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On the Symmetry of Solutions to a *k*-Hessian Type Equation

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Abstract

In this note we prove that if u is a negative solution to a nonlinear elliptic equation involving a Hessian operator, and u is zero on the boundary of a ball, then u is radially symmetric and increasing along the radii.

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1 Introduction

In the famous paper [13] the authors investigate symmetry properties of negative solutions to elliptic equations, vanishing on the boundary. The simplest example in a bounded, connected open set is the following Dirichlet problem

$$\begin{cases} \Delta u = f(u) & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.1}$$

where B is a ball in $\mathbb{R}^n$ ($n \geq 2$) and $f \in C^1(\mathbb{R})$. By using the maximum principle and the moving planes method it is proved that u is radially symmetric and increasing with respect to $|x|$. Later, many mathematicians have been interested in this topic and many papers concerning semilinear elliptic and parabolic equations or fully nonlinear, uniformly elliptic equations may be found in literature (see, for example, [2], [3], [16], [1]). Anyway, in all these papers, all the monotonicity and symmetry results are shown by using fine versions of the strong maximum principle. In [17] P.-L. Lions gave an elegant proof of the symmetry of a negative solution to problem (1.1) for $n = 2$, whose main ingredients are the classical isoperimetric inequality and the well-known Pohožaev identity (see

[18]). Apart from its simplicity, this proof has the advantage that it does not make a direct use of the maximum principle, so it works under weaker assumptions on u and f. In [15] a similar approach has been used to treat the case of the n-Laplacian for $n \geq 2$. The aim of this paper is to prove the symmetry of a negative solution to a k-Hessian equation that seems to be a natural generalization of (1.1) to the case $n \geq 2$. We recall that for a twice differentiable function u defined in a bounded, connected, open set Ω in $\mathbb{R}^n$ the k-Hessian operator ($k = 1, ..., n$) is defined by

$$S_k(D^2u) = S_k(\lambda(D^2u)) = \sum_{1\leq i_1<...<i_k\leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.2}$$

where the λ's are the eigenvalues of the Hessian matrix D^2u of u and S_k is the k^{th}-elementary symmetric function. For example, for $k = 1$, $S_1(D^2u) = \Delta u$, while, for $k = n$, $S_n(D^2u) = \det D^2u$. Denoted by $\kappa_1, ..., \kappa_{n-1}$ the principal curvatures of a generic level set $\{u = t\}$ and by $H_{k-1}(x) = S_{k-1}(\kappa_1, ..., \kappa_{n-1})$ the $(k-1)^{th}$ curvature of the level set of u passing through x, our main result is contained in the following

Theorem 1.1 *Let B_1 be the unitary ball (centered at the origin) in $\mathbb{R}^n$ ($n \geq 2$, n even) and let f be a positive, C^1 function. If u is a C^3 negative solution to*

$$\begin{cases} S_{\frac{n}{2}}(D^2u) = H_{\frac{n}{2}-1} f(u) & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.3}$$

having convex sublevel sets, then necessarily u is radially symmetric and it is increasing with respect to $|x|$.

We finally observe that, in different contexts, many authors have been investigated symmetry questions and, in general, shape optimization problems arising in physics or geometry. A classical reference is the monograph [19]; for recent results and a more up to date bibliography we refer the interested reader to [14] (see also [4], [5], [7], [8], [9], [12]).

2 Notation and preliminaries

In this section we recall a few facts about k-Hessian operators that will be useful in the sequel. Let Ω be a bounded, connected, open set in $\mathbb{R}^n$. Denoting by

$$S_k^{ij}(D^2u) = \frac{\partial}{\partial u_{ij}} S_k(D^2u),$$

Euler identity for homogeneous functions gives

$$S_k(D^2u) = \frac{1}{k} S_k^{ij}(D^2u) u_{ij},$$

where subscripts stand for partial differentiations. For instance, when $k = 1$ we have $S_1^{ij} = \delta_{ij}$ and $S_1(D^2u) = \delta_{ij}u_{ij}$. Thus, from definition (1.2) it can be easily deduced that, for $k > 1$, the k-Hessian operators are fully nonlinear and elliptic if restricted to the class of k-convex functions

$$\Phi_k^2(\Omega) = \left\{u \in C^2(\Omega) \ : \ S_i(D^2u) \geq 0 \text{ in } \Omega, i = 1, 2, ..., k\right\}.$$

Notice that $\Phi_n^2(\Omega)$ coincides with the class of $C^2(\Omega)$ convex functions.

