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# Global Existence Results for Near Triangular Nonlinear Parabolic Systems

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**Abstract**

We study the global existence and regularity of weak solutions to strongly coupled parabolic systems whose diffusion matrices are almost triangular.

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## 1 Introduction

We consider weak solutions to nonlinear parabolic systems of the form

$$u_t = \operatorname{div}(a(u)Du) + f(u, Du), \tag{1.1}$$

in a domain $Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$, with $\Omega$ being an open subset of $\mathbb{R}^n$, $n \geq 1$. The *vector valued* functions $u, f$ take values in $\mathbb{R}^m$, $m \geq 1$. $u_t, Du$ respectively denote the temporal and spatial derivatives of $u$. Here, $a(u) = (a_{ij}(u))$ is an $m \times m$ matrix satisfying, for some positive constants $\lambda, \Lambda$, the following ellipticity condition.

$$\lambda|\xi|^2 \leq a_{ij}(u)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall u \in \mathbb{R}^m. \tag{1.2}$$

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A weak solution $u$ to (1.1) is a function $u \in W_2^{1,0}(Q, \mathbb{R}^m)$ such that

$$\iint_Q [-u\phi_t + a(u)DuD\phi]\, dz = \iint_Q f(u, Du)\phi\, dz$$

for all $\phi \in C_c^1(Q, \mathbb{R}^m)$. Here, we write $dz = dxdt$.

We assume that $u$ takes prescribed value $g$ on the parabolic boundary $S_Q$ of $Q$, i.e. $S_Q = \Omega \cup \partial\Omega \times (0, T)$, for some Hölder continuous function $g$.

Global existence of solutions to systems like (1.1) has long been a fundamental problem in theory as well as in applications (for examples, see [15]) of partial differential equations. Had a good global existence theory been investigated and settled then important studies on dynamics of solutions, stability, etc. would have been carried out and developed. For the scalar case, i.e. $m = 1$, and under fairly general assumptions on $a, f$, it is well known that weak solutions with bounded $L^\infty$ norms exist globally (see [8, 14]). However, this is not the case for systems of more than one equation as counter examples were reported in [5, 6]: weak solutions to a strongly coupled systems, i.e. $a$ is not diagonal, could develop singularities in finite time without blowing up in their $L^\infty$ norm.

From Amann's paper [1], one could notice that global existence follows if we could control or establish a priori estimates for not only the $L^\infty$ but also Hölder norms of weak solutions. Thus, regularity theory for weak solutions to strongly coupled parabolic systems, which is a fundamental issue in the theory of partial differential equations, must be further investigated and developed. However, classical methods of Moser and De Giorgi (see [14]), while successfully applicable to the scalar case, are no longer working for systems. The most general result we know about the regularity of weak solutions to systems of equations is *partial* and comes from [3] which shows that they are Hölder continuous on a large open set whose complement, the singular set, has zero Lebesgue measure and its Hausdorff dimension can be estimated. Of course, this result does not give global existence results.

Few attempts have been made concerning *everywhere* regularity of bounded weak solutions. The results in [7, 16] give positive answers under very restrictive structure on the matrix $a$. The methods in [10, 11, 12, 13] work for systems of two equations with more general structures of $a$ but do not seem to be applicable for larger systems ($m > 2$).

Our main goal in this work is to establish global existence results for weak solutions to (1.1) under suitable assumptions on the matrix $a$ and $f$ but no restriction on the domain dimension $n$ and the size $m$ of the system. In particular, we will be interested in systems with $a$ being close to a triangular matrix.

The first global existence result concerns triangular systems with $f = 0$.

**Theorem 1.1** *Assume that $f \equiv 0$ and $a_{ij} = 0$ if $i > j$. Moreover, the entries $a_{ij}(u)$ are bounded for all $u \in \mathbb{R}^m$, $|a_{ij}(u)| \le \Lambda$ for all $u$ in $\mathbb{R}^m$. If $u$ is a weak solution to (1.1) with its boundary data $g$ in $C^\mu(S_Q)$ then there exist positive constants $\nu, C$, depending on $\lambda, \Lambda$ and $\|g\|_{C^\mu(S_Q)}$, such that $\|u\|_{C^\nu(Q)} \le C$. Moreover, if $a(u)$ is $C^1$ then $u$ is classical and exists globally.*

We then have a similar result for (1.1) with full matrix $a$ whose lower left entries are small, i.e. $a(u)$ is close to an upper triangular matrix.

