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Boundary Behavior of Solutions to Singular Boundary Value Problems for Nonlinear Elliptic Equations

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Abstract

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$ and $h \in C((0,\infty),\ (0,\infty))$ with $\lim_{s\to 0^+} h(s) = \Upsilon \in (0,\infty)$. By the perturbation method, which is due to García Melián, and nonlinear transformations and comparison principles, we derive the exact boundary behavior of solutions to a singular Dirichlet problem $-\Delta v + \frac{h(v)}{v}|\nabla v|^2 = b(x),\ v > 0,\ x \in \Omega,\ v|_{\partial\Omega} = 0$. Then, applying the result, combining two kinds of nonlinear transformations, we derive the exact boundary behavior of solutions to a boundary blow-up elliptic problem and a singular Dirichlet problem, where the weight b is positive in Ω and may be (rapidly) vanishing or blow up on the boundary.

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1 Introduction and the main results

Consider the following singular boundary value problem

$$-\Delta v + \frac{h(v)}{v}|\nabla v|^2 = b(x),\ v > 0,\ x \in \Omega,\ v|_{\partial\Omega} = 0, \tag{P}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N (N \geq 1)$, b satisfies

(b_1) $b \in C^{\alpha}_{\mathrm{loc}}(\Omega)$, for some $\alpha \in (0,1)$, is positive in Ω,

and h satisfies

(h_1) $h \in C((0,\infty),\ (0,\infty))$ and $\lim_{s\to 0^+} h(s) = \Upsilon \in (0,\infty)$.

The problem is related to the following:

I. A boundary blow-up elliptic problem

$$\triangle u = b(x)f(u),\ x \in \Omega,\ u|_{\partial\Omega} = \infty, \tag{1.1}$$

where the last condition means that $u(x) \to +\infty$ as $d(x) = \mathrm{dist}(x, \partial\Omega) \to 0$, and the solution is called " a large solution" or " an explosive solution", f satisfies

(f_1) $f \in C^1[0,\infty)$, $f(0) = 0$, f is increasing on $[0,\infty)$ (or $f \in C^1(\mathbb{R})$, $f(t) > 0,\ \forall t \in \mathbb{R}$, f is increasing on $\mathbb{R}$);

(f_2) there exist $p > 1$ and $T_0 > 0$ such that $f(t)/t^p$ is increasing for $t \geq T_0$;

(f_3) $\lim_{t\to+\infty} f'(t) \int_t^{\infty} \frac{d\nu}{f(\nu)} = C_f$.

Let

$$v = \Phi_1(u) := \int_u^{\infty} \frac{d\nu}{f(\nu)},\ u > 0 \text{ (or } u \in \mathbb{R}). \tag{1.2}$$

We see that problem (1.1) is equivalent to problem (P) with $h(v) = vf'(\phi_1(v))$, where ϕ_1 is the inverse function of Φ_1, i.e., ϕ_1 satisfies

$$-\phi_1'(t) = f(\phi_1(t)),\ t > 0,\ \phi_1(0) := \lim_{t\to 0^+} \phi_1(t) = +\infty, \tag{1.3}$$

or

$$\int_{\phi_1(t)}^{\infty} \frac{d\nu}{f(\nu)} = t,\ t > 0, \tag{1.4}$$

and

$$\lim_{s\to 0^+} h(s) = \lim_{t\to+\infty} \Phi_1(t)f'(t) = \lim_{t\to+\infty} f'(t) \int_t^{\infty} \frac{d\nu}{f(\nu)} = C_f.$$

In particular,

(1) when $f(u) = u^{\rho+1}$ $\rho > 0$, $u \geq 0$, $\Phi_1(u) = \rho^{-1}u^{-\rho}$, $u > 0$, $\phi_1(t) = (\rho t)^{-1/\rho}$, $t > 0$, $h(v) \equiv C_f = \frac{\rho+1}{\rho} > 1$;

(2) when $f(u) = e^u$, $u \in \mathbb{R}$, $\Phi_1(u) = e^{-u}$, $u \in \mathbb{R}$, $\phi_1(t) = -\ln t$, $t > 0$, $h(v) \equiv 1$;