A direct computation yields that $(S_k^{1j}(D^2u),\dots,S_k^{nj}(D^2u))$ is divergence free for every $j = 1,...,n$, i.e.

$$\frac{\partial}{\partial x_i} S_k^{ij} = 0;$$

hence $S_k(D^2u)$ can be written in divergence form

$$S_k(D^2u) = \frac{1}{k} S_k^{ij}(D^2u)u_{ij} = \frac{1}{k}(S_k^{ij}(D^2u)u_j)_i\,. \tag{2.1}$$

Suppose now that $\partial\Omega \in C^2$, with principal curvatures $\kappa = (\kappa_1,...,\kappa_{n-1})$ and outer unit normal ν_x. For $k = 1,...,n-1$ we define the k^{th} curvature of $\partial\Omega$ by

$$H_k(\partial\Omega) = S_k(\kappa_1,...,\kappa_{n-1});$$

moreover we set

$$H_0 = S_0 \equiv 1, \qquad H_n \equiv 0.$$

For example, H_1 is equal to $n-1$ times the mean curvature of $\partial\Omega$, while H_{n-1} is the Gauss curvature of $\partial\Omega$.

If Ω is convex, we can define the quermassintegrals of Ω by

$$V_{n-j}(\Omega) = \frac{1}{n\binom{n-1}{n-j}} \int_{\partial\Omega} H_{j-1}, \quad j = 1,...,n \tag{2.2}$$

while, for $j = 0$, we take $V_n(\Omega) = |\Omega|$ (see, for instance, [10], [21]). The constant in (2.2) is chosen so that for a ball B_R with radius R we have $V_{n-j}(B_R) = \omega_n R^{n-j}$ (hereafter ω_n means the measure of the unitary ball in $\mathbb{R}^n$). The following Aleksandrov-Fenchel inequalities hold true

$$\left(\frac{V_{n-j}(\Omega)}{\omega_n}\right)^{\frac{1}{n-j}} \le \left(\frac{V_{n-l}(\Omega)}{\omega_n}\right)^{\frac{1}{n-l}}, \quad 0 \le j \le l \le n \tag{2.3}$$

and include the classical isoperimetric inequality for $j = 0$ and $l = 1$. Moreover, the following identities, known in differential geometry and in theory of convex bodies as Minkowskian integral formulae, hold true

$$\int_{\partial\Omega} \langle x,\nu_x\rangle H_{j-1} = (n-j+1)\binom{n}{j-1} V_{n-j+1}(\Omega), \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ stands for the euclidean inner product.

Finally, for $1 \le k \le n$, the following pointwise identity can be proved (see [20] for instance)

$$H_{k-1} = \frac{S_k^{ij}(D^2u)u_iu_j}{|Du|^{k+1}}. \tag{2.5}$$

3 Proof of Theorem 1.1

Let $t < 0$ and let us integrate the equation in problem (1.3) on the convex sublevel set $\{u < t\}$. By (2.1), (2.5) and Hölder inequality we can write

$$\begin{aligned}\int_{u<t} H_{\frac{n}{2}-1} f(u) = \int_{u<t} S_{\frac{n}{2}}(D^2u) &= \frac{2}{n}\int_{u=t} H_{\frac{n}{2}-1}|Du|^{\frac{n}{2}} \\ &\geq \frac{2}{n}\frac{\left(\int_{u=t} H_{\frac{n}{2}-1}\right)^{\frac{n+2}{2}}}{\left(\int_{u=t}\frac{H_{\frac{n}{2}-1}}{|Du|}\right)^{\frac{n}{2}}}.\end{aligned}$$

From Reilly's formula (see [20])

$$\frac{d}{dt}\int_{u=t} H_{\frac{n}{2}-2} = \left(\frac{n}{2}-1\right)\int_{u=t}\frac{H_{\frac{n}{2}-1}}{|Du|}$$

and Aleksandrov-Fenchel inequality (2.3) with $j = \frac{n}{2} - 1$ and $l = \frac{n}{2}$, denoting $\varphi(t) = V_{\frac{n}{2}+1}(\{u < t\})$, we get

$$c_n\,\varphi(t) \leq \varphi'(t)\left(\int_{u<t} H_{\frac{n}{2}-1} f(u)\right)^{\frac{2}{n}}, \tag{3.1}$$

where

$$c_n = \left(\frac{n+2}{2}\right)\left(2\omega_n\binom{n-1}{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

From co-area formula we easily deduce

$$\begin{aligned}\int_{u<t} H_{\frac{n}{2}-1} f(u) &= \int_{-\infty}^{t} f(s)\left(\int_{u=s}\frac{H_{\frac{n}{2}-1}}{|Du|}\right)ds \\ &= \binom{n}{\frac{n}{2}-1}\int_{-\infty}^{t} f(s)\varphi'(s)ds.\end{aligned} \tag{3.2}$$