**Theorem 1.2** *Assume that $f \equiv 0$ and $a_{ij}(u)$ are bounded for all $u \in \mathbb{R}^m$, $|a_{ij}(u)| \le \Lambda$ for all $u$ in $\mathbb{R}^m$. There exists $\varepsilon(\lambda, \Lambda) > 0$ such that if $|a_{ij}(u)| \le \varepsilon(\lambda, \Lambda)$ for $i > j$ then any weak solution $u$ to (1.1) with its boundary data $g$ in $C^{\mu}(S_Q)$ satisfies $\|u\|_{C^{\nu}(Q)} \le C$ for some positive constants $\nu, C$ depending on $\lambda, \Lambda$ and $\|g\|_{C^{\mu}(S_Q)}$. In addition, if $a(u)$ is $C^1$ then $u$ is classical and exists globally.*

The above result can be extended to the case of nonzero $f$ as in the following.

**Theorem 1.3** *Assume that there are positive continuous functions $C_0(|u|), C_1(|u|), C_2(|u|)$ such that the upper entries of $a$ and $f$ satisfy*

$$|a_{ij}(u)| \le C_0(|u|) \text{ if } j > i \text{ and } |f(u, Du)| \le C_1(|u|)|Du| + C_2(|u|) \tag{1.3}$$

*for all $u : Q \to \mathbb{R}^m$. There exists $\varepsilon(\lambda, \Lambda) > 0$ such that if $|a_{ij}(u)| \le \varepsilon(\lambda, \Lambda)$ for $i > j$ then any* **bounded** *weak solution $u$ to (1.1) with its boundary data $g$ in $C^{\mu}(S_Q)$ satisfies $\|u\|_{C^{\nu}(Q)} \le C$ for some positive constants $\nu, C$ depending on $\|u\|_{L^{\infty}(Q)}$ and $\|g\|_{C^{\mu}(S_Q)}$. Furthermore, if $a$ and $f$ are $C^1$ then $u$ is classical and exists globally.*

We should remark that the smallness of the lower entries in the above theorem is measured in terms of $\lambda, \Lambda$ and independently of $\|u\|_{L^{\infty}(Q)}$.

In Section 2, we gather our notations and some basic facts from calculus , which will be frequently used in the next sections. We then provide the main vehicles of our proof in Section 3 consisting of estimates for weak solutions of scalar equations. We conclude our paper with the proof of our main theorems in Section 4.

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## 2 Preliminaries

We gather some basic and technical lemmas in this section. Throughout this paper, for some $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$, $Q_R(z_0)$ denotes the parabolic cylinder centered at $z_0$ with radius $R$. That is, $Q_R(z_0) = B_R(x_0) \times [t_0 - R^2, t_0]$. We also abbreviate by $S_R(z_0)$, the parabolic boundary of $Q_R(z_0)$. If the center $z_0$ is understood, we simply write $Q_R, S_R$ for $Q_R(z_0)$ and $S_R(z_0)$ respectively.

First of all, we need the following Poincaré inequality. Let $\chi_R(x)$ be a function in $x$ with compact support in $B_{2R}(x_0)$ and $u$ be a vector valued function on $Q_{2R}(z_0)$. We define

$$u^{\chi}_{z_0,R}(t) = \int_{B_{2R}(x_0)} u(x,t)\chi(x)dx \Big/ \int_{B_{2R}(x_0)} \chi(x)dx.$$

**Lemma 2.1** *Assume that $u$ weakly satisfies in $Q_{2R}(z_0)$ the following system*

$$u_t = div(A) + f. \tag{2.1}$$

*Let $\chi(x)$ be a $C^1$ function on $B_{2R}(x_0)$ such that $|D\chi| \le c/R$. For any $s,t$ in the interval $I_{2R} = (-4R^2 + t_0, t_0)$ and $s < t$, we have*

$$|u^{\chi}_{z_0,R}(t) - u^{\chi}_{z_0,R}(s)|^2 \le C\frac{t-s}{R^{n+2}} \int_{Q_{2R}\cap supp\chi} |A|^2 dz + C\frac{t-s}{R^n} \int_{Q_{2R}\cap supp\chi} |f|^2 dz. \tag{2.2}$$

*Moreover,*

$$\iint_{Q_R} |u - u_R|^2 \, dz \le CR^2 \iint_{Q_{2R}} |Du|^2 \, dz + CR^2 \iint_{Q_{2R}} |A|^2 \, dz + CR^4 \iint_{Q_{2R}} |f|^2 \, dz. \tag{2.3}$$