(3) when $f(t) = c_0 t^{\rho+1} \exp\left(\int_{T_0}^t \frac{y(\tau)}{\tau} d\tau\right)$, $t \geq T_0$, $C_f = (\rho+1)/\rho$, where $\rho > 0$, $c_0 > 0$, $y \in C([T_0,\infty)$ and for $\tau \to \infty$, $y(\tau) \to 0$;

(4) (García Melián [12], Lemma 7) when f satisfies (f_1) on $[0,\infty)$ and (f_2), (f_3), $1 \le C_f \le p/(p-1)$;

(5) if $\lim_{t\to\infty} f'(t) = \infty$, then $\lim_{t\to\infty} \frac{(f'(t))^2}{f(t)f''(t)} = 1$ implies (f_3) with $C_f = 1$. Moreover, $f(t) = c_0 \exp\left(\int_{T_0}^t \frac{d\tau}{\zeta_1(\tau)}\right)$ $\forall t > T_0$ satisfies (f_2) and (f_3) with $C_f = 1$, where ζ_1 is a positive C^1 function on $[T_0,\infty)$ and satisfies $\lim_{\tau\to\infty} \zeta_1'(\tau) = 0$.

Problem (1.1) arises from many branches of mathematics and applied mathematics and has been discussed by many authors and in many contexts; see, for instance, [3], [6], [8], [10], [12], [13], [14], [17], [19], [20], [22], [23], [24], [26], [27], [28] and the complete references therein.

II. A singular Dirichlet problem

$$-\triangle w = b(x)g(w),\ w > 0,\ x \in \Omega,\ w|_{\partial\Omega} = 0, \tag{1.5}$$

where g satisfies

(g_1) $g \in C^1((0,\infty),(0,\infty))$, $\lim_{t\to 0^+} g(t) = +\infty$ and g is decreasing on $(0,\infty)$;

(g_2) $-\lim_{t\to 0^+} g'(t) \int_0^t \frac{d\mu}{g(\mu)} = C_g$.

Now let

$$v = \Phi_2(w) := \int_0^w \frac{d\mu}{g(\mu)},\ w > 0. \tag{1.6}$$

We see that problem (1.5) is equivalent to problem (P) with $h(v) = -vg'(\phi_2(v))$, where ϕ_2 is the inverse function of Φ_2, i.e., ϕ_2 satisfies

$$\phi_2'(t) = g(\phi_2(t)),\ \phi_2(t) > 0,\ t \in (0,\infty),\ \phi_2(0) = 0, \tag{1.7}$$

or

$$\int_0^{\phi_2(t)} \frac{d\mu}{g(\mu)} = t,\ t \in [0,\infty), \tag{1.8}$$

and

$$\lim_{s\to 0^+} h(s) = -\lim_{t\to 0^+} \Phi_2(t)g'(t) = -\lim_{t\to 0^+} g'(t) \int_0^t \frac{d\mu}{g(\mu)} = C_g.$$

In particular,

(1) when $g(u) = u^{-\gamma}$, $u > 0$, $\gamma > 0$, $\Phi_2(w) = \frac{w^{\gamma+1}}{\gamma+1}$, $\phi_2(t) = ((\gamma+1)t)^{1/(\gamma+1)}$, $t > 0$, $h(v) \equiv C_g = \frac{\gamma}{\gamma+1} \in (0,1)$;

(2) when $g(t) = c_0 t^{-\gamma} \exp\left(\int_t^{t_0} \frac{y_1(\tau)}{\tau} d\tau\right)$, $0 < t \le t_0$, $C_g = \gamma/(\gamma+1)$, where $c_0 > 0$, $\gamma > 0$, $y_2 \in C[0,t_0]$ with $y_2(0) = 0$;

(3) (Cîrstea [8], Lemma 3.1)) if $\lim_{t\to 0^+} g'(t) = -\infty$, then $\lim_{t\to 0^+} \frac{(g'(t))^2}{g(t)g''(t)} = 1$ implies (g_2) with $C_g = 1$. Moreover, $g(t) = c_0 \exp\left(\int_t^{t_0} \frac{d\nu}{\zeta_2(\nu)}\right)$ $\forall t \in (0,t_0]$ satisfies (g_1) on $(0,t_0)$ and (g_2) with $C_g = 1$, where $t_0 > 0$, ζ_2 is a positive C^1 function on $(0,t_0)$ such that $\lim_{\nu\to 0^+} \zeta_2(\nu) = \lim_{\nu\to 0^+} \zeta_2'(\nu) = 0$;

(4) when g satisfies (g_1) and (g_2), $C_g \in [0,1]$.