Gathering (3.1) and (3.2) we have

$$\varphi(t) \leq d_n\left(\int_{-\infty}^{t} f(s)\varphi'(s)ds\right)^{\frac{2}{n}}\varphi'(t),$$

where

$$d_n = \frac{2}{n+2}\left(\frac{n}{(n+2)\omega_n}\right)^{\frac{2}{n}}.$$

If we multiply by $f(t)$ and integrate on $]-\infty, 0]$, we obtain

$$\begin{aligned}\int_{-\infty}^{0} f(t)\varphi(t)dt &\leq d_n\int_{-\infty}^{0} f(t)\varphi'(t)\left(\int_{-\infty}^{t} f(s)\varphi'(s)ds\right)^{\frac{2}{n}}dt \\ &= d_n\frac{n}{n+2}\left(\int_{-\infty}^{0} f(t)\varphi'(t)dt\right)^{\frac{n+2}{n}}.\end{aligned} \tag{3.3}$$

On the other hand, integrating the equation on B_1, by (2.1) and Hölder inequality we get

$$\begin{aligned}\binom{n}{\frac{n}{2}-1}\int_{-\infty}^{0} f(s)\varphi'(s)ds &= \int_{B_1} H_{\frac{n}{2}-1}f(u)\\ &= \int_{B_1} S_{\frac{n}{2}}(D^2u) = \frac{2}{n}\int_{\partial B_1} H_{\frac{n}{2}-1}|Du|^{\frac{n}{2}}\\ &= \frac{2}{n}\binom{n-1}{\frac{n}{2}-1}\int_{\partial B_1}|Du|^{\frac{n}{2}}\\ &\le \frac{2}{n}\binom{n-1}{\frac{n}{2}-1}\left(\int_{\partial B_1}|Du|^{\frac{n+2}{2}}\right)^{\frac{n}{n+2}}(n\omega_n)^{\frac{2}{n+2}}.\end{aligned}$$

Then

$$\int_{-\infty}^{0} f(t)\varphi(t)dt \le \frac{2}{n(n+2)}\int_{\partial B_1}|Du|^{\frac{n+2}{2}}. \tag{3.4}$$

By a Pohožaev type identity (see (3.6) below) and (2.4), it holds

$$\begin{aligned}\int_{\partial B_1}|Du|^{\frac{n+2}{2}} &= \frac{1}{\binom{n-1}{\frac{n}{2}-1}}\int_{\partial B_1} H_{\frac{n}{2}-1}|Du|^{\frac{n+2}{2}}\\ &= \frac{1}{\binom{n-1}{\frac{n}{2}-1}}\int_{\partial B_1}\langle x, Du\rangle H_{\frac{n}{2}-1}|Du|^{\frac{n}{2}}\\ &= \frac{n+2}{2\binom{n-1}{\frac{n}{2}-1}}\int_{B_1}\langle x, Du\rangle H_{\frac{n}{2}-1}f(u)\\ &= \frac{n+2}{2\binom{n-1}{\frac{n}{2}-1}}\int_{-\infty}^{0} f(s)\left(\int_{u=s} H_{\frac{n}{2}-1}\langle x,\nu\rangle\right)ds\\ &= \frac{n(n+2)}{2}\int_{-\infty}^{0} f(s)\varphi(s)ds.\end{aligned} \tag{3.5}$$

From (3.3), (3.4) and (3.5) we deduce

$$\int_{-\infty}^{0} f(t)\varphi(t)dt \le \frac{2}{(n+2)\omega_n^{\frac{2}{n}}}\left(\frac{n}{n+2}\right)^{\frac{n+2}{n}}\left(\int_{-\infty}^{0} f(t)\varphi'(t)dt\right)^{\frac{n+2}{n}} \le \int_{-\infty}^{0} f(t)\varphi(t)dt.$$

Thus all the inequalities we used must hold as equalities and u must be radially symmetric, i.e. $u(x) = u(r)$, $r = |x|$. Moreover, being $S_{\frac{n}{2}}(D^2u) = \frac{2}{n}\binom{n-1}{\frac{n}{2}-1} r^{-n+1}\left[r^{\frac{n}{2}}(u')^{\frac{n}{2}}\right]' \ge 0$ (see, for instance, [22]), u is increasing with respect to r.