*Proof.* . We test (2.1) with $(u^{\chi}_{z_0,R}(t) - u^{\chi}_{z_0,R}(s))\chi \cdot 1_{[s,t]}$ to get

$$\left(\int_{B_{2R}} u(x,t)\chi dx - \int_{B_{2R}} u(x,s)\chi dx\right)(u^{\chi}_{z_0,R}(t) - u^{\chi}_{z_0,R}(s)) \le - \int_s^t \int_{B_{2R}} (AD\chi + f\chi)(u^{\chi}_{z_0,R}(t) - u^{\chi}_{z_0,R}(s))dz.$$

The left-hand side can be estimated from below by $cR^n(u^{\chi}_{z_0,R}(t) - u^{\chi}_{z_0,R}(s))^2$, whereas the right-hand side can be bounded by

$$\frac{\varepsilon}{t-s}(u^{\chi}_{z_0,R}(t) - u^{\chi}_{z_0,R}(s))^2 \int_s^t \int_{B_{2R}} 1dz + C(\varepsilon)(t-s) \int_{Q_{2R}\cap supp\chi} [|A|^2|D\chi|^2 + f^2\chi^2]dz.$$

Choosing $\varepsilon$ sufficiently small, we prove (2.2). On the other hand, as $u_R$ minimizes the integrals of $|u-c|^2$ over $Q_R$ for any constant vector $c$, we have

$$\iint_{Q_R} |u - u_R|^2 \, dz \le \int_{-R^2}^0 \int_{B_{2R}(x_0)} |u - u^{\chi}_{z_0,R}(s)|^2 dxdt \le \int_{-R^2}^0 \int_{B_{2R}(x_0)} |u - u^{\chi}_{z_0,R}(t)|^2 dxdt + R^{n+2} \sup_{t,s\in I_{2R}} |u^{\chi}_{z_0,R}(t) - u^{\chi}_{z_0,R}(s)|^2.$$

Applying the Poincaré inequality in $x$ to the first quantity on the right and using (2.2), we obtain (2.3). ∎

We will also frequently use the following elementary lemma in [2].

**Lemma 2.2** *Suppose that $\phi : (0, R_0) \to \mathbb{R}_+$ is increasing and satisfies*

$$\phi(\tau R) \le A(\tau^{\alpha} + \varepsilon)\phi(R) + BR^{\beta} + C, \quad 0 < R < R_0 \tag{2.4}$$

*for any $\tau \in (0, \tau_0)$ with $\tau_0 < 1$ and some $\alpha, \beta$ such that $0 < \beta < \alpha$. Then, for any given $\nu > 0$ such that $\alpha - \nu \ge \beta$, there exists $K(\nu, \tau_0, A)$ such that for $\rho \in (0, \tau_0 R)$ and $\varepsilon$ sufficiently small (depending on $\nu, \tau_0, A$) we have*

$$\phi(\rho) \le K(\nu, \tau_0, A)\left[\left(\frac{\rho}{R}\right)^{\alpha-\nu} \phi(R) + B\rho^{\beta} + C\right], \quad 0 < R < R_0. \tag{2.5}$$

The proof of the above lemma is an elementary iteration argument if we choose $\tau$ and then $\varepsilon$ such that $A(\tau^{\alpha} + \varepsilon) \le \tau^{\alpha-\nu}$.

# 3 Technical lemmas

In this section, we collect estimates for weak solutions of scalar equations. The proof of our main results are based on these estimates.

Fix a cylinder $Q_{R_0}$. We consider the following scalar equation in $Q_{R_0}$.

$$u_t = \operatorname{div}(A(x,t)Du) + \operatorname{div}(F) \tag{3.1}$$

and $u = g$ on $S_{R_0}$.

For any $z \in Q_{R_0}$, we can find $\bar{z} \in S_{R_0}$ such that $d = d(z,\bar{z}) = dist(z, S_{R_0})$. Here, $d(z,\bar{z})$ denotes the parabolic distance between $z$ and $\bar{z}$ in $\mathbb{R}^{n+1}$. We denote $\widetilde{Q}_R(\bar{z}) = Q_R(\bar{z}) \cap Q_{R_0}$ and $\widetilde{S}_R(\bar{z})$ the parabolic boundary of $\widetilde{Q}_R$. We will always use this notation in the rest of this paper.

To begin, we quote the following well known result concerning weak solutions of (3.1) when $F$ is zero.

