)}(1-a)}{N-2}\int_a^1 u'(r)^2 r^{N-1}dr \quad \textit{when } N \ge 3,$$

$$\int_0^1 u(r)^2 r dr < 2a^{-3}(1-a)\int_a^1 u'(r)^2 r^3 dr \quad \textit{when } N = 2.$$

Proof. Let $N \ge 3$. Using the Schwarz inequality, we get

$$
\begin{aligned}
u(a) &= -\int_a^1 u'(r)dr \\
&\le \left(\int_a^1 u'(r)^2 dr\right)^{1/2}\left(\int_a^1 dr\right)^{1/2} \\
&\le (1-a)^{1/2}a^{-(N-1)/2}\left(\int_a^1 u'(r)^2 r^{N-1}dr\right)^{1/2}.
\end{aligned}
$$

It follows from Lemma 4.3 that

$$u(0)^2 \le 4a^{-(N-1)}(1-a)\int_a^1 u'(r)^2 r^{N-1}dr.$$

Since $u(0)$ is the maximum of $u(r)$, we see

$$\int_0^1 u(r)^2 r^{N-3}dr < \frac{u(0)^2}{N-2} \le \frac{4a^{-(N-1)}(1-a)}{N-2}\int_a^1 u'(r)^2 r^{N-1}dr.$$

Let $N = 2$ and repeat the argument above. Then we see

$$u(a) \le (1-a)^{1/2}a^{-3/2}\left(\int_a^1 u'(r)^2 r^3 dr\right)^{1/2},$$

which with Lemma 4.3 yields

$$u(0)^2 \le 4a^{-3}(1-a)\int_a^1 u'(r)^2 r^3 dr.$$

Therefore

$$\int_0^1 u(r)^2 r dr < \frac{u(0)^2}{2} \le 2a^{-3}(1-a)\int_a^1 u'(r)^2 r^3 dr.$$

□

We next consider the case where $h(r)$ changes its sign.

Lemma 4.5 *Assume that $h(r) \le 0$ in $(0,a)$ and $h_+(r) \not\equiv 0$ in $(a,1)$. Let u be a positive radial solution. Then it holds that*

$$\int_0^1 u^2 r^{N-3} dr < \frac{a^{-(N-1)}(1-a)}{N-2}\int_a^1 u'(r)^2 r^{N-1} dr \quad \text{when } N \ge 3,$$

$$\int_0^1 u^2 r dr < \frac{a^{-3}(1-a)}{2}\int_a^1 u'(r)^2 r^3 dr \quad \text{when } N = 2.$$

Proof. Let $N \ge 3$. Since $h(r) \le 0$ in $(0,a)$, we see

$$(u'(r)r^{N-1})' = -hu^p r^{N-1} \ge 0,$$

which implies that $u'(r)r^{N-1}$ is nondecreasing in $(0,a)$, hence $u'(r) \ge 0$. Thus $u(r)$ is nondecreasing in $(0,a)$ and therefore it attains its maximum at a point in $[a,1)$. Denote it by $b \in [a,1)$ and write $M := u(b)$. We use the Schwarz inequality to get

$$\begin{aligned} M &= u(b) = \left|\int_b^1 u'(r)dr\right| \le \sqrt{1-b}\left(\int_b^1 u'(r)^2 dr\right)^{1/2} \\ &\le b^{-(N-1)/2}\sqrt{1-b}\left(\int_b^1 u'(r)^2 r^{N-1} dr\right)^{1/2}. \end{aligned}$$

Since M is the maximum of u, we have

$$\int_0^1 u(r)^2 r^{N-3} dr < \frac{M^2}{N-2}.$$

Combining two inequalities above, we obtain

$$\begin{aligned} \int_0^1 u^2 r^{N-3} dr &< \frac{b^{-(N-1)}(1-b)}{N-2}\int_b^1 u'(r)^2 r^{N-1} dr \\ &\le \frac{a^{-(N-1)}(1-a)}{N-2}\int_a^1 u'(r)^2 r^{N-1} dr, \end{aligned}$$

where we have used that $a \le b$.