Lemma 3.1 *Let $f \in C^1(\mathbb{R})$ and let u be a C^3 solution to the following problem*

$$\begin{cases} S_k(D^2u) = H_{k-1}f(u) & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega\end{cases} \qquad (1 \le k \le n)$$

in a bounded, connected, open set Ω *in* $\mathbb{R}^n$. *Then*

$$\int_\Omega \langle x, Du\rangle H_{k-1} f(u) + \frac{n-2k}{k+1}\int_\Omega H_{k-1} u f(u) = \frac{1}{k+1}\int_{\partial\Omega} \langle x, Du\rangle H_{k-1}|Du|^k. \tag{3.6}$$

Proof. First of all let us multiply the equation by $\langle x, Du\rangle$. Reasoning as in the appendix of [6] we get

$$\begin{aligned}\int_{\partial\Omega} \langle x, Du\rangle H_{k-1}|Du|^k \quad &+ \quad (n-2)\int_\Omega S_k^{ij}(D^2u)u_iu_j + (1-k)\int_\Omega u x_l S_k^{ij}(D^2u)u_{ijl}\\ &= \quad \int_\Omega \langle x, Du\rangle H_{k-1} f(u).\end{aligned} \tag{3.7}$$

On the other hand

$$\frac{\partial}{\partial x_l} S_k(D^2u) = \frac{\partial}{\partial u_{ij}} S_k(D^2u)\frac{\partial u_{ij}}{\partial x_l} = S_k^{ij}(D^2u)u_{ijl} = \frac{\partial}{\partial x_l}\left(H_{k-1} f(u)\right).$$

Then, by divergence theorem,

$$\int_\Omega u x_l S_k^{ij}(D^2u)u_{ijl} = -\int_\Omega \left[\langle x, Du\rangle + nu\right] H_{k-1} f(u). \tag{3.8}$$

Finally (3.7) and (3.8) imply the claim.

References

[1] M. Badiale, and E. Nabana, *A note on radiality of solutions of p-Laplacian equation*, Appl. Anal. **52** (1994), 35–43.

[2] H. Berestycki, and L. Nirenberg, *Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations*, J. Geom. Phys. **5** (1988), 237–275.

[3] H. Berestycki, and L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. (N.S.) **22** (1991), 1–37.

[4] M. F. Betta, F. Chiacchio, and A. Ferone, *Isoperimetric estimates for the first eigenfunction of a class of linear elliptic problems*, Z. Angew. Math. Phys. **58** (2007), 37–52.

[5] B. Brandolini, F. Chiacchio, and C. Trombetti, *Sharp estimates for eigenfunctions of a Neumann problem*, Comm. Partial Differential Equations **34** (2009), 1317–1337.

[6] B. Brandolini, C. Nitsch, P. Salani, and C. Trombetti, *Serrin-type overdetermined problems: an alternative proof*, Arch. Ration. Mech. Anal. **190** (2008), 267–280.

[7] B. Brandolini, C. Nitsch, and C. Trombetti, *New isoperimetric estimates for solutions to Monge-Ampère equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 1265–1275.

[8] B. Brandolini, C. Nitsch, and C. Trombetti, *Shape optimization for Monge-Ampère equations via domain derivative*, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), 825–831.

[9] D. Bucur, *Minimization of the k-th eigenvalue of the Dirichlet Laplacian*, Arch. Ration. Mech. Anal. **206** (2012), 1073–1083.

[10] Yu. D. Burago, and V. A. Zalgaller, Geometric inequalities, Springer-Verlag, Berlin, 1988.

[11] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian*, Acta Math. **155** (1985), 261–301.

[12] F. Chiacchio, and G. di Blasio, *Isoperimetric inequalities for the first Neumann eigenvalue in Gauss space*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), 199–216.

[13] B. Gidas, Wei Ming Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[14] A. Henrot, and P. Michel, Variation et optimisation de formes. Une analyse géométrique. Mathématiques & Applications (Berlin), 48. Springer, Berlin, 2005.

[15] S. Kesavan, and F. Pacella, *Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequalities*, Appl. Anal. **54** (1994), 27–37.

[16] Congming Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on bounded domains*, Comm. Partial Differential Equations **16** (1991), 491–526.

[17] P.-L. Lions, *Two geometrical properties of solutions of semilinear problems*, Applicable Anal. **12** (1981), 267–272.

[18] S. I. Pohožaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR **165** (1965), 36–39.

[19] G. Pólya, and G. Szegö, Isoperimetric Inequalities in Mathematical Physics. Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, N. J., 1951.

[20] R. C. Reilly, *On the Hessian of a function and the curvatures of its graph*, Michigan Math. J. **20** (1973), 373–383.

[21] R. Schneider, Convex bodies: the Brunn-Minkowski theory, Cambridge University Press, Cambridge, 1993.

[22] K. Tso, *On symmetrization and Hessian equations*, J. Analyse Math. **52** (1989), 94–106.