In fact, by (g_1), it is clear that $C_g \geq 0$ and

$$0 < \int_0^s \frac{d\mu}{g(\mu)} < \frac{s}{g(s)}, \ \forall s > 0,$$

i.e.,

$$\lim_{s\to 0^+} g(s) \int_0^s \frac{d\mu}{g(\mu)} = 0.$$

Let

$$I(t) = -g'(t) \int_0^t \frac{d\mu}{g(\mu)}, \ \forall t > 0.$$

Integrate $I(s)$ from 0 to t and integrate by parts, we obtain that

$$\int_0^t I(s)ds = -g(t) \int_0^t \frac{ds}{g(s)} + t,$$

i.e.,

$$\frac{\int_0^t I(s)ds}{t} \leq 1, \ \forall t > 0.$$

It follows from the l'Hospital's rule that

$$\lim_{t\to 0^+} \frac{\int_0^t I(s)ds}{t} = \lim_{t\to 0^+} I(t) = C_g \leq 1.$$

Problem (1.5) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials, and was discussed in a number of works; see, for instance, [4], [5], [9], [11], [14], [18], [28], [29] and the references therein.

Now let's return to problem (P).

By a perturbation method, Arcoya and Martínez-Aparicio [2] (Theorem 1.1), Arcoya, Carmona and Martínez-Aparicio [1] (Theorem 1.3) showed that if h satisfies

(h_2) $h \in C((0,\infty), \ (0,\infty))$ and $\lim_{s\to 0^+} \sup h(s) \in (0,\infty)$,

and $b \in L^\infty(\Omega)$ (or $b \in L^q(\Omega)$ with $q > N/2$) satisfies

(b_2) $\inf\{ \ b(x) > 0, x \in \omega\} > 0, \ \forall \omega \subset\subset \Omega$,

then problem (P) has at least a solution $v \in H_0^1(\Omega) \cap C^\alpha(\Omega)$ (or $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$).

By another perturbation method, one class of nonlinear transformations and a comparison principle, García Melián [12] derived the following results:

If h satisfies (h_1) with $\Upsilon \geq 1$, b satisfies (b_1) and

(b_3) there exist $b_0 > 0$ and $\sigma \in (0,2)$ such that

$$\lim_{d(x)\to 0} b(x)(d(x))^\sigma = b_0, \tag{1.9}$$

then

(i) when $\Upsilon > 1$, for any solution v of problem (P)

$$\lim_{d(x)\to 0} \frac{v(x)}{(d(x))^{2-\sigma}} = A, \tag{1.10}$$

where $A = \frac{b_0}{(2-\sigma)(1+(2-\sigma)(\Upsilon-1))}$;

(ii) when $\Upsilon = 1$ and $h(t) \geq 1$ for small t, (i) continues to hold.

Next let's introduce a class of functions to be used in the following.

Let Λ denote the set of all positive monotonic functions $k \in L^1(0,\delta_0) \cap C^1(0,\delta_0)$ $(\delta_0 > 0)$ which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) := C_k \in [0,\infty), \tag{1.11}$$

where

$$K(t) = \int_0^t k(s)ds. \tag{1.12}$$

We note that for each $k \in \Lambda$, $\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0$, $C_k \in [0,1]$ if k is non-decreasing and $C_k \in [1,\infty)$ if k is non-increasing, and

$$\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0;\ \lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - \lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) = 1 - C_k. \tag{1.13}$$

The set Λ was first introduced by Cîrstea and Rădulescu [6] for non-decreasing functions and by Mohammed [23] for non-increasing functions to study the boundary behaviour of solutions to problem (1.1).