**Lemma 3.1** *Assume that u is a weak solution*

$$u_t = div(A(x,t)Du)$$

*in $Q_{R_0}$ for some function $A(x,t)$ satisfying $\lambda \le A(x,t) \le \Lambda$ for some positive constants $\lambda, \Lambda$. There exist positive constants $\alpha, C$ depending only on $\lambda, \Lambda$ such that for any $z$ in $Q_{R_0}$ we have the following estimate for any $\rho, R$ satisfying $0 < \rho < R < dist(z,\bar{z})/4$.*

$$\iint_{Q_\rho(z)} |u - u_\rho|^2 \, dz \le C\left(\frac{\rho}{R}\right)^{n+2+\alpha} \iint_{Q_R(z)} |u - u_R|^2 \, dz + C\rho^{n+2+\alpha}, \tag{3.2}$$

$$\iint_{Q_\rho(z)} |Du|^2 \, dz \le C\left(\frac{\rho}{R}\right)^{n+\alpha} \iint_{Q_R(z)} |Du|^2 \, dz + C\rho^{n+\alpha}. \tag{3.3}$$

*Moreover, if $\bar{z}$ belong to $S_{R_0}$ and the boundary data of u is Hölder continuous then*

$$\iint_{\widetilde{Q}_\rho(\bar{z})} |Du|^2 \, dz \le C\left(\frac{\rho}{R}\right)^{n+\alpha} \iint_{\widetilde{Q}_R(\bar{z})} |Du|^2 \, dz + C\rho^{n+\alpha} \tag{3.4}$$

*for $0 < \rho < R < R_0$. The constants $C, \alpha$ in (3.4) also depend on the Hölder norm of the boundary data.*

For the proof of this fact, we simply observe that $u$ belongs to De Giorgi classes (see [8]) and $u$ is locally Hölder continuous. We then follow the argument in [4][pages 231-234], to get the above estimates for the parabolic case.

We now go back to (3.1) with nonzero $F$. Concerning the data $A(u), F$ and $g$ of (3.1), we assume that

**A.1)** The diffusion coefficient $A(x,t)$ satisfies $\lambda \le A(x,t) \le \Lambda$ for some positive $\lambda, \Lambda$.

**A.2)** There exist constants $\nu \ge 0$ and $C$ such that the data $F$ satisfies

$$\iint_{Q_\rho(z) \cap Q_{R_0}} |F|^2 \, dz \le C\rho^{n+\nu} \text{ for all } z \in Q_{R_0}. \tag{3.5}$$

**A.3)** The boundary data $g$ is bounded and satisfies

$$g_t = \operatorname{div}(G) \text{ in } Q_{R_0} \text{ for some function } G \text{ such that } |G| \le C|Dg|, \tag{3.6}$$

and

$$\iint_{Q_\rho(z)\cap Q_{R_0}} |Dg|^2 \, dz \le C\rho^{n+\nu} \text{ for all } z \in Q_{R_0}. \tag{3.7}$$

First of all, by subtracting the equations of $u$ and $g$, we see that $U = u - g$ satisfies

$$U_t = \operatorname{div}(A(x,t)Du) + \operatorname{div}(F - G) \text{ and } U = 0 \text{ on } S_{R_0}. \tag{3.8}$$

If we test (3.8) with $U$ then it is easy to see that

$$\sup_t \int_{B_{R_0}} U^2 \, dx + \iint_{Q_{R_0}} |Du|^2 \, dz \le CR_0^{n+\nu}.$$

This implies

$$\iint_{Q_{R_0}} |u|^2 \, dz \le CR_0^{n+2+\nu} \text{ and } \iint_{Q_{R_0}} |Du|^2 \, dz \le CR_0^{n+\nu}. \tag{3.9}$$

In the proof below, we will use $C_1, C_2$, etc. to denote various positive and finite constants which depend on $\lambda, \Lambda$ and $C$ stated in Lemma 3.1 and the above conditions A.1)-A.3).

We then have the following estimates for weak solutions to (3.1).

**Lemma 3.2** *Let $u$ be a weak solution to (3.1). Assume A.1)-A.3). For any $z \in Q_{R_0}$ and $\bar{z} \in S_{R_0}$ such that $d = d(z,\bar{z}) = dist(z, S_{R_0})$, we have for $\nu_0 = \min\{\nu, \alpha\}$ that*

$$\iint_{Q_{2d}(\bar{z})} |Du|^2 \, dz \le C_1 d^{n+\nu_0}, \tag{3.10}$$

$$\iint_{Q_\rho(z)} |Du|^2 \, dz \le C_1 \rho^{n+\nu_0} \quad 0 < \rho < d, \tag{3.11}$$

$$\iint_{\widetilde{Q}_{2d}(\bar{z})} |u - u_{\widetilde{Q}_{2d}}|^2 \, dz \le C_1 d^{n+2+\nu_0}. \tag{3.12}$$