Let $N = 2$. Then the method above works well. We leave the details to the reader and the proof is complete. □

5 Proof of main results

In this section, we prove Theorem 2.1 and Corollary 2.1. For simplicity of notation, we put $O_n := O(n) \times O(N-n)$ and define

$$H_0^1(B, O_n) := \{u \in H_0^1(B) : \ u(gx) = u(x) \quad \text{for } g \in O_n\},$$

which is a closed subspace of $H^1_0(B)$. Moreover, we define

$$D(R, O_n) = D(R) \cap H^1_0(B, O_n), \quad \mathcal{N}(O_n) = \mathcal{N} \cap H^1_0(B, O_n).$$

These sets are nonempty because $\emptyset \neq \mathcal{N}_r \subset \mathcal{N}(O_n) \subset D(R, O_n)$. Put

$$L(O_n) := \inf\{R(u) : \ u \in D(R, O_n)\} = \inf\{R(u) : \ u \in \mathcal{N}(O_n)\}.$$

We call u an $O(n) \times O(N-n)$ *invariant least energy solution* if $u \in \mathcal{N}(O_n)$ and $R(u) = L(O_n)$. The minimizer u of $R(u)$ among $\mathcal{N}(O_n)$ exists, which can be proved in the standard argument. Moreover, it is a critical point of R in $H^1_0(B, O_n)$. Then it becomes a critical point of R in $H^1_0(B)$ by the principle of symmetric criticality due to Palais [17]. Therefore to prove Theorem 2.1, it is enough to show that an $O(n) \times O(N-n)$ invariant least energy solution is not radial.

Proof of Theorem 2.1. Let $N \geq 2$ and $1 \leq n \leq N-1$. Let u be a positive radial least energy solution and v be defined by (3.1) for $N \geq 3$ or (3.21) for $N = 2$. Observe Propositions 3.1, 3.2 and Lemmas 4.4, 4.5. If a is sufficiently close to 1, then u satisfies (3.6) or (3.22). Therefore $R(v) < R(u)$ for $\varepsilon > 0$ small enough. Since v is $O(n) \times O(N-n)$ invariant, the $O(n) \times O(N-n)$ invariant least energy is less than the radial least energy. Thus an $O(n) \times O(N-n)$ invariant least energy solution is not radial. Since the global least energy is clearly less than or equal to the $O(n) \times O(N-n)$ invariant least energy, a global least energy solution is not radial. The proof is complete. □

We conclude the paper by proving Corollary 2.1.

Proof of Corollary 2.1. Let $h(r)$ be as in Corollary 2.1. Fix $a \in (1-\varepsilon, 1)$, where ε is determined by Theorem 2.1. Let us show that h satisfies $(\mathrm{A})_a$ if $\lambda > 0$ is large enough. To this end, it is enough to prove that $\mu(h, a) \to 0$ as $\lambda \to \infty$. We use the same method as in [14, Corollary 2.2]. Denote the maximum of $g(r)$ on $[0, a]$ by m. Since $g(1)$ is the maximum of $g(r)$, we choose $b \in (a, 1)$ such that

$$M := \min_{b \leq r \leq 1} g(r) > m.$$

Then

$$\begin{aligned}
&\left(\int_0^a h(r) r^{-1+\nu} dr\right)\left(\int_a^1 h(r)(1-r)^{p+1} r^{N-1} dr\right)^{-1} \\
&\leq \left(\int_0^a h(r) r^{-1+\nu} dr\right)\left(\int_b^1 h(r)(1-r)^{p+1} r^{N-1} dr\right)^{-1} \\
&\leq C(m/M)^{\lambda} \to 0 \quad \text{as } \lambda \to \infty,
\end{aligned}$$

where $C > 0$ is independent of m, M and λ. Hence $\mu(h, a) \to 0$ and the proof is complete. □

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