Some basic examples of non-increasing functions in Λ are

(i) $k \equiv C_0 > 0$, $K(t) = C_0 t$, $C_k = 1$;

(ii) $k(t) = t^{-\sigma/2}$ with $\sigma \in (0,2)$, $K(t) = \frac{2t^{(2-\sigma)/2}}{2-\sigma}$, $C_k = \frac{2}{2-\sigma}$;

(iii) $k(t) = (-\ln t)^\sigma$, $\sigma > 0$, $C_k = 1$;

and some basic examples of non-decreasing functions in Λ are

(i) $k(t) = t^{\sigma/2}, \sigma > 0$, $C_k = 2/(2+\sigma)$;

(ii) $k(t) = 1/(-\ln t)^\sigma, \sigma > 0$, $C_k = 1$;

(iii) $k(t) = t^\sigma/\ln(1+t^{-1}), \sigma > 0$, $C_k = 1/(1+\sigma)$;

(iv) $k(t) = e^{-t^{-\sigma}}$, $\sigma > 0$, $C_k = 0$;

(v) $k(t) = e^{-e^{t^{-\sigma}}}$, $\sigma > 0$, $C_k = 0$;

(vi) $k(t) = e^{-(-\ln t)^\sigma}$, $\sigma > 1$, $C_k = 0$.

Inspired by the idea of García Melián [12], in this paper we consider the exact boundary behavior of solutions to problem (P) for the following three cases:

(I) $\Upsilon \in (1,\infty)$ and the more general weight b;

(II) $\Upsilon = 1$ and $b \equiv b_0 > 0$ on Ω;

(III) $\Upsilon \in (1/2, 1)$ and $b \equiv b_0 > 0$ on Ω.
Then, applying the result, we derive the exact boundary behavior of solutions to problems (1.1) and (1.5).

Our main results are summarized in the following.

Theorem 1.1 *Let h satisfy* (h_1), *b satisfy* (b_1) *and* (b_4), *There exist some $k \in \Lambda$ and $b_0 > 0$ such that*

$$\lim_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} = b_0.$$

If

$$\Upsilon > 1 \text{ and } C_k + 2\Upsilon > 2, \tag{1.14}$$

then for any solution $v \in C(\bar{\Omega}) \cap C^2(\Omega)$ of problem (P)

$$\lim_{d(x)\to 0} \frac{v(x)}{K^2(d(x))} = \tau_0, \tag{1.15}$$

where

$$\tau_0 = \frac{b_0}{2(C_k + 2\Upsilon - 2)}. \tag{1.16}$$

Remark 1.1 Since $\Upsilon > 1$ in (1.14), it is clear that C_k can be equal to zero.

Theorem 1.2 *Let h satisfy* (h_1), *$b \equiv b_0$ on Ω. If*

$$\frac{1}{2} < \Upsilon \leq 1, \tag{1.17}$$

then the result of Theorem 1.1 holds, where $\tau_0 = \frac{b_0}{2(2\Upsilon-1)}$.

Corollary 1.1 *Let f satisfy* (f_1)-(f_3).
(i) *If b satisfies* (b_1), (b_4), *and* (1.14) *with $\Upsilon = C_f$ holds, then for any solution $u \in C^2(\Omega)$ of problem* (1.1)

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi_1(\tau_0 K^2(d(x)))} = 1. \tag{1.18}$$

(ii) *If $b \equiv b_0$ on Ω and $\Upsilon = C_f = 1$, then the result of* (i) *holds.*

Remark 1.2 García Melián [13] showed that (f_3) implies (f_2).

Corollary 1.2 *Let g satisfy* (g_1) *and* (g_2), *$b \equiv b_0$ on Ω. If* (1.17) *with $\Upsilon = C_g$ holds, then for the unique solution $w \in C(\bar{\Omega}) \cap C^2(\Omega)$ of problem* (1.5)

$$\lim_{d(x)\to 0} \frac{w(x)}{\phi_2(\tau_0 (d(x))^2)} = 1. \tag{1.19}$$

Remark 1.3 When $C_g = \gamma/(1+\gamma)$ with $\gamma > 0$, (1.17) reads $\gamma > 1$.

The outline of this paper is as follows. In Section 2 we prove Theorems 1.1-1.2. The proof of Corollaries 1.1 and 1.2 is given in Section 3.

2 Boundary behaviour of solutions to problem (P)

First we need some preliminary considerations.