*Proof.* Let $W$ be the the weak solution to

$$\begin{cases} W_t = \operatorname{div}(A(x,t)DW) & \text{in } \widetilde{Q}_{2d}(\bar{z}), \\ W = U & \text{on } \widetilde{S}_{2d}(\bar{z}). \end{cases} \tag{3.13}$$

Since $W = 0$ on $Q_{2d}(\bar{z}) \cap S_{R_0}$, we see that $W$ is Hölder continuous near $\bar{z}$ so that (3.4) of Lemma 3.1 can be applied here to give

$$\iint_{\widetilde{Q}_\rho(\bar{z})} |DW|^2 \, dz \le C_2 \left(\frac{\rho}{R}\right)^{n+\alpha} \iint_{\widetilde{Q}_R(\bar{z})} |DW|^2 \, dz + C_2\rho^{n+\alpha}, \quad 0 < \rho < R < d.$$

Here, the constants $C_2, \alpha$ are positive and depending on $\lambda, \Lambda$ in A.1).

Testing the equation

$$(U - W)_t = \operatorname{div}[A(x,t)(DU - DW) + A(x,t)Dg + F - G]$$

on $\widetilde{Q}_{2d}(\bar{z})$ with $U - W$ ($U = u - g$) and using A.1) and Young's inequality we get

$$\iint_{\widetilde{Q}_{2d}(\bar{z})} |DU - DW|^2 \, dz \le C_3 \iint_{\widetilde{Q}_{2d}(\bar{z})} (|Dg|^2 + |F|^2 + |G|^2) \, dz \le C_4 d^{n+\nu}.$$

Hence,

$$\begin{aligned} \iint_{\widetilde{Q}_{2d}(\bar{z})} |DW|^2 \, dz &\le \iint_{\widetilde{Q}_{2d}(\bar{z})} |DU - DW|^2 \, dz + \iint_{\widetilde{Q}_{2d}(\bar{z})} |DU|^2 \, dz \\ &\le C_4 d^{n+\nu} + \iint_{\widetilde{Q}_{2d}(\bar{z})} |DU|^2 \, dz. \end{aligned}$$

We now have for $\rho \in (0, d)$ that

$$\begin{aligned} \iint_{\widetilde{Q}_{\rho}(\bar{z})} |DU|^2 \, dz &\le \iint_{\widetilde{Q}_{2d}(\bar{z})} |DU - DW|^2 \, dz + \iint_{\widetilde{Q}_{\rho}(\bar{z})} |DW|^2 \, dz \\ &\le C_2 \left(\frac{\rho}{d}\right)^{n+\alpha} \iint_{\widetilde{Q}_{d}(\bar{z})} |DW|^2 \, dz + C_5 d^{n+\alpha} \\ &\le C_2 \left(\frac{\rho}{d}\right)^{n+\alpha} \iint_{\widetilde{Q}_{2d}(\bar{z})} |DU|^2 \, dz + C_6 d^{n+\alpha}. \end{aligned}$$

Letting $\rho = \tau d$, we get

$$\iint_{\widetilde{Q}_{\rho}(\bar{z})} |DU|^2 \, dz \le C_2 \tau^{n+\alpha} \iint_{\widetilde{Q}_{2d}(\bar{z})} |DU|^2 \, dz + C_6 d^{n+\alpha} \quad \forall d > 0.$$

Let $\nu_0 = \min\{\alpha, \nu\}$. Using Lemma 2.2, we get

$$\iint_{\widetilde{Q}_{\rho}(\bar{z})} |DU|^2 \, dz \le C_7 \left(\frac{\rho}{R}\right)^{n+\nu_0} \iint_{\widetilde{Q}_{R}(\bar{z})} |DU|^2 \, dz + C_8 \rho^{n+\nu_0} \quad \text{for all } 0 < \rho < R.$$

As $U = u - g$, we use the assumption on $g$ and obtain a similar estimate for $Du$. Taking $R = R_0$ and using (3.9), we obtain the desired estimate (3.10).

The proof of (3.11) is similar. Indeed, for $R \in (0, d)$, we now consider $W$ satisfying

$$\begin{cases} W_t = \operatorname{div}(A(x,t)DW) & \text{in } Q_R(z), \\ W = U & \text{on } S_R(z). \end{cases} \tag{3.14}$$

Using (3.3) and following the above lines, we obtain

$$\iint_{Q_{\tau R}(z)} |DU|^2 \, dz \le C_9 \tau^{n+\alpha} \iint_{Q_R(z)} |DU|^2 \, dz + C_{10} R^{n+\nu} \quad \text{for all } 0 < R < d/4.$$

As before, this implies, for all $\rho$ such that $0 < \rho < d/4$,

$$\iint_{Q_{\rho}(z)} |DU|^2 \, dz \le C_{11} \left(\frac{\rho}{d}\right)^{n+\nu_0} \iint_{Q_{\frac{d}{4}}(z)} |DU|^2 \, dz + C_{12} \rho^{n+\nu_0}.$$

Because

$$\frac{1}{d^{n+\nu_0}}\iint_{Q_{\frac{d}{4}}(z)}|DU|^2\,dz \le C(n)\frac{1}{d^{n+\nu_0}}\iint_{Q_{2d}(\bar z)}|DU|^2\,dz,$$

we can use (3.10) in the previous estimate to obtain (3.11) if $0<\rho<d/4$. If $d/4\le\rho<d$ then (3.11) follows from the above and (3.10).

Finally, assuming that $\bar z$ belong to the base of $Q_{R_0}$ or $\widetilde{Q}_{2d}$, as $U$ is defined on $\widetilde{Q}_{2d}$ and is zero on the base, we can apply Lemma 2.1 with $A$ being $A(x,t)Du+F-G$ and the Poincaré inequality with respect to $x$ to get

$$\begin{aligned}\iint_{\widetilde{Q}_{2d}}|U|^2\,dz &\le \iint_{\widetilde{Q}_{2d}}|U-U^\chi(t)|^2\,dz + d^{n+2}\sup_t|U^\chi(t)|^2\\ &\le C_{13}d^2\iint_{\widetilde{Q}_{2d}}|DU|^2\,dz + C_{14}d^2\iint_{\widetilde{Q}_{2d}}(|Du|^2+|F|^2+|G|^2)\,dz\\ &\le C_{15}d^2\iint_{\widetilde{Q}_{2d}}|Du|^2\,dz + C_{16}d^{n+2+\nu}.\end{aligned}\tag{3.15}$$

If $\bar z$ belongs to the lateral boundary then we can simply use the Poincaré inequality with respect to $x$ to obtain the same result.

Using the minimizing property of $u_{\widetilde{Q}_{2d}}$ again, we now have

$$\begin{aligned}&\iint_{\widetilde{Q}_{2d}(\bar z)}|u-u_{\widetilde{Q}_{2d}}|^2\,dz\\ &\quad\le \iint_{\widetilde{Q}_{2d}(\bar z)}|u-g_{\widetilde{Q}_{2d}}|^2\,dz \le \iint_{\widetilde{Q}_{2d}(\bar z)}(|U|^2+|g-g_{\widetilde{Q}_{2d}}|^2)\,dz\\ &\quad\le C_{15}d^2\iint_{\widetilde{Q}_{2d}(\bar z)}|Du|^2\,dz + C_{16}d^{n+2+\nu} + \iint_{\widetilde{Q}_{2d}(\bar z)}|g-g_{\widetilde{Q}_{2d}}|^2\,dz.\end{aligned}$$

Using (3.10) and applying the Poincaré inequality to $g$ to estimate the last term, we prove (3.12) and complete our proof. ∎

An easy consequence of the above lemma is the following estimate for the mean oscillation of $u$.

**Lemma 3.3** *Assume A.1)-A.3). Let u be a weak solution to (3.1). Let $z\in Q_{R_0}$ and $\nu_0=\min\{\nu,\alpha\}$. For any $\rho>0$, we have the following estimate for the mean oscillation of u.*

$$\iint_{Q_\rho(z)\cap Q_{R_0}}|u-u_{Q_\rho(z)\cap Q_{R_0}}|^2\,dz \le C_{17}\rho^{n+2+\nu_0}.\tag{3.16}$$

*Proof.* If $Q_\rho(z)\subset Q_{R_0}$ then the Poincaré inequality (2.3) and (3.10) yield (3.16). Otherwise, we have $Q_\rho(z)\subset Q_{2\rho}(\bar z)$ so that the minimizing property of average and (3.12), with $d=\rho$, imply

$$\iint_{Q_\rho(z)\cap Q_{R_0}}|u-u_{Q_\rho(z)\cap Q_{R_0}}|^2\,dz \le \iint_{\widetilde{Q}_{2\rho}(\bar z)}|u-u_{\widetilde{Q}_{2\rho}}|^2\,dz \le C_1\rho^{n+2+\nu_0}.$$

The proof is complete. ∎

# 4 Proof of the main theorems

We present the proof of our main theorems in this section. Without loss of generality we can assume that the domain $Q$ in (1.1) is a cylinder $Q_{R_0}$ centered at the origin.