Lemma 2.1 (Berhanu, Gladiali and Porru [4], Lemma 2.1) *If $\gamma > 1$, then the unique solution $w \in C(\bar{\Omega}) \cap C^2(\Omega)$ of the following problem*

$$-\triangle w = b_0 w^{-\gamma},\ w > 0,\ x \in \Omega,\ w|_{\partial\Omega} = 0, \tag{2.1}$$

satisfies

$$\lim_{d(x)\to 0} \frac{w(x)}{(d(x))^{2/(1+\gamma)}} = \left(\frac{b_0(1+\gamma)^2}{2(\gamma-1)}\right)^{1/(1+\gamma)}. \tag{2.2}$$

Lemma 2.2 (Cîrstea and Rădulescu [6], Theorem 1 and Mohammed [23], Theorem 3.4) *Let $\rho > 0$. If b satisfies* (b_1) *and* (b_4)*, then any solution $u \in C^2(\Omega)$ of the following problem*

$$\triangle u = b(x)u^{1+\rho},\ u > 0 \text{ in } \Omega,\ u|_{\partial\Omega} = \infty \tag{2.3}$$

satisfies

$$\lim_{d(x)\to 0} u(x)(K(d(x)))^{2/\rho} = \left(\frac{2(2+\rho C_k)}{b_0\rho^2}\right)^{1/\rho}. \tag{2.4}$$

Lemma 2.3 (the comparison principle)(Diaz and Letelier [15], Theorem 2.2) *Let b satisfy* (b_1) *and f satisfy* (f_1). If $u_1, u_2 \in C^2(\Omega)$ satisfies $\Delta u_1 - b(x)f(u_1) \geq \Delta u_2 - b(x)f(u_2)$ in Ω and $\limsup_{d(x)\to 0} u_1(x)/u_2(x) \leq 1$, then $u_1 \leq u_2$ in Ω.

Proof of Theorem 1.1. Since $\Upsilon > 1$, let $\varepsilon \in (0, \min\{\Upsilon - 1,\ (C_k + 2\Upsilon - 2)/2\})$, and we see by (h_1) and problem (P) that there exists $\delta > 0$ such that

$$\Upsilon - \varepsilon < h(v(x)) < \Upsilon + \varepsilon,\ \forall x \in \Omega_\delta$$

and

$$-\Delta v + (\Upsilon + \varepsilon)\frac{|\nabla v|^2}{v} \geq b(x),\ x \in \Omega_\delta, \tag{2.5}$$

$$-\Delta v + (\Upsilon - \varepsilon)\frac{|\nabla v|^2}{v} \leq b(x),\ x \in \Omega_\delta. \tag{2.6}$$

Set

$$u_{1\varepsilon} = v^{-q_{1\varepsilon}},\ q_{1\varepsilon} = \Upsilon - \varepsilon - 1.$$

By (2.6) and a direct computation, we have

$$\Delta u_{1\varepsilon} \le q_{1\varepsilon} b(x) u_{1\varepsilon}^{1+q_{1\varepsilon}^{-1}},\ x \in \Omega_\delta.$$

Since $q_{1\varepsilon} > 0$, let $U_{1\varepsilon m}$ be the unique solution to the following problem (see [26])

$$\Delta u = q_{1\varepsilon} b(x) u^{1+q_{1\varepsilon}^{-1}},\ u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = m, \tag{2.7}$$

where $m = 1, 2, 3, \cdots$, $\{U_{1\varepsilon m}(x)\}$ is nondecreasing for $x \in \Omega$ and $\lim_{m\to\infty} U_{1\varepsilon m}(x) = U_{1\varepsilon}(x),\ x \in \Omega$ and $U_{1\varepsilon} \in C^2(\Omega)$ which is one solution to the problem

$$\Delta u = q_{1\varepsilon} b(x) u^{1+q_{1\varepsilon}^{-1}},\ u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = \infty. \tag{2.8}$$

It follows by Lemma 2.2 that

$$\lim_{d(x)\to 0} U_{1\varepsilon}(x)(K(d(x))^{2q_{1\varepsilon}} = \left(\frac{2(2q_{1\varepsilon} + C_k)}{b_0}\right)^{q_{1\varepsilon}}.$$