First of all, let $\widetilde{g}$ be the weak solution to the heat equation

$$\begin{cases} \widetilde{g}_t = \operatorname{div}(D\widetilde{g}) & \text{in } Q_{R_0}, \\ \widetilde{g} = g & \text{on } S_{R_0}. \end{cases}$$

Since $g$ is Hölder continuous on the boundary, $g \in C^{\mu}(S_{R_0})$, it is well known that $\widetilde{g}$ belongs to the Campanato space $\mathcal{L}^{2,n+2+\nu}(Q_{R_0})$ (see [2] for the definition) for some $\nu > 0$ and

$$\|\widetilde{g}\|_{\mathcal{L}^{2,n+2+\nu}(Q_{R_0})} \le C_g$$

for some constant $C_g$ depending only on the Hölder norm $\|g\|_{C^{\mu}(S_{R_0})}$ of $g$. Taking $G = D\widetilde{g}$ and using the Caccioppoli inequality and the above estimate, we see that the assumption (3.7) in A.3) of Section 3 is satisfied.

Again, in the proof below, we will use $C_1, C_2$, etc. to denote various positive and finite constants which depend on $\lambda, \Lambda$ and $C_g$ as stated in our theorems.

**Proof of Theorem 1.1.** By induction or working up-ward, we need only consider the following system of two equations in $Q_{R_0}$.

$$\begin{cases} u_t &= \operatorname{div}(P(u,v)Du) + \operatorname{div}(Q(u,v)Dv) \\ v_t &= \operatorname{div}(R(u,v)Dv) \end{cases} \tag{4.1}$$

and $(u, v) = g$ on $S_{R_0}$.

Applying (3.4) of Lemma 3.1 (see also (3.9)) to the equation of $v$, with $A(x,t) = R(u(x,t), v(x,t))$, we find $\nu_1 > 0$ such that

$$\iint_{Q_\rho} |Dv|^2 \, dz \le C(\|g\|_{C^{\mu}(S_{R_0})})\rho^{n+\nu_1} \text{ for all } Q_\rho \subset Q_{R_0}. \tag{4.2}$$

Letting $F = Q(u,v)Dv$ and using the fact that $Q(u,v)$ is bounded, we can apply Lemma 3.3 to find $\nu_0 > 0$ such that $\|u\|_{\mathcal{L}^{2,n+2+\nu_0}(Q_{R_0})}$ is bounded by a constant depending on $\|g\|_{C^{\mu}(S_{R_0})}$. By [4, Theorem 2.9], $u$ is Hölder continuous and its norm is uniformly bounded for all $t > 0$. Global existence then follows from Amann [1] if the matrix $a$ is $C^1$ continuous. ∎

We now consider the near triangular cases.

**Proof of Theorem 1.2.** We consider again the equation

$$U_t = \operatorname{div}(A(x,t)Du) + \operatorname{div}(F - G) \text{ and } U = 0 \text{ on } S_{R_0}$$

and compare $U$ against the weak solution $W$ of

$$\begin{cases} W_t = \operatorname{div}(A(x,t)DW) & \text{in } \widetilde{Q}_{2d}(\bar{z}), \\ W = U & \text{on } \widetilde{S}_{2d}(\bar{z}). \end{cases}$$

Arguing as in Lemma 3.2, we obtain for all $d > 0$, that

$$\iint_{\widetilde{Q}_\rho(\bar{z})} |DU|^2\, dz \le C_1 \left(\frac{\rho}{d}\right)^{n+\alpha} \iint_{\widetilde{Q}_{2d}(\bar{z})} |DU|^2\, dz + C_2 \iint_{\widetilde{Q}_{2d}(\bar{z})} |F|^2\, dz + C_2 d^{n+\nu}.$$

Going back to our systems, we consider the following system for $u = (u_1, \dots, u_m)$

$$\begin{cases} (u_k)_t = \operatorname{div}(\sum_{i=1}^m a_{ki}(u) Du_i) & \text{in } Q_{R_0}, \\ u = g & \text{on } S_{R_0}. \end{cases} \tag{4.3}$$

We define

$$\Phi_k(\rho) = \iint_{\widetilde{Q}_\rho(\bar{z})} |Du_k|^2\, dz, \quad 1 \le k \le m.$$

Using the assumption on $g$ and that the lower left entries are small, $|a_{ki}(u)| \le \varepsilon_0^k$ for $i < k$ and some small $\varepsilon_0$ to be determined later, we see that the above argument applies to the equation of $u_k$ with

$$A(x,t) = a_{kk}(u(x,t)), \quad F = \sum_{i \ne k} a_{ki}(u) Du_i.$$

We infer that

$$\Phi_k(\tau d) \le C_1 \tau^{n+\alpha} \Phi_k(d) + C_3 \sum_{k<i\le m} \Phi_i(d) + C_3 \varepsilon_0^k \sum_{1\le i<k} \Phi_i(d) + C_4 d^{n+\nu}. \tag{4.4}$$