Let $M > 0$ such that $U_{1\varepsilon} > M$ and $U_{1\varepsilon m} - M \le U_{1\varepsilon} - M \le u_{1\varepsilon}$ on $\Gamma_\delta := \{x \in \Omega : d(x) = \delta\}$. We see by $u_{1\varepsilon}|_{\partial\Omega} = +\infty$ and Lemma 2.3 that $U_{1\varepsilon m} - M \le u_{1\varepsilon}$ on Ω_δ, thus $U_{1\varepsilon} - M \le u_{1\varepsilon}$ on Ω_δ, i.e.,

$$v \le (U_{1\varepsilon} - M)^{-1/q_{1\varepsilon}},\ x \in \Omega_\delta.$$

It follows that

$$\limsup_{d(x)\to 0} \frac{v(x)}{K^2(d(x))} \le \frac{b_0}{2(2q_{1\varepsilon} + C_k)}.$$

Letting $\varepsilon \to 0$, we obtain

$$\limsup_{d(x)\to 0} \frac{v(x)}{K^2(d(x))} \le \tau_0. \tag{2.9}$$

Next let

$$u_{2\varepsilon} = v^{-q_{2\varepsilon}},\ q_{2\varepsilon} = \Upsilon + \varepsilon - 1.$$

By (2.5) and a direct computation, we have

$$\Delta u_{2\varepsilon} \ge q_{2\varepsilon} b(x) u_{2\varepsilon}^{1+q_{2\varepsilon}^{-1}},\ x \in \Omega_\delta.$$

Since $q_{2\varepsilon} > 0$, let $U_{2\varepsilon}$ be the unique solution to the following problem

$$\Delta u = q_{2\varepsilon} b(x) u^{1+q_{2\varepsilon}^{-1}},\ u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = \infty. \tag{2.10}$$

It follows by Lemma 2.2 that

$$\lim_{d(x)\to 0} U_{2\varepsilon}(x)(K(d(x)))^{2q_{2\varepsilon}} = \left(\frac{2(2q_{2\varepsilon} + C_k))}{b_0}\right)^{q_{2\varepsilon}}.$$

Let $M > 1$ such that $U_{2\varepsilon}+M \geq u_{2\varepsilon}$ on Γ_δ. We see by Lemma 2.3 that $U_{2\varepsilon}+M \geq u_{2\varepsilon}$ on Ω_δ, i.e.,

$$v \geq (U_{2\varepsilon} + M)^{-1/q_{2\varepsilon}},\ x \in \Omega_\delta.$$

It follows that

$$\liminf_{d(x)\to 0} \frac{v(x)}{K^2(d(x))} \geq \frac{b_0}{2(2q_{2\varepsilon} + C_k)}.$$

Letting $\varepsilon \to 0$, we derive

$$\liminf_{d(x)\to 0} \frac{v(x)}{K^2(d(x))} \geq \tau_0. \tag{2.11}$$

Thus (1.15) holds.

Proof of Theorem 1.2. (i) $\Upsilon \in (1/2, 1)$. Let $\varepsilon \in (0, \min\{\ 1-\Upsilon,\ (2\Upsilon-1)/2\})$ and

$$w_{1\varepsilon} = v^{\gamma_{1\varepsilon}},\ \gamma_{1\varepsilon} = 1 - \Upsilon - \varepsilon.$$

By (2.5) and a direct computation, we see that

$$-\Delta w_{1\varepsilon} \geq b_0\gamma_{1\varepsilon} w_{1\varepsilon}^{-(\gamma_{1\varepsilon}{}^{-1}-1)},\ x \in \Omega_\delta,\ w_{1\varepsilon}|_{\partial\Omega} = 0,\ w_{1\varepsilon}|_{\Gamma_\delta} = v^{\gamma_{1\varepsilon}}, \tag{2.12}$$

By $\gamma_{1\varepsilon} \in (0, 1/2)$, $\gamma_{1\varepsilon}^{-1} > 2$, let $W_{1\varepsilon}$ be the unique solution to the following problem

$$-\Delta w = b_0\gamma_{1\varepsilon} w^{-(\gamma_{1\varepsilon}{}^{-1}-1)},\ w > 0,\ x \in \Omega,\ w|_{\partial\Omega} = 0,\ w|_{\Gamma_\delta} = v^{\gamma_{1\varepsilon}}. \tag{2.13}$$

It follows by Lemma 2.1 that

$$\lim_{d(x)\to 0} \frac{W_{1\varepsilon}(x)}{(d(x))^{2\gamma_{1\varepsilon}}} = \left(\frac{b_0}{2(1-2\gamma_{1\varepsilon})}\right)^{\gamma_{1\varepsilon}}. \tag{2.14}$$

By the comparison principle ([16], Theorem 2.3), we can show that

$$W_{1\varepsilon} \leq w_{1\varepsilon} \text{ in } \Omega_\delta,$$

i.e., $(W_{1\varepsilon})^{\frac{1}{\gamma_{1\varepsilon}}} \leq v$ in Ω_δ. Thus, we obtain that

$$\liminf_{d(x)\to 0} \frac{v(x)}{(d(x))^2} \geq \frac{b_0}{2(C_k - 2\gamma_{1\varepsilon})}.$$

It follows that for $\varepsilon \to 0$

$$\liminf_{d(x)\to 0} \frac{v(x)}{(d(x))^2} \geq \tau_0. \tag{2.15}$$

Subsequently, let

$$w_{2\varepsilon} = v^{\gamma_{2\varepsilon}},\ \gamma_{2\varepsilon} = 1 - \Upsilon + \varepsilon,$$

by (2.6) and a direct computation, we see that

$$-\Delta w_{2\varepsilon} \leq b_0\gamma_{2\varepsilon} w_{2\varepsilon}^{-(\gamma_{2\varepsilon}{}^{-1}-1)},\ x \in \Omega_\delta,\ w_{2\varepsilon}|_{\partial\Omega} = 0,\ w_{2\varepsilon}|_{\Gamma_\delta} = v^{\gamma_{2\varepsilon}}. \tag{2.16}$$

Since $\varepsilon < (2\Upsilon - 1)/2$, we see that $\gamma_{2\varepsilon}^{-1} > 2$. Let $W_{2\varepsilon}$ be the unique solution to the following problem

$$-\Delta w = b_0\gamma_{2\varepsilon} w^{-(\gamma_{2\varepsilon}{}^{-1}-1)},\ w > 0,\ x \in \Omega,\ w|_{\partial\Omega} = 0,\ w|_{\Gamma_\delta} = v^{\gamma_{2\varepsilon}}. \tag{2.17}$$

It follows by Lemma 2.1 that

$$\lim_{d(x)\to 0} \frac{W_{2\varepsilon}(x)}{(d(x))^{2\gamma_{2\varepsilon}}} = \left(\frac{b_0}{2(1-2\gamma_{2\varepsilon})}\right)^{\gamma_{2\varepsilon}}. \tag{2.18}$$

By the comparison principle ([16], Theorem 2.3), we can show that

$$W_{2\varepsilon}(x) \geq w_{2\varepsilon}(x),\ \forall x \in \Omega_\delta$$

and

$$\limsup_{d(x)\to 0} \frac{v(x)}{(d(x))^2} \leq \tau_0. \tag{2.19}$$

Thus (1.15) holds.

(ii) $\Upsilon = 1$. Let $\varepsilon \in (0, 1/2)$ and

$$w_{3\varepsilon} = v^{\varepsilon}.$$

By (2.6) and a direct computation,

$$-\Delta w_{3\varepsilon} \leq \varepsilon b_0 w_{3\varepsilon}^{-(\varepsilon^{-1}-1)},\ x \in \Omega_\delta,\ w_{3\varepsilon}|_{\partial\Omega} = 0,\ w_{3\varepsilon}|_{\Gamma_\delta} = v^{\varepsilon}. \tag{2.20}$$

Since $\varepsilon^{-1} > 2$, in the same proof of (2.19), we can obtain

$$\limsup_{d(x)\to 0} \frac{v(x)}{(d(x))^2} \leq \tau_0 = \frac{b_0}{2}. \tag{2.21}$$

Finally let

$$u_{3\varepsilon} = v^{-\varepsilon}.$$

By (2.5) and a direct computation, we have

$$\Delta u_{3\varepsilon} \geq \varepsilon b_0 u_{3\varepsilon}^{1+\varepsilon^{-1}},\ x \in \Omega_\delta.$$

In the same proof of (2.11), we can obtain

$$\liminf_{d(x)\to 0} \frac{v(x)}{(d(x))^2} \geq \tau_0 = \frac{b_0}{2}. \tag{2.22}$$

Thus (1.15) holds and the proof is finished.