We then multiply this inequality by ${\varepsilon_0}^{m-k}$ and sum the results to get

$$\phi(\tau d) \le C_5(\tau^{n+\alpha} + \varepsilon_0)\phi(d) + C_6 d^{n+\nu} \tag{4.5}$$

for $\phi(\rho) = \sum_{k=1}^m \varepsilon_0^{m-k} \Phi_k(\rho)$ and $d > 0$. Here, the constants $C_5, \alpha$ are positive and depending only on the upper and lower bounds $\lambda, \Lambda$ of the diagonal entries $a_{ii}$.

Thus, by Lemma 2.2, if $\varepsilon_0$ is sufficiently small, depending on $C_5$, then we have for any positive $\nu_0 < \min\{\alpha, \nu\}$

$$\phi(\rho) \le C_7 \left(\frac{\rho}{d}\right)^{n+\nu_0} \phi(d) + C_8 \rho^{n+\nu_0}.$$

This gives

$$\Phi_m(\rho) \le \phi(\rho) \le C_7 \left(\frac{\rho}{d}\right)^{n+\nu_0} \phi(d) + C_8 \rho^{n+\nu_0}. \tag{4.6}$$

By subtracting the systems for $u$ and the boundary data $g$ and testing the result with $u - g$, we easily see that $\phi(R_0)$ is bounded by $C_9 R_0^{n+\nu_0}$. Choosing $d = R_0$ in (4.6), we obtain $\Phi_m(\rho) \le C_{10} \rho^{n+\nu_0}$. Using this in the inequality for $\Phi_k$ in (4.4) for $k < m$ and repeating the argument, we obtain $\Phi_k(\rho) \le C_{11} \rho^{n+\nu_0}$ for all $k$. We have thus proved that

$$\iint_{\widetilde{Q}_\rho(\bar{z})} |Du_k|^2\, dz \le C_{11} \rho^{n+\nu_0} \quad 1 \le k \le m. \tag{4.7}$$

We now consider an arbitrary cylinder $Q_\rho(z)$ with $z \in Q_{R_0}$ and define

$$\widehat{\Phi}_k(\rho) = \iint_{Q_\rho(z)} |Du_k|^2\, dz \text{ and } \widehat{\phi}(\rho) = \sum_{k=1}^m \varepsilon_0^{m-k} \widehat{\Phi}_k(\rho), \quad 0 < \rho < d = dist(z, S_{R_0}).$$

By comparing $u_k$ with the weak solution $W$ of

$$\begin{cases} W_t = \operatorname{div}(A(x,t)DW) & \text{in } Q_R(z), \\ W = U & \text{on } S_R(z), \end{cases}$$

it is easy to see that the above argument can be repeated to give a similar estimate for $\widehat{\Phi}, \widehat{\phi}$ as in (4.6). Because $\widehat{\phi}(d) \le \phi(2d)$, we can use (4.7) to obtain

$$\iint_{Q_\rho(z)} |Du_k|^2 \, dz \le C_{12}\rho^{n+\nu_0}. \tag{4.8}$$

The estimates (4.7) and (4.8) allow us to use the Poincaré inequality as in Lemma 3.3 to conclude that the $\mathcal{L}^{n+2+\nu_0}(Q_{R_0})$ norms of $u_k$ are bounded. Hence $u_k$ are Hölder continuous with norms uniformly bounded by that of $g$. The global existence result then follows by Amann [1]. Our proof is complete. ∎

Finally, we have

**Proof of Theorem 1.3.** Consider a bounded weak solution $u$ to (1.1), $\|u\|_{L^\infty(Q)} \le M$, with $f$ satisfying (1.3). It is easy to see that our previous argument can be extended to this case. In fact, because $u$ is bounded, the upper entries of $a(u)$ are bounded and a simple use of Young's inequality in our previous argument yields some extra terms which are bounded by $d^{n+2}$, and thus by $d^{n+\nu}$, and can be repeated. As noted in the proof of Theorem 1.2, the smallness of $\varepsilon_0$ is measured in terms of $C_5, \alpha$ in (4.5) so that $\varepsilon_0$ depends only on the upper and lower bounds $\lambda, \Lambda$ of the diagonal entries $a_{ii}$. We then see that the Hölder norm of $u$ is bounded. The global existence then follows if $a, f$ are $C^1$ continuous. ∎

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