3 Proof of Corollaries 1.1 and 1.2

In this section, applying Theorem 1.1, we prove Corollaries 1.1 and 1.2.

Lemma 3.1 (García Melián [13], Lemma 6 and Remark 1) *Under the assumptions on f in Corollary* 1.1*, for $\varepsilon, a \in (0,\infty)$ with $\varepsilon < a$*

$$\lim_{\varepsilon\to 0}\lim_{t\to 0}\frac{\phi_1((a-\varepsilon)t)}{\phi_1(at)} = 1.$$

Lemma 3.2 *Under the assumptions on g in Corollary* 1.2*, for $\varepsilon, a \in (0,\infty)$ with $\varepsilon < a$*

$$\lim_{\varepsilon\to 0}\lim_{t\to 0}\frac{\phi_2((a-\varepsilon)t)}{\phi_2(at)} = 1.$$

Proof. Since ϕ_2 is increasing on $[0,\infty)$, we see that

$$0 < \phi_2((a-\varepsilon)t) \le \phi_2(at),\ \frac{\phi_2((a-\varepsilon)t)}{\phi_2(at)} \le 1,\ \forall t > 0.$$

Moreover, by (g_1), $g(s)$ and $g(s)/s^q$ $(q \in (0,1))$ are dereasing for $s > 0$, we also see that

$$\frac{\phi_2'((a-\varepsilon)s)}{(\phi_2((a-\varepsilon)s))^q} = \frac{g(\phi_2(a-\varepsilon)s)}{(\phi_2((a-\varepsilon)s))^q} \ge \frac{g(\phi_2(as))}{(\phi_2(as))^q} = \frac{\phi_2'(as)}{(\phi_2(as))^q},\ \forall s > 0.$$

Integrate the inequality from 0 to t, we obtain that

$$\frac{\phi_2((a-\varepsilon)t)}{\phi_2(at)} \ge \left(\frac{a-\varepsilon}{a}\right)^{1/(1-q)},\ \forall t > 0.$$

Thus the result follows.

Proof of Corollary 1.1. By Theorems 1.1-1.2 and the definition of $v(u)$, we see that

$$\lim_{d(x)\to 0}\frac{\Phi_1(u(x))}{K^2(d(x))} = \tau_0.$$

Taking sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ small enough such that

$$(\tau_0-\varepsilon)K^2(d(x)) \le \Phi_1(u(x)) \le (\tau_0+\varepsilon)K^2(d(x)),\ x \in \Omega_\delta.$$

Since Φ_1 and ϕ_1 is decreasing, we see that

$$\frac{\phi_1((\tau_0+\varepsilon)K^2(d(x)))}{\phi_1(\tau_0K^2(d(x)))} \le \frac{u(x)}{\phi_1(\tau_0K^2(d(x)))} \le \frac{\phi_1((\tau_0-\varepsilon)K^2(d(x)))}{\phi_1(\tau_0K^2(d(x)))},\ x \in \Omega_\delta.$$

Thus the result follows by Lemma 3.1.

Proof of Corollary 1.2. By Theorem 1.2 and the definition of $v(w)$ we see that

$$\lim_{d(x)\to 0} \frac{\Phi_2(w(x))}{(d(x))^2} = \tau_0.$$

Taking sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ small enough such that

$$(\tau_0 - \varepsilon)(d(x))^2 < \Phi_2(w(x)) < (\tau_0 + \varepsilon)(d(x))^2,\ x \in \Omega_\delta.$$

Since Φ_2 and ϕ_2 is increasing, we see

$$\frac{\phi_2((\tau_0 - \varepsilon)(d(x))^2)}{\phi_2(\tau_0(d(x))^2)} < \frac{w(x)}{\phi_2(\tau_0(d(x))^2)} < \frac{\phi_2((\tau_0 + \varepsilon)(d(x))^2)}{\phi_2(\tau_0(d(x))^2)},\ x \in \Omega_\delta.$$

Thus the result follows by Lemma 3.2. and the proof is finished.

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